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On the genus of some total graphs

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Abstract

Let R be a commutative ring with a proper ideal I. A generalization of total graph is introduced and investigated. It is the (undirected) graph with all elements of R as vertices, that two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in S_H(I)$ where $S_H(I) = \{a \in R : ra \in I$ for some $r \in H\}$ and H is a multiplicatively closed subset of R. This version of total graph is denoted by $T(\Gamma_H^I(R))$. We in addition characterize certain lower and upper bounds for the genus of the total graph, and compute genus $T(\Gamma_H^I(R))$ on finite ring R, with respect to some special ideal I.

Keywords: Commutative rings; Multiplicatively closed subset; Total graph; Genus

1. Introduction

Throughout, all rings will be commutative with non-zero identity. Let R be a ring and I a proper ideal of R. The *total graph* of a commutative ring R, denoted by $T(\Gamma(R))$, was introduced by Anderson and Badawi in [1] and studied by several authors ([2–4], etc.), where the authors in [3,4] obtained some facts on the genus of total graphs. They considered a total graph with all elements of R as vertices, that two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$ where $Z(R)$ denotes the set of all zero-divisors of R. The total graph is then extended in joint papers [5,6] of the second author in rings and modules, respectively. Furthermore, a generalized total graph was introduced in [7]. For a proper submodule N of M, there is a generalization of the graph of modules relative N under multiplicatively closed subset H denoted by $T(\Gamma_H^N(M))$ which was studied by present authors in [8]. The vertex set of $T(\Gamma_H^N(M))$ is M, that two distinct vertices m and m' are adjacent if and only if $m + m' \in M_H(N)$ where $M_H(N) = \{m \in M : rm \in N$ for some $r \in H\}$ and H is a multiplicatively closed subset of R, i.e. $ab \in H$ for all $a, b \in H$. As N is a proper submodule of M and $N \subseteq M_H(N)$, $M_H(N)$ is not empty.

We define a generalized total graph over ring R, denoted by $T(\Gamma_H^I(R))$, with all elements of R as vertices, that two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in S_H(I)$ where $S_H(I) = \{a \in R : ra \in I$ for some $r \in H\}$, I is an ideal of R and H is a multiplicatively closed subset of R.

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It follows from the definition that if $S_H(I) = R$, (for example, if $I = R$, $0 \in H$, $H \bigcap (S_H(I) : R) \neq \emptyset$, $H \bigcap (0 : R) \neq \emptyset$ or $H \bigcap (I : R) \neq \emptyset$, by [8]), then $T(\Gamma_H^I(R))$ is complete; so we suppose that $S_H(I) \neq R$. We denote by $\Gamma_H^I(S_H(I))$ and $\Gamma_H^I(S_H^C(I))$ the (induced) subgraphs of $T(\Gamma_H^I(R))$ with vertices in $S_H(I)$ and $R - S_H(I)$, respectively. Based on our assumption, $S_H(I) \neq R$ and so $\Gamma_H^I(S_H^C(I))$ is always nontrivial.

Let G be a simple graph. We say that G is *totally disconnected* if none of two vertices of G are adjacent. We use K_n to denote complete graph with n vertices. A *bipartite graph* G is a graph whose vertex set $V(G)$ can be partitioned into subsets V_1 and V_2 such that the edge set consists of precisely those edges which join vertices in V_1 to vertices of V_2. In particular, if E consists of all possible such edges, then G is called the *complete bipartite graph* and denoted by $K_{m,n}$ when $|V_1| = m$ and $|V_2| = n$. Two subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (resp., G_2) is adjacent (in G) to any vertex not in G_1 (resp., G_2). The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2$ whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$. The *Cartesian product* of graphs G_1 and G_2 is defined as the graph $G_1 \times G_2$ which the vertex set is $V(G_1) \times V(G_2)$ and the edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$. Two graphs G and H are said to be *isomorphic* to each another, written $G \cong H$, if there exists a bijection $f : V(G) \to V(H)$ such that for each pair x, y of vertices of G, $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. For a vertex v of graph G, $deg(v)$ is the degree of vertex v and $\delta(G) := \min\{deg(v)$: v is a vertex of $G\}$. For a nonnegative integer k, a graph G is called $k-regular$ if every vertex of G has degree k. The *genus* of a graph G, denoted by $g(G)$, is the minimal integer n such that the graph can be embedded in S_n, where S_n denotes a sphere with n handles. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. For such graphs the genus is zero. A graph with genus one is called a *toroidal* graph. If G' is a subgraph of G, then $g(G') \leq g(G)$. For details on the notion of embedding of a graph in a surface, see White [9, Chapter 6].

In Section 2, we remind some facts and give a lower bound for genus of the graph $T(\Gamma_H^I(R))$. We proceed in Section 3 by determining all isomorphism classes of finite rings R whose $T(\Gamma_H^I(R))$ has genus at most one (i.e. a planar or toroidal graph). Also, we compute genus of the graph over $R = \mathbb{Z}_n$ under some well-known multiplicatively closed subsets of R.

2. Background problem and some comments

Throughout, $\lceil x \rceil$ denotes the least integer that is greater than or equal to x. In the following theorem we give some well-known formulas, see, e.g., [9–11]:

Theorem 2.1. *The following statements hold:*

(1) *For $n \geq 3$ we have $g(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$.*
(2) *For $m, n \geq 2$ we have $g(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$.*
(3) *Let G_1 and G_2 be two graphs and for each i, p_i be the number of vertices of G_i. Then $\max \{p_1 g(G_2) + g(G_1), p_2 g(G_1) + g(G_2)\} \leq g(G_1 \times G_2)$.*
(4) *The genus of a graph is the sum of the genuses of its components.*

According to Theorem 2.1 we have $g(K_n) = 0$ for $1 \leq n \leq 4$, $g(K_n) = 1$ for $5 \leq n \leq 7$ and $g(K_n) \geq 2$, for other value of n.

Corollary 2.2. *If G is a graph with n vertices, then $g(G) \leq \lceil \frac{(n-3)(n-4)}{12} \rceil$.*

In the following of the section, we characterize a lower bound for the genus of the graph $T(\Gamma_H^I(R))$. Considering the fact that $\Gamma_H^I(S_H(I))$ is in the form of $K_{|S_H(I)|}$ (see [8, Remark 3.1]), in view of Theorem 2.1, it is enough for us to obtain a lower bound for genus of the graph $\Gamma_H^I(S_H^C(I))$.

Theorem 2.3 ([8, Corollary 3.7]). *Let $| S_H(I) |= \alpha$, $| R/S_H(I) |= \beta$, and $2 \in (S_H(I) : R)$.Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\beta - 1$ copies of K_α.*

Theorem 2.4 ([8, Theorem 3.10]). *Let $| S_H(I) |= \alpha$ and $| R/S_H(I) |= \beta$. If H, a multiplicatively closed subset of R containing some even elements, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$.*

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Corollary 2.5. *Let $|S_H(I)| = \alpha$ and $|R/S_H(I)| = \beta$. Then the following hold:*

1. *If $2 \in (S_H(I) : R)$, then $g(\Gamma_H^I(S_H^C(I))) = (\beta - 1)\lceil \frac{(\alpha-3)(\alpha-4)}{12} \rceil$.*
2. *If $2r \in H$ for some $r \in R$, then $g(\Gamma_H^I(S_H^C(I))) = \frac{\beta-1}{2}\lceil \frac{(\alpha-2)^2}{4} \rceil$.*

Proof. It is obvious by Theorems 2.3 and 2.4. □

Lemma 2.6 *([12, Proposition .2.1]). If G is a graph with n vertices and genus g, then $\delta(G) \leq 6 + \frac{12g-12}{n}$.*

Theorem 2.7 *([8, Theorem 3.13]). Suppose that the edge set of $\Gamma_H^I(S_H^C(I))$ is not empty and x is a vertex of the graph. Then the degree of x is either $\mid S_H(I) \mid$ or $\mid S_H(I) \mid -1$.*

Proposition 2.8. *Let $\Gamma_H^I(S_H^C(I))$ with t' vertices have a nonempty edge set, and let $|S_H(I)| = t$. Then $\frac{(t-7)t'}{12} + 1 \leq g(\Gamma_H^I(S_H^C(I)))$.*

Proof. By Lemma 2.6 and Theorem 2.7, $t - 1 = |S_H(I)| - 1 = \delta$, so $t - 1 \leq 6 + \frac{(12g-12)}{t'}$. Then $(t-7)t' \leq 12g - 12$. Hence $\frac{(t-7)t'}{12} + 1 \leq g(\Gamma_H^I(S_H^C(I)))$. □

Corollary 2.9. *If R is infinite, then $g(\Gamma_H^I(S_H^C(I)))$ is infinite for all ideals I and all closed subsets H of R.*

3. The genus of $T(\Gamma_H^I(R))$

Considering Corollary 2.9, the genus will be infinite if R is not a finite ring. In order to compute the genus, we consider a finite ring R.

In view of Theorem 2.1, it is enough for us to study the genus of $\Gamma_H^I(S_H^C(I))$.

Remark 3.1. If $H \cap I \neq \emptyset$, then $S_H(I) = R$ and $\Gamma_H^I(S_H^C(I))$ is trivial so, in the following, we suppose that $H \cap I = \emptyset$. It should be noted that if $H \cap I = \emptyset$, then $H \cap S_H(I) = \emptyset$.

Theorem 3.2. *Let R be a finite ring such that $R = R_1 \times R_2 \times \cdots \times R_t$ with $t \geq 4$, $I = 0 \times R_2 \times \cdots \times R_t$ and H be a multiplicatively closed subset of R. Then $\gamma(\Gamma_H^I(S_H^C(I))) \geq 2$.*

Proof. It is enough to show that there is a subgraph L of $\Gamma_H^I(S_H^C(I))$ with $\gamma(L) \geq 2$; this implies that $\gamma(\Gamma_H^I(S_H^C(I))) \geq 2$. So, we proceed for $t = 4$.

1. Let $H \cap (Z(R_1) \times R_2 \times \cdots \times R_t) = \emptyset$. Then $I = S_H(I)$. By way of contradiction, let there exists $(r_1, \ldots, r_t) \in R - I$ such that $(r_1, \ldots, r_t)(h_1, \ldots, h_t) \in I$ for some $(h_1, \ldots, h_t) \in H$ (note that h_1 has inverse in R_1). Then $r_1 h_1 = 0$ implies that $r_1 = 0$, a contradiction.
   (1′) If $R_1 = \mathbb{Z}_2$, then $K_8 \subseteq \Gamma_H^I(S_H^C(I))$.
   (2′) Let $|R_1| > 2$.
   (a′) If $2 \in Z(R_1)$, then considering the vertices $\{(a_1, a_2, a_3, a_4)|a_i \in R_i$ for $i \geq 2\}$, there is $a_1 \neq 0$ belonging to R_1 such that $2a_1 = 0$. So, $K_8 \subseteq \Gamma_H^I(S_H^C(I))$.
   (b′). Let $2 \notin Z(R_1)$. For a non zero element $l_1 \in R_1$, set $X_1 = \{(l_1, l_2, l_3, l_4) \in R | l_i \in R_i$ for $2 \leq i \leq 4\}$ and $Y_1 = \{(-l_1, l_2, l_3, l_4) \in R | l_i \in R_i$ for $2 \leq i \leq 4\}$. Then X_1, Y_1 is a bipartition for $K_{n,n}$ for $n \geq 8$. Hence, $K_8 \subseteq \Gamma_H^I(S_H^C(I))$ or $K_{8,8} \subseteq \Gamma_H^I(S_H^C(I))$; so $\gamma(\Gamma_H^I(S_H^C(I))) \geq 2$.
2. Let $H \cap (Z(R_1) \times R_2 \times \cdots \times R_t) \neq \emptyset$. Set $T = H \cap (Z(R_1) \times R_2 \times \cdots \times R_t)$. For $(d_1, d_2, \ldots, d_t) \in T$, put $K_{(d_1,d_2,\ldots,d_t)} = \{(b_1, b_2, \ldots, b_t) \in R | b_1 \neq 0, b_1 d_1 = 0\}$ and define $K = \bigcup_{(d_1,d_2,\ldots,d_t) \in T} K_{(d_1,d_2,\ldots,d_t)}$.
   Claim. $S_H(I) = I \cup K$.
   Let there exists $(r_1, \ldots, r_t) \in R - I$ such that $(r_1, \ldots, r_t)(h_1, \ldots, h_t) \in I$ for some $(h_1, \ldots, h_t) \in H$. Then $r_1 h_1 = 0$ implies that $h_1 \in Z(R_1) - \{0\}$ (by $I \cap H = \emptyset$). So $(r_1, \ldots, r_t) \in K$. Conversely, let $(b_1, n_2, \ldots, n_t) \in K$. Then there exists $(d', n'_2 \ldots, n'_t) \in T$ such that $b_1 d' = 0$. So $(b_1, n_2, \ldots, n_t)(d', n'_2 \ldots, n'_t) \in I$ implies that $(b_1, n_2, \ldots, n_t) \in S_H(I)$, hence $K \subseteq S_H(I)$. Therefore, $S_H(I) = I \cup K$.

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(i) If $R_1 = \mathbb{Z}_2$, then $K_8 \subseteq \Gamma^I_H(S^C_H(I))$.
(ii) Suppose $|R_1| > 2$.
(i′) Let there exists $(2, m_2, m_3, m_4) \in T_4$, where $T_4 = H \cap (Z(R_1) \times R_2 \times R_3 \times R_4)$. Then $2 \in Z(R_1) - \{0\}$ (by $H \cap I = \emptyset$). So $2a_1 = 0$ for non zero element $a_1 \in R_1$ and $\{(a_1, n_2, n_3, n_4)|n_i \in R_i \text{ for } 2 \le i \le 4\} \subseteq K \subseteq S_H(I)$, then $a_1 \ne 2$ (otherwise, $S_H(I) \cap H \ne \emptyset$), also $a_1 \ne 1$ since $2 \ne 0$ (by $I \cap H = \emptyset$). Considering distinct sets $\{(a_1 - 1, a_2, a_3, a_4)|a_i \in R_i \text{ for } i \ge 2\}$, one has $K_{8,8} \subseteq \Gamma^I_H(S^C_H(I))$.
(ii′) Let $(2, m_2, m_3, m_4) \notin T_4$ for every $m_i \in R_i$ with $i \ge 2$. By the similar argument of case 1., if $2 \in Z(R_1)$, then $K_8 \subseteq \Gamma^I_H(S^C_H(I))$ and if $2 \notin Z(R_1)$, then $K_{8,8} \subseteq \Gamma^I_H(S^C_H(I))$.
Hence, $\gamma(\Gamma^I_H(S^C_H(I))) \ge 2$. □

Remark 3.3. It should be noted that, Theorem 3.2 is satisfied for every $I = R_1 \times R_2 \times \cdots \times R_{n-1} \times 0 \times R_{n+1} \times \cdots \times R_t$ with $n > 1$ and $t \ge 4$.

Theorem 3.4. *Let $R = R_1 \times R_2 \times R_3$ where every R_i is a finite ring for $1 \le i \le 3$, $I = 0 \times R_2 \times R_3$ and H be a multiplicatively closed subset of R.*

1. *Let $H \cap (Z(R_1) \times R_2 \times R_3) = \emptyset$ or $H \cap (Z(R_1) \times R_2 \times R_3) \ne \emptyset$ with $(2, m_2, m_3) \notin H \cap (Z(R_1) \times R_2 \times R_3)$ for every $m_2 \in R_2$ and $m_3 \in R_3$.*
   (i) *If $2 \in Z(R_1)$ and $|R_2||R_3| \ge 8$, then $\gamma(\Gamma^I_H(S^C_H(I))) \ge 2$.*
   (ii) *If $2 \notin Z(R_1)$ and $|R_2||R_3| \ge 4$, then $\gamma(\Gamma^I_H(S^C_H(I))) \ge 1$.*
2. *Let there exists $(2, m_2, m_3) \in H \cap (Z(R_1) \times R_2 \times R_3)$ and $|R_2||R_3| \ge 4$. Then $\gamma(\Gamma^I_H(S^C_H(I))) \ge 1$.*

Proof.

1. (i) Consider $\{(a_1, a_2, a_3)|a_i \in R_i\}$ which is in the form of K_n for $n \ge 8$, where $2a_1 = 0$ for some $a_1 \ne 0$ belonging to R_1.
   (ii) For a non zero element $l_1 \in R_1$, consider $\{(l_1, l_2, l_3)|l_i \in R_i, i = 2, 3\} \cup \{(-l_1, m_2, m_3)|m_i \in R_i, i = 2, 3\}$ which is in the form of $K_{n,n}$ for $n \ge 4$.
2. If there is $(2, m_2, m_3) \in H \cap (Z(R_1) \times R_2 \times R_3)$, then $2 \in Z(R_1) - \{0\}$, so $2a_1 = 0$ for non zero element $a_1 \in R_1$ and $\{(a_1, n_2, n_3)|n_i \in R_i, i = 2, 3\} \subseteq S_H(I)$, then $a_1 \ne 2$ (otherwise, $S_H(I) \cap H \ne \emptyset$); furthermore, $a_1 \ne 1$ since $2 \ne 0$ (by $I \cap H = \emptyset$). Hence the vertices $\{(a_1 - 1, a_2, a_3)|a_i \in R_i, i = 2, 3\} \cup \{(1, a_2, a_3)|a_i \in R_i, i = 2, 3\}$ are in the form of $K_{n,n}$ for $n \ge 4$. □

Corollary 3.5. *Let $R = R_1 \times R_2$ where every R_i is a finite ring for $i \in \{1, 2\}$, $I = 0 \times R_2$ and H be a multiplicatively closed subset of R.*
It is easily proved that the following statements hold.

1. *Let $H \cap (Z(R_1) \times R_2) = \emptyset$ or $H \cap (Z(R_1) \times R_2) \ne \emptyset$ with $(2, m_2) \notin H \cap (Z(R_1) \times R_2)$ for every $m_2 \in R_2$.*
   (i) *If $2 \in Z(R_1)$ and $|R_2| \ge 8$, then $\gamma(\Gamma^I_H(S^C_H(I))) \ge 2$.*
   (ii) *If $2 \notin Z(R_1)$ and $|R_2| \ge 4$, then $\gamma(\Gamma^I_H(S^C_H(I))) \ge 1$.*
2. *Let there exists $(2, m_2) \in H \cap (Z(R_1) \times R_2)$ and $|R_2| \ge 4$. Then $\gamma(\Gamma^I_H(S^C_H(I))) \ge 1$.*

Proof. It is obvious by Theorem 3.4. □

Example 3.6. Let $R = \mathbb{Z}_6 \times \mathbb{Z}_4$, $I = 0 \times \mathbb{Z}_4$ and H be a multiplicatively closed subset of R. Then $Z(R) = (Z(\mathbb{Z}_6) \times \mathbb{Z}_4) \cup (\mathbb{Z}_6 \times Z(\mathbb{Z}_4))$. If $H \cap (Z(\mathbb{Z}_6) \times \mathbb{Z}_4) = \emptyset$, then $S_H(I) = I$. Let $H \cap (Z(\mathbb{Z}_6) \times \mathbb{Z}_4) \ne \emptyset$ and let $(2, n_s) \in H$ (so for every $n_i \in \mathbb{Z}_4$, $(3, n_i) \notin H$, otherwise $H \cap I \ne \emptyset$). Then $S_H(I) = I \cup \{(3, t_2)|t_2 \in \mathbb{Z}_4\}$. Considering the vertices of $\Gamma^I_H(S^C_H(I))$ in the form of $\{(1, b_2), (4, t_2)|b_2, t_2 \in \mathbb{Z}_4\} \cup \{(2, m_2), (5, l_2)|m_2, l_2 \in \mathbb{Z}_4\}$, one has $\Gamma^I_H(S^C_H(I))$ is a $K_{8,8}$.

Theorem 3.7. *Let $R = \mathbb{F}_q \times R_n$, $I = 0 \times R_n$ where R_n is a ring with $|R_n| = n$, $\mathbb{F}_q$ is a field with q elements and let H be a multiplicatively closed subset of R.*

1. *Let $q = 2^m$.Then $\Gamma^I_H(S^C_H(I))$ is planar if and only if $2 \le n \le 4$ and it is toroidal if and only if $m = 1$ and $5 \le n \le 7$.*

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2. *If for every $m \in \mathbb{N}$, $q \neq 2^m$, then $\Gamma^I_H(S^C_H(I))$ is planar if and only if $n = 2$ and it is toroidal if and only if $q = 3$ and $n \in \{3, 4\}$.*

Proof. Claim; for every $n \in \mathbb{N}$, $I = S_H(I)$. By way of contradiction, let there exists $(m, d) \in R - I$ such that $(m, d)(t, l) \in I$ for some $(t, l) \in H$. So $mt = 0$ which implies that $m = 0$; so $m = 0$, a contradiction.

1. Let $q = 2^m$.
   (a) If $m = 1$, for every $(1, b) \in S^C_H(I)$, we have $(1, b) + (1, b') \in S_H(I)$ where b, b' are disjoint elements of R_n. So $\Gamma^I_H(S^C_H(I))$ is K_n.
   (b) Let $m > 1$. Considering vertices of $\Gamma^I_H(S^C_H(I))$ in the form of $\{(l, a)|l \in \mathbb{F}_{2^m}, a \in R_n\}$ and by the fact that $char(\mathbb{F}_{2^m}) = 2$, in the same way of proof of Theorem 3.2, $\Gamma^I_H(S^C_H(I))$ is a disjoint union of $2^m - 1$ copies of K_n. Hence, $\Gamma^I_H(S^C_H(I))$ is planar if and only if $2 \leq n \leq 4$ and it is toroidal if and only if $m = 1$ and $5 \leq n \leq 7$.
2. Let for all $m \in \mathbb{N}$, one has $q \neq 2^m$. For non zero element $l \in F_q$, let $X_l = \{(l, a)|a \in R_n\}$ and $Y_l = \{(-l, a)|a \in R_n\}$. Then X_l, Y_l is a bipartition of $K_{n,n}$. So $\Gamma^I_H(S^C_H(I))$ is a disjoint union of $\frac{q-1}{2}$ copies of $K_{n,n}$. Hence $\Gamma^I_H(S^C_H(I))$ is planar if and only if $n = 2$ and it is toroidal if and only if $q = 3$ and $n \in \{3, 4\}$. □

Theorem 3.8. *Let R be a finite ring for which $R = R_1 \times R_2 \times \cdots \times R_t$ with $t \geq 2$, $I = 0 \times R_2 \times \cdots \times R_t$, $H = R - Z(R)$ and let $2 \notin Z(R_1)$. If $\gamma(\Gamma^I_H(S^C_H(I))) \leq 1$, then R is isomorphic to the one of the following rings:*

$$\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2, R_1 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2).$$

Proof. Note that $t < 4$, by Theorem 3.2.

1′. Let $t = 3$.
If $\gamma(\Gamma^I_H(S^C_H(I))) < 1$, then $|R_2||R_3| < 4$, by Theorem 3.4. For $\gamma(\Gamma^I_H(S^C_H(I))) = 1$, consider $|R_2||R_3| = 4$, by the proof of Theorem 3.4. So, for $\gamma(\Gamma^I_H(S^C_H(I))) \leq 1$, $|R_2||R_3| \leq 4$. Hence $R_2 = R_3 = \mathbb{Z}_2$. For every non zero element $b_1 \in R_1$, set $X_{b_1} = \{(b_1, n_2, n_3)|n_i \in R_i, i = 2, 3\}$. Then X_{b_1}, X_{-b_1} is a bipartition of $K_{4,4}$. If $|R_1| > 3$, then $\gamma(\Gamma^I_H(S^C_H(I))) \geq 2$ (since there exist at least two bipartition of $K_{4,4}$ in $\Gamma^I_H(S^C_H(I))$). Hence, $|R_1| \leq 3$, so $R_1 = \mathbb{Z}_3$ (since $2 \notin Z(R_1)$). Therefore, $R = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

2′. Let $t = 2$.
For every non zero element $b_1 \in R_1$, set $X_{b_1} = \{(b_1, n_2)|n_2 \in R_2\}$. Then X_{b_1}, X_{-b_1} is a bipartition of $K_{n,n}$ for $n = |R_2|$. Since $2 \notin Z(R_1)$ and $\gamma(\Gamma^I_H(S^C_H(I))) \leq 1$, then by Corollary 3.5 and by the same way of case 1′, $|R_2| \leq 4$.
(i) Let $|R_2| = 2$. Then X_{b_1} and X_{-b_1}, for nonzero $b_1 \in R_1$, are in the form of $K_{2,2}$ and $\Gamma^I_H(S^C_H(I))$ is a disjoint union of some copies of $K_{2,2}$. So for every $R = R_1 \times R_2$ with $|R_2| = 2$, $\Gamma^I_H(S^C_H(I))$ is planar.
(ii) Let $|R_2| = 3, 4$. For every non zero element $b_1 \in R_1$, X_{b_1}, X_{-b_1} is a bipartition of $K_{3,3}$ or $K_{4,4}$ and $\gamma(K_{3,3}) = \gamma(K_{4,4}) = 1$. So, by $2 \notin Z(R_1)$ and the same way of case 1′, $|R_1| = 3$ and $R_1 = \mathbb{Z}_3$.
Hence, for $t = 2$, R is isomorphic to the one of the following rings:

$$R_1 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2).$$ □

Proposition 3.9. *Let R be a finite ring for which $R = R_1 \times R_2 \times \cdots \times R_t$ with $t \geq 2$, $I = 0 \times R_2 \times \cdots \times R_t$, $H = R - Z(R)$ and let $2 \in Z(R_1)$. If $\gamma(\Gamma^I_H(S^C_H(I))) \leq 1$, then R is isomorphic to the one of the following rings:*

$$R_1 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_7,$$

or

$$R_1 \times \mathbb{Z}_2 \times \mathbb{Z}_2, R_1 \times \mathbb{Z}_3, R_1 \times R_2 \; with \; |R_2| = 4.$$

Proof. Note that $t < 4$, by Theorem 3.2. Put $P = \{a_1 \in R_1|a_1 \neq 0, 2a_1 = 0\}(|P| \geq 1$, by $2 \in Z(R_1))$ and $R_1^{\star} = R_1 - \{0\}$.

1. Let $t = 3$. Put $l = |R_2||R_3|$. For every non zero element $a_1 \in R_1$, set $X_{a_1} = \{(a_1, n_2, n_3)|n_i \in R_i, i = 2, 3\}$. If $a_1 \in P$, then X_{a_1} is in the form of K_l. If $a_1 \notin P$, then X_{a_1}, X_{-a_1} is a bipartition of $K_{l,l}$.
Since $2 \in Z(R_1)$, then by Theorem 3.4, $|R_2||R_3| < 8$. So $|R_2||R_3| = 4$ or 6.

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(a) Let $|R_2||R_3| = 4$.
(i′) If $char(R_1) = 2$ ($R_1^\star = P$), then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of some copies of K_4, which is planar.
(ii′) Let $char(R_1) \neq 2$. Consider $\overline{P} = R_1{}^\star - P$ (note that $|\overline{P}| \geq 2$).
Suppose that $|\overline{P}| > 2$. We claim that $|\overline{P}| \geq 4$. If $|\overline{P}| = 3$ and $a_1, -a_1, a_2 \in \overline{P}$, then $-a_2 \in \overline{P}$. So $|\overline{P}| = 4$, a contradiction.
Hence, if $|\overline{P}| > 2$, considering the sets X_{a_1}, X_{-a_1} and X_{a_2}, X_{-a_2} where $a_1, -a_1, a_2, -a_2 \in \overline{P}$, there are at least two copies of $K_{4,4}$ in $\Gamma_H^I(S_H^C(I))$ that implies, $\gamma(\Gamma_H^I(S_H^C(I))) \geq 2$.
So, $|\overline{P}| = 2$. Then for $b_1, -b_1 \in \overline{P}$, X_{b_1}, X_{-b_1} is in the form of $K_{4,4}$. Thus, in this case, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of some copies of K_4 and one copy of $K_{4,4}$, which is toroidal.
Therefore, if $t = 3$ and $|R_2||R_3| = 4$, then $R = R_1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with $char(R_1) = 2$ ($R_1^\star = P$) or $|\overline{P}| = 2$.
(b) Let $|R_2||R_3| = 6$. If $char(R_1) \neq 2$, then $K_{6,6}$ is a subgraph of $\Gamma_H^I(S_H^C(I))$ which implies that $\gamma(\Gamma_H^I(S_H^C(I))) > 1$. So $char(R_1) = 2$. Also, by $\gamma(\Gamma_H^I(S_H^C(I))) \leq 1$, we should have $|P| = 1$, since for every $l \in P$, $K_6 \subseteq \Gamma_H^I(S_H^C(I))$ and $\gamma(K_6) = 1$. Hence, in this case we have $R_1 = Z_2$ and R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$.

2. Let $t = 2$. For every non zero element $a_1 \in R_1$, set $X_{a_1} = \{(a_1, n_2) | n_2 \in R_2\}$. Since $2 \in Z(R_1)$, then by Corollary 3.5, $n = |R_2| < 8$. For every non zero element $a_1 \in P$, X_{a_1} is in the form of K_n for $n \leq 7$. If $a_1 \notin P$, then X_{a_1}, X_{-a_1} is a bipartition of $K_{n,n}$.
(a′) Let $|R_2| = 2$. If $char(R_1) = 2$, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of some copies of K_2, which is planar. If $char(R_1) \neq 2$ ($R_1^\star \neq P$), then for every $a_1 \in \overline{P}$, X_{a_1}, X_{-a_1} is a bipartition of $K_{2,2}$. So $\Gamma_H^I(S_H^C(I))$ is a disjoint union of some copies of K_2 and $K_{2,2}$, which is planar. Hence, in this case, $R = R_1 \times \mathbb{Z}_2$.
(b′) Let $|R_2| = 3$. If $char(R_1) = 2$, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of some copies of K_3, which is planar. Let $char(R_1) \neq 2$ ($R_1^\star \neq P$). If $|\overline{P}| > 2$, then $|\overline{P}| \geq 4$. So, there exist at least two copies of $K_{3,3}$ in $\Gamma_H^I(S_H^C(I))$ that implies that $\Gamma_H^I(S_H^C(I)) \geq 2$.
Hence, $|\overline{P}| = 2$ and $\Gamma_H^I(S_H^C(I))$ is a disjoint union of some copies of K_3 and one copy of $K_{3,3}$, which is toroidal. Thus, in this case, $R = R_1 \times \mathbb{Z}_3$ with $char(R_1) = 2$ or $|\overline{P}| = 2$.
(c′) If $|R_2| = 4$, then by the same way of case 1(a), $R = R_1 \times R_2$, where $|R_2| = 4$ with $char(R_1) = 2$ or $|\overline{P}| = 2$.
(d′). Let $|R_2| \in \{5, 6, 7\}$. If $|P| \geq 2$, then $\gamma(\Gamma_H^I(S_H^C(I))) \geq 2$ (since for every $a_1 \in P$, $K_n \subseteq \Gamma_H^I(S_H^C(I))$ for $n \in \{5, 6, 7\}$). So $|P| = 1$. If $char(R_1) \neq 2$, then $K_{n,n} \subseteq \Gamma_H^I(S_H^C(I))$ for $n \in \{5, 6, 7\}$, that implies that $\gamma(\Gamma_H^I(S_H^C(I))) > 1$. So $char(R_1) = 2$ and $R_1 = \mathbb{Z}_2$, since $|P| = 1$.
Hence, in this case, R is isomorphic to the one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_7. \quad \square$$

Remark 3.10. Let R be a ring with $I = 0$ and H be a multiplicatively closed subset of R. Then $\Gamma_H^I(S_H^C(I))$ is a subgraph of $T(\Gamma(R))$, since $S_H(I) = \{r \in R : rs = 0 \text{ for some } s \in H (0 \notin H)\} \subseteq Z(R)$. Recall that $T(\Gamma(R))$ is the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$ where $Z(R)$ denotes the set of all zero-divisors of R, so $\gamma(\Gamma_H^I(S_H^C(I))) \leq \gamma(T(\Gamma(R)))$.

For finite ring R with ideal $I = 0 \times 0 \times \cdots \times R_n \times \cdots \times R_t$, $n \geq 1$, $t \geq 2$ and multiplicatively closed subset H of R, $\Gamma_H^I(S_H^C(I))$ is a subgraph of $T(\Gamma(R))$, since $S_H(I) \subseteq Z(R)$. For the recent inclusion, let $(r_1, r_2, \ldots, r_t) \in S_H(I)$. Then $(r_1, r_2, \ldots, r_t)(s_1, s_2, \ldots, s_t) \in I$ for some $(s_1, s_2, \ldots, s_t) \in H$, so $r_i s_i = 0$ for every $1 \leq i \leq n - 1$. If for every $1 \leq i \leq n - 1$, $s_i = 0$, then $H \cap I \neq \emptyset$, a contradiction, so there exists $s_l \neq 0$ for $1 \leq l \leq n - 1$. Then $r_l s_l = 0$ implies that $r_l \in Z(R_l)$, hence $S_H(I) \subseteq Z(R)$.

Theorem 3.11. *Let $I = k\mathbb{Z}_n$ with $d = (k, n)$ and $H = \{a | a \in \mathbb{Z}_n, (a, n) = 1\}$.*

1. *Let n be an even integer.*
(i) *If d is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $K^{\frac{n}{d}}$ and $\frac{d-2}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$. Hence $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(\frac{n}{d}-3)(\frac{n}{d}-4)}{12} \rceil + \frac{d-2}{2} \lceil \frac{(\frac{n}{d}-2)^2}{4} \rceil$.*
(ii) *If d is an odd integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{d-1}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{d-1}{2} \lceil \frac{(\frac{n}{d}-2)^2}{4} \rceil$.*
2. *If n is an odd integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{d-1}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{d-1}{2} \lceil \frac{(\frac{n}{d}-2)^2}{4} \rceil$.*

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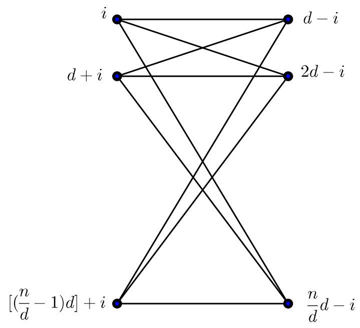

Fig. 1. The total graph of Theorem 3.11.

Proof. We at first show that $d\mathbb{Z}_n = I = S_H(I)$. There are $t_1, t_2 \in \mathbb{Z}$ such that $t_1 n + t_2 k = d$, so $d \in I$. Hence $I = d\mathbb{Z}_n$. We claim that $I = S_H(I)$. By way of contradiction, if there exists $m \in \mathbb{Z}_n - I$ such that $tm \in I$ for some $t \in H$, then $d|tm$. But $(d, t) = 1$ (since $d|n$), so $d|m$, a contradiction. Hence $I = S_H(I)$.

1. Let n be an even integer. Then
(i) If $d = 2$, then $I = 2\mathbb{Z}_n$; so for every $m' \in S_H^C(I)$, $2m' \in S_H(I)$. Then by Theorem 2.3, $\Gamma_H^I(S_H^C(I))$ is in the form of $K_{\frac{n}{2}}$. Therefore, $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(\frac{n}{2}-3)(\frac{n}{2}-4)}{12} \rceil$, by Theorem 2.1.
Let d be an even integer greater than 2, then $S_H^C(I)$ has the following elements:
$1, 2, \ldots, \frac{d}{2}, \ldots, d-1$
$d+1, \ldots, \frac{3d}{2}, \ldots, 2d-1$
.
.
.
$[(\frac{n}{d}-1)d]+1, \ldots, (2\frac{n}{d}-1)\frac{d}{2}, \ldots, \frac{n}{d}d-1.$
Hence, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $K_{\frac{n}{d}}$ and $\frac{d-2}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$. So by Theorem 2.1, $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(\frac{n}{d}-3)(\frac{n}{d}-4)}{12} \rceil + \frac{d-2}{2} \lceil \frac{(\frac{n}{d}-2)^2}{4} \rceil$.
(ii) Let d be an odd integer, then $S_H^C(I)$ has the following elements: $1, 2, \ldots, \frac{d-1}{2}, \frac{d+1}{2}, \ldots, d-1$
$d+1, \ldots, \frac{3d-1}{2}, \frac{3d+1}{2}, \ldots, 2d-1$
.
.
.
$[(\frac{n}{d}-1)d]+1, \ldots, \frac{[(2\frac{n}{d}-1)d]-1}{2}, \frac{[(2\frac{n}{d}-1)d]+1}{2}, \ldots, \frac{n}{d}d-1.$
Hence, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{d-1}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$ as in Fig. 1, where $1 \le i \le \frac{d-1}{2}$. Then by Theorem 2.1, $g(\Gamma_H^I(S_H^C(I))) = \frac{d-1}{2} \lceil \frac{(\frac{n}{d}-2)^2}{4} \rceil$.
2. Let n be an odd integer; so, $2 \in H$. Then by Theorem 2.4, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{d-1}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$ and by Theorem 2.1, $g(\Gamma_H^I(S_H^C(I))) = \frac{d-1}{2} \lceil \frac{(\frac{n}{d}-2)^2}{4} \rceil$. □

Example 3.12. Consider $R = \mathbb{Z}_{18}$, $I = 12\mathbb{Z}_{18}$ and $H = \{a | a \in \mathbb{Z}_{18}, (a, 18) = 1\}$. Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of K_3 and two copies of $K_{3,3}$. Then $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(3-3)(3-4)}{12} \rceil + \frac{6-2}{2} \lceil \frac{(3-2)^2}{4} \rceil = 2$.

Theorem 3.13. *Let $R = \mathbb{Z}_n \times \mathbb{Z}_n$, $I = 0 \times \mathbb{Z}_n$ and $H = \{(a, b) | a, b \in \mathbb{Z}_n \times \mathbb{Z}_n, (a, n) = 1\}$. Then:*

1. *If n is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of K_n and $\frac{n-2}{2}$ copies of $K_{n,n}$. In this case, $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(n-3)(n-4)}{12} \rceil + \frac{n-2}{2} \lceil \frac{(n-2)^2}{4} \rceil$.*
2. *If n is an odd integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{n-1}{2}$ copies of $K_{n,n}$. In this case $g(\Gamma_H^I(S_H^C(I))) = \frac{n-1}{2} \lceil \frac{(n-2)^2}{4} \rceil$.*

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Proof.

1. Let n be an even integer. We claim that $I = S_H(I)$. By way of contradiction, if there is $(m, d) \in \mathbb{Z}_n \times \mathbb{Z}_n - (0 \times \mathbb{Z}_n)$ such that $(t, l)(m, d) \in 0 \times \mathbb{Z}_n$ for some $(t, l) \in H$, then $n|tm$, but $(t, n) = 1$; so $n|m$, a contradiction. Hence $I = S_H(I)$. The number of elements of $(\mathbb{Z}_n \times \mathbb{Z}_n) - S_H(I)$ is $n^2 - n$. For every element $(a, b) \in S_H^C(I)$, we have $(a, b) + (-a, b) \in S_H(I)$. Let $(a, b) + (c, d) \in S_H(I)$ for (a, b) and $(c, d) \in S_H^C(I)$. Then $a + c = 0$ and this implies that $a = -c$. Hence, each element (a, b) of $S_H^C(I)$ is just adjacent to $(-a, d)$ for every element $d \in \mathbb{Z}_n$. Because n is even, so just for one element $0 \neq a = \frac{n}{2} \in \mathbb{Z}_n$, $(a, b) \in S_H^C(I)$ is adjacent to (a, c) for every $b, c \in \mathbb{Z}_n$. Hence, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of K_n and $\frac{n-2}{2}$ copies of $K_{n,n}$. Now by Theorem 2.1, $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(n-3)(n-4)}{12} \rceil + \frac{n-2}{2} \lceil \frac{(n-2)^2}{4} \rceil$.
2. If n is an odd integer, then $2 \in H$. By Theorem 2.4, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{n-1}{2}$ copies of $K_{n,n}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{n-1}{2} \lceil \frac{(n-2)^2}{4} \rceil$, by Theorem 2.1. □

Example 3.14. Consider $R = \mathbb{Z}_8 \times \mathbb{Z}_8$, $I = 0 \times \mathbb{Z}_8$ and $H = \{(a, b) | (a, b) \in \mathbb{Z}_8 \times \mathbb{Z}_8, (a, 8) = 1\}$. Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of K_8 and 3 copies of $K_{8,8}$. Hence $g(\Gamma_H^I(S_H^C(I))) = 29$.

Theorem 3.15. *Let $I = k\mathbb{Z}_n$, $H = \mathbb{Z}_n - p\mathbb{Z}_n$ where p is a prime number with $1 < p \leq n$, $| S_H(I) | = \alpha$ and $| \mathbb{Z}_n / S_H(I) | = \beta$.*

1. *If $p \neq 2$, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{\beta-1}{2} \lceil \frac{(\alpha-2)^2}{4} \rceil$.*
2. *Let $p = 2$. Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $K_{\frac{n}{2^{l'}}}$ and $\frac{2^{l'}-2}{2}$ copies of $K_{\frac{n}{2^{l'}}, \frac{n}{2^{l'}}}$. Hence*
$$g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(\frac{n}{2^{l'}} - 3)(\frac{n}{2^{l'}} - 4)}{12} \rceil + \frac{2^{l'} - 2}{2} \lceil \frac{(\frac{n}{2^{l'}} - 2)^2}{4} \rceil,$$
where $(k, n) = 2^{l'} r'$ with $l', r' \in \mathbb{N}$ and $(2, r') = 1$.

Proof. By the proof of Theorem 3.11, $I = d\mathbb{Z}_n$, where $d = (k, n)$.

1. If $p \neq 2$, then $2 \in \mathbb{Z}_n - p\mathbb{Z}_n$. So, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$ by Theorem 2.4 and $g(\Gamma_H^I(S_H^C(I))) = \frac{(\beta-1)}{2} \lceil \frac{(\alpha-2)^2}{4} \rceil$, by Theorem 2.1.
2. Let $p = 2$. If n is odd, then $0 \in H$ and this is impossible; so we assume that n is even. It should be noted that d is even; since if d is odd, then $d \in H$ and $\emptyset \neq I \cap H$ that this implies that $\mathbb{Z}_n = S_H(I)$, which is not in our assumptions.
Let $n = 2^l r$ for $l, r \in \mathbb{N}$ where $(2, r) = 1$. We can assume that $d = 2^{l'} r'$ where $l', r' \in \mathbb{N}$ and $l' \leq l, r' \leq r$. Here, we claim that $S_H(I) = 2^{l'}\mathbb{Z}_n$. Suppose there is $m \in \mathbb{Z}_n - 2^{l'}\mathbb{Z}_n$ such that $d|tm$ for some $t \in H$, so $2^{l'}|tm$. Hence $2^{l'}|m$, a contradiction. Therefore, $S_H(I) \subseteq 2^{l'}\mathbb{Z}_n$. Now, let $2^{l'}t' \in 2^{l'}\mathbb{Z}_n$ for some $t' \in \mathbb{Z}_n$. It is clear that $2^{l'}t'r' \in I$, so $2^{l'}t' \in S_H(I)$. Hence, by an argument similar to the proof of Theorem 3.11, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $K_{\frac{n}{2^{l'}}}$ and $\frac{2^{l'}-2}{2}$ copies of $K_{\frac{n}{2^{l'}}, \frac{n}{2^{l'}}}$. So $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(\frac{n}{2^{l'}}-3)(\frac{n}{2^{l'}}-4)}{12} \rceil + \frac{2^{l'}-2}{2} \lceil \frac{(\frac{n}{2^{l'}}-2)^2}{4} \rceil$, by Theorem 2.1. □

Example 3.16. Consider $R = \mathbb{Z}_{64}$, $I = 4\mathbb{Z}_{64}$ and $H = \mathbb{Z}_{64} - 2\mathbb{Z}_{64}$. Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $K_{16,16}$ and K_{16} such that $g(\Gamma_H^I(S_H^C(I))) = 62$.

Theorem 3.17. *Let $I = k\mathbb{Z}_n$, $H = \{a^s | s \geq 0\}$ such that $a|n$ with $(\frac{n}{a}, a) = 1$, $| S_H(I) | = \alpha$ and $| \mathbb{Z}_n / S_H(I) | = \beta$.*

1. *If there is at least one even number in H, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$. In this case $g(\Gamma_H^I(S_H^C(I))) = \frac{\beta-1}{2} \lceil \frac{(\alpha-2)^2}{4} \rceil$.*
2. *Let there be no even number in H, $(d, \frac{n}{a}) = l$ where $d = (k, n)$ and $n = rl$ for some $r \in \mathbb{N}$.*
(1′). If l is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of K_r and $\frac{l-2}{2}$ copies of $K_{r,r}$. In this case $g(\Gamma_H^I(S_H^C(I))) = \lceil \frac{(r-3)(r-4)}{12} \rceil + \frac{l-2}{2} \lceil \frac{(r-2)^2}{4} \rceil$.
(2′). If l is an odd integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{l-1}{2}$ copies of $K_{r,r}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{l-1}{2} \lceil \frac{(r-2)^2}{4} \rceil$.

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Proof. Note that by the proof of Theorem 3.11, $I = d\mathbb{Z}_n$.

1. If there is at least one even element in H, then by Theorem 2.4, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$. Therefore, $g(\Gamma_H^N(S_H^C(N))) = \frac{(\beta-1)}{2}\lceil\frac{(\alpha-2)^2}{4}\rceil$, by Theorem 2.1.
2. Let there be no even element in H. If $(d, \frac{n}{a}) = 1$, then $d|a$, since $n = \frac{n}{a}a$ where $(\frac{n}{a}, a) = 1$. Hence $\emptyset \neq I \cap H$ and this implies that $\mathbb{Z}_n = S_H(I)$, which is not the case. So, we assume that $(d, \frac{n}{a}) = l \neq 1$. Put $I' = l\mathbb{Z}_n$. We claim that $I' = S_H(I)$. Suppose there is $m' \in \mathbb{Z}_n - I'$ such that $d|a^g m'$, where $a^g \in H$ and $g \geq 0$, then $l|a^g m'$. So, $l|m'$, a contradiction.

   Conversely, we show that $d|al$ and this implies that $I' \subseteq S_H(I)$. Let $d = n_1 l$ and $\frac{n}{a} = n_2 l$, for some $n_1, n_2 \in \mathbb{N}$ (so $(n_1, n_2) = 1$). We show that $n_1|a$; this implies that $n_1 l|al$ and the proof is complete.

   Let $(n_1, a) = h$ (so $n_1 = g_1 h$ and $a = g_2 h$, for some $g_1, g_2 \in \mathbb{N}$ where $(g_1, g_2) = 1$). By the fact that $d = n_1 l$ and $d|n$, one has $g_1 hl|g_2 hn_2 l$; so $g_1|g_2 n_2$ and $(g_1, g_2) = 1$ implies that $g_1|n_2$. This yields $g_1 = 1$ and so $n_1|a$.

   (1′). Let l be an even integer, then by the same way as the proof of Theorem 3.11, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of K_r and $\frac{l-2}{2}$ copies of $K_{r,r}$. Hence, $g(\Gamma_H^I(S_H^C(I))) = \lceil\frac{(r-3)(r-4)}{12}\rceil + \frac{l-2}{2}\lceil\frac{(r-2)^2}{4}\rceil$, by Theorem 2.1.

   (2′). Let l be an odd integer. By the same way as the proof of Theorem 3.11, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{l-1}{2}$ copies of $K_{r,r}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{l-1}{2}\lceil\frac{(r-2)^2}{4}\rceil$, by Theorem 2.1. □

Example 3.18. Consider $R = \mathbb{Z}_{60}$, $I = 15\mathbb{Z}_{60}$ and $H = \{3^s | s \geq 0\}$. Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of 2 copies of $K_{12,12}$ and $g(\Gamma_H^I(S_H^C(I))) = 50$.

Theorem 3.19. *Let $I = k\mathbb{Z}_n$, $H = \{1, t\}$ with $n|t^2 - 1$, $|\ S_H(I)\ | = \alpha$ and $|\ \mathbb{Z}_n/S_H(I)\ | = \beta$.*

1. *If t is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{\beta-1}{2}\lceil\frac{(\alpha-2)^2}{4}\rceil$.*
2. *let t be an odd integer and $(k, n) = d$.*
   (1′). If d is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $K_{\frac{n}{d}}$ and $\frac{d-2}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$. In this case $g(\Gamma_H^I(S_H^C(I))) = \lceil\frac{(\frac{n}{d}-3)(\frac{n}{d}-4)}{12}\rceil + \frac{d-2}{2}\lceil\frac{(\frac{n}{d}-2)^2}{4}\rceil$.
   (2′). If d is an odd integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{d-1}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{d-1}{2}\lceil\frac{(\frac{n}{d}-2)^2}{4}\rceil$.

Proof.

1. Let t be an even integer. In view of Theorem 2.4, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{\beta-1}{2}\lceil\frac{(\alpha-2)^2}{4}\rceil$.
2. By the proof of Theorem 3.11, $I = d\mathbb{Z}_n$. Let t be an odd integer. We claim that $I = S_H(I)$. Otherwise, there is $a \in \mathbb{Z}_n - I$ such that $d|at$. So $d|at^2$. By the assumption $d|t^2 - 1$, hence $d|a$ and this implies that $a \in I$, a contradiction. Thus, $I = S_H(I)$.
   (1′) If d is an even integer, then we proceed in the proof of Theorem 3.11. So $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $K_{\frac{n}{d}}$ and $\frac{d-2}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$. Then
   $$g(\Gamma_H^I(S_H^C(I))) = \lceil\frac{(\frac{n}{d}-3)(\frac{n}{d}-4)}{12}\rceil + \frac{d-2}{2}\lceil\frac{(\frac{n}{d}-2)^2}{4}\rceil,$$
   by Theorem 2.1.
   (2′) If d is an odd integer, then we proceed in the proof of Theorem 3.11. Hence, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{d-1}{2}$ copies of $K_{\frac{n}{d},\frac{n}{d}}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{d-1}{2}\lceil\frac{(\frac{n}{d}-2)^2}{4}\rceil$, by Theorem 2.1. □

Example 3.20. Consider $R = \mathbb{Z}_{50}$, $I = 5\mathbb{Z}_{50}$ and $H = \{1, 49\}$. Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of two copies of $K_{10,10}$ and $g(\Gamma_H^I(S_H^C(I))) = 32$.

Theorem 3.21. *Let $R = \mathbb{Z}_n \times \mathbb{Z}_n$, $I = 0 \times \mathbb{Z}_n$, $H = \{(1, 1), (t, t)\}$ with $n|t^2 - 1$, $|\ S_H(I)\ | = \alpha$ and $|\ \mathbb{Z}_n/S_H(I)\ | = \beta$.*

1. *If t is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{\beta-1}{2}\lceil\frac{(\alpha-2)^2}{4}\rceil$.*

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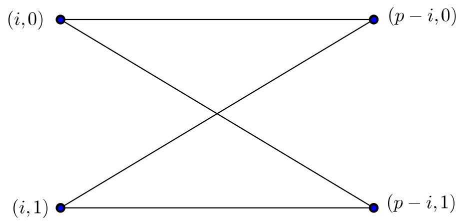

Fig. 2. The total graph of Theorem 3.23(1).

2. *Let t be an odd integer.*
   (i) *If n is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of K_n and $\frac{n-2}{2}$ copies of $K_{n,n}$. In this case, $g(\Gamma_H^I(S_H^C(I))) = \lceil\frac{(n-3)(n-4)}{12}\rceil + \frac{n-2}{2}\lceil\frac{(n-2)^2}{4}\rceil$.*
   (ii) *If n is an odd integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{n-1}{2}$ copies of $K_{n,n}$. In this case $g(\Gamma_H^I(S_H^C(I))) = \frac{n-1}{2}\lceil\frac{(n-2)^2}{4}\rceil$.*

Proof.

1. If t is an even integer, then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{\beta-1}{2}$ copies of $K_{\alpha,\alpha}$ by Theorem 2.4 and $g(\Gamma_H^I(S_H^C(I))) = \frac{\beta-1}{2}\lceil\frac{(\alpha-2)^2}{4}\rceil$.
2. Let t be an odd integer. We claim that $I = S_H(I)$. Otherwise, there is $(a, b) \in \mathbb{Z}_n \times \mathbb{Z}_n$ such that $(a, b)(t, t) \in I$ where $a \neq 0$ and b is an arbitrary element of $\mathbb{Z}_n$. So $n|at$; then $n|at^2$. By the assumption $n|t^2 - 1$, so $n|a$ and this implies that $a = 0$, a contradiction. Hence $I = S_H(I)$. The remaining is similar to proof of Theorem 3.13. □

Example 3.22. Consider $R = \mathbb{Z}_{21} \times \mathbb{Z}_{21}$, $I = 0 \times \mathbb{Z}_{21}$ and $H = \{(1, 1), (13, 13)\}$. Then $\Gamma_H^I(S_H^C(I))$ is a disjoint union of 10 copies of $K_{21,21}$ and $g(\Gamma_H^I(S_H^C(I))) = 910$.

Theorem 3.23. *Let $\mathbb{Z}_p$ denote the field of p elements where $p > 2$ is a prime number, $R = \mathbb{Z}_p \times \mathbb{Z}_{2^m}$ where $m \in \mathbb{N}$ and $I = 0 \times \mathbb{Z}_{2^m}$. Let H be one of the following sets: $\{(1, 1)\}, \{(1, 1), (2, 1)\}, \ldots, \{(1, 1), (2, 1), \ldots, (p-1, 1)\}, \{(1, 0)\}, \{(1, 0), (2, 0)\}, \ldots, \{(1, 0), (2, 0), \ldots, (p-1, 0)\}$.*

1. *If $m = 1$, then $\Gamma_H^I(S_H^C(I))$ is planar.*
2. *If $m > 1$, then $g(\Gamma_H^I(S_H^C(I))) = \frac{p-1}{2}\lceil\frac{(2^m-2)^2}{4}\rceil$.*

Proof. We note that $I = S_H(I)$ for every positive integer m and all cases of H. Otherwise, there exists $(a, c) \in \mathbb{Z}_p \times \mathbb{Z}_{2^m}$ such that $a \neq 0$ and $p|at$ for some $(t, b) \in H$, where $b \in \mathbb{Z}_2$. Because $(p, t) = 1$, so $p|a$, a contradiction. Hence, in all cases, $I = S_H(I)$.

1. If $m = 1$, for every $(a, b) \in S_H^C(I)$, we have $(a, b)+(-a, b) \in S_H(I)$, where $a \in Z_p$ and $b \in Z_2$. So, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{p-1}{2}$ copies of $K_{2,2}$, where $R = \mathbb{Z}_p \times \mathbb{Z}_2$ and $I = 0 \times \mathbb{Z}_2$, as Fig. 2, where $1 \leq i \leq \frac{p-1}{2}$. Hence $g(\Gamma_H^I(S_H^C(I))) = 0$, by Theorem 2.1.
2. In the same way of the case 1, for $R = \mathbb{Z}_p \times \mathbb{Z}_{2^m}$ with $m \in \mathbb{N}$ and $m > 1$, $\Gamma_H^I(S_H^C(I))$ is a disjoint union of $\frac{p-1}{2}$ copies of $K_{2^m,2^m}$ and $g(\Gamma_H^I(S_H^C(I))) = \frac{p-1}{2}\lceil\frac{(2^m-2)^2}{4}\rceil$, where $I = 0 \times \mathbb{Z}_{2^m}$. □

References

[1] D.F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra 320 (7) (2008) 2706–2719.
[2] S. Akbari, D. Kiani, F. Mohammadi, S. Moradi, The total graph and regular graph of a commutative ring, J. Pure Appl. Algebra 213 (12) (2009) 2224–2228.
[3] H.R. Maimani, C. Wickham, S. Yassemi, Rings whose total graphs have genus at most one, Rocky Mountain J. Math. 42 (5) (2010) 1551–1560.

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[4] T. Tamizh Chelvam, T. Asir, On the genus of the total graph of a commutative ring, Comm. Algebra 41 (2013) 142–153.
[5] A. Abbasi, . Sh. Habibi, The total graph of a commutative ring with respect to proper ideals, J. Korean Math. Soc. 49 (1) (2012) 85–98.
[6] A. Abbasi, . Sh. Habibi, The total graph of a module over a commutative ring with respect to proper submodules, J. Algebra Appl. 11 (3) (2012) 1250048 13 pages.
[7] D.F. Anderson, A. Badawi, The generalized total graph of a commutative ring, J. Algebra Appl. 12 (5) (2013) 1250212 18 pages.
[8] A. Abbasi, L. Hamidian Jahromi, A generalization of total graphs of modules over commutative rings under multiplicatively closed subsets, J. Math. Ext. 11 (3) (2017) 87–102.
[9] A.T. White, Graphs, Groups and Surfaces, North-Holland, Amsterdam, 1973.
[10] J. Battle, F. Harary, Y. Kodama, J.W.T. Youngs, Additivity of the genus of a graph, Bull. Amer. Math. Soc. 68 (6) (1962) 565–568.
[11] A.T. White, The genus of the Cartesian product of two graphs, J. Combin. Theory Ser. B 11 (2) (1971) 89–94.
[12] C. Wickham, Classification of rings with genus one zero-divisor graphs, Comm. Algebra 36 (2) (2008) 325–345.

Further reading

[1] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.

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