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ON A TYPE OF SUPERLINEAR GROWTH VARIATIONAL PROBLEMS

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Abstract. In this note, we propose an elementary method to study the existence and uniqueness of solutions to a type of variational problems which arise naturally in the theory of large deviations. This type of problems involves a movable boundary and may not have the coercivity condition in general. Our method is elementarily based on direct analysis over the space of absolutely continuous functions and specific properties of the underlying functional.

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Key words: Calculus of variation, maximizer, large deviation.

1. Introduction. Let H be a real-valued function on $\mathbb{R}$, we study in this note the following type of variational problems

$$\max_{\phi \in C_0^1[0,1]} \int_0^1 L(\phi(t), \phi'(t))dt$$

with $L(v,u) = v - v^2 - H(u)$ under appropriate assumptions on H, where $C_0^1[0,1]$ denotes the space of continuously differentiable functions on $[0,1]$ vanishing at 0. Here $\phi(1)$ is variable as a movable boundary. The main motivation of investigating this type of variational problems is from the study of large deviations in probability theory. More precisely, let us use $\mathrm{AC}_0[0,1]$ to denote the space of absolutely continuous functions on $[0,1]$ vanishing at 0, then one way of characterizing a *rough large deviation principle* (see [8] and [9]) is through a variational problem

$$\max_{\phi \in \mathrm{AC}_0[0,1]} [F(\phi) - S(\phi)]$$

where F is a continuous functional under some restrictions and S is the so called rate functional describing how difficult a stochastic process falls into a neighborhood of ϕ. This rough large deviation principle is also referred as Varadhan's integral

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lemma (see [4] for example). When one tries to establish precise large deviations, it is in general required that the maximizer in the above variational problem is unique and has C^1 smooth regularity (see [6] and [9]).

Indeed, in a recent paper [10] the author considered a functional $F(\phi) = \int_0^1 \left(\phi(t) - \phi(t)^2\right) dt$ and a special family of pure jump stochastic processes depending on a small parameter $\epsilon > 0$ whose trajectories are step functions with finitely many steps on time interval $[0, 1]$. Those jump processes make jumps of size $\pm\epsilon$ according to the rate $\epsilon/2$, and in this case the rate functional $S(\phi)$ for this family turns out to be $\int_0^1 \left(\phi'(t) \ln\left(\phi'(t) + \sqrt{\phi'(t)^2 + 1}\right) + 1 - \sqrt{\phi'(t)^2 + 1}\right) dt$ for all absolutely continuous ϕ. This leads to the problem $\max_{\phi \in C_0^1[0,1]} \int_0^1 L(\phi(t), \phi'(t))dt$ for $L(v, u) = v - v^2 - H_0(u)$ with $H_0(u) = u \ln\left(u + \sqrt{u^2 + 1}\right) + 1 - \sqrt{u^2 + 1}$.

In this note, we consider a more general H and establish the following main result.

THEOREM 1.1. *Suppose that a function $H(u)$ is non-negative and strictly convex such that*

$$\lim_{|u| \to \infty} \frac{H(u)}{|u|} = \infty. \tag{1.1}$$

Then the variational problem

$$\max_{\phi \in C_0^1[0,1]} \int_0^1 \left[\phi(t) - \phi(t)^2 - H(\phi'(t))\right] dt \tag{1.2}$$

has a unique maximizer.

The uniqueness follows from standard arguments which is included in Section 2. The existence of $\max \int_0^1 L(\phi(t), \phi'(t))dt$ in reference was given, in general, with two fixed boundaries and with the coercivity condition, namely,

$$-L(v, u) \geq \alpha|u|^q - \beta, \text{ for some } \alpha > 0, \beta \geq 0 \text{ and } q > 1.$$

We refer the reader to [7], [5, Section 8.2] and [2, Chapters 11–16] for precise arguments together with suitable refinements and extensions. Our assumption (1.1) indicates that $-L(v, u)$ grows in u faster than $|u|$, but not necessarily faster than $|u|^q$ for $q > 1$. This can be also seen for the special H_0 defined above. The assumption (1.1) is called *superlinear growth* according to [1, Section 3.2]. Although some more complicated method based on Sobolev spaces may be applied to derive the existence (see for example [1] and [2]), an elementary proof of the existence is given in this note mainly based on nice properties of the functional $\int_0^1 \left[\phi(t) - \phi(t)^2 - H(\phi'(t))\right] dt$ and the analysis over the space $\mathrm{AC}_0[0, 1]$.

2. Uniqueness. With $-L(v, u) = H(u) + v^2 - v$, the variational problem can be written as

$$\alpha = \max_{\phi \in C_0^1[0,1]} \int_0^1 \left[\phi(t) - \phi(t)^2 - H(\phi'(t))\right] dt = - \min_{\phi \in C_0^1[0,1]} \int_0^1 -L(\phi(t), \phi'(t))dt. \tag{2.1}$$

The function $-L$ from $\mathbb{R}^2$ to $\mathbb{R}$ is convex. Now suppose ϕ_1 and ϕ_2 are two minimizers of the problem (2.1). Let $w(t) = [\phi_1(t) + \phi_2(t)]/2$, then on one hand

$$\int_0^1 -L(w(t), w'(t))dt \geq -\alpha.$$

On the other hand, the convexity of $-L$ yields

$$\begin{aligned}\int_0^1 -L(w(t), w'(t))dt &= \int_0^1 -L\left(\frac{1}{2}(\phi_1(t), \phi_1'(t)) + \frac{1}{2}(\phi_2(t), \phi_2'(t))\right) dt \\ &\leq \frac{1}{2}\int_0^1 -L(\phi_1(t), \phi_1'(t))dt + \frac{1}{2}\int_0^1 -L(\phi_2(t), \phi_2'(t))dt = -\alpha,\end{aligned}$$

which indicates that $w(t)$ is also a minimizer of (2.1). From the equality

$$\begin{aligned}\int_0^1 &\left[-\frac{1}{2}L(\phi_1(t), \phi_1'(t)) - \frac{1}{2}L(\phi_2(t), \phi_2'(t)) + L(w(t), w'(t))\right] dt \\ &= -\frac{1}{2}\alpha - \frac{1}{2}\alpha + \alpha = 0 \quad (2.2)\end{aligned}$$

and the fact that the integrand of (2.2) is always non-positive, we have

$$-\frac{1}{2}L(\phi_1(t), \phi_1'(t)) - \frac{1}{2}L(\phi_2(t), \phi_2'(t)) = -L(w(t), w'(t)), \qquad \text{for all } t \in [0,1].$$

We rewrite the above identity as follows

$$\begin{aligned}\frac{1}{2}\phi_1^2(t) &+ \frac{1}{2}\phi_2^2(t) - \left(\frac{\phi_1(t) + \phi_2(t)}{2}\right)^2 \\ &= H\left(\frac{\phi_1'(t) + \phi_2'(t)}{2}\right) - \left(\frac{1}{2}H(\phi_1'(t)) + \frac{1}{2}H(\phi_2'(t))\right). \quad (2.3)\end{aligned}$$

If there were a point $t_0 \in [0,1]$ such that $\phi_1(t_0) \neq \phi_2(t_0)$, then the left hand side of (2.3) would be strictly less than zero which is from the strict convexity of the function x^2. But the right hand side is always non-negative. This produces a contradiction.

3. Existence. Let us denote

$$v(\phi) = \int_0^1 \left[\phi(t) - \phi(t)^2 - H(\phi'(t))\right] dt.$$

The existence of our variational problem is proved in the following way. We first prove the existence of $\max_{\phi \in \mathrm{AC}_0[0,1]} v(\phi)$. To get the C^1 smoothness of the maximizer, we identify our original problem with the one having two fixed boundaries

$$\max_{\substack{\phi \in \mathrm{AC}_0[0,1] \\ \phi(1)=c}} v(\phi), \text{ for some } c \in \mathbb{R}.$$

Then we show C^1 regularity of the maximizer by means of two fixed boundary variational results.

3.1. Existence of $\max_{\phi\in \mathrm{AC}_0[0,1]} v(\phi)$. Let us first define a subset $\mathcal{A}$ of $\mathrm{AC}_0[0,1]$ as follows

$$\mathcal{A} = \left\{ \phi \in \mathrm{AC}_0[0,1] : \int_0^1 H(\phi'(t))dt \le \frac{1}{4} + H(0) \right\}.$$

We need the following lemma regarding the properties of the subset $\mathcal{A}$ and $v(\phi)$.

Lemma 3.1. *The subset $\mathcal{A}$ is compact in $AC_0[0,1]$ and $-v(\phi)$ is lower semi-continuous in $AC_0[0,1]$ in the uniform topology.*

This important and purely technical lemma will be proved at the end of this section. Now we claim that

$$\sup_{\phi\in \mathrm{AC}_0[0,1]} v(\phi) = \sup_{\phi\in\mathcal{A}} v(\phi). \tag{3.1}$$

To see (3.1), we notice that for any $\phi \notin \mathcal{A}$,

$$v(\phi) < -H(0) = v(0)$$

from which it follows that such ϕ can not be a maximizer. Let us write $\alpha = \sup_{\phi\in\mathcal{A}} v(\phi)$ and choose a sequence $\{\phi_n(t)\}_{n\ge1} \subseteq \mathcal{A}$ such that

$$\lim_{n\to\infty} v(\phi_n) = \alpha, \text{ and } \lim_{n\to\infty} \max_{0\le t\le 1} |\phi_n(t) - \phi_0(t)| = 0 \text{ for some } \phi_0 \in \mathrm{AC}_0[0,1].$$

The reason that we can choose such a sequence ϕ_n is from the fact that $\mathcal{A}$ is compact in $\mathrm{AC}_0[0,1]$ according to Lemma 3.1 (after passing to a subsequence). To achieve the existence, we will show $v(\phi_0) = \alpha$. First it is trivial that $v(\phi_0) \le \alpha$. What is more, the lower semi-continuity of $-v(\cdot)$ in Lemma 3.1 implies that

$$v(\phi_0) \ge \limsup_{n\to\infty} v(\phi_n) = \alpha,$$

which proves the existence.

Proof of Lemma 3.1. The proof will be given in several steps. First we show that $\mathcal{A}$ is an absolutely euqicontinuous family of functions: for any $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that whenever finitely many non-overlapping intervals $\sum_i (t_i - s_i) \le \delta$, then

$$\sum_i |\phi(t_i) - \phi(s_i)| < \epsilon, \qquad \text{for any } \phi \in \mathcal{A}. \tag{3.2}$$

To see (3.2), we first note that the assumption (1.1) implies that there exists some $C(\epsilon) > 0$, such that whenever $|u| > C$,

$$\frac{H(u)}{|u|} \ge \frac{2\left[\frac{1}{4} + H(0)\right]}{\epsilon}.$$

Now we set $\delta(\epsilon) = \epsilon/(2C)$, then for all $\phi \in \mathcal{A}$,

$$\begin{aligned}\frac{1}{4} + H(0) &\geq \int_0^1 H(\phi'(t))dt \geq \sum_i \int_{s_i}^{t_i} H(\phi'(t))dt \\ &\geq \sum_i \int_{s_i}^{t_i} \frac{H(\phi'(t))}{|\phi'(t)|} |\phi'(t)| 1_{\{|\phi'(t)|>C\}}(t)dt \\ &\geq \frac{2\left[\frac{1}{4} + H(0)\right]}{\epsilon} \cdot \sum_i \int_{s_i}^{t_i} |\phi'(t)| 1_{\{|\phi'(t)|>C\}}(t)dt.\end{aligned}$$

This further implies

$$\begin{aligned}\epsilon/2 &\geq \sum_i \int_{s_i}^{t_i} |\phi'(t)| 1_{\{|\phi'(t)|>P\}}(t)dt \\ &= \sum_i \int_{s_i}^{t_i} |\phi'(t)|dt - \sum_i \int_{s_i}^{t_i} |\phi'(t)| 1_{\{|\phi'(t)|\leq C\}}(t)dt,\end{aligned}$$

from which it follows

$$\begin{aligned}\sum_i |\phi(t_i) - \phi(s_i)| &\leq \sum_i \int_{s_i}^{t_i} |\phi'(t)|dt \leq \epsilon/2 \\ &+ \sum_i \int_{s_i}^{t_i} |\phi'(t)| 1_{\{|\phi'(t)|\leq C\}}(t)dt \leq \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

The second step is to prove the lower semi-continuity of $-v(\cdot)$. To this end, let us choose a sequence of absolutely continuous functions $\{\phi_n\} \subseteq \mathcal{A}$ such that

$$\max_{0\leq t\leq 1} |\phi_n(t) - \phi_0(t)| \to 0 \text{ as } n \to \infty.$$

We show that ϕ_0 is also absolutely continuous. More precisely, according to absolute equicontinuity (3.2), for any $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that if $\sum_i (t_i - s_i) < \delta$, then $\sup_n \sum_i |\phi_n(t_i) - \phi_n(s_i)| < \epsilon$. By sending $n \to \infty$ we get $\sum_i |\phi_0(t_i) - \phi_0(s_i)| < \epsilon$, which proves the absolute continuity of ϕ_0. Let $0 = t_0 < t_1 \cdots < t_k = 1$, then Jensen's inequality implies

$$\begin{aligned}&\liminf_{n\to\infty} \int_0^1 H(\phi_n'(t))dt = \liminf_{n\to\infty} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} H(\phi_n'(t))dt \\ &\geq \liminf_{n\to\infty} \sum_{i=0}^{k-1} (t_{i+1} - t_i) H\left(\frac{\phi_n(t_{i+1}) - \phi_n(t_i)}{t_{i+1} - t_i}\right) \\ &= \sum_{i=0}^{k-1} (t_{i+1} - t_i) H\left(\frac{\phi_0(t_{i+1}) - \phi_0(t_i)}{t_{i+1} - t_i}\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} H\left(\Psi(t)\right) dt, \quad \text{where } \Psi(t) = \frac{\phi_0(t_{i+1}) - \phi_0(t_i)}{t_{i+1} - t_i} \text{ for } t_i \le t < t_{i+1} \\
&= \int_0^1 H\left(\Psi(t)\right) dt.
\end{aligned}
$$

If a sequence $\triangle_m$ of partitions is infinitely small, then the corresponding functions $\Psi_m(t)$ converge to $\phi_0'(t)$ almost everywhere (because of absolute continuity of ϕ_0). By combining the continuity of H and Fatou's lemma we get

$$
\int_0^1 H\left(\phi_0'(t)\right) dt \le \liminf_{n\to\infty} \int_0^1 H(\phi_n'(t))dt,
$$

which proves the lower semi-continuity of $\int_0^1 H(\phi(t))dt$. The lower semi-continuity of $-v(\cdot)$ now follows from that of $\int_0^1 H(\phi(t))dt$.

The last step is to prove the compactness of $\mathcal{A}$ in $\mathrm{AC}_0[0,1]$. The lower semi-continuity of $\int_0^1 H(\phi(t))dt$ yields that $\mathcal{A}$ is closed in $\mathrm{AC}_0[0,1]$. What is more, the equicontinuity in the first step and the fact that all functions in $\mathcal{A}$ have zero initial value imply that $\mathcal{A}$ is pre-compact in $C_0[0,1]$. Thus $\mathcal{A}$ is compact in $\mathrm{AC}_0[0,1]$. □

3.2. C^1 regularity of the maximizer. For one movable boundary variational problem $\max_{\phi\in \mathrm{AC}_0[0,1]} v(\phi)$, there seems to be no ready results on C^1 regularity which can be used to our problem, but this can be achieved by identifying our original problem with the one having two fixed boundaries:

$$
g(c) := \max_{\substack{\phi\in \mathrm{AC}_0[0,1] \\ \phi(1)=c}} v(\phi).
$$

Firstly, we note that $g(c)$ is well defined (i.e. the maximum is reached) for some c as the value $\phi(1)$ of the maximizer of $\max_{\phi\in \mathrm{AC}_0[0,1]} v(\phi)$. Secondly, two variational problems are identical

$$
\max_{\phi\in \mathrm{AC}_0[0,1]} v(\phi) = g(c) \ \text{ for some (possibly not unique) } c \in \mathbb{R} \tag{3.3}
$$

because of the existence of $\max_{\phi\in \mathrm{AC}_0[0,1]} v(\phi)$. Without this existence, (3.3) may fail. Thus if we can prove C^1 regularity for any maximizer of $g(c)$, then the same regularity holds for the maximizer of $\max_{\phi\in \mathrm{AC}_0[0,1]} v(\phi)$. To this end, we shall, for example, apply the regularity results in [3] under mild hypotheses by using nonsmooth analysis. More precisely, under the conditions of Theorem 1.1, the function $\phi(t) - \phi(t)^2 - H(\phi'(t))$ satisfies all the hypotheses of Theorem 2.1 and Corollary 3.1 in [3], which yields the required C^1 regularity of the maximizer.

References

1. G. Buttazzo, M. Giaquinta, and S. Hildebrandt, *One-dimensional variational problems: An introduction,* Oxford University Press, New York, 1998.
2. L. Cesari, *Optimization-theory and applications,* Springer-Verlag, New York, 1983.
3. F. Clarke and R. Vinter, Regularity properties of solutions to the basic problem in the calculus of variations, *Transactions of the American Mathematical Society* **289** (1985), 73–98.
4. A. Dembo and O. Zeitouni, *Large deviations techniques and applications,* 2nd. ed., Springer, Berlin/Heidelberg/New York, 2010.
5. L. Evans, *Partial differential equations,* AMS, Providence, RI, 1997.
6. M. Schilder, Some asymptotic formulas for Wiener integrals, *Transactions of the American Mathematical Society* **125**(1) (1966), 63–85.
7. L. Tonelli, Sur une méthode directe du calcul des variations, *Rend. Circ. Mat. Palermo* **39** (1915), 233–264.
8. S. Varadhan, Asymptotic probabilities and differential equations, *Communications on Pure and Applied Mathematics* **19** (1969), 261–286.
9. A. Wentzell, *Limit theorems on large deviations for Markov stochastic processes,* Kluwer, Dordrecht, 1990.
10. X. Yang, Asymptotic expansions for Laplace transforms of Markov processes, *Journal of Mathematical Analysis and Applications* **457**(1) (2018), 694–721.

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