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On Analytic Nonlinear Input-output Systems: Expanded Global Convergence and System Interconnections

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ON ANALYTIC NONLINEAR INPUT-OUTPUT SYSTEMS: EXPANDED GLOBAL CONVERGENCE AND SYSTEM INTERCONNECTIONS

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ABSTRACT

ON ANALYTIC NONLINEAR INPUT-OUTPUT SYSTEMS:
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SYSTEM INTERCONNECTIONS

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Functional series representations of nonlinear systems first appeared in engineering in the early 1950's. One common representation of a nonlinear input-output system are Chen-Fliess series or Fliess operators. Such operators are described by functional series indexed by words over a noncommutative alphabet. They can be viewed as a noncommutative generalization of a Taylor series. A Fliess operator is said to be globally convergent when its radius of convergence is infinite, in other words, when there is no a priori upper bound on both the L_1-norm of an admissible input and the length of time over which the corresponding output is well defined. If such bounds are required to ensure convergence, then the Fliess operator is said to be locally convergent with a finite radius of convergence. However, in the literature, a Fliess operator is classified as locally convergent or globally convergent based solely on the growth rate of the coefficients in its generating series. The existing growth rate bounds provide sufficient conditions for global convergence which are very conservative. Therefore, the first main goal of this dissertation is to develop a more exact relationship between the coefficient growth rate and the nature of convergence of the corresponding Fliess operator. This first goal is accomplished by introducing a new topological space of formal power series which renders a Fréchet space instead of

the more commonly used ultrametric space. Then, a direct relationship is developed between the nature of convergence of a Fliess operator and its generating series. The second main goal of this dissertation is to show that the global convergence of Fliess operators is preserved under the nonrecursive interconnections, namely the parallel sum and product connections and the cascade connection. This fact had only been understood previously in a narrow sense based on the more conservative tests for global convergence.

To my beloved parents, Leonor & Jorge,

my dear husband, Ivan

and my little daughter, Isabella.

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"Un cronopio tiene un hijo, y enseguida lo invade la maravilla y está seguro de que su hijo es el pararrayos de la hermosura y que por su venas corre la química completa."

– Julio Cortázar,

Historias de cronopios y de famas[0]

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[0]A cronopio has a son, and immediately invades him the wonder, and he is sure that his son is the lightning rod of beauty and that through his veins runs the complete chemistry.

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NOMENCLATURE

$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$C[t_0, t_1]$	Set of all continuous functions over the time interval $[t_0, t_1]$
X^*	Set of all words formed under the alphabet X
$\mathbb{R}\langle X\rangle$	Set of all polynomials generated by the alphabet X
$\mathbb{R}^\ell\langle X\rangle$	Set of all ℓ-dimensional vector–valued polynomials generated by the alphabet X
$\mathbb{R}\langle\langle X\rangle\rangle$	Set of all formal power series generated by the alphabet X
$\mathbb{R}^\ell\langle\langle X\rangle\rangle$	Set of all ℓ-dimensional vector–valued formal power series generated by the alphabet X
γ_c	The minimum of the Gevrey orders associated with the series c
$\mathbb{R}_\gamma\langle\langle X\rangle\rangle$	Set of all formal power series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ where $\gamma_c = \gamma$
$\mathbb{R}_{LC}\langle\langle X\rangle\rangle$	Set of all formal power series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ where $0 \leq \gamma_c \leq 1$
$\mathbb{R}_{GC}\langle\langle X\rangle\rangle$	Set of all formal power series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ where $0 \leq \gamma_c < 1$
$\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$	Closure of the space $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ in the semi-norm topology
$\partial\mathbb{R}_{GC}\langle\langle X\rangle\rangle$	Border of the space $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ in the semi-norm topology
$\mathscr{C}$	Catenation product
$\sqcup\!\sqcup$	Shuffle product
$\circ$	Composition product
E_η	Iterated integral associated with the word η
F_c	Fliess operator associated with the formal power series c

$S_{\infty,e}$	Space of all series whose Fliess operators are globally or localy convergent
S_{∞}	Space of all series whose Fliess operators are globally convergent

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CHAPTER 1

INTRODUCTION

"Lo verdaderamente nuevo da miedo o maravilla."

– Julio Cortázar,
Historias de cronopios y de famas[1]

This chapter provides the background and motivation for the dissertation followed by the main goals of the research. Finally, the basic outline of this dissertation is presented.

1.1 BACKGROUND AND MOTIVATION

Functional series representations of nonlinear systems first appeared in engineering in the early 1950's. The most relevant functional series are the ones of Volterra [38, 42], Wiener [38, 44], and Fliess [14, 15]. Fliess, motived by Chen's work on path integrals [3, 4], introduced an algebraic description of functional expansions now known as Chen-Fliess series or Fliess operators [13–15, 27, 43]. These operators form a very general class of nonlinear input-output systems and can be viewed as a noncommutative generalization of a Taylor series. Their algebraic nature is especially well suited for describing system interconnections [10, 11, 22], feedback invariants [18, 20, 24], and solving system inversion problems in a nonlinear setting.

1.1.1 Fliess operators and their convergence

Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the set comprised of all words over X including the empty word, $\emptyset$, under the catenation product. A formal power series c is a mapping $c : X^* \to \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The value of c at $\eta \in X^*$ is denoted by (c, η), and is called the *coefficient* of η in c. Specifically, one can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, as described next. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq$

[1] What is truly new gives fear or wonder.

$i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L_{\mathfrak{p}}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B_{\mathfrak{p}}^m(R_u)[t_0, t_1] := \{u \in L_{\mathfrak{p}}^m[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R_u\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau, \tag{1.1.1}$$

$x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \tag{1.1.2}$$

[14, 15]. Properties of Fliess operators, such as continuity, local convergence, global convergence, differentiability, and analyticity have been extensively studied [14, 15, 25,41,43]. In the classical literature, the word "convergence" of a formal power series describes the growth rate of the coefficients of the generating series. For example, if there exist real numbers $K, M > 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^*, \tag{1.1.3}$$

where $|\eta|$ denotes the length of the word η and $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$, then c is said to be *locally convergent*, and the set of all locally convergent formal power series is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$. This result implies that F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R_u)[t_0,\, t_0{+}T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0,\, t_0{+}T]$ for sufficiently small $R_u, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p}{+}1/\mathfrak{q} = 1$ [25]. The least upper bound on $R := \max\{R_u, T\}$, say $\rho(F_c)$, is called the *radius of convergence* of the operator. It was shown in [7, 9] that

$$0 < \frac{1}{M(m+1)} \leq \rho(F_c).$$

A Fliess operator is said to be *locally convergent* when its radius of convergence is finite or infinite (i.e., $\rho(F_c) \leq \infty$). On the other hand, a Fliess operator is said to be *globally convergent* when its radius of convergence is infinite (i.e., $\rho(F_c) = \infty$), in other words, when there is no a priori upper bound on both the L_1-norm of an admissible input and the length of time over which the corresponding output is well defined. Finally, if such bounds are required to ensure convergence, then the Fliess operator is said to be *locally convergent with finite radius of convergence* (i.e., $\rho(F_c) < \infty$). It is important to observe that the definitions of globally convergent and locally convergent with finite radius of convergence used to describe a Fliess operator are mutually exclusive. Figure 1 shows typical operator outputs for these two cases, well defined for all time (left) and well defined only over a finite interval of time (right).

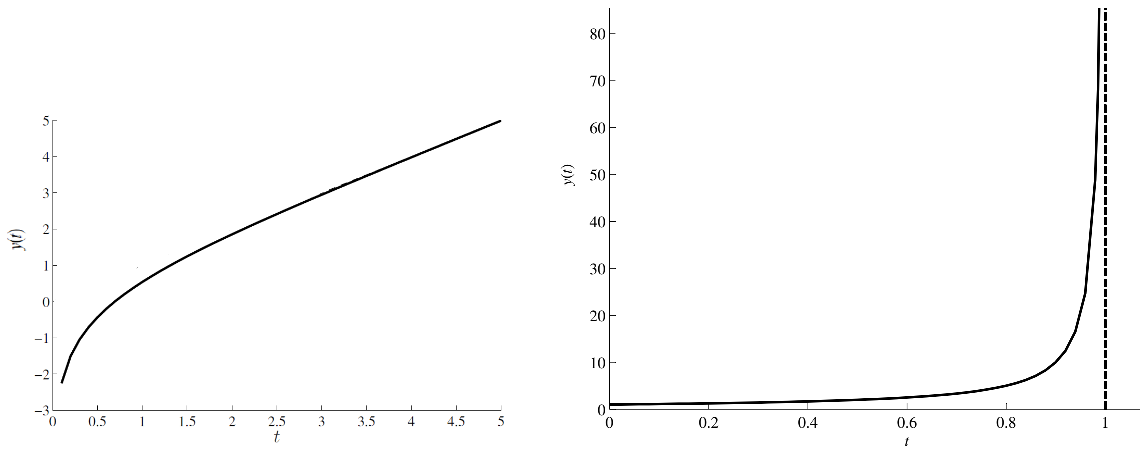

Fig. 1: Typical outputs of two Fliess operators: well defined for all time $t > 0$ and well defined only over a finite interval of time.

Note that when a series c is locally convergent then its corresponding Fliess operator F_c is at least locally convergent. The following example gives a particular generating series and the nature of its Fliess operator.

Example 1.1.1. Consider the locally convergent series $c = \sum_{k=0}^{\infty} k!\, x_1^k$. Observe

$$F_c[u](t) = \sum_{k=0}^{\infty} k! E_{x_1^k}[u](t) = \sum_{k=0}^{\infty} E_{x_1}^k[u](t) = \frac{1}{1 - E_{x_1}[u](t)}.$$

Setting $u = 1$ gives $F_c[1](t) = 1/(1-t)$, which has a finite escape time at $t = 1$. Thus, F_c is locally convergent with finite radius of convergence. □

When c satisfies the stronger condition

$$|(c, \eta)| \le K M^{|\eta|}, \quad \forall \eta \in X^*, \tag{1.1.4}$$

the series is said to be *globally convergent*. The set of all such series is denoted by

$\mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$. It was shown in [25] for this case that the series (1.1.2) defines an operator from the extended space $L^m_{\mathfrak{p},e}(t_0)$ into $C[t_0,\infty)$, where

$$L^m_{\mathfrak{p},e}(t_0) := \{u : [t_0,\infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L^m_{\mathfrak{p}}[t_0,t_1], \forall t_1 \in (t_0,\infty)\},$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0,t_1]$. Hence, its corresponding Fliess operator F_c is globally convergent. The following is a global version of Example 1.1.1.

Example 1.1.2. Consider the globally convergent series $c = \sum_{k=0}^{\infty} x_1^k$. Observe

$$F_c[u](t) = \sum_{k=0}^{\infty} E_{x_1^k}[u](t) = \sum_{k=0}^{\infty} \frac{1}{k!} E^k_{x_1}[u](t) = \exp\left(E_{x_1}[u](t)\right).$$

It is clear that for any input and length of time, the output above is always well defined (In particular, $u = 1$ gives $F_c[1](t) = e^t$, which is an entire function). Therefore, F_c is globally convergent. □

It is important to observe that the definitions of local and global convergence used to describe a generating series are *not* mutually exclusive, in fact, every globally convergent series is also a locally convergent series since

$$|(c,\eta)| \le KM^{|\eta|} \le KM^{|\eta|}\,|\eta|!, \quad \forall \eta \in X^*.$$

Therefore, the series $c = \sum_{k=0}^{\infty} x_1^k$ in Example 1.1.2 is also locally convergent. This example illustrates some of the ambiguity between the growth rate of the coefficients of a generating series and the convergence behavior of its corresponding Fliess operator. In particular, the condition (1.1.4) was shown in [25] to be a sufficient condition for global convergence of a Fliess operator. At present, a necessary condition is not given in the literature. Similarly, if a Fliess operator is locally convergent with finite radius of convergence, a precise claim about the growth rate of its corresponding generating series is not immediately evident. So the first goal of this dissertation is to develop an exact relationship between the growth rate of the coefficients of a generating series and the nature of the convergence of the corresponding Fliess operator. In particular, it will be shown that (1.1.4) is very conservative as a test for global Fliess operator convergence.

1.1.2 Interconnection of Fliess operators

Given two input-output systems F_c and F_d, there are three fundamental *non-recursive* system interconnections normally encountered in engineering applications: the parallel sum, the parallel product, and the cascade connection. For any admissible input, u, the parallel sum and parallel product connections as shown in Figures 2 and 3 correspond to

$$y = F_c[u] + F_d[u]$$

and

$$y = F_c[u]F_d[u],$$

respectively [14]. When Fliess operators are interconnected in a cascade fashion as

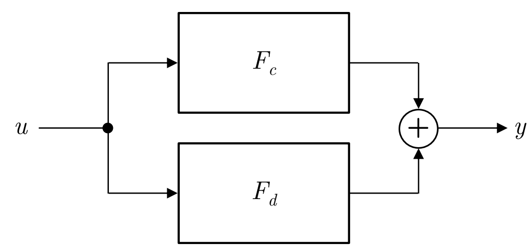

Fig. 2: Parallel sum connection of two Fliess operators.

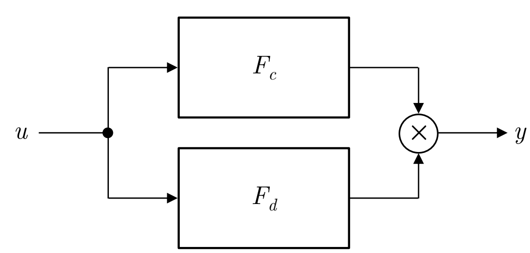

Fig. 3: Parallel product connection of two Fliess operators.

shown in Figure 4, the composite system is described by

$$y = F_c[F_d[u]]$$

[10, 11]. It is known that the nonrecursive system interconnections of two Fliess operators with locally convergent generating series are always *well–posed* and yield

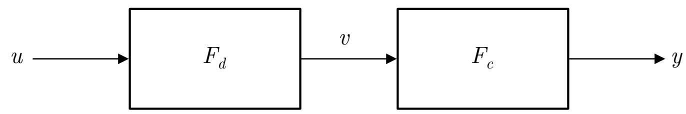

Fig. 4: Cascade connection of two Fliess operators.

another Fliess operator in this class [5, 22]. In addition, two Fliess operators with globally convergent generating series interconnected in a parallel sum or parallel product manner always yield another Fliess operator in this class [41]. However, in general this claim does *not* hold for the cascade connection [8, 10, 11]. The following discussion serves as a motivating example related to the second main goal of this dissertation.

Example 1.1.3. Consider the globally convergent series used in Example 1.1.2

$$c = \sum_{k=0}^{\infty} x_1^k.$$

It is easy to verify that $y = F_c[u]$ has a bilinear state space realization [8]

$$\begin{aligned} \dot{z} &= zu, \quad z(0) = 1 \\ y &= z. \end{aligned}$$

Cascading two such realizations gives the state space system

$$\begin{aligned} \dot{z}_1 &= z_1 z_2, \quad z_1(0) = 1 \\ \dot{z}_2 &= z_2 u, \quad z_2(0) = 1 \\ y &= z_1. \end{aligned}$$

The resulting input-output system is therefore

$$y(t) = F_c[F_c[u]](t) = F_{c \circ c}[u](t),$$

where $c \circ c$ represents the generating series of the composite system. It was shown explicitly in [10, 11] that $c \circ c$ has a subseries of coefficients growing faster than the global rate given by (1.1.4). However, since $y(t)$ can be written in terms of a composition of the functional $F_c[u](t) = \exp\left(\int_0^t u(\tau)\,d\tau\right)$, it can be shown that if the input is well defined and absolutely integrable over *any* finite time interval, then the output of the composite system is also well defined over the same interval [41]. Thus, the Fliess operator $y(t)$ is globally convergent. For example, if the input is set to be zero, the output is a double exponential function as shown in Figure 5, which is real analytic.

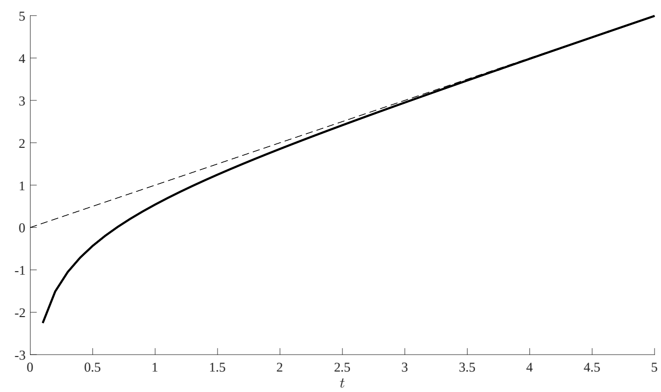

Fig. 5: Zero-input response of the cascade system $F_{c\circ c}$ on a double logarithmic scale (solid line) and the function t (dotted line).

Therefore $F_{c\circ c}$ is a specific example of a Fliess operator which is globally convergent, but whose generating series does not satisfy (1.1.4).

□

In light of the example above, the second goal of this dissertation is to precisely describe when the nonrecursive interconnection of two globally convergent Fliess operators is again globally convergent. In particular, the question of whether this phenomenon could be predicted from the generating series alone is answered. It should be stressed that this question is broader than the one addressed in [41], where global convergence of a Fliess operator was preserved when (1.1.4) was satisfied. Here the class of globally convergent operators is significantly enlarged. It should also be noted in closing that this second goal is not relevant for feedback (recursive) interconnections as global convergence is known to not be preserved in general [21, Example 3].

1.2 PROBLEM STATEMENT

The main goals of this dissertation are listed below.

i. Give an exact relationship between the growth rate of the coefficients of a generating series and the nature of the convergence of its corresponding Fliess operator.

ii. Describe precisely when the nonrecursive interconnection of two globally convergent Fliess operators is again globally convergent.

1.3 DISSERTATION OUTLINE

This dissertation is organized as follows. In Chapter 2, the necessary mathematical tools needed throughout the dissertation are presented. First, some elements from the fundamental theory of formal power series and the nonrecursive interconnections of two Fliess operators are described. Then, a section on topology is introduced, some basic notation, definitions, and properties related to topological vector spaces and in particular, locally convex topological vector spaces are summarized.

In Chapter 3, the first main goal of the dissertation is addressed. First, new and stronger sufficient conditions on the growth rate of the coefficients of the generating series are given in order to ensure global convergence of its corresponding Fliess operator. Then, a new space of formal power series is introduced. It requires one to view the set of formal power series as a locally convex topological vector space with a family of semi-norms instead of the more common ultrametric space setting. Subsequently, an example is introduced in order to illustrate how to classify a generating series in this new space of formal power series. Finally, this new space is proved to be a Fréchet space, and a straightforward relationship between the nature of convergence of a Fliess operator and its generating series is given.

In Chapter 4, the second main goal of the dissertation is answered. First, the two types of parallel interconnections, sum and product, are presented, and it is proved that the new space of formal power series is closed under addition and shuffle product. Finally, the cascade interconnection is addressed, and it is shown that the new space of formal power series is closed under the composition product. These results allow one to show that all nonrecursive interconnection of two globally convergent Fliess operators preserve the global convergence property.

In Chapter 5, the main conclusions of the dissertation are summarized, and future research topics are given.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

"Las cosas invisibles necesitan encarnarse, las ideas caen a la tierra como palomas muertas."

– Julio Cortázar,
Historias de cronopios y de famas[2]

This chapter presents all the mathematical tools needed throughout the dissertation. First, some elements from the theory of formal power series are presented. Then, the nonrecursive interconnections of two Fliess operators are described. Finally, a section on topology is provided in order to make possible the development of an exact relationship between the growth rate of the coefficients of a generating series and the nature of convergence of its corresponding Fliess operator in Chapter 3.

2.1 FORMAL POWER SERIES

The generating series of Fliess operators are characterized by noncommutative formal power series. Thus, this section presents some basic notation and definitions related to them. First, the definition of formal languages and formal power series are introduced. Then two products of formal power series are defined: the shuffle and composition products, along with their basic properties. These properties will be used in Chapter 3 and Chapter 4. The majority of the presentation is based on [2, 19].

2.1.1 Formal languages and formal power series

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. Let $|\eta|_{x_i}$ denote the number of times the letter $x_i \in X$ appears in the word η. The set of all words of length k is denoted by X^k. The set of all words

[2]Invisible things need to be incarnated, ideas fall to the ground like dead doves.

including the empty word, $\emptyset$, is designated by X^*. A *language* is any subset of X^*. The catenation product is defined as follows.

Definition 2.1.1. The **catenation product** is the associative mapping

$$\begin{aligned} \mathscr{C} \ : \ & X^* \times X^* \to X^* \\ & (\eta, \xi) \mapsto \eta\xi. \end{aligned}$$

Clearly, for any $\eta, \xi, \nu \in X^*$ it holds that

$$(\eta\xi)\nu = \eta(\xi\nu).$$

Also, the empty word $\emptyset$ is the identity element for $\mathscr{C}$ since

$$\eta\emptyset = \emptyset\eta = \eta, \ \forall \eta \in X^*.$$

The triple $(X^*, \mathscr{C}, \emptyset)$ is a *free monoid* of X.

Given a finite $\ell \in \mathbb{N}$, a *formal power series* in X is any mapping of the form

$$c : X^* \to \mathbb{R}^\ell.$$

The value of c for a specific word $\eta \in X^*$ is denoted by (c, η) and is called the *coefficient* of η in c. Typically, c is represented as the formal sum

$$c = \sum_{\eta \in X^*} (c, \eta)\eta.$$

The coefficient $(c, \emptyset)$ is referred to as the *constant term*. When the constant term is zero, c is called *proper*. The *support* of c is the language

$$\text{supp}(c) = \{\eta : (c, \eta) \neq 0\}.$$

The *order* of a series c is defined as

$$\text{ord}(c) = \begin{cases} \min\{|\eta| : \eta \in \text{supp}(c)\} & : \ c \neq 0, \\ \infty & : \ c = 0. \end{cases}$$

A series $\hat{c}$ is said to be a *subseries* of c if $\text{supp}(\hat{c}) \subseteq \text{supp}(c)$ and $(\hat{c}, \eta) = (c, \eta), \ \forall \eta \in$

supp($\hat{c}$). The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. In addition, the set of all series with finite support is denoted by $\mathbb{R}^\ell\langle X\rangle$. Its elements are called *polynomials*. The sets $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $\mathbb{R}^\ell\langle X\rangle$ have considerable algebraic structure, each admits a vector space structure over $\mathbb{R}$. If $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, their *sum* is defined by

$$c + d = \sum_{\eta\in X^*} (c + d, \eta)\eta = \sum_{\eta\in X^*} ((c, \eta) + (d, \eta))\eta,$$

and their *scalar product* is given by

$$\alpha c = \sum_{\eta\in X^*} (\alpha c, \eta)\eta = \sum_{\eta\in X^*} \alpha(c, \eta)\eta, \quad \forall\alpha \in \mathbb{R}.$$

2.1.2 Generating series for parallel connections

The following theorem relates the sum of the generating series to the parallel sum connection of the corresponding Fliess operators.

Theorem 2.1.1. *[14] Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the parallel sum connection $F_c + F_d$ shown in Figure 2 has the generating series $c + d$. That is,*

$$F_c + F_d = F_{c+d}.$$

The definition of shuffle product is given below [2,14,31,35]. This product is used to describe the parallel product connection of Fliess operators.

Definition 2.1.2. The **shuffle product** of two words $\eta, \xi \in X^*$ is the $\mathbb{R}$-bilinear mapping inductively defined as

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$.

The next example shows the key role of the shuffle product when working with the product of iterated integrals.

Example 2.1.1. Let u be a piecewise continuous, real-valued function defined over the finite interval $[t_0, t_1]$. The iterated integral E_η was defined inductively in (1.1.1) as

$$E_\eta[u](t, t_0) = E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau$$

with $E_\emptyset[u] = 1$ and $u_0 = 1$, since each non empty word $\eta \in X^*$ can be written as $\eta = x_i\bar{\eta}$ where $x_i \in X$ and $\bar{\eta} \in X^*$. Observe that for $x_i, x_j \in X$ the integration by parts formula gives

$$\begin{aligned}
E_{x_i}[u]E_{x_j}[u] &= \int u_i(\tau)d\tau \int u_j(\tau)d\tau \\
&= \int \left(u_i(\tau) \int u_j(\tau_1)d\tau_1\right) d\tau + \int \left(u_j(\tau) \int u_i(\tau_1)d\tau_1\right) d\tau \\
&= \int u_i(\tau)E_{x_j}[u]d\tau + \int u_j(\tau)E_{x_i}[u]d\tau \\
&= E_{x_ix_j}[u] + E_{x_jx_i}[u] \\
&= E_{x_ix_j+x_jx_i}[u] = E_{x_i \sqcup\!\sqcup\, x_j}[u].
\end{aligned}$$

As a consequence, any product of two iterated integrals is a linear combination of iterated integrals, and it can be expressed in terms of the shuffle product. □

The shuffle product definition is linearly extended to any two series $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ by

$$c \sqcup\!\sqcup\, d = \sum_{\eta,\xi \in X^*} (c,\eta)(d,\xi)\eta \sqcup\!\sqcup\, \xi.$$

An equivalent expression is

$$c \sqcup\!\sqcup\, d = \sum_{\nu \in X^*} (c \sqcup\!\sqcup\, d, \nu)\nu,$$

where

$$(c \sqcup\!\sqcup\, d, \nu) = \sum_{\eta,\xi \in X^*} (c,\eta)(d,\xi)(\eta \sqcup\!\sqcup\, \xi, \nu).$$

Observe that, the shuffle product is always well defined since the product is locally finite [2]. Also, $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ forms an associative $\mathbb{R}$-algebra under the catenation product and an associative and commutative $\mathbb{R}$-algebra under the shuffle product. The next lemma assigns upper bounds to the product of multiple iterated integrals, its proof requires the use of shuffle products. It will be essential in Chapter 3.

Lemma 2.1.1. *[7, 9] Let $X = \{x_0, x_1, \ldots x_m\}$. For any $u \in L_1^m[0,T]$ and $\eta \in X^*$*

$$|E_\eta[u](t)| \le E_\eta[\bar{u}](t),\ 0 \le t \le T, \tag{2.1.1}$$

where $\bar{u} \in L_1^m[0,T]$ has components $\bar{u}_j := |u_j|$, $j = 1, 2, \ldots m$. Furthermore, for any *integers* $r_j \geq 0$ *it follows that*

$$\left|\prod_{j=0}^{m} E_{x_j{}^{r_j}}[u](t)\right| \leq \prod_{j=0}^{m} \frac{U_j^{r_j}(t)}{r_j!},\ 0 \leq t \leq T,$$

where $U_j(t) := \int_0^t |u_j(s)|ds$. *In particular, if on* $[0,T]$ *it is assumed that* $\max\{\|u\|_1, T\} \leq R$ *then*

$$\prod_{j=0}^{m} E_{x_j{}^{r_j}}[\bar{u}](t) \leq \frac{R^k}{\prod_{j=0}^{m} r_j!},\quad 0 \leq t \leq T, \tag{2.1.2}$$

where $k = \sum_j r_j$.

The following theorem relates the shuffle product of the generating series to the parallel product connection of the corresponding Fliess operators.

Theorem 2.1.2. *[14] Given Fliess operators* F_c *and* F_d*, where* $c, d \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$*, the parallel product connection* F_cF_d *shown in Figure* 3 *has the generating series* $c ⧢ d$*. That is,*

$$F_cF_d = F_{c ⧢ d}.$$

2.1.3 Composition product and the cascade connection

The composition product can be traced back to Ferfera's work in [10,11]. However, the interpretation utilized here first appeared in [22] . This product is used to describe the cascade connection of Fliess operators.

Definition 2.1.3. The **composition product** of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(1). \tag{2.1.3}$$

The mapping ψ_d is the algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to the set of vector space endomorphism on $\mathbb{R}\langle\langle X\rangle\rangle$, $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with

$$\psi_d(x_i)(e) = x_0(d_i ⧢ e),$$

$i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := 1$). $\psi_d(\emptyset)$ is defined to be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$.

This composition product is associative and $\mathbb{R}$-linear in its left argument. A commonly used metric on $\mathbb{R}\langle\langle X\rangle\rangle$ is the ultrametric metric

$$\text{dist} : (c,d) \mapsto \sigma^{\text{ord}(c-d)},$$

where σ is any real number such that $0 < \sigma < 1$ [2]. $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ forms a complete ultrametric space under the mapping dist. It is important to note that the mapping ψ_d is continuous in the ultrametric sense. In Chapter 4, the following subset of $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ described will be useful.

Definition 2.1.4. [12, 14] A series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is said to be **exchangeable** if for arbitrary $\eta, \xi \in X^*$

$$|\eta|_{x_i} = |\xi|_{x_i}\,,\ i = 0, 1, \cdots \ell \implies (c,\eta) = (c,\xi).$$

The next lemma will be essential when analyzing the cascade interconnection of two globally convergent Fliess operators in Chapter 4.

Lemma 2.1.2. *[23] If $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is an exchangeable series, and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$ is arbitrary, then the composition product can be written in the form*

$$c \circ d = \sum_{k=0}^{\infty} \sum_{\substack{r_0, r_1, \ldots, r_m \geq 0 \\ r_0 + r_1 + \cdots + r_m = k}} (c, x_0^{r_0} \cdots x_m^{r_m}) \psi_d(x_0^{r_0})(1) \sqcup\!\sqcup \cdots \sqcup\!\sqcup \psi_d(x_m^{r_m})(1).$$

The following theorem relates the composition product of the generating series to the cascade connection of the corresponding Fliess operators.

Theorem 2.1.3. *Given Fliess operators F_c and F_d, where $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$, the cascade connection $F_c \circ F_d$ shown in Figure 4 has the generating series $c \circ d$. That is,*

$$F_c \circ F_d = F_{c \circ d},$$

where the composition product of c and d is given by (2.1.3).

2.2 TOPOLOGICAL FRAMEWORK

The first main goal of this dissertation is to development an exact relationship between the growth rate of the coefficients of a generating series and the nature of convergence of its corresponding Fliess operator. In order to achieve this goal a new topological space of formal power series is introduced. It requires one to view the set of formal power series as a Fréchet space instead of the more common ultrametric space setting given in Section 2.1.3. Thus, this section presents some basic notation, definitions, and properties related to topological vector spaces and in particular, locally convex topological vector spaces. First, a few preliminaries concerning real analysis are summarized in order to make this dissertation more self-contained. Then, topological vector spaces and locally convex topological vector spaces are introduced along with their Cauchy criterion and completeness properties. These will be used throughout Chapter 3 and Chapter 4. The majority of the concepts presented in this section have been taken from [16, 28, 33, 34, 36, 37].

2.2.1 Preliminaries

The concepts of metric, normed, and semi-normed spaces provide the foundation concerning real analysis for this whole section. The aim is to introduce these spaces and give some specific examples, but their theory is too extensive to describe here in any detail. The proofs are deferred to the references.

Definition 2.2.1. A **metric** d on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, the following hold:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (positivity);
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A **metric space** (X, d) is a set X with a metric d defined on X.

A subspace of a metric space is a subset whose metric is obtained by restricting the metric to the subset. The following examples illustrates these concepts.

Example 2.2.1. Define the absolute–value $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, y) = |x - y| .$$

Then d is a metric on $\mathbb{R}$. The natural numbers $\mathbb{N}$ and the rational numbers $\mathbb{Q}$ with the absolute–value metric are metric subspaces of $\mathbb{R}$, as is any other subspace $A \subseteq \mathbb{R}$.

□

Example 2.2.2. Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then d is a metric on $\mathbb{R}^2$, called the Euclidean, or ℓ^2, metric. It corresponds to the usual notion of distance between points in the plane. $\square$

The concepts of open and closed balls are introduced next.

Definition 2.2.2. Let (X, d) be a metric space. The **open ball** $B_r(x)$ of radius $r > 0$ and center $x \in X$ is the set of points contained in

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

Similarly, the **closed ball** $\bar{B}_r(x)$ of radius $r > 0$ and center $x \in X$ is the set of points contained in

$$\bar{B}_r(x) = \{y \in X : d(x, y) \leq r\}.$$

A ball in a metric space is analogous to an interval in $\mathbb{R}$. The next examples illustrate this idea.

Example 2.2.3. Consider $\mathbb{R}$ with its standard absolute–value metric defined in Example 2.2.1. Then the open ball is the open interval of radius r centered at x, i.e., $B_r(x) = \{y \in \mathbb{R} : |x - y| < r\}$, and the closed ball is the closed interval of radius r centered at x, i.e., $\bar{B}_r(x) = \{y \in \mathbb{R} : |x - y| \leq r\}$. $\square$

Example 2.2.4. Consider $\mathbb{R}^2$ with the ℓ^2 metric defined in Example 2.2.2. Then, the open ball $B_r(x)$ is an open disc of radius r centered at x, and the closed ball $\bar{B}_r(x)$ is the closed disc of radius r centered at x. $\square$

In general, there are no algebraic operations defined on a metric space, only a distance function. Most of the vector spaces that have a metric on them are usually derived from a norm. It is assumed that the reader is familiar with the basic theory of vector spaces. The definition of semi-norm is given first. It will be used extensively in this dissertation.

Definition 2.2.3. Given a vector space X , a **semi-norm** is a function $\|\cdot\| : X \to \mathbb{R}$, such that

1. $\|x\| \geq 0$;
2. $\|kx\| = |k|\,\|x\|$;
3. $\|x+y\| \leq \|x\| + \|y\|$

for all $x, y \in X$ and $k \in \mathbb{R}$.

Note that it is possible for $\|x\|$ to be zero even when x is nonzero. The following concept plays a key role when analyzing the relationship between semi-norms and local convexity in the final subsection of this chapter.

Definition 2.2.4. A family $\mathscr{P}$ of semi-norms on a vector space X is said to be **separating** if for each $x \in X$, $x \neq 0$ there corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.

The definitions of a norm and a normed space are given next.

Definition 2.2.5. A **normed vector space** $(X, \|\cdot\|)$ is a vector space X together with a function $\|\cdot\| : X \to \mathbb{R}$, called a **norm** on X, such that

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
2. $\|kx\| = |k|\,\|x\|$;
3. $\|x+y\| \leq \|x\| + \|y\|$

for all $x, y \in X$ and $k \in \mathbb{R}$.

It is easy to see that a semi-norm is an example of a norm. The following example shows the relationship between a metric and normed spaces.

Example 2.2.5. Given a normed vector space $(X, \|\cdot\|)$, define $d : X \times X \to \mathbb{R}$ as

$$d(x, y) = \|x - y\|.$$

The positivity of d follows immediately from the first property of a norm in Definition 3.2.1. Also,

$$d(x, y) = \|x - y\| = \|y - x\| = d(y, x),$$

which proves the symmetry of d. Finally,

$$d(x, y) = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(y, z)$$

proves the triangle inequality. Therefore, d is a metric on X, and (X, d) is a metric space. □

The next examples give some classical normed vector spaces.

Example 2.2.6. The set of real numbers $\mathbb{R}$ with the absolute–value norm $|\cdot|$ is a one-dimensional normed vector space. □

Example 2.2.7. The space $\mathbb{R}^2$ with the norm define for $x = (x_1, x_2)$ by

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2},$$

is a normed vector space. This norm is referred to as the ℓ^2-norm or the Euclidean norm. The corresponding metric is the Euclidean ℓ^2 metric. □

The concepts of open and closed sets are introduced next.

Definition 2.2.6. Let X be a metric space. A set $U \subset X$ is **open** if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subset U$. A set $V \subset X$ is **closed** if $V^c := X \setminus V$ is open.

The next example illustrates the previous concept.

Example 2.2.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider U a subset of $\mathbb{R}$ defined as

$$U = \{x \in \mathbb{R} : 0 < f(x)\}.$$

Suppose $y \in U$, then $f(y) > 0$. Since, f is continuous, there exists some $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < f(y)$. Which implies $-f(y) < f(x) - f(y)$ and hence $0 < f(x)$. That is, if $x \in B_\delta(y)$, then $f(x) > 0$. Therefore, $B_\delta(y) \subset U$ and U is an open subset of $\mathbb{R}$. □

As expected open balls are open and closed balls are closed.

Lemma 2.2.1. *Let X be a metric space. If $x \in X$ and $r > 0$, then the open ball $B_r(x)$ is open and the closed ball $\bar{B}_r(x)$ is closed.*

2.2.2 Topological vector spaces

The notion of a topological space is given first.

Definition 2.2.7. A **topology** on X is a family τ of subsets of X that contains $\emptyset$ and X and is closed under arbitrary (countable or uncountable) unions and finite intersections. The pair (X, τ) is called a **topological space**.

There are topological spaces whose topology is derived from a metric. The following lemma gives a characterization of a metric space using open sets. It can be proved using Definition 2.2.6 and Lemma 2.2.1.

Lemma 2.2.2. *Let X be a metric space. The following properties hold:*

1. *The empty set $\emptyset$ and the whole set X are open.*

2. *An arbitrary (countable or uncountable) union of open sets is open.*

3. *A finite intersection of open sets is open.*

The three properties of a metric space described in the previous lemma let one to see that every metric space is a topological space. It is enough to see that Lemma 2.2.2 verifies the definition of a topology. The resulting family of open sets is called *the metric topology of the metric space* . The concepts of open and closed sets in this topological context are introduced next.

Definition 2.2.8. Let (X, τ) be a topological space. Then a set $U \subset X$ is **open** with respect to τ if $U \in \tau$, and a set $V \subset X$ is **closed** with respect to τ if $V^c \in \tau$.

The following gives an example of a topology and illustrates the previous concept.

Example 2.2.9. Let X be any set. Then $T = \{\emptyset, X\}$ is a topology on X, called the trivial topology. The empty set and the whole set are both open and closed, and no other subsets of X are either open or closed. □

It is important to also define the neighborhood of a point.

Definition 2.2.9. Let (X, τ) be a topological space and $x \in X$. Then a set $V \subset X$ is a **neighborhood** of x if it contains an open set U such that

$$x \in U \subset V.$$

The previous concept can be generalized to sets.

Definition 2.2.10. Let (X, τ) be a topological space and $B \subset X$. Then a set $V \subset X$ is a **neighborhood** of B if it contains an open set U such that

$$B \subset U \subset V.$$

The following gives a classical example of a topology.

Example 2.2.10. The standard topology τ on $\mathbb{R}$ is defined as

$$\tau = \{D \subset \mathbb{R} : \text{D is a union of open intervals}\}.$$

The term open interval is suggestive of the fact that every such interval is an open set. In a similar way, every closed interval is closed in this topology. □

Similar to a vector space, every topological space has a basis associated to it. The formal definition for a topological basis is given next.

Definition 2.2.11. A **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X (called **basis elements**) such that X and the intersection of any two basis elements can be represented as the union of some basis elements.

A classic example that illustrates the previous definition is given next.

Example 2.2.11. It is easy to verify that the set of all open intervals is a basis for the standard topology τ on $\mathbb{R}$, defined in Example 2.2.10. □

The following lemma shows, in general, how to find such a basis.

Lemma 2.2.3. *[33, Lemma 2.3] Let X be a topological space. Suppose that $\mathscr{B}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element B of $\mathscr{B}$ such that $x \in B \subset U$. Then $\mathscr{B}$ is a basis for the topology of X.*

The next definitions are classic in topology.

Definition 2.2.12. A topological space is a **Hausdorff space** if given $x, y \in X$ such that $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$.

Definition 2.2.13. A space X is said to have a **countable basis** at x if there is a countable collection $\mathscr{B}$ of neighborhoods of x such that any neighborhood of x contains at least one of the elements of $\mathscr{B}$. A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**.

An example of a first countable space is given next.

Example 2.2.12. Let (X, d) be a metric space. For each $x \in X$ consider the neighborhood basis

$$\mathscr{B}_x = \{B_r(x) : r > 0, r \in \mathbb{Q}\}$$

consisting of open balls of rational radius r around x. Clearly, $\mathscr{B}_x$ is a countable neighborhood basis at x. Hence, X with the metric topology is a first countable space.

□

The following definition describes another notion of countability.

Definition 2.2.14. A topological space X is said to satisfy the **second countability axiom** if X has a countable basis for its topology.

The relationship between these countability axioms is given next.

Lemma 2.2.4. *If a topological space X is second countable, then it is first countable.*

It is important to mention that Lemma 2.2.3 and 2.2.4 will be used for describing the new space of formal power series introduced in Chapter 3. The next definition is crucial in order to properly define a topological vector space, a central concept used in this work. First, it must be noted that on the basic definition of a vector space V, the scalars are members of a field F, in which case V is called a *vector space over F*. In particular, if the field F is $\mathbb{R}$ then V is called a *real vector space*. On this context, the definition of a topological field is given next.

Definition 2.2.15. A **topological field** is a field equipped with a topology such that the field operations of addition, multiplication, and non-zero inversion are continuous.

The following is a main concept used throughout this work.

Definition 2.2.16. A **topological vector space** X over a topological field F is a vector space which is provided with a topology such that the maps $(x, y) \to x + y$ of $X \times X$ into X, and $(\alpha, x) \to \alpha x$ of $F \times X$ into X, are continuous.

The following gives an example of a topological vector space.

Example 2.2.13. Let $(X, \|\cdot\|)$ be a normed vector space. Every norm on a vector space generates a metric by the formula $d(x,y) = \|x-y\|$ where $x, y \in X$, as shown in Example 2.2.5. Since, X is a metric space, the metric topology endowed in X makes it a topological space. To see that X with this metric topology is a topological vector space, one must verify that $(x,y) \to x+y$ and $(\alpha, x) \to \alpha x$ are continuous. First, note that the product topology on $X \times X$ is the topology generated by the Euclidean product metric $d \times d$, i.e.,

$$(d \times d)((x_1, y_1), (x_2, y_2)) = \sqrt{\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2}.$$

In order to check if the addition is continuous, let $(x_0, y_0) \in X \times X$ and $\epsilon > 0$. Set $\delta = \epsilon/\sqrt{2}$. If $(x,y) \in X \times X$ satisfies $(d \times d)((x,y),(x_0,y_0)) < \delta$, then

$$\|x + y - x_0 - y_0\| \leq \|x - x_0\| + \|y - y_0\| \leq \sqrt{2}(d \times d)((x,y),(x_0,y_0)) < \sqrt{2}\delta = \epsilon.$$

On the other hand, the product topology on $F \times X$ is the topology generated by the Euclidean product metric $d_F \times d$, where d_F is the usual Euclidean metric on $F = \mathbb{R}$ or $F = \mathbb{C}$. In order to check if the scalar multiplication is continuous, let $x_0 \in X$, $\alpha_0 \in F$ and $\epsilon > 0$. Set $\delta = \min\left\{1, \frac{\epsilon}{1+|\alpha_0|+\|x_0\|}\right\}$. If $x \in X$ and $\alpha \in F$ satisfy $(d_F \times d)((\alpha, x), (\alpha_0, x_0)) < \delta$, then

$$\begin{aligned}\|\alpha x - \alpha_0 x_0\| &\leq \|\alpha(x - x_0)\| + \|(\alpha - \alpha_0)x_0\| = |\alpha|\,\|x - x_0\| + |\alpha - \alpha_0|\,\|x_0\| \\ &< (|\alpha| + \delta)\delta + \|x_0\|\delta \leq (1 + |\alpha_0| + \|x_0\|)\delta \leq \epsilon.\end{aligned}$$

Thus, every normed vector space endowed with the topology given by the metric induced by the norm is a topological vector space. In general, an arbitrary metric space is not a topological vector space. Indeed, there exist metrics for which both the vector space operations of addition and scalar multiplication are discontinuous.

□

The concept of a topological vector space given in Definition 2.2.16 uses the notion of continuity of functions. Checking for continuity, in general, is not an easy task. However, when the space has certain characteristics, it is possible to use the notion of convergence instead of continuity. First, the following definition is necessary.

Definition 2.2.17. A **directed set** is a set A equipped with a binary relation $\lesssim$ such that

(a) $\alpha \lesssim \alpha$ for all $\alpha \in A$,

(b) if $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ then $\alpha \lesssim \gamma$ for all $\alpha, \beta, \gamma \in A$,

(c) for any $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \lesssim \gamma$ and $\beta \lesssim \gamma$.

The next examples illustrate this concept.

Example 2.2.14. The set of natural numbers $\mathbb{N}$ with the usual order $\leq$ is a directed set. □

Example 2.2.15. Let X be a set. Consider its power set 2^X ordered by set inclusion. Note that $A \subseteq A$ for all $A \in 2^X$. Also, if $A, B, C \in 2^X$ with $A \subseteq B$ and $B \subseteq C$, it is immediate that $A \subseteq C$. Finally, if $A, B \in 2^X$, then $A \cup B \in 2^X$ and it follows that

$$A \subseteq A \cup B \text{ and } B \subseteq A \cup B.$$

Therefore, the set 2^X ordered by set inclusion is a directed set. □

The next example generalizes the previous one.

Example 2.2.16. Any set that is closed under binary intersections and ordered by reverse inclusion i.e., $A \lesssim B$ if and only if $A \subseteq B$, is a directed set. In particular, let X be a topological space and pick any point $x \in X$. Then, the set formed by the collection of all neighborhoods of x, i.e.,

$$\mathscr{N}_x := \{V \subseteq X : V \text{ is a neighborhood of } x\}$$

ordered by reverse inclusion is a directed set. □

Directed sets are used to define nets, which generalizes the notion of a sequence.

Definition 2.2.18. A **net** in X is a mapping $\alpha \to x_\alpha$ from a directed set A into X. It is usually denoted as $\langle x_\alpha \rangle_{\alpha \in A}$, and it is said that $\langle x_\alpha \rangle$ is indexed by A.

In a general topological space, sequences cannot be used to fully characterize the topology. Sequences associate a point in X to every natural number. Nets are more general, they associate a point to every element in a directed set. As will be shown shortly, under certain conditions a net can be used to describe the topology. The next examples illustrate the idea of a net using the directed sets defined above.

Example 2.2.17. Consider $\mathbb{N}$ equipped with $\leq$ as in Example 2.2.14. Note that every function $f : \mathbb{N} \to \mathbb{N}$ is a net. □

Example 2.2.18. Let X be any set. It is easy to check that any net in X indexed by $(\mathbb{N}, \leq)$ is a sequence in X. Specifically, any sequence x_n is a function on $\mathbb{N}$, and thus, every sequence on $\mathbb{N}$ is a net. □

Nets are one of the tools used in topology to generalize certain concepts from a metric space to a topological point of view. For example, the notion of convergence in a metric space is defined using sequences. However, in a topological vector space it is described using nets. The next definition is needed in order to properly define convergence in this context.

Definition 2.2.19. Let X be a topological vector space and E a subset of X. A net $\langle x_\alpha \rangle_{\alpha \in A}$ in X is **eventually** in E if there exists $\alpha_0 \in A$, such that $x_\alpha \in A$ for all $\alpha \gtrsim \alpha_0$.

The general definition of convergence in a topological vector space is as follows.

Definition 2.2.20. A net $\langle x_\alpha \rangle_{\alpha \in A}$ in X is said to **converge** to x if for every neighborhood U of x, $\langle x_\alpha \rangle_{\alpha \in A}$ is eventually in U.

First, countable spaces have the convenient property that such concepts as closure and continuity can be characterized in terms of sequential convergence as noted in [16, p. 116]. Therefore, the following theorem can be used in conjunction with Definition 2.2.16 in order to use the notion of convergence instead of continuity in the context of topological vector spaces.

Theorem 2.2.1. *[33, Theorem 1.1 (b)] Let X be a space satisfying the first countability axiom. Then the function $f : X \to Y$ is continuous if and only if for every convergent sequence (x_n) in X converging to x, the sequence $(f(x_n))$ converges to $f(x)$.*

2.2.3 Locally convex topological vector spaces

In functional analysis, locally convex spaces are by far the most important class of topological vector spaces. First consider the basic definition of a convex subset.

Definition 2.2.21. A set A is said to be **convex**, if $x, y \in A$ then $tx + (1-t)y \in A$ for $0 \leq t \leq 1$. In addition, A is called an **absolutely convex** set if given $x, y \in A$ then $tx + \sigma y \in A$, when $t + \sigma \leq 1$ and $0 \leq \sigma \leq 1$.

In the context of topological vector spaces, this concept yields the following definition.

Definition 2.2.22. A **locally convex topological vector space** is a topological vector space such that there is a base for the topology consisting of convex sets.

An example of a locally convex topological vector space is given below.

Example 2.2.19. Consider the normed vector space $(X, \|\cdot\|)$ endowed with the topology given by the metric induced by the norm. It was shown in Example 2.2.13 that this is a topological vector space. In addition, it is easy to see that the collection of open balls in X,

$$B_r(x) = \{y \in X : \|x - y\| < r\},$$

is a basis for the topology of X. Furthermore, if $y, z \in B_r(x)$ and $0 \leq t \leq 1$, then it follows that

$$\begin{aligned} \|ty + (1-t)z\| &= \|t(y-x) + (1-t)(z-x)\| \\ &\leq t\|y-x\| + (1-t)\|z-x\| \leq tr + (1-t)r = r. \end{aligned}$$

Therefore, $ty + (1-t)z \in B_r(x)$, which shows that open balls in X are convex sets. Hence, the space X with the topology induced by the norm is a locally convex topological vector space. □

The following notion will play a key part when analyzing the relationship between semi-norms and local convexity.

Definition 2.2.23. Let A be a convex set in a topological vector space X. The **Minkowski functional** or **gauge functional**, μ_A associated with A is

$$\mu_A(x) = \inf\{t > 0 : x \in tA\} \ (x \in X),$$

where $x \in A$, and the notation tA denotes the set $tA := \{ta : a \in A\}$.

The next example illustrates a special case of a Minkowski functional.

Example 2.2.20. Consider a vector space X with the norm $\|\cdot\|$. Let A be the open ball of radius one in X. Define the function mu_A from X to the real numbers, by

$$\mu_A(x) = \inf\{r > 0 : x \in rA\} = \inf\{r > 0 : x \in rB_x(1)\} = \inf\{r > 0 : x \in B_x(r)\},$$

where $x \in X$. It is easy to see that $\mu_A(x)$ is actually the norm in the space X, i.e., $\mu_A(x) = \|x\|$. □

As noted in [37, p. 25], semi-norms are closely related to local convexity in two ways: In every locally convex space there exists a separating family of semi-norms. Conversely, if $\mathscr{P}$ is a separating family of semi-norms on a vector space X, then $\mathscr{P}$ can be used to define a locally convex topology on X. The first statement, related to locally convex spaces, implies the next theorem.

Theorem 2.2.2. *[37] Every locally convex topology X is induced by a family of semi-norms.*

The key idea behind the proof is to consider the Minkowski functionals associated with each absolutely convex open neighborhood of zero in X. Then, the semi-norms on X will turn out to be precisely the Minkowski functionals (see Theorem 1.36 part (b) in [37] for details). On the other hand, the second statement noted in [37, p. 25], related to a family of semi-norms on a vector space, implies the following theorem.

Theorem 2.2.3. *[16, Theorem 5.14] Let $\{p_\alpha\}_{\alpha \in A}$ be a family of semi-norms on the vector space X. For $x \in X, \alpha \in A$, and $\epsilon > 0$, let*

$$U_{x,\alpha}(\epsilon) = \{y \in X : p_\alpha(y - x) < \epsilon\},$$

and let τ be the topology generated by the basis elements $U_{x,\alpha}(\epsilon)$. Then the following statements hold:

(a) For each $x \in X$, the finite intersections of the sets $U_{x,\alpha}(\epsilon)$ $(\alpha \in A, \epsilon > 0)$ form a neighborhood base at x.

(b) If $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in X, then $x_i \to x$ if and only if $p_\alpha(x_i - x) \to 0$ for all $\alpha \in A$.

(c) (X, τ) is a locally convex space.

When a vector space has a family of semi-norms, then the Definition 2.2.12 related to the Hausdorff property can be rewritten as follows.

Theorem 2.2.4. *[16, Theorem 5.16 (a)] Let X be a vector space with the topology defined by a family $\{p_\alpha\}_{\alpha \in A}$ of semi-norms. Then X is Hausdorff if and only if for each $x \neq 0$ there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$.*

The next example shows how Theorem 2.2.3 helps one to recognize a locally convex topological vector space when a family of semi-norms is given without having to use the formal definition in Definition 2.2.22. Two definitions needed for the following example are given next.

Definition 2.2.24. A collection $\mathscr{A}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathscr{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition 2.2.25. A space X is said to be **compact** if every open covering $\mathscr{A}$ of X contains a finite subcollection that also covers X.

Example 2.2.21. Let X be topological vector space. Consider the vector space $C(X, \mathbb{R})$ of all continuous real-valued functions on X. For a compact set $K \subseteq X$, define the semi-norm on $C(X, \mathbb{R})$ by

$$p_K(f) = \sup\{|f(x)| : x \in K\}.$$

The topology induced by the family of semi-norms $\{p_K : K \subseteq X \text{ compact}\}$ is called the compact-open topology. Furthermore, $C(X, \mathbb{R})$ endowed with this topology is a locally convex space by Theorem 2.2.3. □

Finally, there are two essential topics to be considered when studying a topological vector space: the Cauchy criterion and completeness. In the context of metric spaces, these concepts can be defined entirely using sequences. However, in the context of topological vector spaces, nets are used in place of sequences. The next definition illustrates this fact for the Cauchy criterion.

Definition 2.2.26. A net $\langle x_\alpha \rangle_{\alpha \in A}$ in topological vector space X is called a **Cauchy net** if the net $\langle x_i - x_j \rangle_{(i,j) \in A \times A}$ converges to zero in X. The set $A \times A$ is a directed set defined as $(i, j) \lesssim (i', j')$ if and only if $i \lesssim i'$ and $j \lesssim j'$.

The next definition regarding completeness is analogous to that for metric spaces.

Definition 2.2.27. A topological vector space is said to be **complete** if every Cauchy net converges.

As described in the previous subsection, for first countable spaces, questions related to closure and continuity reduce to questions about sequences. Furthermore, in a first countable space (for example, a metric space) every Cauchy net reduces to a Cauchy sequence, as shown in the next theorem.

Theorem 2.2.5. *Let X be a locally convex topological vector space. If X is first countable and $\langle x_\alpha \rangle_{\alpha \in A}$ is a Cauchy net, then there exists a map $f : \mathbb{N} \to A$, such that $\langle x_{f(n)} \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence.*

Hence, in first countable spaces, analyzing convergence of nets is equivalent to analyzing convergence of sequences. Also, as expected, the completeness of these spaces is completely characterized by Cauchy sequences. In this context, it is important to observe that *not* every net corresponds to a sequence. The next example illustrates this fact.

Example 2.2.22. Consider $\mathbb{R}$ with the usual topology. Let $I = [1, +\infty)$ and for all $i \in I$ set

$$c_i = \frac{1}{i}.$$

Then, $\{c_i\}_{i \in I}$ is a Cauchy net using the usual order $\leq$ in the interval I. Clearly, this net converges to 0. However, $\{c_i\}_{i \in I}$ is not a Cauchy sequence. □

The following definition is classical.

Definition 2.2.28. A complete normed vector space with respect to the metric induced by its norm is called a **Banach space**.

Two standard examples are given next.

Example 2.2.23. The space $\mathbb{R}^2$ with the Euclidean, ℓ^2–norm is a normed vector space as shown in Example 2.2.7. Let $(x_{n,1}, x_{n,2})$ with $n \in \mathbb{N}$ be a Cauchy sequence in $\mathbb{R}^2$. Pick $\epsilon > 0$. Since the real line $\mathbb{R}$ is complete, there exist $y_1, y_2 \in \mathbb{R}$ and $N_1, N_2 \in \mathbb{N}$ such that

$$|x_{n,k} - y_k| < \frac{\epsilon}{2}, \forall n_k > N_k,$$

where $k \in \{1,2\}$. Thus, for all $n_1 > N_1$ and $n_2 > N_2$ it follows that

$$\begin{aligned}\|(x_{n,1},x_{n,2})-(y_1,y_2)\|_2 &= \sqrt{(x_{n,1}-y_1)^2+(x_{n,2}-y_2)^2}\\ &\leq |x_{n,1}-y_1|+|x_{n,2}-y_2| < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

Therefore, every Cauchy sequence converges, that implies that $\mathbb{R}^2$ with the ℓ^2–norm is a Banach space. □

Example 2.2.24. Let $\mathfrak{p} \geq 1$ and f a measurable real–valued function defined on the interval $[t_0, t_1] \subseteq \mathbb{R}$. Define the norm

$$\|f\|_{\mathfrak{p}} = \left(\int_{t_0}^{t_1} |f|^{\mathfrak{p}}\, dt\right)^{1/\mathfrak{p}}.$$

This norm is referred to as the $L_{\mathfrak{p}}$-norm. In addition, the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ is denoted by $L_{\mathfrak{p}}[t_0, t_1]$. An important property is that $L_{\mathfrak{p}}[t_0, t_1]$ is a Banach space. The proof, which is rather long, can be found in [36]. □

The most important type of locally convex topological vector space is a Fréchet space.

Definition 2.2.29. A complete Hausdorff topological vector space whose topology is defined by a countable family of semi-norms is called a **Fréchet space**.

It is important to note that every Banach space with the metric induced by its norm is a Fréchet space, since the space is complete with respect to this metric. However, not every Fréchet space is a Banach space, even more, some Fréchet spaces do not have a metric associated to them. The following are known examples of Fréchet spaces.

Example 2.2.25. Consider the Example 2.2.21. If X is a compact space, then the locally convex topological vector space $C(X, \mathbb{R})$ is a Fréchet space. □

Example 2.2.26. Consider the vector space $\mathcal{C}^{\infty}[0,1]$ of all infinitely differentiable functions $f : [0,1] \to \mathbb{R}$ and define the following semi-norms

$$p_k(f) = \sup_{0\leq x\leq 1} \left|f^{(k)}(x)\right|,\ k = 0, 1, \cdots,$$

where $f^{(k)}$ denotes the k–th derivative of f and $f^{(0)} := f$. The space $\mathcal{C}^{\infty}[0,1]$ with the family of semi-norms $p_k(f)_{k\in\mathbb{N}}$ is a Fréchet space. □

Example 2.2.27. Consider the extended space $L^m_{\mathfrak{p},e}(t_0)$ defined as

$$L^m_{\mathfrak{p},e}(t_0) := \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L^m_{\mathfrak{p}}[t_0, t_1], \forall t_1 \in (t_0, \infty)\},$$

where $u_{[t_0,t_1]}$ denotes the restriction of the measurable real–valued function u to $[t_0, t_1]$, and $L_{\mathfrak{p}}[t_0, t_1]$ is as defined in Example 2.2.24. An important property is that $L^m_{\mathfrak{p},e}(t_0)$ is a Fréchet space. The proof, which is rather long, can be found in [26]. □

CHAPTER 3

EXPANDED SET OF GLOBALLY CONVERGENT FLIESS OPERATORS

"Un cronopio pequeñito buscaba la llave de la puerta de la calle en la mesa de luz, la mesa de luz en el dormitorio, el dormitorio en la casa, la casa en la calle. Aquí se detenía el cronopio, pues para salir a la calle precisaba la llave de la puerta"

– Julio Cortázar,
Historias de cronopios y de famas [3]

The goal of this chapter is to develop an exact relationship between the growth rate of the coefficients of a generating series and the nature of the convergence of its corresponding Fliess operator. As noted in Chapter 1, in the classical literature, the word "convergence" of a formal power series is not trivially related to the convergence of its corresponding Fliess operator. The "convergence" of a formal power series describes only the growth rate of its coefficients, while the convergence of a Fliess operator is related to the bounds on both the L_1-norm of an admissible input and the length of time over which the corresponding output is well defined.

In particular, observe that the Fliess operator's definitions of globally convergent and locally convergent with finite radius of convergence related to the operator are mutually exclusive. A Fliess operator cannot be both globally and locally convergent with finite radius of convergence at the same time, it will always fall into exactly one category. However, the same cannot be said regrading the classical definition of convergence concerning generating series. In this case, the definitions of locally and globally convergence are *not* mutually exclusive. It is clear that every globally convergent series is also a locally convergent series as shown at the end of Subsection 1.1.1 in Chapter 1. In the presence of all this confusing terminology, this dissertation aims

[3] A little cronopio looked for the key to the street door on the night table, the night table in the bedroom, the bedroom in the house, the house in the street. Here the cronopio stopped, because to go out to the street he needed the key to the door.

to clarify the situation by giving a straightforward relationship between the nature of convergence of a Fliess operator and its generating series.

In order to achieve this objective, the chapter is organized as follows: In Section 3.1 sufficient conditions on the growth rate of the coefficients of the generating series are given in order to ensure convergence of its corresponding Fliess operator. The result significantly expands the class of globally convergent Fliess operators based on the condition (1.1.4). Next, Section 3.2 introduces the new space of formal power series. It requires one to view the set of formal power series as a locally convex topological vector space with a family of semi-norms instead of the more common ultrametric space setting. Subsequently, in Section 3.3 an example is introduced in order to illustrate how to classify a generating series in the new space of formal power series. Finally, Section 3.4 gives the precise relationship between the growth rate of the coefficients generating series and the nature of the convergence of its corresponding Fliess operator via the two main theorems.

3.1 SUFFICIENT CONDITIONS FOR GLOBAL CONVERGENCE

In [25], the sufficient condition (1.1.4) for global convergence of a Fliess operator was given. In this section, a less restrictive sufficient condition is developed in terms of the Gevrey order of the generating series. The following definition based on [1,6,17] is needed. It is assumed throughout that $X = \{x_0, x_1, \ldots, x_m\}$. Also, there is no loss of generality in assuming $\ell = 1$.

Definition 3.1.1. A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is **Gevrey of order** $s \in [0,\infty)$ if there exists constants $K, M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^s, \quad \forall \eta \in X^*. \tag{3.1.1}$$

Clearly a Gevrey series of order s is also of order s' if $s' > s$.

The following example illustrates the previous definition.

Example 3.1.1. Consider the series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ and $K, M > 0$ such that

$$|(c,\eta)| = KM^{|\eta|}(|\eta|!)^2, \quad \forall \eta \in X^*.$$

Clearly, c is Gevrey of order $s \in [2,\infty)$. □

It is important to note that if a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has a growth rate given by (1.1.3) and no other information is known (for example, a more restrictive growth condition), then the series c is Gevrey of order $s \in [0,\infty)$. To see this assertion consider the next three examples.

Example 3.1.2. Consider the series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ and $K, M > 0$ such that

$$|(c,\eta)| = KM^{|\eta|}\,|\eta|!, \quad \forall \eta \in X^*.$$

Clearly, the growth rate of the coefficients of the series satisfies (1.1.3). Hence, c is Gevrey of order $s \in [1,\infty)$. □

Example 3.1.3. Consider the series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ and $K, M > 0$ such that

$$|(c,\eta)| = KM^{|\eta|}(|\eta|!)^{1/2}, \quad \forall \eta \in X^*.$$

In which case, the growth rate of the coefficients of the series satisfies (1.1.3) since

$$|(c,\eta)| = KM^{|\eta|}(|\eta|!)^{1/2} \le KM^{|\eta|}\,|\eta|!, \quad \forall \eta \in X^*.$$

In fact, c is Gevrey of order $s \in [1/2,\infty)$. □

The examples above illustrate that the stricter the growth condition on the coefficient, the larger the interval of the Gevrey order. Also, it is important to note that the Gevrey order is always an interval with upper bound being infinity. However, its lower bound is a number greater or equal to zero, this lower bound will play a key role in the following subsection.

3.1.1 Classification of formal power series using Gevrey order

The following definition plays a key role in the classification of formal power series henceforth.

Definition 3.1.2. Given a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$, define γ_c as the minimum of all s for which c is Gevrey of order s, i.e.,

$$\gamma_c := \min\{s \in [0,\infty) : s \text{ satisfies } (3.1.1)\}.$$

For example, if c satisfies the growth condition (1.1.4) then $\gamma_c = 0$. If it satisfies (1.1.3), it is only known that $\gamma_c \in [0,1]$. The set of all generating series with minimum Gevrey order γ is denoted by $\mathbb{R}_\gamma\langle\langle X\rangle\rangle$. Specifically, a series $c \in \mathbb{R}_\gamma\langle\langle X\rangle\rangle$ if and only if $\gamma_c = \gamma$. When $c \in \mathbb{R}_1\langle\langle X\rangle\rangle$, the series provides for a type of local convergence for F_c, while the condition $c \in \mathbb{R}_0\langle\langle X\rangle\rangle$ provides global convergence for F_c. This new concept can be used to introduce a new definition for the symbol $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$: Fix a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$. If $0 \leq \gamma_c < 1$ then the set of all such generating series will be designated by $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. On the other hand, when $0 \leq \gamma_c \leq 1$ then F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R_u)[t_0,\, t_0+T]$ into $B_{\mathfrak{q}}^{(}S_u)[t_0,\, t_0+T]$ for sufficiently small $R_u, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [25]. The set of all such generating series will be denoted by $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$.

Henceforth, this dissertation will avoid referring to the "old terminology". Meaning, phrases like global convergent series and local convergent series will not be used. Also, the new interpretation given to the symbols $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ and $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ will be used for the remainder of the dissertation. In order to better visualize the situation, see Figure 6. It summarizes the classification of formal power series using Gevrey order under the new nomenclature.

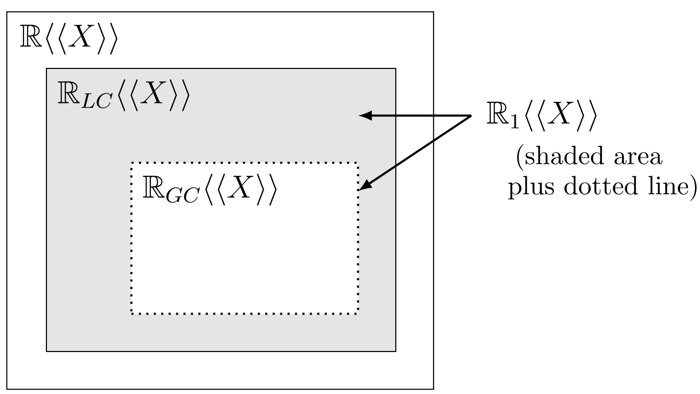

Fig. 6: Classification of generating series by Gevrey order.

Note that

$$\bigcup_{0\leq s\leq 1} \mathbb{R}_\gamma\langle\langle X\rangle\rangle = \mathbb{R}_{LC}\langle\langle X\rangle\rangle,$$

all $\mathbb{R}_\gamma\langle\langle X\rangle\rangle$ with $0 \leq s \leq 1$ are pairwise disjoint, and

$$\bigcap_{0\leq s\leq 1} \mathbb{R}_\gamma\langle\langle X\rangle\rangle = \emptyset.$$

This later fact implies that the new classification using the minimum Gevrey order (specifically, γ_c) gives a partition of the set $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$. In particular,

$$\bigcup_{0\leq s<1} \mathbb{R}_{\gamma}\langle\langle X\rangle\rangle = \mathbb{R}_{GC}\langle\langle X\rangle\rangle.$$

The next subsection justifies the new notation $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

3.1.2 Convergence in the $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ space

It is known that when $c \in \mathbb{R}_0\langle\langle X\rangle\rangle \subset \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, its associated Fliess operator F_c is globally convergent [25]. However, in general, when $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ it is only known that F_c is at least locally convergent since $\mathbb{R}_{GC}\langle\langle X\rangle\rangle \subset \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. The next theorem makes a stronger claim.

Theorem 3.1.1. *If $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, then for any $u \in L_1^m[0,T]$ the series*

$$y(t) = F_c[u](t) = \sum_{\eta\in X^*} (c,\eta)E_\eta[u](t)$$

converges absolutely and uniformly on $[0,T]$ for any $T > 0$.

Proof: If $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, then there exist constants $K, M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^{\gamma_c}, \ \ \forall \eta \in X^*,$$

where $0 \leq \gamma_c < 1$. Fix some $T > 0$. Pick any $u \in L_1^m[0,T]$ and let $R = \max\{\|u\|_1, T\}$. Observe that from (2.1.1)

$$\begin{aligned} F_c[u](t) &= \sum_{\eta\in X^*} (c,\eta)E_\eta[u](t) \\ &\leq \sum_{\eta\in X^*} |(c,\eta)|\,|E_\eta[u](t)| \\ &\leq \sum_{k=0}^{\infty} KM^k(k!)^{\gamma_c} \sum_{\eta\in X^k} E_\eta[\bar{u}](t) \\ &= \sum_{k=0}^{\infty} KM^k(k!)^{\gamma_c} \sum_{\substack{r_0,r_1,\ldots,r_m\geq 0 \\ r_0+r_1+\cdots+r_m=k}} \prod_{j=0}^{m} E_{{x_j}^{r_j}}[\bar{u}](t), \end{aligned}$$

where the identities

$$\sum_{\eta\in X^k}\eta = \sum_{\substack{r_0,r_1,\ldots,r_m\geq 0\\ r_0+r_1+\cdots+r_m=k}} x_0^{r_0} \sqcup\!\sqcup\, x_1^{r_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, x_m^{r_m},\quad k\geq 0$$

and $E_\eta E_\xi = E_{\eta \sqcup\!\sqcup \xi}$ were used in the last step. Then from (2.1.2) it follows that

$$\begin{aligned}
F_c[u](t) &\leq \sum_{k=0}^{\infty} KM^k(k!)^{\gamma_c} \sum_{\substack{r_0,r_1,\ldots,r_m\geq 0\\ r_0+r_1+\cdots+r_m=k}} \frac{R^k}{\prod_{j=0}^m r_j!}\\
&= \sum_{k=0}^{\infty} K(MR)^k(k!)^{\gamma_c} \sum_{\substack{r_0,r_1,\ldots,r_m\geq 0\\ r_0+r_1+\cdots+r_m=k}} \frac{1}{\prod_{j=0}^m r_j!}\\
&= \sum_{k=0}^{\infty} K(MR)^k(k!)^{\gamma_c}\frac{(m+1)^k}{k!}\\
&= \sum_{k=0}^{\infty} K(MR(m+1))^k\frac{1}{(k!)^{1-\gamma_c}}.
\end{aligned} \tag{3.1.2}$$

Applying the ratio test to the sequence

$$a_k := K\frac{(MR(m+1))^k}{(k!)^{1-\gamma_c}},\ k\geq 0$$

and using the fact that $0 < 1-\gamma_c \leq 1$:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\frac{(MR(m+1))^{k+1}}{((k+1)!)^{1-\gamma_c}}\frac{(k!)^{1-\gamma_c}}{(MR(m+1))^k}\\
&= (MR(m+1))\lim_{k\to\infty}\frac{1}{(k+1)^{1-\gamma_c}}\\
&= 0.
\end{aligned}$$

Thus, the series $F_c[u](t)$ converges absolutely and uniformly on $[0,T]$. ■

The following result it is an immediate consequence of Theorem 3.1.2.

Theorem 3.1.2. *If $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, then the series (1.1.2) defines an operator from the extended space $L_{\mathfrak{p},e}^m(0)$ into $C[0,\infty)$ and its corresponding Fliess operator F_c is globally convergent.*

The notion of a *bounding function* has proved useful in computing the radius of convergence for Fliess operators [41].

Definition 3.1.3. Let B and $f$be real–valued functions defined on the interval $[0,T]$, $T>0$. B is a **bounding function** of f if and only if f is bounded pointwise by B, i.e.,

$$B(t) \leq f(t),\ \forall t \in [0,T].$$

The next corollary follows directly from (3.1.2). It describes a class of bounding functions for Fliess operators with generating series in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

Corollary 3.1.1. *Suppose $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ with $0 \leq \gamma_c < 1$ and growth constants $K, M > 0$. Then for any $u \in L_1^m[0,T]$ and $T > 0$, the function*

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t)$$

has a bounding function

$$B(t) = \sum_{k=0}^{\infty} K(MR(t)(m+1))^k \frac{1}{(k!)^{1-\gamma_c}},$$

where $R(t) := \max\{\|u\|_{1,[0,t]}, t\}$, $t \in [0,T]$. (Here $\|\cdot\|_{1,[0,t]}$ denotes the 1-norm restricted to the interval $[0,t]$.)

The next example illustrates the use of Theorem 3.1.2 and Corollary 3.1.1.

Example 3.1.4. Consider the single-input, single-output (SISO) Wiener system as shown in Figure 7, where $\dot{z} = u$ with $z(0) = 0$ and $h(z) = e^z$. In which case, direct

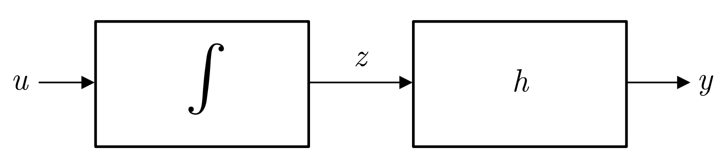

Fig. 7: SISO Wiener system.

substitution of z into h gives

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} \frac{(z(t))^k}{k!} = \sum_{k=0}^{\infty} \frac{(E_{x_1}[u](t))^k}{k!} = \sum_{k=0}^{\infty} \frac{E_{x_1 {\sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup} k}[u](t)}{k!} = \sum_{k=0}^{\infty} \frac{k! E_{x_1^k}[u](t)}{k!} \\ &= \sum_{k=0}^{\infty} E_{x_1^k}[u](t) = F_c[u](t). \end{aligned}$$

Note that the generating series is $c = \sum_{k=0}^{\infty} {x_1}^k$, and F_c is globally convergent as shown in Example 1.1.2. Moreover, c is Gevrey of order $s \in [0,\infty)$ with $\gamma_c = 0$, and its growth constants are $K = M = 1$. Then, $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. For any $T > 0$ and $u \in L_1[0,T]$, let $R(t) = \max\{\|u\|_{1,[0,t]}, t\}$ on $[0,T]$. It follows from Corollary 3.1.1 that for any $t \in [0,T]$

$$F_c[u](t) = \sum_{k=0}^{\infty} E_{{x_1}^k}[u](t) \leq \sum_{k=0}^{\infty} \frac{2^k R(t)^k}{k!} = e^{2R(t)} = B(t).$$

This example is consistent with the fact that a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ with growth condition given by (1.1.4), i.e, $c \in \mathbb{R}_0\langle\langle X\rangle\rangle$, is known to have exponential bounding function [25].

□

The final theorem of this section is a generalization of the previous example. That is, in light of Theorem 3.1.2, when $0 \leq \gamma_c < 1$, the corresponding Fliess operator is well defined on $[0,T]$ for any $T > 0$. Therefore, as in the $\gamma_c = 0$ case, it may also have a bounding function which is entire. To develop this result, the following technical lemma and definition are needed first.

Lemma 3.1.1. *For any integer $l \geq 0$ and $0 < r \leq 1$ such that $lr \gg 1$ it follows that*

$$(lr)! \leq K_r M_r^l (l!)^r,$$

where $K_r = ((2\pi)^{1-r} r)^{1/2}$ and $M_r = r^r$.

Proof: Using Stirling's formula, $\mathcal{O}(k!) = \mathcal{O}(\sqrt{2\pi k}\left(\frac{k}{e}\right)^k)$, observe

$$\mathcal{O}((lr)!) = \mathcal{O}\left(\sqrt{2\pi lr}\left(\frac{lr}{e}\right)^{lr}\right) = \mathcal{O}\left(\frac{\sqrt{2\pi lr}}{\left(\sqrt{2\pi l}\right)^r}(r^r)^l (l!)^r\right)$$
$$= \mathcal{O}\left(\frac{1}{l^{(1-r)/2}}((2\pi)^{1-r} r)^{1/2}(r^r)^l (l!)^r\right).$$

Noting that $1/l^{(1-r)/2} \leq 1$, it follows that

$$\mathcal{O}((lr)!) \leq \mathcal{O}(((2\pi)^{1-r} r)^{1/2}(r^r)^l (l!)^r)$$

■

The following definition introduces the concept of a Mittag-Leffler function as a generalization of an exponential function.

Definition 3.1.4. [30] The **Mittag-Leffler function** is

$$\mathscr{E}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)},$$

where t, α, β are real numbers and $\alpha, \beta > 0$. In particular, $\mathscr{E}_{1,1}(t) = \mathrm{e}^t$.

Finally, in the following theorem, a general bounding function for the Fliess operator with any generating series in $\mathbb{R}_{GC}\langle\langle X \rangle\rangle$ is given in terms of a Mittag-Leffler function.

Theorem 3.1.3. *Suppose* $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ *with* $0 \leq \gamma_c < 1$ *and growth constants* $K, M > 0$*. Let* $R(t) = \max\{\|u\|_{1,[0,t]}, t\}$ *on the interval* $[0, T]$*. Then a bounding function for* $F_c[u](t)$ *is*

$$B(t) = KK_{\bar{s}}\mathscr{E}_{-\bar{s},1}(M_{\bar{s}}A(t)),$$

where $\bar{s} = \gamma_c - 1$, $A(t) = MR(t)(m+1)$, $K_{\bar{s}} = (-(2\pi)^{1+\bar{s}}\bar{s})^{1/2}$ *and* $M_{\bar{s}} = (-\bar{s})^{-\bar{s}}$.

Proof: Setting $A(t) = MR(t)(m+1)$, it follows from Corollary 3.1.1 that the bounding function

$$B(t) = K \sum_{k=0}^{\infty} \frac{1}{k!^{-\bar{s}}} A(t)^k$$

applies. Using Lemma 3.1.1 with $l = k$ and $r = -\bar{s}$ when $k \gg 1$ gives

$$B(t) \leq K \sum_{k=0}^{\infty} \frac{K_{\bar{s}} M_{\bar{s}}^k}{(-k\bar{s})!} A(t)^k = KK_{\bar{s}} \sum_{k=0}^{\infty} \frac{1}{\Gamma(-k\bar{s}+1)} (M_{\bar{s}}A(t))^k = KK_{\bar{s}}\mathscr{E}_{-\bar{s},1}(M_{\bar{s}}A(t)),$$

where $K_{\bar{s}} = (-(2\pi)^{1+\bar{s}}\bar{s})^{1/2}$ and $M_{\bar{s}} = (-\bar{s})^{-\bar{s}}$. ■

Note that, in particular, when $c \in \mathbb{R}_0\langle\langle X \rangle\rangle \subset \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ then $\gamma_c = 0$. Therefore, using Theorem 3.1.3 the bounding function for $F_c[u](t)$ is as expected an exponential function,

$$KK_{\bar{s}}\mathscr{E}_{-\bar{s},1}(M_{\bar{s}}A(t)) = K\mathscr{E}_{1,1}(A(t)) = K\mathrm{e}^{MR(t)(m+1)}.$$

3.2 SPACES OF FORMAL POWER SERIES $S_{\infty,E}$ AND S_∞

A new space of formal power series, denoted by S_∞, is needed in order to develop a clear relationship between a generating series and its Fliess operator in the context of convergence. First, in subsection 3.2.1 a set of normed linear spaces $S_\infty(R), R > 0$ is defined. The main properties related to these spaces are then presented. Next, in subsection 3.2.2 the topological properties of the S_∞ are given. In particular, it is shown that S_∞ is a locally convex topological vector space with a family of semi-norms. In subsection 3.2.3 a description of the relationship between the topology induced on the S_∞ space and the usual ultrametric topology is illustrated via examples. Finally, in subsection 3.2.4, the main relationships between the various spaces are fully development.

3.2.1 The $S_\infty(R)$ spaces

The following norm is of central importance in this work.

Definition 3.2.1. Let $c \in \mathbb{R}\langle\langle X\rangle\rangle$. For any real number $R > 0$,

$$\|c\|_{\infty,R} := \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\}.$$

Define the following family of normed linear spaces:

$$S_\infty(R) := \{c \in \mathbb{R}\langle\langle X\rangle\rangle : \|c\|_{\infty,R} < \infty\},$$

where the addition and scalar multiplication are defined as in Subsection 2.1.1. It is clear that $S_\infty(R)$ is closed under addition and scalar multiplication since

$$\begin{aligned}\|c+d\|_{\infty,R} &= \sup_{\eta\in X^*} \left\{ |(c+d,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} \leq \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} + \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} \\ &= \|c\|_{\infty,R} + \|d\|_{\infty,R}\end{aligned}$$

for all $c, d \in S_\infty(R)$, and

$$\|\alpha c\|_{\infty,R} = \sup_{\eta\in X^*} \left\{ |(\alpha c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} \leq |\alpha| \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} = |\alpha|\, \|c\|_{\infty,R}$$

for all $c \in S_\infty(R), \alpha \in \mathbb{R}$. Also, define:

$$S_{\infty,e} := \bigcup_{R>0} S_\infty(R)$$

and

$$S_\infty := \bigcap_{R>0} S_\infty(R).$$

It is easy to verify that each space $S_\infty(R)$ is Hausdorff, since every metric space is Hausdorff. The next theorem shows how two different norms for the same series are related.

Theorem 3.2.1. *Let $0 < R < R'$. For any $c \in \mathbb{R}\langle\langle X\rangle\rangle$,*

$$\|c\|_{\infty,R} \leq \|c\|_{\infty,R'}.$$

Proof: If $0 < R < R'$, then

$$\|c\|_{\infty,R} = \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} \leq \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R'^{|\eta|}}{|\eta|!} \right\} = \|c\|_{\infty,R'}.$$

■

An immediate consequence of this theorem is that the spaces $S_\infty(R), R > 0$ are nested as shown in Figure 8.

Corollary 3.2.1. *If $0 < R < R'$, then, $S_\infty(R') \subset S_\infty(R)$.*

Proof: If $c \in S_\infty(R')$, then $\|c\|_{\infty,R'} < \infty$. Thus, by Theorem 3.2.1 it follows that $\|c\|_{\infty,R} < \infty$, which implies $c \in S_\infty(R)$. ■

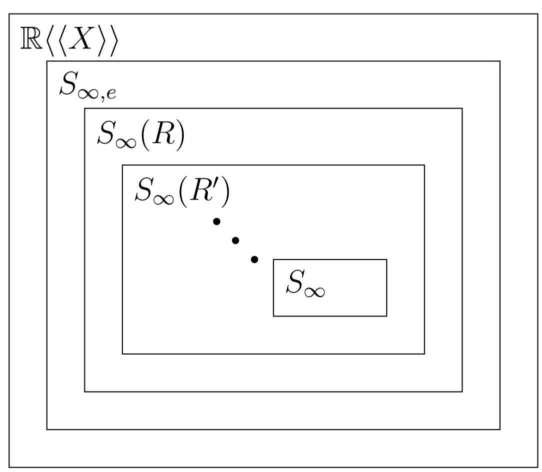

Fig. 8: The spaces $S_{\infty,e}, S_\infty(R)$ and S_∞ are nested.

The following example gives a series in the space S_∞.

Example 3.2.1. Consider the series

$$c = \sum_{n=0}^{\infty} x_0^n.$$

For any $R > 0$

$$\|c\|_{\infty,R} = \sup_{n\geq 0} \left\{ |(c, x_0^n)| \frac{R^n}{n!} \right\} = \sup_{n\geq 0} \frac{R^n}{n!} = k < \infty,$$

where k is a constant depending on R. Therefore, $c \in S_\infty(R) \subset S_{\infty,e}$. In addition, $c \in S_\infty$ since $c \in S_\infty(R)$, $\forall R > 0$. □

The next example describes a series in the complement of S_∞, i.e, $S_{\infty,e} \setminus S_\infty$.

Example 3.2.2. Consider the series

$$c = \sum_{n=0}^{\infty} 2^n n!\, x_0^n.$$

For any $R > 0$

$$\|c\|_{\infty,R} = \sup_{n\geq 0} \left\{ |(c, x_0^n)| \frac{R^n}{n!} \right\} = \sup_{n\geq 0}\ (2R)^n = \begin{cases} \infty & : \ R > 1/2 \\ 1 & : \ R \leq 1/2. \end{cases}$$

Therefore, $c \in S_\infty(R)$ when $R \leq 1/2$, so that $c \in S_{\infty,e}$. In addition, $c \in S_{\infty,e} \setminus S_\infty$ □

The next lemma provides the relationship between the norms of a series in $S_{\infty,e}$ and its subseries.

Lemma 3.2.1. *Let $c \in S_{\infty,e}$. If $\hat{c}$ is a subseries of c, then for any $R > 0$*

$$\|\hat{c}\|_{\infty,R} \leq \|c\|_{\infty,R}.$$

Proof: For fixed $R > 0$ observe

$$\begin{aligned}\|c\|_{\infty,R} &= \sup_{\eta\in X^*}\left\{|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\}\\ &= \max\left\{\sup_{\eta\in \mathrm{supp}(\hat{c})}\left\{|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\},\ \sup_{\eta\notin \mathrm{supp}(\hat{c})}\left\{|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\}\right\}\\ &= \max\left\{\|\hat{c}\|_{\infty,R},\ \sup_{\eta\notin \mathrm{supp}(\hat{c})}\left\{|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\}\right\}.\end{aligned}$$

Therefore, $\|\hat{c}\|_{\infty,R} \leq \|c\|_{\infty,R}$. ■

As the spaces $S_\infty(R), R > 0$ are nested, it is natural to ask if a sequence converges in one space, will it also converge in a larger space? Furthermore, would the limit point be the same in both spaces? Such questions are addressed in the next lemma.

Lemma 3.2.2. *Let $0 < R < R'$. If $c_i \to c$ as a sequence in $S_\infty(R')$, then $c_i \to c$ as a sequence in $S_\infty(R)$.*

Proof: Since $c_i \to c$ in $S_\infty(R')$ then, for any $\epsilon > 0$ there exists a natural number N such that if $i > N$ then $\|c_i - c'\|_{\infty,R'} < \epsilon$. On the other hand, since $R < R'$, it follows by Theorem 3.2.1 that

$$\|c_i - c'\|_{\infty,R} < \|c_i - c'\|_{\infty,R'}.$$

Thus, $\|c_i - c'\|_{\infty,R} < \epsilon$, which leads to the conclusion that $c_i \to c'$ in the larger space $S_\infty(R) \supset S_\infty(R')$. ■

The final theorem of this subsection shows that each space $S_\infty(R)$ is complete.

Theorem 3.2.2. *$(S_\infty(R), \|\cdot\|_{\infty,R})$ is a Banach space for any $R > 0$.*

Proof: The proof parallels the classical proof for the completeness of l^∞ [29, p. 33]. Fix $R > 0$ and let $\{c_i\}_{i\geq 0}$ be a Cauchy sequence in the normed linear space $S_\infty(R)$. Then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $i, j > N$

$$\|c_i - c_j\|_{\infty,R} = \sup_{\eta\in X^*}\left\{|(c_i - c_j,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\} < \epsilon.$$

Therefore, given any word $\eta \in X^*$

$$|(c_i - c_j,\eta)|\,\frac{R^{|\eta|}}{|\eta|!} = \left|(c_i,\eta)\frac{R^{|\eta|}}{|\eta|!} - (c_j,\eta)\frac{R^{|\eta|}}{|\eta|!}\right| < \epsilon, \tag{3.2.1}$$

implying that $\{(c_i,\eta)R^{|\eta|}/\,|\eta|!\}_{i\geq 0}$ is a Cauchy sequence in $\mathbb{R}$. For each $\eta \in X^*$ define

$$c_\eta = \lim_{i\to\infty}(c_i,\eta)\frac{R^{|\eta|}}{|\eta|!},$$

and let $c := \sum_{\eta\in X^*}(c,\eta)\eta$, where $(c,\eta) := c_\eta\,|\eta|!/R^{|\eta|}$. The claim now is that $c \in S_\infty(R)$. Letting $j \to \infty$ in (3.2.1) gives

$$\left|(c_i,\eta)\frac{R^{|\eta|}}{|\eta|!} - c_\eta\right| < \epsilon,\quad i > N. \tag{3.2.2}$$

For any fixed i, since $c_i \in S_\infty(R)$, there exists a real number $B_i > 0$ such that $|(c_i,\eta)|\,R^{|\eta|}/\,|\eta|! \leq B_i$ for all $\eta \in X^*$. Therefore, if $i > N$ then for every $\eta \in X^*$

$$|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!} \leq \left|c_\eta - (c_i,\eta)\frac{R^{|\eta|}}{|\eta|!}\right| + |(c_i,\eta)|\,\frac{R^{|\eta|}}{|\eta|!} \leq \epsilon + B_i.$$

Hence, $c \in S_\infty(R)$. To show completeness, it is only necessary to show that $c_i \to c$ as a sequence in $S_\infty(R)$. From (3.2.2) it follows that for any $\eta \in X^*$

$$|(c_i,\eta) - (c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!} < \epsilon,\quad i > N.$$

Therefore, $\|c_i - c\|_{\infty,R} < \epsilon$ when $i > N$, implying that $c_i \to c$ as desired. ■

3.2.2 The semi-norm topology on S_∞

The space

$$S_\infty = \bigcap_{R>0} S_\infty(R)$$

is of particular interest here as there exists a topology which makes it a locally convex topological vector space. The space S_∞ cannot in any obvious way be viewed as a normed linear space. First consider each $S_\infty(R)$ with the topology induced by the norm $\|\cdot\|_{\infty,R}$. Define the topology τ on the space S_∞ as the one generated by the basis elements

$$B_{c,R}(\epsilon) := \{d \in S_\infty : \|c - d\|_{\infty,R} < \epsilon\}, \tag{3.2.3}$$

where $c \in S_\infty$ and $\epsilon, R > 0$. It is easy to check that τ contains and S_∞, and is closed under arbitrary unions and finite intersections. Thus, (S_∞,τ) is a topological space. However, to see that is a locally convex topological vector space a little more work is needed. It is first necessary to show that (S_∞, τ) is second countable, and therefore first countable. Then, Definition 2.2.16 and Lemma 2.2.1 are used to show that (S_∞, τ) is a topological vector space and τ is usually called the semi-norm topology. Finally, it is proved to be a locally convex topological vector space using Theorem 2.2.3. The next theorem addresses the second countability axiom.

Theorem 3.2.3. *The space (S_∞, τ) is second countable.*

Proof: The proof parallels the classical proof for the second countability of $\mathbb{R}$ [34, p. 56]. Let $\mathscr{B}$ be the collection of open sets $B_{c_q,R_q}(r)$, where $R_q, r \in \mathbb{Q}_+$, and $c_q \in S_\infty$ with $(c_q, \eta) \in \mathbb{Q}, \forall \eta \in X^*$. In order to prove that $\mathscr{B}$ is a countable basis, one can use Lemma 2.2.3. In which case it is necessary to prove that for any set $U \in S_\infty$ and $c \in U$ there exists an open set $B_{c_q,R_q}(r) \in \mathscr{B}$ such that

$$c \in B_{c_q,R_q}(r) \subseteq U.$$

First, select any open set $U \in S_\infty$ and a series $c \in U$. Since τ is the assumed topology on S_∞, there exists an open set $B_{c,R}(\epsilon) \subset U$. On the other hand, there exists rational numbers $R_q, r \in \mathbb{Q}_+$ such that

$$\frac{\epsilon}{4} < r < \frac{\epsilon}{2} \tag{3.2.4}$$

and

$$R_q > R. \tag{3.2.5}$$

In addition, there exists a series $c_q \in S_\infty$ with $(c_q, \eta) \in \mathbb{Q}, \forall \eta \in X^*$ such that

$$|(c, \eta) - (c_q, \eta)| < \left(\frac{\epsilon}{4}\right) \frac{|\eta|!}{{R_q}^{|\eta|}},\ \forall \eta \in X^*. \tag{3.2.6}$$

Since $(c, \eta), \left(\frac{\epsilon}{4}\right) \frac{|\eta|!}{{R_q}^{|\eta|}} \in \mathbb{R}$ for all $\eta \in X^*$, and the set $\mathbb{Q}$ is dense in $\mathbb{R}$, then, from (3.2.6) it follows that

$$|(c, \eta) - (c_q, \eta)| \frac{{R_q}^{|\eta|}}{|\eta|!} < \frac{\epsilon}{4},\ \forall \eta \in X^*.$$

The definition of the supremum gives

$$\sup_{\eta \in X^*} \left\{ |(c - c_q, \eta)| \frac{{R_q}^{|\eta|}}{|\eta|!} \right\} < \frac{\epsilon}{4}.$$

Using the norm as in Definition 3.2.1 yields

$$\|c - c_q\|_{\infty,R_q} < \frac{\epsilon}{4}.$$

Therefore, $c_q \in B_{c,R_q}\left(\frac{\epsilon}{4}\right)$. In addition, since $\epsilon/4 < r$ from (3.2.4), it follows that

$$\|c - c_q\|_{\infty,R_q} < \frac{\epsilon}{4} < r.$$

This implies that $c \in B_{c_q,R_q}(r) \in \mathscr{B}$. Next, taking any $d \in B_{c_q,R_q}(r)$, note that since $R < R_q$ from (3.2.5), it follows that

$$\|c - d\|_{\infty,R} \leq \|c - d\|_{\infty,R_q} \leq \|c - c_q\|_{\infty,R_q} + \|c_q - d\|_{\infty,R_q}, \tag{3.2.7}$$

where the triangle inequality was used in the last step. Observe that

$$\|d - c_q\|_{\infty,R_q} < r,$$

since $d \in B_{c_q,R_q}(r)$. Applying the previous results to (3.2.7) gives

$$\|c - d\|_{\infty,R} \leq \|c - d\|_{\infty,R_q} \leq \|c - c_q\|_{\infty,R_q} + \|c_q - d\|_{\infty,R_q} < \frac{\epsilon}{4} + r.$$

In addition, since $r < \frac{\epsilon}{2}$ from (3.2.4), it follows that

$$\|c - d\|_{\infty,R} < \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon. \tag{3.2.8}$$

Thus, $d \in B_{c,R}(\epsilon)$, and therefore, $B_{c_q,R_q}(r) \subseteq B_{c,R}(\epsilon)$. Finally, it is concluded that

$$c \in B_{c_q,R_q}(r) \subseteq B_{c,R}(\epsilon) \subseteq U,$$

which completes the prove. ∎

By Lemma 2.2.4, second countability implies first countability. Thus, the following result concerning first countability of S_∞ is immediate.

Corollary 3.2.2. *The space* (S_∞, τ) *is first countable.*

As discussed in Chapter 2 and noted in [16, p. 116], first countable spaces have the convenient property that such concepts as closure and continuity can be characterized in terms of sequential convergence. In which case, only sequences are needed henceforth instead of nets. In addition, to prove that (S_∞, τ) is a topological vector space, the notion of convergence can be used instead of continuity. A common convergence criterion when dealing with semi-norms is the following.

Definition 3.2.2. A sequence $\{c_i\}_{i\in\mathbb{N}}$ in S_∞ **converges** to $c \in S_\infty$ in the τ topology on the space S_∞ if and only if $\|c_i - c\|_{\infty,R} \to 0$ as $i \to \infty$ for all $R > 0$.

This criterion is selected since each $S_\infty(R)$ is a Banach space, and therefore, it is Hausdorff and first countable. Then, as described in [28, p. 20] sequentially continuous maps are continuous.

The following theorem describes the primary space of formal power series used in this work.

Theorem 3.2.4. *The space (S_∞, τ) is a topological vector space.*

Proof: Since the space (S_∞, τ) is first countable, Theorem 2.2.1 is used in order to use the notion of convergence instead of continuity in Definition 2.2.16. Thus, it is necessary to check two claims: First, if $c_n \to c$ and $d_n \to d$ in S_∞, then $c_n+d_n \to c+d$ in S_∞. Second, if $c_n \to c$ in S_∞ and $\alpha_n \to \alpha$ in $\mathbb{R}$, then $\alpha_n c_n \to \alpha c$ in S_∞. Consider the first claim. Fix any $\epsilon > 0$. Then for all $R > 0$ it must hold that

$$\|c_n - c\|_{\infty,R} < \frac{\epsilon}{2},\ n > N$$

and

$$\|d_n - d\|_{\infty,R} < \frac{\epsilon}{2},\ n > M,$$

for some $N, M > 0$. For $n > \max\{N, M\}$ and any $R > 0$ it follows that

$$\|(c_n + d_n) - (c + d)\|_{\infty,R} \leq \|c_n - c\|_{\infty,R} + \|d_n - d\|_{\infty,R} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $c_n + d_n \to c + d$ in S_∞. Concerning the second claim, suppose $c_n \to c$ in S_∞ and $\alpha_n \to \alpha$ in $\mathbb{R}$. Fix any $\epsilon > 0$. Then for all $R > 0$, there exists some $N > 0$ such that

$$|\alpha_n - \alpha| < \min\left\{\frac{\epsilon}{2\|c\|_{\infty,R}}, 1\right\},\ n > N.$$

In addition, there exists some $M > 0$ such that

$$\|c_n - c\|_{\infty,R} < \frac{\epsilon}{2(|\alpha| + 1)},\ n > M.$$

Thus, for $n > \max\{N, M\}$ and for any $R > 0$, it follows that

$$\begin{aligned}\|(\alpha_n c_n - \alpha c)\|_{\infty,R} &\leq \|(\alpha_n c_n - \alpha_n c)\|_{\infty,R} + \|(\alpha_n c - \alpha c)\|_{\infty,R} \\ &= |\alpha_n|\, \|c_n - c\|_{\infty,R} + |\alpha_n - \alpha|\, \|c\|_{\infty,R} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore, $\alpha_n c_n \to \alpha c$ in S_∞, which completes the proof. ■

Since the space S_∞ is a topological vector space equipped with a family of semi-norms $\|\cdot\|_{\infty,R}$, $R > 0$, the topology τ is usually called the *semi-norm topology.* The Hausdorff property of S_∞ is verified next.

Theorem 3.2.5. *The space S_∞ with the semi-norm topology is Hausdorff.*

Proof: For each $R > 0$, the norm property on $S_\infty(R)$ ensures that if $c \neq 0$ then $\|c\|_{\infty,R} \neq 0$. Thus, from Theorem 2.2.4 it follows that S_∞ with the semi-norm topology is Hausdorff. ■

The next result shows that S_∞ with the semi-norm topology is a *locally convex* topological vector space.

Theorem 3.2.6. *The space S_∞ with the semi-norm topology is a locally convex topological vector space.*

Proof: It follows directly from Theorem 2.2.3 part (c), since the semi-norm topology in S_∞ is induced by the family of semi-norms $\|\cdot\|_{\infty,R}$, $R > 0$. ■

Finally, it is only natural to wonder if a sequence converges in the space $S_\infty(R)$ and in the semi-norm topology, will it converge to the same point? This question is answered in the next lemma.

Lemma 3.2.3. *Fix $R > 0$. If $c_i \to c$ as a sequence in $S_\infty(R)$ and $c_i \to c'$ in the semi-norm topology, then $c = c'$.*

Proof: Since $c_i \to c'$ in the semi-norm topology, then $\|c_i - c\|_{\infty,\hat{R}} \to 0$ as $i \to \infty$ for all $\hat{R} > 0$. Then, in particular, for $\hat{R} = R$ it follows that $\|c_i - c\|_{\infty,R} \to 0$ as $i \to \infty$. This is equivalent to $c_i \to c$ as a sequence in $S_\infty(R)$, therefore $c = c'$. ■

3.2.3 The semi-norm topology versus the ultrametric topology

In the previous subsections, a new norm $\|\cdot\|_{\infty,R}$, $R > 0$ on $\mathbb{R}\langle\langle X\rangle\rangle$ was introduced and used to define several spaces. Furthermore, S_∞ was endowed a related semi-norm topology. As mentioned in Chapter 2, a more common metric on $\mathbb{R}\langle\langle X\rangle\rangle$ is the ultrametric metric. The objective of this section is to demonstrate that convergence in the semi-norm topology is in general *unrelated* to convergence in the ultrametric sense. The following example illustrates the case where a series converges in the semi-norm topology but fails to converge in the ultrametric sense.

Example 3.2.3. Consider the sequence of constants in Example 2.2.22

$$\left\{c_i = \frac{1}{i}\right\}_{i\geq 1}$$

as polynomials in $\mathbb{R}\langle\langle X\rangle\rangle$. Clearly,

$$\|c_i - 0\|_{\infty,R} = \frac{1}{i}$$

for all $R > 0$. Thus, $c_i \in S_\infty$ and $c_i \to 0$ as $i \to \infty$ in the semi-norm topology. On the other hand, this sequence does *not* approach zero in the ultrametric sense because $\text{dist}(c_i, 0) = 1$ for every $i \geq 1$. In fact, this sequence is not even Cauchy because $\text{dist}(c_i, c_{i+1}) = 1$ for every $i \geq 1$. □

The next example illustrates the case where a series converges in the ultrametric sense but not in the semi-norm topology.

Example 3.2.4. Consider the sequence of polynomials

$$c_i = 1 + M1!\,x_0 + M^2 2!\,x_0^2 + \cdots + M^i i!\,x_0^i, \quad i \geq 0,$$

where $M > 0$ is fixed. It is easily verified that $\text{dist}(c_i, c) \to 0$ as $i \to \infty$ when

$$c = \sum_{n=0}^{\infty} M^n n!\,x_0^n.$$

Therefore, $c_i \to c$ in the ultrametric sense. Next observe that

$$\|c_i\|_{\infty,R} = \begin{cases} (MR)^i & : \ MR > 1 \\ 1 & : \ MR \leq 1, \end{cases}$$

and thus, each $c_i \in S_{\infty,e}$. Similarly, $c \in S_{\infty,e}$ because $\|c\|_{\infty,R} < \infty$ when $MR \leq 1$. In addition, $c \in S_{\infty,e} \setminus S_\infty$. Note that, if $M = 2$ the series c is the one considered in Example 3.2.2. On the other hand,

$$\|c_i - c\|_{\infty,R} = \sup_{n>i} (MR)^n = \begin{cases} (MR)^{i+1} & : \ MR < 1 \\ 1 & : \ MR = 1 \\ \infty & : \ MR > 1, \end{cases}$$

which implies that

$$\lim_{i\to\infty} \|c_i - c\|_{\infty,R} = \lim_{i\to\infty} (MR)^{i+1} = 0,$$

only when $MR < 1$. Therefore, the sequence $\{c_i\}_{i\geq 1}$ converges to c in the normed linear space $S_\infty(R)$ when $R < 1/M$, but *not* to c in the semi-norm topology. In fact, $\|c_i - c_{i-1}\|_{\infty,R} = (MR)^i$ can not be made arbitrarily small for sufficient large i when $MR \geq 1$. So the sequence is not Cauchy in the semi-norm topology. □

3.2.4 Relationships between $S_{\infty,e}$, S_∞, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$, and $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$

The main relationships between the spaces $S_{\infty,e}$, S_∞, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$, and $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ are presented. First, the following concept is needed in order to make the comparison easier. Given $c \in S_{\infty,e}$, define $\bar{R}_c$ as the supreme of all R for which $c \in S_\infty(R)$, i.e.,

$$\bar{R}_c := \sup_{\substack{\|c\|_{\infty,R}<\infty \\ R>0}} R.$$

As shown in Section 3.1.1, when $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, then there exist constants $K, M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^{\gamma_c}, \ \ \forall \eta \in X^*,$$

where $0 \leq \gamma_c \leq 1$. Also, F_c constitutes a well defined mapping for sufficiently small $R_u, T > 0$. The least upper bound on $\max\{R_u, T\}$ is $\rho(F_c)$. Note that if

$R \leq 1/M \leq \rho(F_c)(m+1)$ then

$$\|c\|_{\infty,R} \leq \sup_{\eta \in X^*} K(MR)^{|\eta|} = K < \infty,$$

otherwise, $\|c\|_{\infty,R}$ is unbounded. Thus, $\bar{R}_c = 1/M$. Which implies that $c \in S_\infty(1/M) \subseteq S_{\infty,e}$. Therefore,

$$\mathbb{R}_{LC}\langle\langle X \rangle\rangle \subseteq S_{\infty,e}. \tag{3.2.9}$$

On the other hand, if $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$, then there exist constants $K, M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^{\gamma_c}, \quad \forall \eta \in X^*,$$

where $0 \leq \gamma_c < 1$. Also, it is not hard to see that for every $R > 0$

$$\|c\|_{\infty,R} \leq \sup_{\eta \in X^*} \frac{K(MR)^{|\eta|}}{(|\eta|!)^{1-\gamma_c}} < \infty,$$

thus $\bar{R}_c = \infty$, which implies that $c \in S_\infty \subset S_{\infty,e}$. Therefore,

$$\mathbb{R}_{GC}\langle\langle X \rangle\rangle \subseteq S_\infty \subset S_{\infty,e}. \tag{3.2.10}$$

The next claim is that, the relation shown in (3.2.9) can be strengthened as shown in the following theorem.

Theorem 3.2.7. $\mathbb{R}_{LC}\langle\langle X \rangle\rangle = S_{\infty,e}$.

Proof: In light of (3.2.9), it only needs to be shown that $S_{\infty,e} \subseteq \mathbb{R}_{LC}\langle\langle X \rangle\rangle$. The proof is by contradiction. If $c \in S_{\infty,e}$, then there exists a finite $\bar{R}_c > 0$ such that $c \in S_\infty(R)$ for all $0 < R < \bar{R}_c$. Now assume $c \notin \mathbb{R}_{LC}\langle\langle X \rangle\rangle$. Then for any constants $K, M > 0$ there is a subseries $\hat{c}$ of c and some $\epsilon > 0$ such that

$$|(\hat{c},\eta)| > KM^{|\eta|}(|\eta|!)^{1+\epsilon}, \quad \forall \eta \in \text{supp}(\hat{c}).$$

On the other hand, for all $0 < R < \bar{R}_c$

$$\|c\|_{\infty,R} = \sup_{\eta \in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} \geq \sup_{\eta \in \text{supp}(\hat{c})} K(MR)^{|\eta|}(|\eta|!)^{\epsilon}.$$

Thus, $\|c\|_{\infty,R} = \infty$ for any $R > 0$. Which contradicts the fact $c \in S_\infty(R)$ for all $0 < R < \bar{R}_c$. Therefore, $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, and the theorem is proved. ■

The relationship between $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and S_∞ is more complicated. First, it is shown in the following section that there exist a series in S_∞ which is not in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. Therefore, the sets are not equivalent. That stated, a result involving the closure of the space $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and S_∞ can be proved. Let $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ denote the closure of $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ in the semi-norm topology on S_∞. This statement makes sense since $\mathbb{R}_{GC}\langle\langle X\rangle\rangle \subseteq S_\infty$, and S_∞ with the semi-norm topology is first countable. Therefore, if $c \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$, then there exists a sequence $\{c_i\}_{i\geq 0}$ in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ which converges to c in the semi-norm topology.

The following theorem illustrates one relationship between $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ and S_∞.

Theorem 3.2.8. $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \subseteq S_\infty$.

Proof: If $c \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$, then there exists a sequence $\{c_i\}_{i\geq 0}$ in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle \subseteq S_\infty \subset S_\infty(R)$ which converges to c in the semi-norm topology. Therefore, $\{c_i\}_{i\geq 0}$ also converges to c as a sequence in the complete normed linear space $S_\infty(R)$ for every $R > 0$. This implies that $c \in S_\infty(R)$ for every $R > 0$. Thus, $c \in S_\infty := \cap_{R>0} S_\infty(R)$. ■

Figure 9 summarizes what relationships have been proved so far concerning the sets of formal power series $S_{\infty,e}$, S_∞, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ and $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

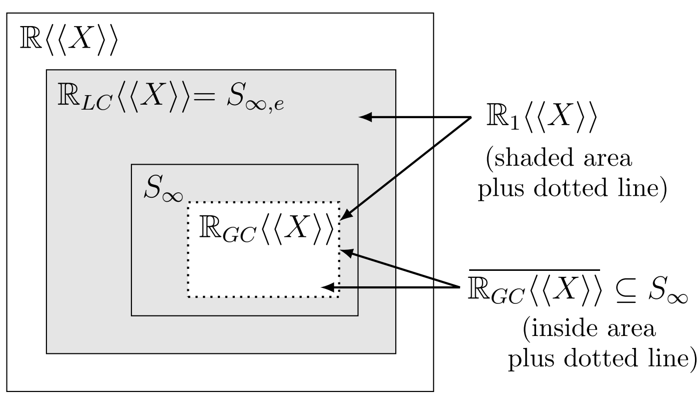

Fig. 9: Relationship between $S_{\infty,e}$, S_∞, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$, and $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

The relationship in Theorem 3.2.8 can be strengthened to an equivalence instead of one sided inclusion. However, the tools needed to prove such a statement will be deferred to Section 3.4.

3.3 EXAMPLE: CLASSIFICATION OF THE FERFERA SERIES

In Chapter 1, Example 1.1.3 mentioned a specific case of a Fliess operator which is globally convergent, but whose generating series does not satisfy (1.1.4). It is only known that such a generating series has a coefficient growth rate strictly between (1.1.3) and (1.1.4) as explained in [41]. As part of the first main goal of this dissertation, a precise growth rate of the coefficients for that generating series is presented. Also, the specific formal power series space to which it belongs is given in this section.

The Ferfera series considered in Example 1.1.3 is analyzed explicitly. Let $X = \{x_0, x_1\}$ and consider the rational series

$$x_1^* := \sum_{k=0}^{\infty} x_1^k.$$

The specific series considered by Ferfera in [10, 11] is

$$c_F := x_1^* \circ x_1^*$$

using the notion of formal power series composition defined in (2.1.3). As shown in Chapter 1, $y = F_{x_1^*}[u]$ has a state space realization $\dot{z} = zu$, $y = z$, $z(0) = 1$ [8]. Cascading two such realizations, the response is similar for different inputs. In fact, it is shown in [41, Theorem 8] that the cascade of any two systems having generating series in $\mathbb{R}_0\langle\langle X\rangle\rangle$ always has a double exponential bounding function. Therefore, the Fliess operator associated to c_F is globally convergent. It was also shown in [10, 11] that c_F does not have a coefficient growth rate satisfying (1.1.4). In fact, the specific growth rate is still unknown. A general formula for the coefficients of c_F is

$$(c_F, x_0^{k_0} x_1^{k_1} \cdots x_0^{k_{l-1}} x_1^{k_l}) = (k_0)^{k_1} (k_0 + k_2)^{k_3} \cdots (k_0 + k_2 + \cdots + k_{l-1})^{k_l} \qquad (3.3.1)$$

for all $l \geq 0$ and $k_i \geq 0$, $i = 0, 1, \ldots, l$ [21]. The following two subseries of c_F are also of interest:

$$c_F^{1/2} := \sum_{k=0}^{\infty} (c_F, x_0^k x_1^k)\, x_0^k x_1^k$$

$$c_F^1 := \sum_{k_0,k_1=0}^{\infty} (c_F, x_0^{k_0} x_1^{k_1})\, x_0^{k_0} x_1^{k_1}.$$

Ferfera's central argument in showing that rationality is not preserved under composition was the observation that the coefficients

$$(c_F^{1/2}, x_0^k x_1^k) = k^k, \quad k \geq 0$$

grow too fast to satisfy (1.1.4). Therefore, c_F can not be rational.

This section is organized as follows: First, in Subsection 3.3.1 the Gevrey order of the Ferfera series is calculated in order to know its precise coefficient growth rate. Subsequently, in Subsection 3.3.2, the specific space to which the Ferfera series belongs is shown.

3.3.1 Gevrey order of the Ferfera series

In order to determine the precise growth rate of the coefficients of the Ferfera series, c_F, it is necessary to calculate its Gevrey order. First, to gain some insight on what to expect, the subseries $c_F^{1/2}$ of c_F is considered and its Gevrey order is calculated.

Theorem 3.3.1. *The series $c_F^{1/2}$ has Gevrey order $s \in [1/2, \infty)$, i.e., $c_F^{1/2} \in \mathbb{R}_{1/2}\langle\langle X \rangle\rangle$.*

Proof: Let $n = \left|x_0^k x_1^k\right| = 2k \geq 0$ and define the sequences

$$a_n = (c_F^{1/2}, x_0^{n/2} x_1^{n/2}) = (n/2)^{(n/2)}$$
$$b_n(s) = KM^n (n!)^s$$

for any fixed $K, M > 0$. Also define the function

$$f_n(s) = \ln\left(\frac{a_n}{b_n(s)}\right) = \frac{n}{2}\ln\left(\frac{n}{2}\right) - \ln(K) - n\ln(M) - s\ln(n!).$$

Using Stirling's approximation

$$\mathcal{O}(n!) = \mathcal{O}\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right), \tag{3.3.2}$$

it follows directly that

$$\mathcal{O}\left(f_n(s)\right) =$$
$$\mathcal{O}\left(-\ln(K) - \frac{n}{2}\ln(2) - n\ln(M) + n\left(\frac{1}{2} - s\right)\ln(n) + ns - \frac{s}{2}\ln(2\pi) - \frac{s}{2}\ln(n)\right).$$

Consider the following cases:

1. If $s < 1/2$ then $\lim_{n\to\infty} f_n(s) \leq \lim_{n\to\infty} n\left(\frac{1}{2} - s\right)\ln(n) + \mathcal{O}(n) = +\infty$.

2. If $s > 1/2$ then $\lim_{n\to\infty} f_n(s) \leq \lim_{n\to\infty} n\left(\frac{1}{2} - s\right)\ln(n) + \mathcal{O}(n) = -\infty$.

3. If $s = 1/2$ then

$$\lim_{n\to\infty} f_n(1/2) =$$
$$-\ln(K) - \frac{1}{4}\ln(2\pi) - \frac{1}{4}\lim_{n\to\infty}\ln(n) + \lim_{n\to\infty} n\left(\frac{1}{2} - \frac{\ln(2)}{2} - \ln(M)\right).$$

Therefore, if $\frac{1}{2} - \frac{\ln(2)}{2} - \ln(M) \leq 0$ then $\lim_{n\to\infty} f_n(1/2) = -\infty$. In summary, when $s \geq 1/2$

$$\lim_{n\to\infty} \frac{a_n}{b_n(s)} = 0,$$

and this implies in particular that $b_n(1/2)$ is growing faster than a_n. On the other hand, if $s < 1/2$ then a_n can not be bounded by a sequence of the form $b_n(s)$. Hence, the coefficients $(c_F^{1/2}, x_0^{n/2}x_1^{n/2})$ must be upper bounded for all $n \geq 0$ by $KM(n!)^{1/2}$ for some $K, M > 0$. That is,

$$(c_F^{1/2}, x_0^{n/2}x_1^{n/2}) < KM(n!)^{1/2}, \quad n \geq 1,$$

which implies that the series $c_F^{1/2}$ has Gevrey order $s \in [1/2, \infty)$, i.e., $c_F^{1/2} \in \mathbb{R}_{1/2}\langle\langle X\rangle\rangle$. ■

As an empirical check, an estimate of the minimum of the Gevrey order of $c_F^{1/2}$ was computed numerically using the nonlinear fitting capabilities of *Mathematica* via the code:

```
nmax=300;
```

```
data=Table[{n,Log[(n/2)^(n/2)]},{n,1,nmax,2}];
nlm=NonlinearModelFit[data,Log[K*M^n*(n!)^s],{K,M,s},n]
Show[ListPlot[data],Plot[nlm[n],{n,1,nmax}],Frame->True]
```

The corresponding growth parameters estimates are shown in Table 1. The quality of the fit for the first 30 coefficients is shown on a semi-logarithmic scale in Figure 10. It is representative of the fit for the entire data set of 300 points.

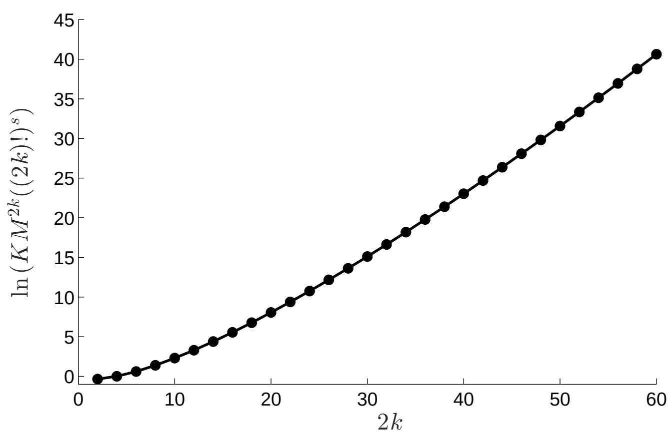

Fig. 10: Empirical fit of $KM^{2k}((2k)!)^s$ (solid line) to the first 30 coefficients (word length $n = 2k \leq 60$) of the series $c_F^{1/2}$ (dots).

TABLE 1: Growth parameters estimates for the series $c_F^{1/2}$.

series coefficients	K	M	s
$(x_1^* \circ x_1^*, x_0^k x_1^k)$	0.39102	1.14373	0.503423

It is important to note that Theorem 3.3.1 has two purposes: First, it shows that $c_F^{1/2} \in \mathbb{R}_{1/2}\langle\langle X \rangle\rangle$, and second, it illustrates a specific example of a series that is in a space $\mathbb{R}_\gamma\langle\langle X \rangle\rangle$ where γ is a number strictly between 0 and 1. Also, since $c_F^{1/2}$ is a subseries of c_F, the minimum value of the Gevrey order of c_F is at least 1/2. The following theorem gives the precise Gevrey order of c_F.

Theorem 3.3.2. *The series c_F has Gevrey order $s \in [1, \infty)$, i.e., $c_F \in \mathbb{R}_1\langle\langle X \rangle\rangle$.*

Proof: It is sufficient to show that c_F^1 has Gevrey order $s \in [1, \infty)$, since c_F contains c_F^1 as a subseries and it is known that c_F could not have a growth rate of its coefficients faster than (1.1.3). In other words, c_F has a minimum Gevrey order $s \leq 1$ (because it has an analytic state space realization [39]). Let $n = \left|x_0^{k_0} x_1^{k_1}\right| = k_0 + k_1 \geq 0$ and

$$d_n := \sum_{k_0=0}^{n} (c_F, x_0^{k_0} x_1^{n-k_0})\, x_0^{k_0} x_1^{n-k_0}.$$

Define the sequences

$$\begin{aligned} a_n(k_0) &= (d_n, x_0^{k_0} x_1^{n-k_0}) = (k_0)^{(n-k_0)},\ 0 \leq k_0 \leq n, \\ b_n(s) &= KM^n (n!)^s,\ K, M > 0, \end{aligned}$$

using (3.3.1) with $l = 1$. Also, define

$$f_n(k_0, s) = \ln\left(\frac{a_n(k_0)}{b_n(s)}\right) = (n - k_0)\ln(k_0) - \ln(K) - n\ln(M) - s\ln(n!),$$

so using (3.3.2) yields

$$\begin{aligned} &\mathcal{O}\left(f_n(k_0, s)\right) = \\ &\quad \mathcal{O}\left((n - k_0)\ln(k_0) - \ln(K) - n\ln(M) + sn - \frac{s}{2}\ln(2\pi n) - sn\ln(n).\right) \end{aligned}$$

Observe that $f_n(k_0, s)$ has a maximum over $\mathbb{R}$ if and only if

$$k_0 = \hat{k}_0 := \exp(W(ne) - 1),$$

since

$$\left.\frac{\partial f_n(k_0, s)}{\partial k_0}\right|_{k_0 = \hat{k}_0} = -\ln(\hat{k}_0) + \frac{(n - \hat{k}_0)}{\hat{k}_0} = 0$$

and

$$\left.\frac{\partial^2 f_n(k_0, s)}{\partial k_0^2}\right|_{k_0 = \hat{k}_0} = -\frac{1}{\hat{k}_0} - \frac{n}{\hat{k}_0^2} < 0,\ \ \forall\, 0 \leq \hat{k}_0 \leq n,$$

where W denotes the *Lambert W–function*, namely, the inverse of the function $g(z) = z\exp(z)$ [5]. Therefore, the goal is to compute $\lim_{n\to\infty} f_n(\hat{k}_0, s)$. Observe that

$$\frac{\partial}{\partial s}\lim_{n\to\infty} f_n(\hat{k}_0, s) = \lim_{n\to\infty}\left(n - \frac{1}{2}\ln(2\pi n) - n\ln(n)\right) < 0,$$

which implies that $\lim_{n\to\infty} f_n(\hat{k}_0, s)$ is a non-increasing function of s. A direct calculation gives

$$\lim_{n\to\infty} f_n(\hat{k}_0, s) = \lim_{n\to\infty} \Big(nW(ne) + W(ne)\exp(W(ne)-1) + \exp(W(ne)-1) \\ +sn - n - n\ln(M) - \frac{s}{2}\ln(2\pi n) - sn\ln(n)\Big).$$

Using the fact that $W(ne)\exp(W(ne)-1) = n$ gives

$$\lim_{n\to\infty} f_n(\hat{k}_0, s) = \lim_{n\to\infty} \Big(nW(ne) + \exp(W(ne)-1) + sn \\ -n\ln(M) - \frac{s}{2}\ln(2\pi n) - sn\ln(n)\Big).$$

This reduces to computing the limit

$$\lim_{n\to\infty} nW(ne) - sn\ln(n).$$

But since $\lim_{n\to\infty} W(ne)/\ln(n) = 1$, it follows that:

1. If $s < 1$ then $\lim\limits_{n\to\infty} f_n(\hat{k}_0, s) = +\infty$.

2. If $s \geq 1$ then $\lim\limits_{n\to\infty} f_n(\hat{k}_0, s) = -\infty$.

Thus, if $s \geq 1$ then

$$\lim_{n\to\infty} \frac{a_n(\hat{k}_0)}{b_n(s)} = 0,$$

which implies that $b_n(1)$ is growing faster than $a_n(\hat{k}_0)$, and thus faster than $a_n(k_0)$ for all $0 \leq k_0 \leq n$. On the other hand, if $s < 1$ then $a_n(k_0)$ can not be bounded by a sequence of the form $b_n(s)$. Hence, the coefficients of c_F^1 for words of length n must be upper bounded by $KM^n n!$ for some $K, M > 0$. Namely,

$$(c_F^1, x_0^{k_0} x_1^{n-k_0}) < KMn! \;\; \forall\, 0 \leq k_0 \leq n < \infty$$

and no smaller Gevrey type bound applies. This, implies that the series c_F^1 has Gevrey order $s \in [1, \infty)$. Therefore, the series c_F has also Gevrey order $s \in [1, \infty)$, i.e., $c_F \in \mathbb{R}_1\langle\langle X\rangle\rangle$. ■

An estimate of the minimum of the Gevrey order of c_F^1 was also computed numerically using *Mathematica* via the code:

```
nmax=300;
data=Reap[For[n=1,n >=nmax,n++,maxn=0;
For[j=0,j  >= i,j++,m=(n-j)*Log[j];maxn=If[m >=maxn,m,maxn]]
Sow[{n,maxn}]];][[2,1]];
nlm=NonlinearModelFit[data,Log[K*M^n*(n!)^s],{K,M,s},n]
Show[ListPlot[data],Plot[nlm[n],{n,1,nmax}],Frame->True]
```

The fit is as shown in Figure 11. A sample of the corresponding data is shown in Table 2. The asymptotic behavior of the estimates of the Gevrey order of c_F^1 as a function of maximum word length is shown on a semi-logarithmic scale in Figure 12. The estimates are monotonically increasing towards $s = 1$ but at an extremely slow rate.

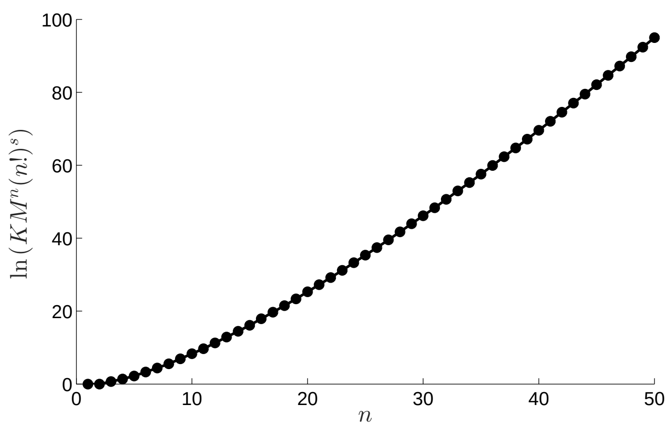

Fig. 11: Empirical fit of $KM^n(n!)^s$ (solid line) to the first 30 coefficients of the series c_F^1 (dots).

3.3.2 Topological aspects of the Ferfera series

It is important to observe that $c_F \notin \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ since $c_F \in \mathbb{R}_1\langle\langle X\rangle\rangle$ as a consequence of Theorem 3.3.2. Thus, c_F can be in either $S_{\infty,e} \setminus S_\infty$ or $S_\infty \setminus \mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

TABLE 2: Growth parameters estimates for the series c_F^1.

maximum word length	K	M	s
50	1.49671	0.696282	0.758571
300	6.60041	0.579581	0.808544
500	19.496	0.545318	0.821234
5000	1.04761×10^9	0.414991	0.865870

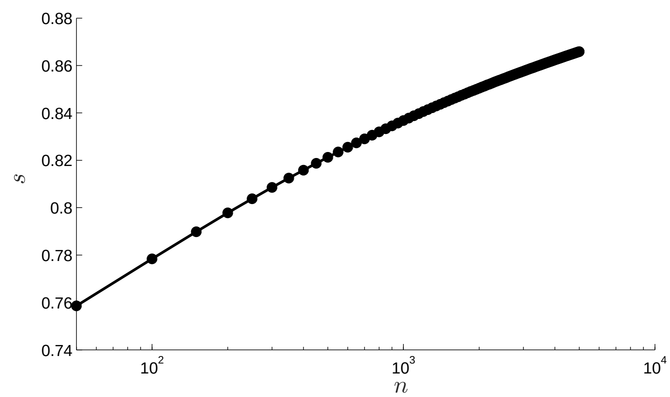

Fig. 12: Gevrey order estimates of c_F^1 as a function of maximum word length n.

However, a more specific set where it belongs cannot be given at this point. Such a set will be presented in the next section. Here the goal is to provide the specific set to which the subseries c_F^1 of c_F belongs. The following theorem says that c_F^1 is in the closure of $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ in the semi-norm topology.

Theorem 3.3.3. *The series* $c_F^1 \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$.

Proof: It is sufficient to construct a sequence in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ that converges to c_F^1 in the semi-norm topology since $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ must contain all its limit points by the definition of closure. Consider the truncation of c_F^1,

$$d_N := \sum_{n=0}^{N} d_n := \sum_{n=0}^{N} \sum_{k_0=0}^{n} (c_F, x_0^{k_0} x_1^{n-k_0})\, x_0^{k_0} x_1^{n-k_0}.$$

Clearly, the polynomial $d_N \in \mathbb{R}_0\langle\langle X \rangle\rangle$. Observe that for any $R > 0$

$$\|d_N\|_{\infty,R} = \sup_{\substack{n \leq N \\ 0 \leq k_0 \leq n}} \left\{ k_0^{n-k_0} \frac{R^n}{n!} \right\} < \infty,$$

and

$$\|d_N - c_F^1\|_{\infty,R} = \sup_{\substack{n > N \\ 0 \leq k_0 \leq n}} \left\{ k_0^{n-k_0} \frac{R^n}{n!} \right\}.$$

Since $k_0 = \hat{k}_0 := \exp(W(ne) - 1)$ maximizes $k_0^{n-k_0}$ over $0 \leq k_0 \leq n$,

$$\|d_N - c_F^1\|_{\infty,R} \leq \sup_{n > N} \left\{ \exp(W(ne) - 1)^{n - \exp(W(ne)-1)} \frac{R^n}{n!} \right\}. \tag{3.3.3}$$

Now define

$$f(n) = \exp(W(ne) - 1)^{n - \exp(W(ne)-1)} \frac{R^n}{n!}.$$

Applying the logarithm to both sides of this equation and using (3.3.2), it follows that

$$\ln(f(n)) = (W(ne) - 1)(n - \exp(W(ne) - 1)) + n\ln(R) + n - \frac{1}{2}\ln(2\pi n) - n\ln(n).$$

The identity $W(ne)\exp(W(ne) - 1)) = n$ then yields

$$\ln(f(n)) = nW(ne) + \frac{n}{W(ne)} + n(\ln(R) - 1) - \frac{1}{2}\ln(2\pi n) - n\ln(n). \tag{3.3.4}$$

Observe that $f(n)$ has a maximum over $\mathbb{R}$ if and only if $n = \hat{n}$, where

$$\left.\frac{\mathrm{d}\ln(f(n))}{\mathrm{d}n}\right|_{n=\hat{n}} = W(\hat{n}e) - \frac{1}{2\hat{n}} - \ln(\hat{n}) + \ln(R) - 1 = 0$$

since

$$\left.\frac{\mathrm{d}^2\ln(f(n))}{\mathrm{d}n^2}\right|_{n=\hat{n}} = \frac{W(\hat{n}e) - 2\hat{n} + 1}{2\hat{n}^2(W(\hat{n}e) + 1)} < 0.$$

Therefore,

$$\sup_{n>N} f(n) = \sup_{n>N} \left\{ \exp(W(ne)-1)^{n-\exp(W(ne)-1)} \frac{R^n}{n!} \right\}$$
$$= \max\{f(N), f(\hat{n})\}.$$

Substituting this bound into (3.3.3) and taking the limit gives

$$\lim_{N\to\infty} \|d_N - c_F^1\|_{\infty,R} \leq \lim_{N\to\infty} \max\{f(N), f(\hat{n})\} = \lim_{N\to\infty} f(N).$$

Now using (3.3.4) and the fact that $\lim_{N\to\infty} \ln f(N) = \ln \lim_{N\to\infty} f(N)$, it follows that

$$\lim_{N\to\infty} \ln(f(N)) = \lim_{N\to\infty} N(W(Ne) - \ln(N) + \ln(R) - 1)) + \frac{N}{W(Ne)} - \frac{1}{2}\ln(2\pi N)$$
$$= -\infty.$$

The identity $\lim_{N\to\infty} W(Ne) - \ln(N) = -\infty$ has also been used above. Thus, for any $R > 0$

$$\lim_{N\to\infty} \|d_N - c_F^1\|_{\infty,R} \leq \lim_{N\to\infty} f(N) = 0.$$

Hence, the sequence $\{d_N\}_{N\geq 0} \in \mathbb{R}_0\langle\langle X\rangle\rangle \subset \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ converges to c_F^1 in the semi-norm topology, and, consequently, $c_F^1 \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$. ■

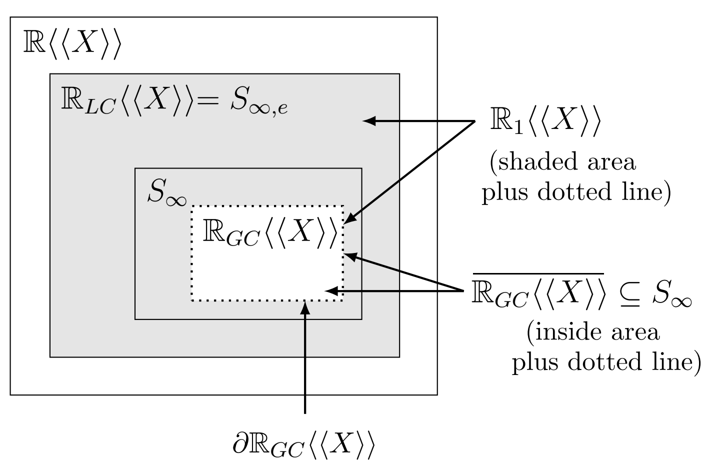

Fig. 13: Relationships between $S_{\infty,e}$, S_∞, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle$, $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and $\partial\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

Finally, by simple set theory, $\partial\mathbb{R}_{GC}\langle\langle X\rangle\rangle = \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \setminus \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ (as indicated in Figure 13), thus one can write $c_F^1 \in \partial\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. On the other hand, $c_F^1 \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \subset S_\infty$ which implies that $c_F^1 \in S_\infty \setminus \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. This later fact confirms that, as expected in Subsection 3.2.4, there exists a series in S_∞ which is not in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$, namely, c_F^1.

3.4 ON THE RADIUS OF CONVERGENCE OF FLIESS OPERATORS

In this section, the first main goal of this dissertation is addressed, namely, developing the precise relationship between the growth rate of the coefficients of a generating series and the nature of the convergence of its corresponding Fliess operator. First, a sufficient condition for a generating series to have a corresponding locally convergent Fliess operator with finite radius of convergence is given. Next, a complete characterization (extending the one given in Section 3.2) of the space S_∞ is given, specifically, the equality $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} = S_\infty$ is proved, and the space S_∞ with the semi-norm topology is shown to be a Fréchet space. Finally, a necessary and sufficient condition for a generating series to have a corresponding globally convergent Fliess operator and a locally convergent Fliess operator with finite radius of convergence is given.

The following classical theorems from complex analysis are used extensively throughout this section.

Theorem 3.4.1. *[45] Consider a power series $f(z) = \sum_{n\geq 0} a_n z^n$ defined on $\mathbb{C}$. There exists a real number $0 \leq R \leq \infty$, called the radius of convergence of the series f, such that the series converges for all values of z with $|z| < R$ and diverges for all z such that $|z| > R$ with $R = 1/\limsup_{n\to\infty} |a_n|^{1/n}$ $(1/0 := \infty,\ 1/\infty := 0)$.*

Theorem 3.4.2. *[45] Let $f(z) = \sum_{n\geq 0} a_n z^n/n!$ be a function which is analytic at $z = 0$. Suppose z_0 is a singularity of f having smallest modulus. Then for any $\epsilon > 0$ there exists an integer $N \geq 0$ such that for all $n > N$, $|a_n| < (1/|z_0| + \epsilon)^n n!$. Furthermore, for infinitely many n, $|a_n| > (1/|z_0| - \epsilon)^n n!$.*

3.4.1 Locally convergent Fliess operators with finite radius of convergence

A sufficient condition for a generating series to have a corresponding locally convergent Fliess operator is given next.

Theorem 3.4.3. *If $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ then the radius of convergence of series (1.1.2) is finite.*

Proof: Since $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$, there exists $R_u, T > 0$ such that for any $u \in B_1^m(R_u)[0,T]$ the operator converges absolutely and uniformly on $[0,T]$. Define the truncation $c_N = \sum_{n=0}^{N}\sum_{\eta\in X^n}(c,\eta)\eta$. Clearly, $c_N \in \mathbb{R}_0\langle\langle X\rangle\rangle \subset \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, and thus, the operator defining $F_{c_N}[u](t)$ converges absolutely and uniformly on $[0,T]$ for any $T > 0$ and $u \in L_{1,e}(0)$. Furthermore, observe that for any fixed $N > 0$, the radius of convergence of the series

$$F_c[u](t) = F_{c_N}[u](t) + F_{c-c_N}[u](t),$$

is finite if and only if

$$F_{c-c_N}[u](t) = \sum_{k=N+1}^{\infty}\sum_{\eta\in X^k}(c-c_N,\eta)E_\eta[u](t)$$

has a finite radius of convergence. The key observation is that the sequence $\{c_N\}_{N\geq 0}$ can not converge to c in the semi-norm topology, otherwise $c \in \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$, which contradicts the assumption that $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$. Using this fact, a finite singularity of $F_{c-c_N}[u](t)$ can be constructed. This implies that $F_c[u](t)$ also has a finite singularity, and therefore, a finite radius of convergence. Following [7, 9, Example 1], it is immediate that

$$|F_{c-c_N}[u](t)| \leq \sum_{n=N+1}^{\infty}\sum_{\eta\in X^n}|(c-c_N,\eta)|\,\frac{\hat{R}^n}{n!} \leq \sum_{n=0}^{\infty} a_n\frac{\hat{R}^n}{n!}, \tag{3.4.1}$$

where $\hat{R} := 2\max\{R_u, T\} > 0$ and

$$a_n := \begin{cases} \max_{\eta\in X^n}|(c-c_N,\eta)| & : \ n > N \\ 0 & : \ n \leq N. \end{cases}$$

Define

$$L = \lim_{N\to\infty}\|c_N - c\|_{\infty,R} = \lim_{N\to\infty}\sup_{\eta\in X^*}\left\{|(c-c_N,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\}. \tag{3.4.2}$$

Note that $L > 0$ for some $R > 0$ since $\{c_N\}_{N\geq 0}$ does not converge to c in the semi-norm topology. In particular, choosing $R = \hat{R}$ gives

$$L = \lim_{n\to\infty} \sup_{n\geq 0} \left\{ |a_n| \frac{\hat{R}^n}{n!} \right\}. \tag{3.4.3}$$

The definition of the limit superior implies that for any $0 < \epsilon < 1$ there exists an integer $N \geq 0$ such that for all $n > N$, $|a_n|\, \hat{R}^n/n! < L+\epsilon$. Furthermore, for infinitely many n, $|a_n|\, \hat{R}^n/n! > L - \epsilon$ [45, p. 46]. From the first inequality

$$|a_n| < \frac{(L+\epsilon)n!}{\hat{R}^n} \leq \left(\frac{L^{1/n}}{\hat{R}} + \epsilon' \right)^n n!,$$

and for infinitely many n,

$$|a_n| > \frac{(L-\epsilon)n!}{\hat{R}^n} \geq \left(\frac{L^{1/n}}{\hat{R}} - \epsilon' \right)^n n!,$$

where $\epsilon' := \epsilon^{1/N}/\hat{R}$. Thus, from Theorems 3.4.1 and 3.4.2 it follows that

$$z_0 := \lim_{n\to\infty} \frac{\hat{R}}{L^{1/n}} = \frac{1}{\limsup_{n\to\infty} \left(|a_n|/n!\right)^{1/n}}.$$

Since $L > 0$, the real number $z_0 \neq 0$ is a finite singularity of $f(z) := \sum_{n=0}^{\infty} a_n z^n/n!$. In light of (3.4.1) then $F_{c-c_N}[u](t)$ must also have a finite singularity, and the theorem is proved. ■

The previous theorem gave a sufficient condition for a generating series to have a locally convergent with finite radius of convergence Fliess operator, namely the generating series must be in the space $\mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$. Therefore, it is important to know as much as possible about this space. The following theorem gives one such characterization involving subseries growing at the factorial rate.

Theorem 3.4.4. *Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ with growth constants $K, M > 0$. Then, there exists a subseries $\hat{c} \in \mathbb{R}_1\langle\langle X\rangle\rangle$ of c whose coefficients are growing exactly at the rate $KM^{|\eta|}\,|\eta|!$.*

Proof: Following the proof of Theorem 3.4.3, for any $\epsilon > 0$ and $L > 0$ defined as in (3.4.3), there must exist an integer $N > 0$ such that

$$\left(\frac{L^{1/n}}{\hat{R}} - \epsilon' \right)^n n! < |a_n| < \left(\frac{L^{1/n}}{\hat{R}} + \epsilon' \right)^n n! \tag{3.4.4}$$

for all $n > N$. Let

$$a_n := \begin{cases} \max_{\eta \in X^n} |(c - c_N, \eta)| & : \ n > N \\ 0 & : \ n \leq N, \end{cases}$$

and for each $n > N$ define

$$\eta_n^* := \arg \max_{\nu \in X^n} |(c, \nu)|.$$

Construct $\hat{c} \in \mathbb{R}\langle\langle X \rangle\rangle$ such that for all $\eta \in X^n$, $n \geq 0$

$$(\hat{c}, \eta) := \begin{cases} (c, \eta_n^*) & : \ \eta = \eta_n^*, \ n > N \\ 0 & : \ \text{otherwise.} \end{cases}$$

Clearly $\hat{c}$ is a subseries of c, and by design $|a_n| = |(\hat{c}, \eta)|$ for all $\eta \in X^n$ since $\text{supp}(\hat{c}) \subset X^* \setminus X^N$. Thus, a direct application of (3.4.4) gives for some $K > 0$

$$|(\hat{c}, \eta)| = KM^{|\eta|} \, |\eta|!, \ \forall \eta \in \text{supp}(\hat{c}),$$

where

$$M := \lim_{n \to \infty} \frac{L^{1/n}}{\hat{R}} = \lim_{N \to \infty} \left(\sup_{\eta \in X^*} \left\{ |(c - c_N, \eta)| \, \frac{1}{|\eta|!} \right\} \right)^{1/N}.$$

∎

Actually, Theorem 3.4.4 can be extended to show that a series is in $\mathbb{R}_{LC}\langle\langle X \rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$ if and *only if* the series has a subseries growing at the factorial rate. However, at this point that conclusion cannot be proved without more tools. In particular, a characterization of the space S_∞ beyond the one given in Section 3.2 is needed. First, Theorem 3.2.8 is strengthened to an equivalence instead of as a one sided inclusion in the following result.

Theorem 3.4.5. $\overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle} = S_\infty$.

Proof: From Theorem 3.2.8 it is known that $\overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle} \subseteq S_\infty$. Thus, it only needs to be shown that $S_\infty \subseteq \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$. The proof is by contradiction. Suppose $c \in S_\infty$ with $c \notin \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$. Then $c \in S_\infty \setminus \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle} \subset \mathbb{R}_{LC}\langle\langle X \rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle}$, and by Theorem 3.4.4 there exists a subseries $\hat{c}$ of c such that

$$|(\hat{c}, \eta)| = KM^{|\eta|} \, |\eta|!, \ \ \forall \eta \in \text{supp}(\hat{c}).$$

Then for $R > 1/M$

$$\|\hat{c}\|_{\infty,R} = \sup_{\eta \in \mathrm{supp}(\hat{c})} K(MR)^{|\eta|} = \infty.$$

Therefore, $\|c\|_{\infty,R} = \infty$ when $R > 1/M$ since by Lemma 3.2.1 it follows that $\|\hat{c}\|_{\infty,R} \leq \|c\|_{\infty,R}$. This is a contradiction since $c \in S_\infty$. ■

Subsequently, a characterization of S_∞ related to its completeness is needed.

Lemma 3.4.1. *The space S_∞ with the semi-norm topology is complete.*

Proof: Given that $S_\infty = \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ is closed in the semi-norm topology, $S_\infty \subset S_\infty(R)$, and $S_\infty(R)$ is a complete metric space for any fixed $R > 0$, then, using [16, Proposition 0.24] it follows that S_∞ is a complete space. ■

Next, the space S_∞ with the semi-norm topology is showed to be a Fréchet space.

Theorem 3.4.6. *The space S_∞ with the semi-norm topology is a Fréchet space.*

Proof: The prove is done by a simple check of all the requirements in Definition 2.2.29. First, the space is known to be Hausdorff by Theorem 3.2.5. The space is also complete from the previous Lemma 3.4.1. Therefore, the space S_∞ is a complete topological vector space whose topology is defined by a countable family of semi-norms, i.e., it is a Fréchet space. ■

Given all the new information about the spaces, Figure 13 can be updated as shown in Figure 14. This summarizes the final relationships between $S_{\infty,e}$, S_∞, and the various notions of convergence.

Next, the anticipated result involving Theorem 3.4.4 as both a necessary and sufficient condition is given.

Theorem 3.4.7. *A series $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ if and only if there exists a subseries $\hat{c} \in \mathbb{R}_1\langle\langle X\rangle\rangle$ of c, whose coefficients are each growing exactly at the rate $KM^{|\eta|}\,|\eta|!$ for some growth constants $K, M > 0$.*

Proof: In light of Theorem 3.4.4, the sufficient condition is already proved. Thus, only the necessary condition needs to be shown. Let $\hat{c} \in \mathbb{R}_1\langle\langle X\rangle\rangle$ be a subseries of c with coefficients growing exactly at the rate $KM^{|\eta|}\,|\eta|!$. Then, it follows that

$$\|\hat{c}\|_{\infty,R} = \sup_{\eta \in \mathrm{supp}(\hat{c})} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} = \sup_{\eta \in \mathrm{supp}(\hat{c})} K(MR)^{|\eta|} = \infty,$$

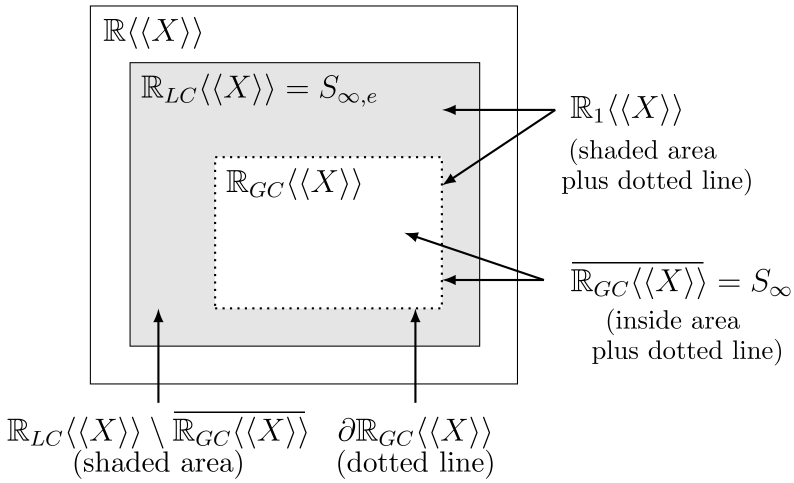

Fig. 14: Relationships between $S_{\infty,e}$, S_∞ and the various notions of convergence.

when $R > 1/M$. Using Lemma 3.2.1, $\|c\|_{\infty,R} = \infty$ when $R > 1/M$. Therefore, $c \in S_{\infty,e} \setminus S_\infty$. Applying Theorems 3.2.7 and 3.4.5 gives $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$, which completes the proof. ■

The next example illustrates Theorem 3.4.7 and Theorem 3.4.3.

Example 3.4.1. Consider the series $c = \sum_{k=0}^{\infty} k!\, x_1^k$. Clearly, it is growing at the factorial rate, in which case, $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ by Theorem 3.4.7. On the other hand,

$$\|c\|_{\infty,R} = \sup_{\eta\in X^*}\left\{|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\} = \sup_{n\geq 0} R^n.$$

Thus, $\|c\|_{\infty,R} < \infty$ if and only if $R < 1$. This indicates that the radius of convergence of $F_c[u](t)$ is unity. To confirm this, apply the identity $k!\, x_1^k = x_1^{\sqcup\!\sqcup\, k}$ so that

$$F_c[u](t) = \sum_{k=0}^{\infty} k! E_{x_1^k}[u](t) = \sum_{k=0}^{\infty} E_{x_1}^k[u](t) = \frac{1}{1 - E_{x_1}[u](t)}.$$

Setting $u = 1$ gives $F_c[1](t) = 1/(1-t)$, which has a finite escape time at $t = 1$ as expected. □

3.4.2 Global convergence of Fliess operators

A sufficient conditions for a generating series to have a corresponding globally convergent Fliess operator is given next.

Theorem 3.4.8. *If $c \in S_\infty$, then the radius of convergence of series (*1.1.2*) is infinite.*

The proof of this theorem is based on the approach taken to prove Theorem 3.4.3. Recall that a series $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ was selected, and then its truncation was defined as

$$c_N = \sum_{n=0}^{N} \sum_{\eta \in X^n} (c,\eta)\eta, \quad N \geq 0.$$

Clearly, $c_N \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. On the other hand, for any fixed $N > 0$,

$$F_c[u](t) = F_{c_N}[u](t) + F_{c-c_N}[u](t),$$

which implied that

$$F_{c-c_N}[u](t) = \sum_{k=N+1}^{\infty} \sum_{\eta \in X^k} (c - c_N, \eta) E_\eta[u](t).$$

Then, a bound for the function $|F_{c-c_N}[u](t)|$ was imposed using (3.4.1). Next, the limit L was defined by (3.4.2) as

$$L = \lim_{N\to\infty} \|c_N - c\|_{\infty,R}.$$

The key observation in this procedure was that $L > 0$ for some $R > 0$. This is because c was in the complement of $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$, and therefore, any sequence in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ would never converge to c in the semi-norm topology. In particular, the sequence $\{c_i\}_{i\geq 0}$ does not converge to c in the semi-norm topology. In this case, the series is in $S_\infty = \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and not in its complement $\mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} = S_{\infty,e} \setminus S_\infty$ as in the previous subsection. Thus, the procedure summarized above until the definition of L still holds in the present context. The hypothesis now is that the sequence $\{c_i\}_{i\geq 0}$ converges to c in the semi-norm topology when $c \in S_\infty$, so that $L = 0$ for all $R > 0$. In order to prove this hypothesis, the following lemma is needed first. It can be viewed as a generalization of Example 3.2.4.

Lemma 3.4.2. *Let $c \in S_{\infty,e}$ and define $c_N = \sum_{n=0}^{N} \sum_{\eta \in X^n} (c,\eta)\eta$, $N \geq 0$. Then there exists an $R > 0$ such that $c_N \to c$ as a sequence in the normed linear space $S_\infty(R)$.*

Proof: If $c \in S_{\infty,e} = \mathbb{R}_{LC}\langle\langle X \rangle\rangle$ then $|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^{\gamma_c}$, $\forall \eta \in X^*$ for some $K, M > 0$ and $0 \leq \gamma_c \leq 1$. Therefore,

$$\|c_N - c\|_{\infty,R} \leq \sup_{n>N} K \frac{(MR)^n}{(n!)^{1-\gamma_c}}.$$

When $0 \leq \gamma_c < 1$ it follows that

$$\begin{aligned} \lim_{N\to\infty} \|c_N - c\|_{\infty,R} &\leq \lim_{N\to\infty} \sup_{n>N} K \frac{(MR)^n}{(n!)^{1-\gamma_c}} \\ &= \lim_{N\to\infty} K \frac{(MR)^{N+1}}{((N+1)!)^{1-\gamma_c}} = 0 \end{aligned} \tag{3.4.5}$$

for all $R > 0$. On the other hand, when $\gamma_c = 1$

$$\lim_{N\to\infty} \|c_N - c\|_{\infty,R} \leq \lim_{N\to\infty} K(MR)^{N+1} = 0,$$

when $R < 1/M$ and infinity otherwise. This implies in both cases that there exists an $R > 0$ such that $c_N \to c$ as a sequence in the normed linear space $S_\infty(R)$. ■

The following corollary describes the particular case when $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle \subset S_{\infty,e}$.

Corollary 3.4.1. *If $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ then $c_N \to c$ in the semi-norm topology.*

Proof: The claim follows directly from (3.4.5). ■

The next lemma provides the exact hypothesis needed in order to prove Theorem 3.4.8, namely, that the sequence $\{c_N\}_{N\geq 0}$ converges to c in the semi-norm topology when $c \in S_\infty$.

Lemma 3.4.3. *Let $c \in S_\infty$ and define $c_N = \sum_{n=0}^{N} \sum_{\eta \in X^n} (c,\eta)\eta$, $N \geq 0$. Then $c_N \to c$ in the semi-norm topology.*

Proof: In light of Corollary 3.4.1, the claim only needs to be shown for series in $\partial\mathbb{R}_{GC}\langle\langle X \rangle\rangle$. The proof is by contradiction. If $c \in S_\infty$ then

$$\|c\|_{\infty,R} < \infty, \ \forall R > 0. \tag{3.4.6}$$

Now suppose $\{c_N\}_{N\geq 0}$ does not converges to c in the semi-norm topology. In which case,

$$L = \lim_{N\to\infty} \|c_N - c\|_{\infty,R} > 0 \tag{3.4.7}$$

for some $R > 0$. Note that the proof of Theorem 3.4.3 uses only the fact that (3.4.2) holds since $\{c_N\}_{N\geq 0}$ does not converge to c in the semi-norm topology. Therefore, L in (3.4.3) is well defined and (3.4.4) also holds. Following the proof of Theorem 3.4.4, a subseries $\hat{c} \in \mathbb{R}_1\langle\langle X\rangle\rangle$ of c whose coefficients are growing exactly at the rate $KM^{|\eta|}\,|\eta|!$ for some $K, M > 0$ is constructed. However, by Theorem 3.4.7 it follows that $c \in\ S_{\infty,e} \setminus S_\infty$. This fact contradicts (3.4.6), which completes the proof. ■

Now, the proof of Theorem 3.4.8 can be given.

Proof of Theorem 3.4.8*:* Following the same approach as in the proof of Theorem 3.4.3, one is led to the conclusion in this case that for any $R > 0$

$$L = \lim_{N\to\infty} \|c_N - c\|_{\infty,R} = 0$$

precisely because the sequence $\{c_N\}_{N\geq 0}$ converges to c in the semi-norm topology via Lemma 3.4.3. Applying Theorems 3.4.1 and 3.4.2 as before now gives

$$z_0 := \lim_{n\to\infty} \frac{\hat{R}}{L^{1/n}} = \frac{1}{\limsup_{n\to\infty} \left(|a_n|/n!\right)^{1/n}} = \infty.$$

Thus, f can not have a finite singularity, implying that $F_c[u](t)$ has a infinite radius of convergence. ■

It is important to note that Theorem 3.4.3 and 3.4.8 can both be extended to give not only a sufficient condition but also a necessary condition. The following two theorems accomplish the first main goal of this dissertation, namely to develop the precise relationship between the growth rate of the coefficients of a generating series and the nature of the convergence of its corresponding Fliess operator.

Theorem 3.4.9. *A series $c \in S_{\infty,e} \setminus S_\infty$ if and only if the radius of convergence of series (*1.1.2*) is finite.*

Proof: The sufficient condition was already proved in Theorem 3.4.3. Thus, only the necessary condition needs to be shown. The proof will be done by contradiction. Given a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ whose associated Fliess operator (1.1.2) has finite radius of convergence, assume $c \in S_\infty$. Thus, by Theorem 3.4.8, the radius of convergence of series (1.1.2) is infinite. This yields a contradiction since the radius of convergence was finite. Therefore, $c \in S_{\infty,e} \setminus S_\infty$, which completes the proof. ■

The following result also follows from the fact that $S_{\infty,e} \setminus S_\infty$ and S_∞ are complements.

Theorem 3.4.10. *A series $c \in S_\infty$ if and only if the radius of convergence of series (1.1.2) is infinite.*

Proof: Analogous to the proof of the previous theorem, only the necessary condition needs to be shown. Given a series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ whose associated Fliess operator (1.1.2) has infinite radius of convergence, assume $c \in S_{\infty,e} \setminus S_{\infty,e}$. Thus, by Theorem 3.4.8, the radius of convergence of series (1.1.2) is finite. This yields a contradiction since the radius of convergence was infinite. Therefore, $c \in S_\infty$, and the theorem is proved. ■

In light of Theorem 3.4.10, it now makes more sense to call $S_\infty = \overline{\mathbb{R}_{GC}\langle\langle X \rangle\rangle} \supset \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ the set of *globally convergent* generating series. This means, of course, that the growth rate of the coefficients in the generating series is no longer the sole indicator of whether the Fliess operator is globally convergent. In Section 3.3 it was proved that $c_F^1 \in \partial\mathbb{R}_{GC}\langle\langle X \rangle\rangle$. Now another series on the border is exhibited.

Corollary 3.4.2. *The series $c_F \in \partial\mathbb{R}_{GC}\langle\langle X \rangle\rangle$.*

Proof: In [41, Theorem 8] it was shown that c_F always has a double exponential bounding function and that ensures that $F_{c_F}[u](t)$ is well defined on $[0, T]$ for any $T > 0$ and $u \in L_{1,e}(0)$. Thus, the radius of convergence of F_{c_F} is infinite. Applying, Theorem 3.4.10 it follows that $c_F \in S_\infty$. However, by Theorem 3.3.2, $c_F \in \mathbb{R}_1\langle\langle X \rangle\rangle$. Hence, $c_F \in \partial\mathbb{R}_{GC}\langle\langle X \rangle\rangle$. ■

The next example illustrates Theorem 3.4.10.

Example 3.4.2. Reconsider the series c_F^1 and the truncated version d_N as defined in the proof of Theorem 3.3.3. From (3.4.1) it follows that

$$\left|F_{c_F^1 - d_N}[u](t)\right| \leq \sum_{n>N} \sum_{k_0=0}^{n} k_0^{n-k_0} \frac{R^n}{n!}.$$

Therefore, $F_{c_F^1 - d_N}[u](t)$ converges for all $R, T > 0$ using the ratio test on the upper bound above. In addition, $F_{c_F^1}[u](t) = F_{d_N}[u](t) + F_{c_F^1 - d_N}[u](t)$ is also bounded, and thus, this further confirms the claim in Theorem 3.3.3 that $c_F^1 \in S_\infty$. □

3.5 SUMMARY

It was proved in Section 3.1 that having $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, i.e., $0 \leq \gamma_c < 1$, is a sufficient condition for global convergence of the corresponding Fliess operator. Section 3.2 introduced the spaces of formal power series, using the norm, $\|\cdot\|_{\infty,R}$, $R > 0$. Then, the space $S_\infty := \cap_{R>0} S_\infty(R)$ was proved to be a locally convex topological vector space with a family of semi-norms. In addition, two important relationships between these sets were proved, specifically, $\mathbb{R}_{LC}\langle\langle X\rangle\rangle = S_{\infty,e} := \cup_{R>0} S_\infty(R)$, and $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \subseteq S_\infty$. Subsequently, in Section 3.3, the example of the Ferfera series c_F was considered. First, a precise growth rate for the coefficients of c_F is presented, in particular is shown that $c_F \in \mathbb{R}_1\langle\langle X\rangle\rangle$. Then, the specific set to which its subseries c_F^1 belongs to is given, namely, $c_F^1 \in \partial\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. Finally, Section 3.4 gives the precise relationship between the growth rate of the coefficients of a generating series and the nature of the convergence of its corresponding Fliess operator via Theorems 3.4.9 and 3.4.10. It also, became clear to call $S_\infty = \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} \supset \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ the space of *globally convergent* generating series, since its associated Fliess operator is always globally convergent. Ultimately, the main consequence of these results is that the set of generating series known to render global convergence has been expanded, and now the growth rate of the coefficients of a generating series is no longer the sole indicator of whether a Fliess operator is global convergent.

CHAPTER 4

APPLICATION TO NONRECURSIVE INTERCONNECTED FLIESS OPERATORS

"Negarse a que el acto delicado de girar el picaporte, ese acto por el cual todo podría transformarse, se cumpla con la fría eficacia de un reflejo cotidiano"

– Julio Cortázar,
Historias de cronopios y de famas [4]

The goal of this chapter is to describe precisely when the nonrecursive interconnection of two globally convergent Fliess operators is again globally convergent. As shown in Chapter 3, the set of generating series known to render global convergence has been expanded, and now the growth rate of the coefficients of a generating series is no longer the sole indicator of whether a given generating series renders a globally convergent Fliess operator. It should be stressed that this is a larger question than the one addressed in [41], where global convergence of a Fliess operator was shown to be preserved for nonrecursive interconnections when (1.1.4) was satisfied.

This chapter is organized as follows. In Section 4.1, the two types of parallel interconnections are presented: sum and product. First, the particular case when both series are in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ is considered, and a condition on the Gevrey order of the resulting series is given. Then, the space S_∞ is showed to be closed under addition and the shuffle product, which implies that both parallel interconnections of two globally convergent Fliess operators preserve the global convergence property. Finally, Section 4.2 addresses the cascade interconnection, and the space S_∞ is shown to be closed under the composition product. This implies that the cascade interconnection of two globally convergent Fliess operators also preserves the global convergence property.

[4] To deny that the delicate act of turning the latch, that act by which everything could be transformed, is fulfilled with the cold efficacy of a daily reflex.

4.1 PARALLEL SUM AND PARALLEL PRODUCT INTERCONNECTIONS

This section is divided in two subsections, each dealing with a type of parallel interconnection. In each subsection, first the particular case when both series are in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ is presented. Then, an upper bound on the minimum of the Gevrey order of the resulting series is given. Lastly, since global convergence of a Fliess operator is completely characterized by its generating series being in the space S_∞, it is only necessary to show that S_∞ is closed under addition and the shuffle product in order to prove that both parallel interconnections of two globally convergent Fliess operators preserve the global convergence property. The following definition and technical results will be needed to do this analysis.

Definition 4.1.1. A series $c \in \mathbb{R}_\gamma\langle\langle X\rangle\rangle$ is said to be **maximal** with growth constants $K, M > 0$ if each component of (c,η) is equal to $KM^{|\eta|}(|\eta|!)^\gamma$, $\forall \eta \in X^*$.

Note that if a series is maximal then using Definition 2.1.4, the series is also exchangeable. The next lemma follows from Stirling's approximation.

Lemma 4.1.1. *For any $K, M, s > 0$, there exists an integer $N > 0$ such that*

$$KM^n \leq (n!)^s, \tag{4.1.1}$$

for all integers $n > N$.

Proof: From Stirling's approximation it follows that $\mathcal{O}(n!) = \mathcal{O}(\sqrt{2\pi n}(n/e)^n)$. Therefore,

$$\lim_{n\to+\infty} K\frac{M^n}{(n!)^s} = \frac{K}{(2\pi)^{s/2}} \lim_{n\to+\infty} \frac{(e^sM)^n}{\sqrt{n}n^{ns}} = 0,$$

which directly leads to (4.1.1). ■

The next lemma presents the neoclassical inequality, a generalization of the binomial theorem.

Lemma 4.1.2. *[32] (Neoclassical Inequality) For any integer $n \geq 0$, $x, y \in \mathbb{R}^+$, and $p \geq 1$ it follows that*

$$\left(\frac{1}{p}\right)\sum_{j=0}^{n} \frac{x^{j/p}}{(j/p)!}\frac{y^{(n-j)/p}}{((n-j)/p)!} \leq \frac{(x+y)^{n/p}}{(n/p)!}.$$

Observe that when $p = 1$ above, the result reduces to the binomial theorem.

4.1.1 The parallel sum interconnection

The particular case when both series are in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ is presented first. In order to give an upper bound on the minimum of the Gevrey order of the resulting series of the parallel sum connection, an analysis of the maximal series case is needed first. The following lemma gives an upper bound on the minimum of the Gevrey order of the sum of two series when both series are maximal.

Lemma 4.1.3. *Let $c, d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ be maximal series with minimum Gevrey order γ_c and γ_d, respectively. If $b := c + d$ then $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ has minimum Gevrey order $\gamma_b = \max\{\gamma_c, \gamma_d\}$.*

Proof: First recall that the minimum Gevrey order of a series b is the minimum s satisfying (3.1.1). Observe for any $\nu \in X^n$, $n \geq 0$, that

$$\begin{aligned}(b, \nu) = (c, \nu) + (d, \nu) &= K_c M_c^n (n!)^{\gamma_c} + K_d M_d^n (n!)^{\gamma_d} \\ &\leq K M^n (n!)^s, \end{aligned} \tag{4.1.2}$$

where $s := \max\{\gamma_c, \gamma_d\}$, $M := \max\{M_c, M_d\}$ and $K := K_c + K_d$. Letting γ_b denote the minimum Gevrey order of b, it is clear from (4.1.2) that $\gamma_b \leq s < 1$, which implies that $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. It is shown now that $\gamma_b \not< s$ since considering otherwise would render a contradiction. Suppose $\gamma_b < s$ and there exist constants $K_b, M_b > 0$ such that $(b, \nu) \leq K_b M_b^n (n!)^{\gamma_b}$, $\forall \nu \in X^n$, $n \geq 0$. There is no loss of generality in assuming $\gamma_c \leq \gamma_d$. In which case, $\gamma_b < s = \max\{\gamma_c, \gamma_d\} = \gamma_d$, and therefore,

$$(b, \nu) = K_c M_c^n (n!)^{\gamma_c} + K_d M_d^n (n!)^{\gamma_d} \leq K_b M_b^n (n!)^{\gamma_b}.$$

In particular, this implies that $K_d M_d^n (n!)^{\gamma_d - \gamma_b} \leq K_b M_b^n$. Hence,

$$(n!)^{\gamma_d - \gamma_b} \leq \frac{K_b}{K_d}\left(\frac{M_b}{M_d}\right)^n. \tag{4.1.3}$$

Substituting $M' = M_b/M_d$, $K' = K_b/K_d$ and $s' = \gamma_d - \gamma_b$ in (4.1.3) gives $K' M'^n \geq (n!)^{s'}$, which contradicts (4.1.1) in Lemma 4.1.1 since by assumption $\gamma_d - \gamma_b > 0$. Therefore, $\gamma_b = \max\{\gamma_c, \gamma_d\}$. ■

It is now possible to compute an upper bound on the minimum Gevrey order of the sum of two *arbitrary* series in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

Theorem 4.1.1. *Let $c, d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ with minimum Gevrey order γ_c and γ_d, respectively. If $b := c+d$ then $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ with minimum Gevrey order $\gamma_b \leq \max\{\gamma_c, \gamma_d\}$.*

Proof: For any $\nu \in X^*$ it follows that

$$|(c+d,\nu)| \leq |(c,\nu)| + |(d,\nu)| \leq (\bar{c},\nu) + (\bar{d},\nu) = (\bar{b},\nu),$$

where $\bar{b}$, $\bar{c}$ and $\bar{d}$, are the maximal series corresponding to b, c, and d, respectively (that is, each pair, for example b and $\bar{b}$, share the same growth constants). From Lemma 4.1.3 it then follows directly that $\gamma_b \leq \max\{\gamma_c, \gamma_d\}$. ■

The fact that the upper bound on the minimum Gevrey order of the sum of two series is the maximum of the minimum Gevrey orders of the component series implies that $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ is closed under addition. The next theorem is one of the main results of this dissertation. It shows that S_∞ is also closed under addition, and thus, the parallel sum connection preserves the global convergence property.

Theorem 4.1.2. *The space S_∞ is closed under addition.*

Proof: Let $c, d \in S_\infty$. Then clearly

$$\|c+d\|_{\infty,R} \leq \|c\|_{\infty,R} + \|d\|_{\infty,R} < \infty$$

for all $R > 0$. Hence, $c + d \in S_\infty$. ■

4.1.2 The parallel product interconnection

Analogous to the approach in Subsection 4.1.1, the particular case when both series are maximal and in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ is presented first. However, the case of the parallel product connection is more difficult than the one faced for the parallel sum connection since the sum is replaced now with shuffle the product. The following lemma gives an upper bound on the minimum of the Gevrey order of the resulting series when both series are maximal. The neoclassical inequality plays a key role in the proof.

Lemma 4.1.4. *Let $c, d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ be maximal series with minimum Gevrey order γ_c and γ_d, respectively. If $b := c \sqcup\!\sqcup\, d$ then $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ has minimum Gevrey order $\gamma_b = \max\{\gamma_c, \gamma_d\}$.*

Proof: Observe that for any $\nu \in X^n$, $n \geq 0$,

$$\begin{aligned}(b,\nu) = (c \sqcup\!\sqcup d, \nu) &= \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (c,\eta)(d,\xi)(\eta \sqcup\!\sqcup \xi, \nu) \\ &= \sum_{j=0}^{n} K_c M_c^j (j!)^{\gamma_c} K_d M_d^{n-j} ((n-j)!)^{\gamma_d} \binom{n}{j} \\ &= K_c K_d n! \sum_{j=0}^{n} M_c^j M_d^{n-j} \frac{1}{(j!)^{1-\gamma_c}((n-j)!)^{1-\gamma_d}}.\end{aligned}$$

Using Lemma 3.1.1 when $j \gg 1$and letting $s := \max\{\gamma_c, \gamma_d\} < 1$, $s' := 1 - s$, $K_s := ((2\pi)^{1-s'} s')^{1/2}$ and $M_s := s'^{s'}$, it follows that

$$\begin{aligned}(c \sqcup\!\sqcup d, \nu) &\leq K_c K_d n! \sum_{j=0}^{n} M_c^j M_d^{n-j} \frac{(K_s)^2 M_s^n}{(js')!((n-j)s')!}, \\ &= K_c K_d (K_s)^2 M_s^n n! \sum_{j=0}^{n} \frac{({M_c}^{1/s'})^{js'} ({M_d}^{1/s'})^{(n-j)s'}}{(js')!((n-j)s')!}.\end{aligned}$$

Now applying Lemma 4.1.2 gives

$$(c \sqcup\!\sqcup d, \nu) \leq \frac{1}{s'} K_c K_d (K_s)^2 M_s^n n! \frac{({M_c}^{1/s'} + {M_d}^{1/s'})^{ns'}}{(ns')!}.$$

In which case, from Lemma 3.1.1 when $n \gg 1$ is it immediate that

$$(c \sqcup\!\sqcup d, \nu) \leq K M^n (n!)^s, \tag{4.1.4}$$

where $M := {M_c}^{1/s'} + {M_d}^{1/s'}$ and $K := K_c K_d K_s / s'$. Since the minimum Gevrey order is the minimum s satisfying (3.1.1), if the minimum Gevrey order of $b = c \sqcup\!\sqcup d$ is γ_b, then it is clear from (4.1.4) that $\gamma_b \leq s < 1$, which automatically implies that $b \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$. It is shown now that $\gamma_b \not< s$ since otherwise a contradiction is obtained. Suppose $\gamma_b < s$ and that there exist constants $K_b, M_b > 0$ such that $(b,\nu) \leq K_b M_b^n (n!)^{\gamma_b}$, $\forall \nu \in X^n$, $n \geq 0$. Without loss of generality assume $\gamma_c \leq \gamma_d$. In which case, $\gamma_b < s = \max\{\gamma_c, \gamma_d\} = \gamma_d$. Thus, for any fixed $n \geq 0$

$$(b,\nu) = (c \sqcup\!\sqcup d, \nu) \leq K_b M_b^n (n!)^{\gamma_b},$$

which implies that

$$(c \sqcup\!\sqcup d, \nu) = \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (c, \eta)(d, \xi)(\eta \sqcup\!\sqcup \xi, \nu) \leq K_b M_b^n (n!)^{\gamma_b}.$$

In particular, the $j = 0$ term in the summation above must satisfy

$$(c, \emptyset)(d, \nu) = K_c K_d M_d^n (n!)^{\gamma_d} \leq K_b M_b^n (n!)^{\gamma_b},$$

which amounts to the inequality

$$(n!)^{\gamma_d - \gamma_b} \leq \frac{K_b}{K_c K_d} \left(\frac{M_b}{M_d} \right)^n. \tag{4.1.5}$$

Letting $M' := M_b/M_d$, $K' := K_b/(K_c K_d)$ and $\bar{s} := \gamma_d - \gamma_b$ in (4.1.5) gives $K' M'^n \geq (n!)^{\bar{s}}$, which contradicts (4.1.1) in Lemma 4.1.1 since by assumption $\bar{s} = \gamma_d - \gamma_b > 0$. Thus, $\gamma_b = \max\{\gamma_c, \gamma_d\}$. ■

It is now possible to compute an upper bound on the minimum Gevrey order of the shuffle product of two arbitrary series in $\mathbb{R}_{GC}\langle\langle X \rangle\rangle$.

Theorem 4.1.3. *Let $c, d \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ with minimum Gevrey order γ_c and γ_d, respectively. If $b := c \sqcup\!\sqcup d$ then $b \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ with minimum Gevrey order $\gamma_b \leq \max\{\gamma_c, \gamma_d\}$.*

Proof: First observe that for all $\nu \in X^*$

$$|(c \sqcup\!\sqcup d, \nu)| \leq (\bar{c} \sqcup\!\sqcup \bar{d}, \nu) = (\bar{b}, \nu),$$

where $\bar{b}$, $\bar{c}$ and $\bar{d}$ are maximal series corresponding to b, c and d, respectively. From Lemma 4.1.4 it then follows directly that $\gamma_b \leq \max\{\gamma_c, \gamma_d\}$. ■

The fact that the upper bound on the minimum Gevrey order of the shuffle product of two series is the maximum of the minimum Gevrey orders of the component series implies that $\mathbb{R}_{GC}\langle\langle X \rangle\rangle$ is closed under the shuffle product. In order to show that S_∞ is also closed under the shuffle product, the next inequality is needed. It is interesting to observe its similarity to the Cauchy-Schwarz inequality but using the shuffle product instead of the inner product.

Lemma 4.1.5. *For every* $c, d \in S_\infty$,

$$\|c \sqcup\!\sqcup\, d\|_{\infty,R} \leq \|c\|_{\infty,R}\|d\|_{\infty,R}$$

for all $R > 0$.

Proof: Starting with the definition of the norm on $S_\infty(R)$:

$$\begin{aligned}
\|c \sqcup\!\sqcup\, d\|_{\infty,R} &= \sup_{\nu\in X^*}\left\{|(c \sqcup\!\sqcup\, d,\nu)|\,\frac{R^{|\nu|}}{|\nu|!}\right\}\\
&\leq \sup_{\substack{\nu\in X^n\\ 0\leq j\leq n\\ n\geq 0}}\left\{\sum_{\eta\in X^j,\xi\in X^{n-j}}|(c,\eta)|\,|(d,\xi)|\,(\eta \sqcup\!\sqcup\, \xi,\nu)\frac{R^n}{n!}\right\}\\
&\leq \sup_{\substack{\nu\in X^n\\ 0\leq j\leq n\\ n\geq 0}}\left\{\left(\max_{\eta\in X^j}|(c,\eta)|\,R^j\right)\left(\max_{\xi\in X^{n-j}}|(d,\xi)|\,R^{n-j}\right)\frac{1}{n!}\sum_{\eta\in X^j,\xi\in X^{n-j}}(\eta \sqcup\!\sqcup\, \xi,\nu)\right\}.
\end{aligned}$$

It is easy to show by induction that

$$\sum_{\substack{\eta\in X^j\\ \xi\in X^{n-j}}}(\eta \sqcup\!\sqcup\, \xi,\nu) = \binom{n}{j},\ \forall\nu\in X^n.$$

Therefore,

$$\|c \sqcup\!\sqcup\, d\|_{\infty,R} \leq \sup_{\substack{\nu\in X^n\\ 0\leq j\leq n\\ n\geq 0}}\left\{\left(\max_{\eta\in X^j}|(c,\eta)|\,\frac{R^j}{j!}\right)\left(\max_{\xi\in X^{n-j}}|(d,\xi)|\,\frac{R^{n-j}}{(n-j)!}\right)\right\}.$$

Since $c, d \in S_\infty$, it is clear that $\|c\|_{\infty,R} < \infty$ and $\|d\|_{\infty,R} < \infty$. This implies that

$$\begin{aligned}
\|c \sqcup\!\sqcup\, d\|_{\infty,R} &\leq \sup_{\substack{\eta\in X^j\\ 0\leq j\leq n\\ n\geq 0}}\left\{|(c,\eta)|\,\frac{R^j}{j!}\right\}\sup_{\substack{\xi\in X^{n-j}\\ 0\leq j\leq n\\ n\geq 0}}\left\{|(d,\xi)|\,\frac{R^{n-j}}{(n-j)!}\right\}\\
&\leq \sup_{\eta\in X^*}\left\{|(c,\eta)|\,\frac{R^{|\eta|}}{|\eta|!}\right\}\sup_{\xi\in X^*}\left\{|(d,\xi)|\,\frac{R^{|\xi|}}{|\xi|!}\right\}\\
&= \|c\|_{\infty,R}\|d\|_{\infty,R},
\end{aligned}$$

which completes the proof. ■

The next theorem is one of the main results of this dissertation. It shows that S_∞

is also closed under the shuffle product, and thus, the parallel product connection preserves the global convergence property.

Theorem 4.1.4. *The space S_∞ is closed under the shuffle product.*

Proof: Let $c, d \in S_\infty$. Then from Lemma 4.1.5 it follows that

$$\|c \sqcup\!\sqcup\, d\|_{\infty,R} \le \|c\|_{\infty,R}\|d\|_{\infty,R} < \infty$$

for all $R > 0$. Hence, $c \sqcup\!\sqcup\, d \in S_\infty$. ■

4.2 CASCADE INTERCONNECTION

In this section the cascade interconnection is addressed. It is instructive to start with a few simple examples.

Example 4.2.1. Let $X_0 = \{x_0\}$ and assume $c \in \mathbb{R}_{GC}\langle\langle X_0\rangle\rangle$ has minimum Gevrey order γ_c. Since $c \circ d = c$ for any $d \in \mathbb{R}\langle\langle X\rangle\rangle$, it follows that the minimum Gevrey order γ_c is preserved for this particular series composition. □

Example 4.2.2. Consider the rational series

$$c = \sum_{n_1,n_2=0}^{\infty} KM^{n_1+n_2}\, x_0^{n_1} x_1 x_0^{n_2} = K(Mx_0)^* x_1 (Mx_0)^*.$$

This series is *input-limited* in the sense that there is a fixed upper bound on $|\eta|_{x_1}$ when $\eta \in \text{supp}(c)$ [10, 11]. In this case, the letter x_1, corresponding to the input u in $F_c[u]$, appears exactly once in every word in the support of c. It is known that the composition product preserves rationality when its left argument is input-limited [8, 10, 11]. Therefore, since all rational series are in $\mathbb{R}_0\langle\langle X\rangle\rangle$, then $c \circ d \in \mathbb{R}_0\langle\langle X\rangle\rangle$ for any $d \in \mathbb{R}\langle\langle X\rangle\rangle$. □

Examples 4.2.1 and 4.2.2 provide specific cases in which the Gevrey order of the composition of two series can be determined exactly. The following theorem shows that an explicit upper bound on the minimum Gevrey order of a composition over $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ can be computed when the left argument of the composition product is input-limited. Unfortunately, at present, no other classes of series are known for which an explicit upper bound on the minimum Gevrey order can be determined.

Theorem 4.2.1. *Let $c, d \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ with minimum Gevrey orders γ_c and γ_d, respectively. If $b := c \circ d$ with c input-limited, then $b \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$, and its minimum Gevrey order is $\gamma_b \leq \max\{\gamma_c, \gamma_d\}$.*

Proof: Since c is input-limited, there exists some fixed $N \in \mathbb{N}$ such $|\eta|_{x_1} \leq N$, $\forall \eta \in \text{supp}\{c\}$. Therefore, the composition product $b = c \circ d$ can be written in terms of a finite number of sums and shuffle products. It then follows from Theorems 4.1.1 and 4.1.3 that the minimum Gevrey order of b must satisfy $\gamma_b \leq \max\{\gamma_c, \gamma_d\}$. ■

In order to show that S_∞ is closed under the composition product, the next theorem involving maximal series is needed.

Theorem 4.2.2. *Let $\bar{c} \in S_\infty$ be a maximal series and $d \in S_\infty$. Then $\bar{c} \circ d \in S_\infty$.*

The first step in proving this theorem is to construct a sequence in S_∞ that converges to the composition product, $\bar{c} \circ d$, in the semi-norm topology. Then, since the space S_∞ is Fréchet (thus, complete) the series $\bar{c} \circ d$ will also be on the space. In order to pick this sequence properly, the following lemma is introduced. It can be seen as an analogue of Lemma 3.4.2 but now including the composition product.

Lemma 4.2.1. *Let $c, d \in S_{\infty,e}$ and define $c_N = \sum_{n=0}^{N} \sum_{\eta \in X^n} (c, \eta)\eta$, $N \geq 0$. If $b := c \circ d$ then $b \in S_{\infty,e}$ and there exists an $R > 0$ such that $c_N \circ d \to b$ as a sequence in the normed linear space $S_\infty(R)$.*

Proof: If $c, d \in S_{\infty,e}$ then it is known that $b := c \circ d \in \mathbb{R}_{LC}\langle\langle X \rangle\rangle = S_{\infty,e}$ [22]. This implies that $|(b, \eta)| \leq KM^{|\eta|}(|\eta|!)^{\gamma_b}$, $\forall \eta \in X^*$ for some $K, M > 0$ and $0 \leq \gamma_b \leq 1$. Therefore, for any $R > 0$

$$\|c_N \circ d - b\|_{\infty,R} = \sup_{\nu \in X^*} \left\{ |((c_N - c) \circ d, \nu)| \frac{R^{|\nu|}}{|\nu|!} \right\}.$$

Since $\text{ord}((c_N - c) \circ d) \geq N + 1$, it follows that

$$\begin{aligned} \|c_N \circ d - b\|_{\infty,R} &= \sup_{\substack{\nu \in X^n \\ n > N}} \left\{ |(b, \nu)| \frac{R^{|\nu|}}{|\nu|!} \right\} \\ &\leq \sup_{n > N} K \frac{(MR)^n}{(n!)^{1-\gamma_b}} \leq \begin{cases} K \dfrac{(MR)^{N+1}}{((N+1)!)^{1-\gamma_b}} & : \ 0 \leq \gamma_b < 1 \\ K(MR)^{N+1} & : \ \gamma_b = 1. \end{cases} \end{aligned}$$

When $0 \leq \gamma_b < 1$ observe that

$$\lim_{N\to\infty} \|c_N \circ d - b\|_{\infty,R} \leq \lim_{N\to\infty} \sup_{n>N} K\frac{(MR)^n}{(n!)^{1-\gamma_b}} = \lim_{N\to\infty} K\frac{(MR)^{N+1}}{((N+1)!)^{1-\gamma_b}} = 0 \quad (4.2.1)$$

for all $R > 0$. On the other hand, when $\gamma_b = 1$ and $R < 1/M$ then

$$\lim_{N\to\infty} \|c_N \circ d - b\|_{\infty,R} \leq \lim_{N\to\infty} K(MR)^{N+1} = 0$$

and infinity otherwise. This implies in both cases that there exists an $R > 0$ such that $c_N \circ d \to b$ as a sequence in the normed linear space $S_\infty(R)$. ■

It is important to observe that the lemma above implies that the composition product is continuous in its left argument, at least with respect to a *specific sequence*. The next corollary explores the particular case when $c \circ d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle \subset S_{\infty,e}$. It can be viewed as an analogue of Corollary 3.4.1 involving the composition product.

Corollary 4.2.1. *If $c \circ d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ then $c_N \circ d \to c \circ d$ in the semi-norm topology.*

Proof: The claim follows directly from (4.2.1). ■

A second important observation regarding the proof of Theorem 4.2.2 is that the sequence in S_∞ has to be Cauchy in the semi-norm topology, again since the space S_∞ is Fréchet (thus, complete). To prove that this sequence is Cauchy in the semi-norm topology, Lemma 2.1.2 is applied to a maximal series. That stated, now the proof of Theorem 4.2.2 can be presented.

Proof of Theorem 4.2.2 The key idea is to define the sequence $\{\bar{c}_N\}_{N\geq 0}$ as in Lemma 4.2.1. Then there exists an $R > 0$ such that $\bar{c}_N \circ d \to \bar{c} \circ d$ as a sequence in the normed space $S_\infty(R)$. In light of Corollary 3.2.3, if $\{\bar{c}_N \circ d\}_{N\geq 0}$ also converges in the semi-norm topology then the (unique) limit point will be $\bar{c} \circ d$. In order to prove that this convergence also holds in the semi-norm topology is sufficient to show that $\{\bar{c}_N \circ d\}_{N\geq 0}$ is a Cauchy sequence in the semi-norm topology since the space S_∞ is Fréchet. First note that $\bar{c} \in S_\infty$ is a maximal series, hence $|(\bar{c}, \eta)| = K_c M_c^{|\eta|}(|\eta|!)^{\gamma_c}, \quad \forall \eta \in X^*$ for some $K_c, M_c > 0$ and $0 \leq \gamma_c \leq 1$. By Lemma 3.4.3, $\bar{c}_N \to \bar{c}$ in the semi-norm topology, which implies that $\{\bar{c}_N\}_{N\geq 0}$ is a Cauchy sequence in the semi-norm topology. Thus, given $N_1 > N_2 \in \mathbb{N}$, there exists a natural number L such that for any $\epsilon > 0$

$$\|\bar{c}_{N_1} - \bar{c}_{N_2}\|_{\infty,R} < \epsilon \quad (4.2.2)$$

when $N_2 > L$ and for all $R > 0$. Applying the definition of the norm in (4.2.2) it follows

$$\|\bar{c}_{N_1} - \bar{c}_{N_2}\|_{\infty,R} = \sup_{N_2<n<N_1} K_c \frac{(M_c R)^n}{(n!)^{1-\gamma_c}} < \frac{\epsilon'}{N_1 - N_2}, \tag{4.2.3}$$

where $N_1 > N_2 > L$, $\epsilon' > 0$, and $R > 0$ is arbitrary. On the other hand, observe that

$$\bar{c}_{N_1} \circ d - \bar{c}_{N_2} \circ d = (\bar{c}_{N_1} - \bar{c}_{N_2}) \circ d = \sum_{k>N_2}^{N_1} \sum_{\eta \in X^k} (\bar{c}, \eta)\, \psi_d(\eta)(1).$$

The key observation here is that $\bar{c}$ is maximal and therefore exchangeable. Thus Lemma 2.1.2 can be applied to the expression above. Following [40, Proof of Theorem 2] yields

$$\begin{aligned} \bar{c}_{N_1} \circ d - \bar{c}_{N_2} \circ d &= \sum_{k>N_2}^{N_1} K_c M_c^k (k!)^{\gamma_c} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} \frac{x_0^{\,\sqcup\!\sqcup\, r_0}}{r_0!} \,\sqcup\!\sqcup \cdots \sqcup\!\sqcup\, \frac{(x_m \circ d)^{\,\sqcup\!\sqcup\, r_m}}{r_m!} \\ &= \sum_{k>N_2}^{N_1} K_c \frac{(M_c(x_0 + m x_0 d))^{\,\sqcup\!\sqcup\, k}}{(k!)^{1-\gamma_c}}. \end{aligned} \tag{4.2.4}$$

Fixed $R > 0$, taking the $\|\cdot\|_{\infty,R}$ on both sides of (4.2.4) and using Theorems 4.1.2 and 4.1.4 gives

$$\begin{aligned} \|\bar{c}_{N_1} \circ d - \bar{c}_{N_2} \circ d\|_{\infty,R} &= \left\| \sum_{k>N_2}^{N_1} K_c \frac{(M_c(x_0 + m x_0 d))^{\,\sqcup\!\sqcup\, k}}{(k!)^{1-\gamma_c}} \right\|_{\infty,R} \\ &\leq \sum_{k>N_2}^{N_1} K_c \frac{\|(M_c(x_0 + m x_0 d))^{\,\sqcup\!\sqcup\, k}\|_{\infty,R}}{(k!)^{1-\gamma_c}}. \end{aligned} \tag{4.2.5}$$

However, again applying Theorems 4.1.2 and 4.1.4, it follows that

$$\begin{aligned} \|(x_0 + m x_0 d)^{\,\sqcup\!\sqcup\, k}\|_{\infty,R} \leq& \|(x_0 + m x_0 \circ d)\|_{\infty,R}^k \\ \leq& (R + Rm\|d\|_{\infty,R})^k = R^k (1 + m\|d\|_{\infty,R})^k, \end{aligned} \tag{4.2.6}$$

since $d \in S_\infty$. Substituting (4.2.6) in (4.2.5) gives

$$\|\bar{c}_{N_1} \circ d - \bar{c}_{N_2} \circ d\|_{\infty,R} \leq \sum_{k>N_2}^{N_1} K_c \frac{(M_c R(1 + m\|d\|_{\infty,R}))^k}{(k!)^{1-\gamma_b}}. \tag{4.2.7}$$

Note that each term on the right hand summation of (4.2.7) is bounded by its supreme, therefore

$$\|\bar{c}_{N_1} \circ d - \bar{c}_{N_2} \circ d\|_{\infty,R} \leq (N_1 - N_2) \sup_{N_2<k<N_1} K_c \frac{(M_c R(1 + m\|d\|_{\infty,R}))^k}{(k!)^{1-\gamma_b}}.$$

Note that if $N_1 > N_2 > L$, $\epsilon' > 0$ and L are selected as in (4.2.3), then

$$\|\bar{c}_{N_1} \circ d - \bar{c}_{N_2} \circ d\|_{\infty,R} \leq (N_1 - N_2) \sup_{N_2<n<N_1} K_c \frac{(M_c R')^n}{(n!)^{1-\gamma_c}} < (N_1 - N_2) \frac{\epsilon'}{N_1 - N_2} = \epsilon',$$

where $R' := (M_c R(1 + m\|d\|_{\infty,R}))$, and $R = R'$ is used in (4.2.3). This implies that $\{\bar{c}_N \circ d\}_{N\geq 0}$ is a Cauchy sequence in the semi-norm topology, therefore $\bar{c} \circ d \in S_\infty$.

■

In the light of Theorem 4.2.2, the last main result of this dissertation is given next.

Theorem 4.2.3. *The space S_∞ is closed under the composition product.*

Proof: Let $c, d \in S_\infty$. Then for any $\nu \in X^*$,

$$|(c \circ d, \nu)| \leq \sum_{\eta \in X^*} |(c, \eta)\,(\psi_d(\eta)(1), \nu)| \leq \sum_{\eta \in X^*} |(\bar{c}, \eta)\,(\psi_d(\eta)(1), \nu)| = |(\bar{c} \circ d, \nu)|\,,$$

where $\bar{c}$ is the maximal series corresponding to c. Thus,

$$|(c \circ d, \nu)| \leq |(\bar{c} \circ d, \nu)|\,.$$

This inequality was first shown in [41] for the cases when c, d are in $\mathbb{R}_1\langle\langle X\rangle\rangle$ or $\mathbb{R}_0\langle\langle X\rangle\rangle$. Multiplying each side by $R^{|\nu|}/\,|\nu|!$ gives

$$|(c \circ d, \nu)|\,\frac{R^{|\nu|}}{|\nu|!} \leq |(\bar{c} \circ d, \nu)|\,\frac{R^{|\nu|}}{|\nu|!}.$$

Taking the supreme on the length of the words on each side yields

$$\sup_{\nu \in X^*} \left\{ |(c \circ d, \nu)|\,\frac{R^{|\nu|}}{|\nu|!} \right\} \leq \sup_{\nu \in X^*} \left\{ |(\bar{c} \circ d, \nu)|\,\frac{R^{|\nu|}}{|\nu|!} \right\}.$$

Thus,

$$\|c \circ d\|_{\infty,R} \leq \|\bar{c} \circ d\|_{\infty,R} < \infty,$$

using Theorem 4.2.2 in the last step. Therefore, $c \circ d \in S_\infty$, which completes the proof. ■

The next example illustrates the main theorem of the subsection.

Example 4.2.3. Consider the bilinear state space system

$$\begin{aligned} \dot{z}_1 &= z_1 z_2, \quad z_1(0) = 1 \\ \dot{z}_2 &= z_2 u, \quad z_2(0) = 1 \\ y &= z_1. \end{aligned}$$

It is easily verified that $y = F_{c_F}[u]$. The operator F_{c_F} has an infinite radius of convergence since it was shown in Chapter 3, that $c_F \in S_\infty$. The cascade of two such systems has the realization

$$\dot{z} = g_0(z) + g_1(z)u, \quad y = h(z), \tag{4.2.8}$$

where

$$g_0(z) = \begin{pmatrix} z_1 z_2 \\ z_2 z_3 \\ z_3 z_4 \\ 0 \end{pmatrix}, \quad g_1(z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_4 \end{pmatrix},$$

$h(z) = z_1$, and $z_i(0) = 1$ for all i. The corresponding generating series $c_F \circ c_F$ can be computed by iterated Lie derivatives (see [27]) to give

$$\begin{aligned} c_F \circ c_F = {} & 1 + x_0 + 2x_0^2 + 6x_0^3 + 23x_0^4 + x_0^3 x_1 + 106x_0^5 \\ & + 9x_0^4 x_1 + 3x_0^3 x_1 x_0 + x_0^3 x_1^2 + 568x_0^6 + 68x_0^4 x_1 \\ & + 34x_0^3 x_1 x_0 + 11x_0^4 x_1^2 + 11x_0^3 x_1 x_0^2 + 3x_0^3 x_1^2 x_0 \\ & + 4x_0^3 x_1 x_0 x_1 + x_0^3 x_1^3 + \cdots \end{aligned}$$

Consistent with Theorem 4.2.3, $c_F \circ c_F$ is also in S_∞ and therefore $F_{c_F \circ c_F}$ would have an infinite radius of convergence. In order to test this claim independently, note that the solution of (4.2.8) can be written in terms of compositions of functionals as

$$y(t) = F_{c_F \circ c_F}[u](t) = F_c[F_c[F_c[F_c[u]]]](t),$$

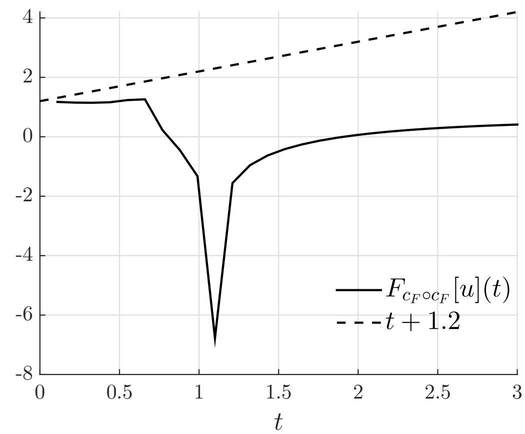

Fig. 15: Response of the operator $F_{c_F \circ c_F}[u]$ when $u = \mathrm{e}^{-t}$ (solid line) on a quadruple logarithmic scale and the bounding function $t + 1.2$ (dotted line).

where

$$F_c[u](t) = \exp\left(\int_0^t u(\tau)\, d\tau\right).$$

Now given any $u \in L_{\mathfrak{p}}^1[0,T]$ for some arbitrary $T > 0$, $F_c[u]$ is clearly well defined on $[0,T]$. Repeating this argument three more times yields the same conclusion for y. A MatLab simulation of (4.2.8) to generate y when $u(t) = \mathrm{e}^{-t}$ is shown in Figure 15. Since the output is four nested exponentials, the response is best viewed by taking four successive logarithms. Note that in the figure the response increases monotonically after approximately $t = 1.1$. The quadruple exponential of $t + 1.2$ (found empirically) bounds the response completely so that there exists no finite escape times. This behavior is consistent with that of a globally convergent Fliess operator.

□

4.3 SUMMARY

In Section 4.1, it was shown that the two parallel interconnections of Fliess operators preserve the global convergence property. Upper bounds on the minimum of the Gevrey orders of the resulting series were also calculated and an interesting Cauchy–Schwarz type inequality but using the shuffle product was presented. It was shown in Section 4.2 that the cascade interconnection of two globally convergent Fliess operators also preserves the global convergence property. Therefore, all nonrecursive interconnection of two globally convergent Fliess operators preserve the global convergence property, which accomplishes the second main goal of this dissertation.

CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH

"Perdón por las manchas de esta página. Son de té con limón, o de naranja. Puede que un día tenga dos mesas, una para comer y otra para escribir"

– Julio Cortázar,
Historias de cronopios y de famas[3]

In this final chapter, the main contributions and conclusions of this dissertation are summarized and future research topics are given.

5.1 MAIN CONCLUSIONS

This dissertation was focused on the solution of two problems.

The first problem was to develop an exact relationship between the growth rate of the coefficient's generating series and the nature of the convergence of its corresponding Fliess operator. This problem was solved through Theorems 3.4.9 and 3.4.10. The set of generating series known to ensure global convergence was expanded, and now the growth rate of the coefficients generating series is no longer the sole indicator of whether a given generating series renders a globally convergent Fliess operator. Specifically, global convergence of a Fliess operator is completely characterized by its generating series being in the Fréchet space $S_\infty = \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$. The Ferfera series c_F was shown to be on the boundary $\partial\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. In which case $c_F \in \mathbb{R}_1\langle\langle X\rangle\rangle$ and F_{c_F} is globally convergent. It is important to mention that, in this dissertation, the approach for the creation of the new set of formal power series was two folded, one considering the properties related to the known spaces on the literature; and on the other hand, considering new topological properties. However, another perspectives for the creation of this new set are also available. In particular, using a concept called projective systems [26].

[3]Sorry for the spots on this page. They are tea with lemon, or orange. One day I may have two tables, one to eat and one to write.

The second problem was to describe precisely when the nonrecursive interconnection of two globally convergent Fliess operators is again globally convergent. This problem was solved through Theorems 4.1.2, 4.1.4, and 4.2.3. Specifically, it was shown that S_∞ is closed under addition, the shuffle product, and the composition product. In the process, explicit upper bounds were given in Theorems 4.1.1 and 4.1.3 for the Gevrey orders of the sum and the shuffle product when generating series were in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle \subset S_\infty$.

5.2 FUTURE RESEARCH

Some interesting future research problems related to this dissertation are proposed. First, it was shown that the new space of *globally convergent* generating series, S_∞, is a Fréchet space. Thus, one could then ask: Are there any other useful properties related to this space? What is the topological structure of the longer space $S_{\infty,e}$? Is there any topology, similar to the semi-norm topology that can be associated to it? Also, what other results can be obtained when using the approach of projective systems [26] when creating the space S_∞? Next, it was proved that the property of global convergence of two Fliess operators is preserved for all nonrecursive interconnections. A follow up question is what type of convergence behavior is possible when interconnecting two systems, one with a finite radius of convergence and one that is globally convergent. The feedback interconnection was not relevant in this dissertation since global convergence is known to not be preserved in general [21, Example 3]. However, the following question can be formulated: Under what special conditions might a feedback interconnection of two Fliess operators preserve the global convergence property?

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2. I. M. Winter-Arboleda, L. A. Duffaut Espinosa, and W. S. Gray "Nonrecursively Interconnected Fliess Operators Preserve Global Convergence: An Expanded View," *Proc.* 22^{nd} *International Symposium on the Mathematical Theory of Networks and Systems*, Minneapolis, Minnesota, 2016.

3. I. M. Winter-Arboleda, W. S. Gray, and L. A. Duffaut Espinosa, "Expanding the Class of Globally Convergent Fliess Operators," *Proc.* 22^{nd} *International Symposium on the Mathematical Theory of Networks and Systems*, Minneapolis, Minnesota, 2016.

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