AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

Binomial trees are graceful

P. Ragukumar & G. Sethuraman

To cite this article: P. Ragukumar & G. Sethuraman (2020) Binomial trees are graceful, AKCE International Journal of Graphs and Combinatorics, 17:1, 632-636, DOI: 10.1016/j.akcej.2018.06.005

To link to this article: https://doi.org/10.1016/j.akcej.2018.06.005

Published online: 16 Jun 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 1, 632–636
https://doi.org/10.1016/j.akcej.2018.06.005

Taylor & Francis
Taylor & Francis Group

AKCE International Journal of Graphs and Combinatorics

Binomial trees are graceful

P. Ragukumar[a,*], G. Sethuraman[b]

[a] *Department of Mathematics, Rajalakshmi Engineering College, Thandalam, Chennai 602 105, India*
[b] *Department of Mathematics, Anna University, Chennai 600 025, India*

Received 6 January 2018; accepted 5 June 2018

Abstract

The binomial tree B_0 consists of a single vertex. The binomial tree B_k is an ordered tree defined recursively. The binomial tree B_k consists of two binomial trees B_{k-1} that are linked together: the root of one is the leftmost child of the root of the other. The popular Graceful Tree Conjecture states that every tree is graceful. In this paper, we show that binomial trees B_k is graceful for every $k \geq 0$.

Keywords: Graceful Tree; Graceful Tree Conjecture; Binomial tree

1. Introduction

All the graphs considered in this paper are finite simple graphs. Terms that are not defined here can be referred from the book [1]. In 1963, at the Smolenice symposium in 1963, Ringel [2] conjectured that K_{2n+1}, the complete graph on $2n+1$ vertices, can be decomposed into $2n+1$ isomorphic copies of a given tree with n edges. In 1965, Kotzig [3] also conjectured that the complete graph K_{2n+1} can be cyclically decomposed into $2n+1$ copies of a given tree with n edges. In an attempt to settle these two conjectures, in 1967, in his classical paper [4] Rosa introduced a hierarchical series of 'valuations' of a graph called ρ, σ, β, α and used these valuations to investigate the cyclic decomposition of complete graphs. Later, β-valuation was called graceful by Golomb [5], and this term is being widely used. A function f is called a graceful labeling of a graph G with m edges, if f is an injective function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$ such that, when every edge uv is assigned the edge label $|f(u) - f(v)|$, then the resulting edge labels are distinct. A graph which admits graceful labeling is called a graceful graph. In the same paper [4] Rosa proved the following significant theorem.

Theorem 1 (*Rosa [4]*)**.** *If a graph T with n edges has a graceful labeling, then K_{2n+1} can be cyclically decomposed into $2n+1$ copies of T.*

Peer review under responsibility of Kalasalingam University.

* Corresponding author.
E-mail addresses: ragukumar2003@gmail.com (P. Ragukumar), sethu@annauniv.edu (G. Sethuraman).

https://doi.org/10.1016/j.akcej.2018.06.005

Please cite this article in press as: P. Ragukumar, G. Sethuraman, Binomial trees are graceful, AKCE International Journal of Graphs and Combinatorics (2018), https://doi.org/10.1016/j.akcej.2018.06.005.

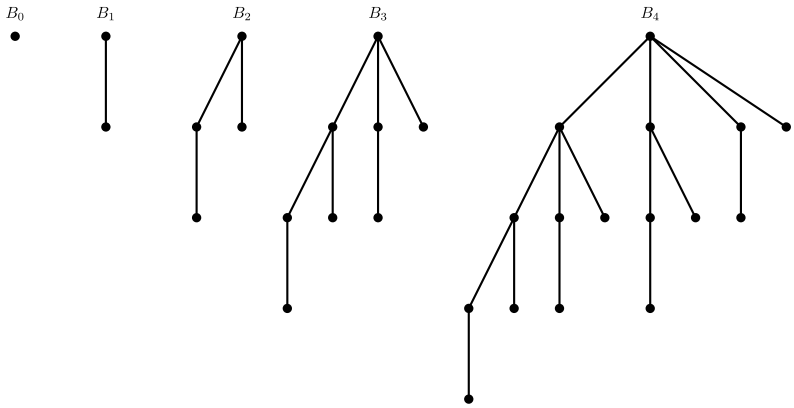

Fig. 1. Binomial Trees of order 0, 1, 2, 3 and 4.

This result led to the Rosa–Kotzig–Ringel Conjecture popularly called Graceful Tree Conjecture, which states that "All trees are graceful". The graceful tree conjecture has been the focus of many papers for over four decades. So far, no proof of the truth or falsity of the conjecture has been found. In the absence of a generic proof, one approach used in investigating the Graceful Tree Conjecture is proving the gracefulness of specialized classes of trees. Aldred and Mckay [6] used computer search to prove that all trees on at most 27 vertices are graceful. Michael Horton [7] has verified that all trees with at most 29 vertices are graceful. Fang [8] has verified that all trees with at most 35 vertices are graceful. Rosa [4] has proved that all paths and caterpillars are graceful. Some special classes of lobsters were shown to be graceful by Ng [9] and Wang et al. [10]. Chen, Lu and Yeh [11] have shown that all firecrackers are graceful. Sethuraman and Jeba Jesintha [12] have shown that all banana trees are graceful. Pastel and Raynaud [13] have shown that olive trees are graceful. Bermond and Sotteau [14] have proved that all symmetrical trees are graceful. One of the general results proved on Graceful Tree Conjecture is the result due to Havier et al. [15] that all trees of diameter five are graceful. Using Havier's branch moving technique, Balbuena et al. [16] have shown that trees having an even or quasi even degree sequence are graceful. Koh, Rogers and Tan [17] gave different methods to construct a bigger graceful tree from smaller graceful trees. Burzio and Ferrarese [18] extended the method of Koh et al. and consequently they have shown that the subdivision of every graceful tree is graceful. Cahit [19] has exhibited canonic labeling technique for proving the gracefulness of a class of rooted trees. Sethuraman and Venkatesh [20] have given a method of attaching caterpillars recursively with other caterpillars to generate graceful trees. For an exhaustive survey on Graceful Tree Conjecture, refer the excellent survey by Gallian [21], for other related results refer [22,23].

The binomial tree B_0 consists of a single vertex. The binomial tree B_k is an ordered tree defined recursively. The binomial tree B_k consists of two binomial trees B_{k-1} that are linked together: the root of one is the leftmost child of the root of the other. Note that there are 2^k vertices in the binomial tree B_k. Fig. 1 shows the binomial trees B_0 through B_4.

For more details about binomial trees refer [24]. In this paper, we show that all binomial trees are graceful.

2. Main result

In this section, we prove that the binomial tree B_k is graceful, for all $k \geq 0$. In fact, we show that the binomial trees B_k admits α-labeling, a stronger version of graceful labeling. A graceful labeling f of a graph G with q edges is said to be an α-valuation if there exists a λ such that $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$ for every edge $uv \in E(G)$, where λ is called the width of the α-valuation.

Theorem 2. *The binomial tree B_k, for all $k \geq 0$ is graceful.*

Proof. We prove that binomial tree of order k, B_k, for all $k \geq 0$ admits a stronger version of the graceful labeling, the α-labeling. More precisely, we prove that the binomial tree B_k admits α-labeling with width $\lambda = 2^{k-1} - 1$, for $k \geq 1$. It is obvious that B_0, a single vertex tree admits α-labeling. We prove this result by induction on k, the order of the

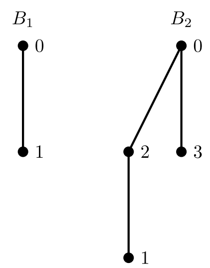

Fig. 2. Gracefully labeled Binomial Trees B_1 and B_2.

Table 1
The First and Second half Sequences of $B_{k-1}^{(i)}$, for $i = 1, 2$ after the relabeling.

$S_1^{(1)}: 2^{k-1}, 2^{k-1}+1, \dots, 2^{k-1}+2^{k-2}-1$ First half sequence of $B_{k-1}^{(1)}$	$S_1^{(2)}: 0, 1, 2, \dots, 2^{k-2}-2, 2^{k-2}-1$ First half sequence of $B_{k-1}^{(2)}$
$S_2^{(1)}: 2^{k-2}, 2^{k-2}+1, \dots, 2^{k-1}-1$ Second half sequence of $B_{k-1}^{(1)}$	$S_2^{(2)}: 2^{k-1}+2^{k-2}, \dots, 2^k-2, 2^k-1$ Second half sequence of $B_{k-1}^{(2)}$

binomial tree B_k. When $k = 1,2$, the α-labelings of B_1 and B_2 respectively having the width $\lambda = 0$, and $\lambda = 1$ are given in Fig. 2.

By induction, we assume that the binomial tree B_{k-1} admits α-labeling with $\lambda = 2^{k-2} - 1$, where $k \geq 2$. By the definition of α-labeling, the vertices of B_{k-1} are labeled with $0, 1, 2, \dots, 2^{k-1}-1$. Under this α-labeling, the sequence of labels $(0, 1, 2, \dots, 2^{k-2}-1)$ and the sequence of labels $(2^{k-2}, 2^{k-2}+1, \dots, 2^{k-1}-1)$ are respectively referred to as the *first half sequence* and the *second half sequence* of the labels of vertices of B_{k-1}.

Consider the binomial tree B_k of order k, $k \geq 2$. For the convenience, we denote B_0, a single vertex tree and we denote $B_k = \left(B_{k-1}^{(1)} \cup B_{k-1}^{(2)}\right) + e$ for $k \geq 1$, where $B_{k-1}^{(1)}$ and $B_{k-1}^{(2)}$ are the two copies of B_{k-1} and e is the new edge added between the roots of $B_{k-1}^{(1)}$ and $B_{k-1}^{(2)}$ (The $B_{k-1}^{(1)}$ is referred to as the first copy of B_{k-1} and $B_{k-1}^{(2)}$ is referred to as the second copy of B_{k-1}). As B_{k-1} has an α-labeling, the two copies of B_{k-1}, $B_{k-1}^{(1)}$ and $B_{k-1}^{(2)}$ also have the same α-labeling. Consider the $B_{k-1}^{(1)}$ and $B_{k-1}^{(2)}$ as α-labeled tree with their α-labeling. Consequently, tree B_k is also a labeled tree labeled from the set $\{0, 1, 2, \dots, 2^{k-1}-1\}$ such that every label is assigned exactly to two vertices of B_k of which one vertex is in $B_{k-1}^{(1)}$ and the other vertex is in $B_{k-1}^{(2)}$. Hereafter, we refer the vertices of $B_{k-1}^{(i)}$, for $i = 1,2$ in B_k with their vertex labels that are defined by the α-labeling of $B_{k-1}^{(i)}$, for $i = 1,2$.

Consider the first half sequence of vertex labels of $B_{k-1}^{(1)}$ and the second half sequence of vertex labels of $B_{k-1}^{(2)}$ and add 2^{k-1} to each label in both of the sequences and retain the original labels for the remaining vertices of $B_{k-1}^{(i)}$, for $i = 1,2$. Thus we have the labels of the vertices of $B_{k-1}^{(i)}$, for $i = 1,2$ after the above relabeling as given in Table 1.

From Table 1, the labels of the vertices of $B_{k-1}^{(i)}$, for $i = 1,2$ can be arranged as the sequence of labels from $S_1^{(2)}$, followed by the sequence of labels from $S_2^{(1)}$, followed by the sequence of labels from $S_1^{(1)}$ and followed by the sequence of labels from $S_2^{(2)}$ as a monotonic sequence of vertex labels $0, 1, 2, \dots, 2^{k-2}-1, 2^{k-2}, 2^{k-2}+1, \dots, 2^{k-1}-1, 2^{k-1}, \dots, 2^k-1$. Hence the labels of the vertices of B_k are distinct.

Since λ of $B_{k-1}^{(i)}$, for $i = 1,2$ is $2^{k-2}-1$, for any edge in $B_{k-1}^{(i)}$, the label of the one end vertex of the edge is in the first half sequence of vertex labels and the label of the other end vertex of the edge is in the second half sequence of vertex labels corresponding to its α-labeling.

Let x be the edge label of an edge $e = uv$ in $B_{k-1}^{(1)}$ with vertex labels $l(u)$, $l(v)$ in $B_{k-1}^{(1)}$. If $l(u) < l(v)$, then $l(u) \in \{0, 1, 2, \dots, 2^{k-2}-1\}$ and $l(v) \in \{2^{k-2}, 2^{k-2}+1, \dots, 2^{k-1}-1\}$. Therefore, by the above relabeling procedure in B_k, u gets the label $l(u) + 2^{k-1}$ and the label of v is retained. Thus, the edge $e = uv$ of $B_{k-1}^{(1)}$ gets the label $|l(u) + 2^{k-1} - l(v)| = |-\{l(u) + 2^{k-1} - l(v)\}| = |-(2^{k-1} - \{l(v) - l(u)\})| = |-(2^{k-1} - x)| = 2^{k-1} - x$ in B_k. Similarly, if $l(v) < l(u)$, then the edge $e = uv$ of $B_{k-1}^{(1)}$ gets the label $2^{k-1} - x$ in B_k.

Let x be the edge label of an edge $e = uv$ in $B_{k-1}^{(2)}$ with vertex labels $l(u), l(v)$ in $B_{k-1}^{(2)}$ corresponding to its α-labeling. If $l(u) < l(v)$, then $l(u) \in \{0, 1, 2, \dots, 2^{k-2}-1\}$ and $l(v) \in \{2^{k-2}, 2^{k-2}+1, \dots, 2^{k-1}-1\}$. Therefore by

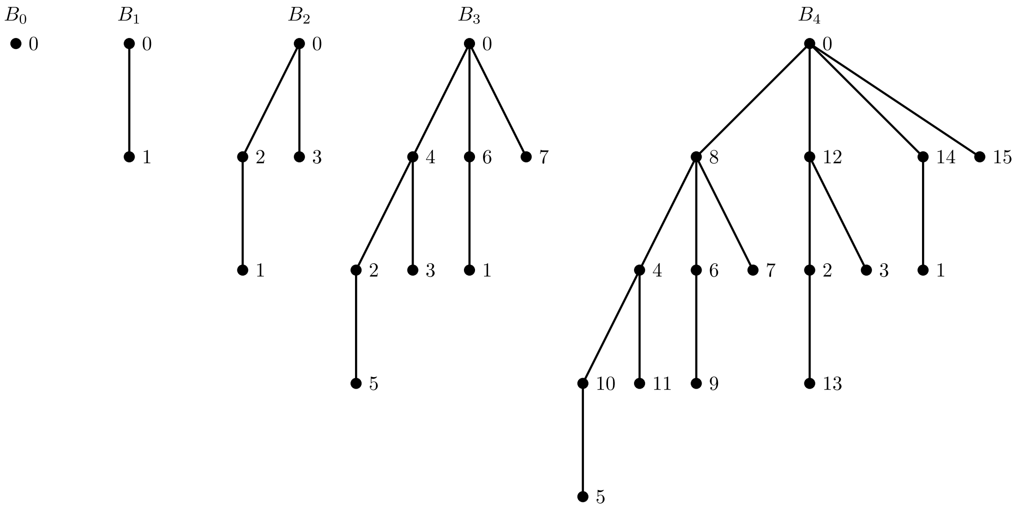

Fig. 3. Gracefully labeled Binomial Trees of order 0, 1, 2, 3 and 4.

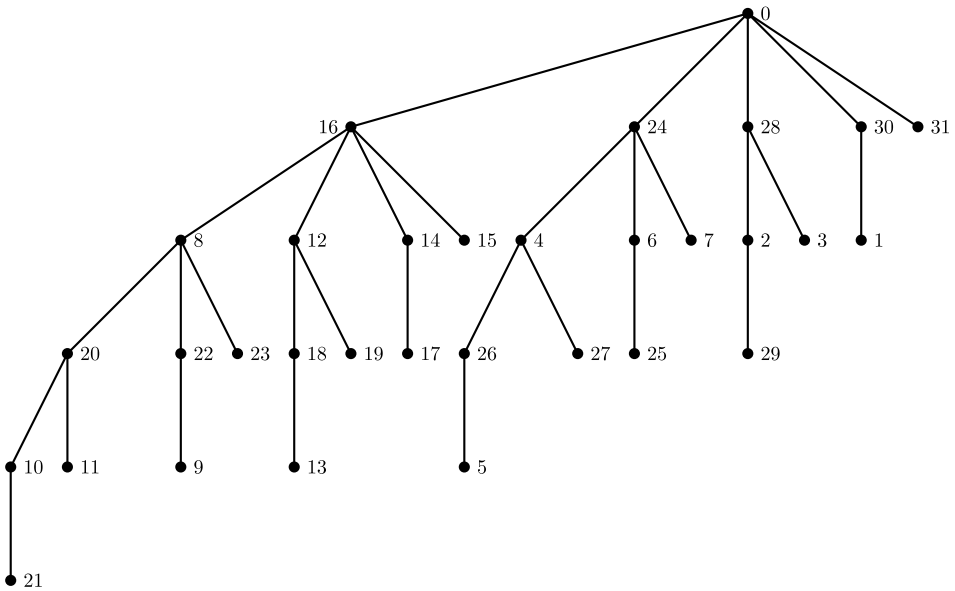

Fig. 4. Gracefully labeled Binomial Tree of order 5.

the above relabeling procedure, v gets the label $l(v) + 2^{k-1}$ and the label of u is retained. Thus, the edge $e = uv$ of $B_{k-1}^{(2)}$ gets the label $|l(v)+2^{k-1}-l(v)| = |2^{k-1}+l(v)-l(u)| = |2^{k-1}+x| = 2^{k-1}+x$ in B_k. Similarly if $l(v) < l(u)$, then the edge $e = uv$ gets the relabel $2^{k-1} + x$ in B_k. Thus, after the relabeling, the $2^{k-1} - 1$ edges in $B_{k-1}^{(1)}$ will get the edge labels as $2^{k-1} - 1, 2^{k-1} - 2, \ldots, 2^{k-1} - (2^{k-1} - 1) = 1$ in B_k. Therefore, after the relabeling, the $2^{k-1} - 1$ edges in $B_{k-1}^{(2)}$ will get the edge labels as $2^{k-1} + 1, 2^{k-1} + 2, \ldots, 2^{k-1} + 2^{k-1} - 1 = 2^k - 1$ in B_k. Also note that after the relabeling process, the label of the root of $B_{k-1}^{(1)}$ relabeled as 2^{k-1} and the label of the root of $B_{k-1}^{(2)}$ is retained as 0 in $B_{k-1}^{(2)}$. Therefore the edge connecting the roots of $B_{k-1}^{(1)}$ and $B_{k-1}^{(2)}$ gets the edge label 2^{k-1} in B_k. Hence the edges of B_k get the distinct edge labels $1, 2, 3, \ldots, 2^k - 1$. Hence the relabeling procedure indeed gives α-labeling for B_k having the width $2^{k-1} - 1$. This completes the induction. □

Please cite this article in press as: P. Ragukumar, G. Sethuraman, Binomial trees are graceful, AKCE International Journal of Graphs and Combinatorics (2018), https://doi.org/10.1016/j.akcej.2018.06.005.

The following two significant theorems indicate that the α-labeled graphs decomposes the complete graphs and complete bipartite graphs.

Theorem 3 (*Rosa [4]*). *If a graph G with q edges has an α-valuation, then there exists a cyclic decomposition of the complete graph K_{2cq+1} into subgraphs isomorphic to G, where c is an arbitrary natural number.*

Theorem 4 (*El-zanati and Vanden Eynden [25]*). *If G has q edges and admits an α-valuation, then $K_{mq,nq}$ can be decomposed into subgraphs isomorphic to G for all positive integers m and n.*

Since the binomial tree B_k, $k \geq 0$ admits α-labeling, the following corollary is a direct consequence of Theorems 3 and 4.

Corollary 4.1. *The complete graph K_{2cq+1} and the complete bipartite graph $K_{mq,nq}$ can be decomposed into isomorphic copies of the binomial tree B_k for every $k \geq 0$, where $m, n, c \in Z^+$.*

Illustration

Gracefully labeled binomial trees B_0, B_1, B_3, B_4 and B_5 are given in Figs. 3 and 4.

References

[1] D.B. West, Introduction to Graph Theory, second ed., Prentice Hall of India, 2001.
[2] G. Ringel, Problem 25, in theory of graphs and its applications, in: Proc. Symposium Smolenice, Prague, 1963, p. 162.
[3] A. Kotzig, Decompositions of a complete graph into 4k-gons, Matematicky Casopis 15 (1965) 229–233 (in Russian).
[4] A. Rosa, On certain valuations of the vertices of a graph, in: Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris, 1966, pp. 349–355.
[5] S.W. Golomb, How to number a graph, in: R.C. Read (Ed.), Graph Theory and Computing, Academic Press, New York, 1972, pp. 23–37.
[6] R.E. Aldred, B.D. Mckay, Graceful and harmonious labelings of trees, Bull. Inst. Combin. Appl. 23 (1998) 69–72.
[7] Horton Michael, Graceful Trees Statistics and Algorithms (Master's thesis). http://eprints.comp.utas.edu.au:81/archieve/00000019/01/.
[8] W. Fang, A computational approach to the graceful tree conjecture. arXiv:1003.3045v1 [cs.DM].
[9] H.K. Ng, Gracefulness of a class of lobsters, Notices AMS 7 (1986) 825-05-294.
[10] J.G. Wang, D.J. Jin, X.G. Lu, D. Zhang, The gracefulness of a class of lobster trees, Math. Comput. Modelling 20 (1994) 105–110.
[11] Chen,W.C., H.I. Lu, Y.N. Yeh, Operations of interlaced trees and graceful trees, Southeast Asian Bull. Math. 21 337–348.
[12] Jesintha J. Jeba, G. Sethuraman, All arbitrary fixed generalized banana trees are graceful, Math. Comput. Sci. 5 (1) (2011) 51–62.
[13] A.M. Pastel, H. Raynaud, Numerotation gracieuse des olivers, in: Colloq. Grenoble, Publications Universite de Grenoble, 1978, pp. 218–223.
[14] J.C. Bermond, D. Sotteau, Graph decompositions and G-design, in: Proc. 5th British Combin. Conf., 53–72 (second series), vol. 12, 1989, pp. 25–28.
[15] Havier Pavel, Havier Alfonz, All trees of diameter five are graceful, Discrete Math. 233 (2001) 133–150.
[16] C. Balbuena, P. Garcia-Vazquez, X. Marcote, J.C. Valenzuela, Trees having an even or quasi even degree sequence are graceful, Appl. Math. Lett. 20 (2007) 370–375.
[17] K.H. Koh, D.G. Rogers, T. Tan, Two theorems on graceful trees, Discrete Math. 25 (1979) 141–148.
[18] M. Burzio, G. Ferrarese, The subdivision graph of a graceful tree is a graceful tree, Discrete Math. 181 (1998) 275–281.
[19] I. Cahit, Graceful labelings of rooted complete trees, preprint, 2002.
[20] G. Sethuraman, S. Venkatesh, Decomposition of complete graphs and complete bipartite graphs into α-labeled trees, Ars Combin. 93 (2009) 371–385.
[21] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 19 (2012) #DS6.
[22] F. Van Bussel, Relaxed graceful labelings of trees, Electron. J. Combin. 9 (2002) #R4.
[23] P.J. Slater, On k-graceful graphs, in: Proc. of the 13th S.E. Conf. on Combinatorics, Graph Theory Comput. (1982) 53–57.
[24] Thomas Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein, Introduction to Algorithms, second ed., MIT Press.
[25] S. El-Zanati, C. Vanden Eynden, Decomposition of $k_{m,n}$ into cubes, J. Combin. Des. 4 (1996) 51–67.

Please cite this article in press as: P. Ragukumar, G. Sethuraman, Binomial trees are graceful, AKCE International Journal of Graphs and Combinatorics (2018), https://doi.org/10.1016/j.akcej.2018.06.005.