Quaestiones Mathematicae

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tqma20

The diameter of strong orientations of strong products of graphs

Irena Hrastnik Ladinek & Simon Špacapan

Quaestiones Mathematicae 2020: 1–14.

https://doi.org/10.2989/16073606.2020.1754958
This is the final version of the article that is published ahead of the print and online issue

THE DIAMETER OF STRONG ORIENTATIONS OF STRONG PRODUCTS OF GRAPHS

Irena Hrastnik Ladinek
University of Maribor, FME, Smetanova 17, 2000 Maribor, Slovenia.
E-Mail irena.hrastnik@um.si

Simon Špacapan*
University of Maribor, FME, Smetanova 17, 2000 Maribor, Slovenia.
E-Mail simon.spacapan@um.si

Abstract. Let G and H be graphs, and $G \boxtimes H$ the strong product of G and H. We prove that for any connected graphs G and H there is a strongly connected orientation D of $G \boxtimes H$ such that $\mathrm{diam}(D) \leq 2r + 15$, where r is the radius of $G \boxtimes H$.

This improves the general bound $\mathrm{diam}(D) \leq 2r^2 + 2r$ for arbitrary graphs, proved by Chvátal and Thomassen.

Mathematics Subject Classification (2010): 05C12, 05C20, 05C76.
Key words: Strong orientation, diameter, strong product.

1. Introduction. The Robbins' theorem states that an undirected graph G admits a strongly connected orientation if and only if G is connected and bridgeless. When orienting the edges of an undirected graph the objective is to obtain an orientation which is strongly connected and, when distances in the obtained digraph are relevant, has some additional metric properties. In this respect two main parameters were subject to investigation, namely the diameter of a (di)graph, and the sum of all distances (or the avarage distance) in a (di)graph. The sum of all distances is known as the Wiener index, introduced by Wiener in 1947 and widely applied in chemistry. The diameter of a digraph is one of the measures of efficiency of a road network with one way streets, which is modeled by a digraph; this topic is discussed in detail in [16], [17] and [18].

In this article we ask what is the minimum diameter of a strongly connected digraph D whose underlying graph is G, where G is a fixed undirected graph subject to this question. Let G be an undirected graph and

$$\mathrm{diam}_{\min}(G) = \min\{\mathrm{diam}(D) \mid D \text{ is a strong orientation of } G\}.$$

In [2] (see also [1]) Chvátal and Thomassen obtained a sharp upper bound for $\mathrm{diam}_{\min}(G)$ of an arbitrary bridgeless connected graph G.

*The author is supported by research grants P1-0297 and J1-9109 of the Ministry of Education of Slovenia.

Theorem 1.1. ([2]) *For every bridgeless connected graph G of radius r we have*

$$\operatorname{diam}_{\min}(G) \leq 2r^2 + 2r.$$

This parameter was later studied in [16] in context of optimizing the traffic flow in city streets which are modeled by grid graphs $P_m \Box P_n$. The authors of [16] construct orientations of $P_m \Box P_n$ which minimize the diameter and compare them to the most commonly used orientations in city streets — orientations where streets and avenues are alternatively turned left and right, or up and down. It is shown that these commonly used orientations are not optimal with respect to diameter and other metric parameters.

Several other classes of graphs have been considered and bounds for $\operatorname{diam}_{\min}(G)$ were obtained, in particular numerous results for products of graphs are known. Cartesian products of trees admit orienatations such that the diameter of the underlying graph is equal to the diameter of the obtained digraph (see [9]). Such orientations are called *optimal* orientations.

Theorem 1.2. ([9]) *If T_1 and T_2 are trees with diameters at least 4, then*

$$\operatorname{diam}_{\min}(T_1 \Box T_2) = \operatorname{diam}(T_1 \Box T_2)\,.$$

The diameter of Cartesian products of complete graphs, products of cycles and products of paths were studied in [5, 6, 7, 8], and in most cases optimal orientations of these products were constructed, except in few cases where the diameter of the obtained digraph is larger than the diameter of the underlying graph by a small constant (we call such orientations *near-optimal*). In [20] a general upper bound for $\operatorname{diam}_{\min}(G \Box H)$ was obtained for arbitrary connected graphs G and H.

A similar type of a problem is the problem where the sum of all distances of the obtained digraph is in question, and not the diameter. The Wiener index of digraphs

$$W(D) = \sum_{(u,v) \in V(D) \times V(D)} d(u,v)$$

has been studied in articles [10, 11] and [14]. In these articles the authors search for the maximum and minimum possible Wiener index of a digraph D whose underlying graph is a fixed graph G (however in these articles, there are no assumptions that the obtained digraph must be strongly connected). In [10] (see also [14]) the maximum Wiener index of a tournament is established, and in [11] the maximum Wiener index of digraphs whose underlying graph is a tree is partly determined; several conjectures are formulated as well. We also mention that the Wiener index of strong products of graphs was determined in [13].

In this article we study strong products of graphs. Let G and H be graphs. The *strong product* of G and H is the graph, denoted as $G \boxtimes H$, with vertex set $V(G \boxtimes H) = V(G) \times V(H)$. Vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \boxtimes H$ if $x_1 = x_2$ and $y_1 y_2 \in E(H)$, or $x_1 x_2 \in E(G)$ and $y_1 = y_2$, or $x_1 x_2 \in E(G)$ and $y_1 y_2 \in E(H)$.

The strong product of graphs is one of the four standard graph products, see [3]. It has attracted considerable attention, especially in the study of Shannon capacity and consequently its application in the information theory. Couriously enough, strong products of graphs were recently applied in a construction of a counterexample to the famous Hedetniemi's conjecture, see [19].

Since $E(G\Box H) \subseteq E(G \boxtimes H)$ any upper bound for $\mathrm{diam}_{\min}(G\Box H)$ is also an upper bound for $\mathrm{diam}_{\min}(G \boxtimes H)$. To obtain a better bound for $\mathrm{diam}_{\min}(G \boxtimes H)$, we have to show how to orient edges in $E(G \boxtimes H) \setminus E(G\Box H)$ so that there will be a shorter path between any pair of vertices in $G \boxtimes H$. This has already been shown for strong products of paths in [12], however here we aim at a general approach which can be applied to any strong product of graphs.

In Section 2 we define near-optimal orienatations of strong products of even cycles, afterwards in Section 3 we generalize the method for products of trees. In particular, in Section 3 we define an orientation of strong product of arbitrary trees by rules A to G. Then, in Section 4, we give several local properties of this orientation (we skip the proofs of this section, because the proving method is rather straightforward, and the results are proved by routine applications of rules A to G; the full version of the paper, including all the proofs, is available in [4]). Finally, in Section 5, the diameter of the orientation defined in Section 3 is established.

In the rest of the introduction we fix the notations and the terminology. Let $D = (V, A)$ be a directed graph, and $u, v \in V$. If $uv \in A$ we write $u \to v$, and we say that u is an *in-neighbor* of v, and that v is an *out-neighbor* of u. A *uv-path* in D is a sequence of pairwise distinct vertices $u = u_0, u_1, \ldots, u_n = v$ such that $u_i u_{i+1} \in A$ for all indices i. We say that D is a *strongly connected* or *strong* digraph if there is a uv-path in D for every $u, v \in V$. For vertices $u, v \in V$ the *distance* from u to v in D is the length of a shortest uv-path in D, if such a path exists, otherwise the distance is ∞. We denote the distance from u to v by $\mathrm{dist}(u, v)$. The *diameter* of D is defined as

$$\mathrm{diam}(D) = \max\{\mathrm{dist}(u, v) \mid u, v \in V\}.$$

For a connected graph G and a vertex v of G, the *shortest path tree* with respect to v is a spanning tree such that for every $x \in V(G)$ we have $d_G(v, x) = d_T(v, x)$ (such a tree exists, and we may obtain it by a BFS algorithm). The *eccentricity* of a vertex $x \in V(G)$ is $ecc(x) = \max\{\mathrm{dist}(x, v) \mid v \in V(G)\}$. A *center* of a graph G is a vertex $v \in V(G)$ with minimal eccentricity. The eccentricity of a central vertex is called the *radius* of G, and is denoted by $\mathrm{rad}(G)$. Clearly, if G is a graph and T is a shortest path tree with respect to a central vertex of G, then $\mathrm{rad}(G) = \mathrm{rad}(T)$. Note also that for any graph G, $\mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$.

Let $G \boxtimes H$ be the strong product of G and H. For a $y \in V(H)$ the *G-layer* G_y is the subgraph of $G \boxtimes H$ induced by $\{(x, y) \mid x \in V(G)\}$, and for an $x \in V(G)$ the *H-layer* H_x is the subgraph of $G \boxtimes H$ induced by $\{(x, y) \mid y \in V(H)\}$. If $e = (x, y)(x', y')$ is an edge of $G \boxtimes H$ such that $x \neq x'$ and $y \neq y'$ then e is called a *direct edge* of $G \boxtimes H$. If an edge of $G \boxtimes H$ is not a direct edge, then it is called a *Cartesian edge.* Note that the edge set of $G \boxtimes H$ is given by

$$E(G \boxtimes H) = E(G \times H) \cup E(G\Box H)\,,$$

where $G \times H$ denotes the direct product of graphs, and $G\Box H$ denotes the Cartesian product of graphs. It is well known (see [3]) that the distance between vertices

(x_1, y_1) and (x_2, y_2) of $G \boxtimes H$ is given by

$$d_{G\boxtimes H}((x_1, y_1)(x_2, y_2)) = \max\{d_G(x_1, x_2), d_H(y_1, y_2)\},$$

and consequently the radius and the diameter of strong products are

$\mathrm{rad}(G\boxtimes H) = \max\{\mathrm{rad}(G), \mathrm{rad}(H)\}$ and $\mathrm{diam}(G\boxtimes H) = \max\{\mathrm{diam}(G), \mathrm{diam}(H)\}$,

respectively.

2. The diameter of strong products of even cycles. Let $G = C_m$ and $H = C_n$, where $m, n \geq 4$ are even. Let $A_1 \cup B_1$ be the bipartition of G and $A_2 \cup B_2$ the bipartition of H. We orient the edges of G and H cyclicly to obtain strong orientations of C_m and C_n, and we denote the obtained digraphs by D_1 and D_2, respectively. Let $-D_1$ and $-D_2$ be directed graphs obtained from D_1 and D_2 by reversing the direction of each arc, respectively. Note that G-layers and H-layers of $G \boxtimes H$ are isomorphic to G and H, respectively. Therefore we may use orientations D_1 and D_2 to orient layers of $G \boxtimes H$ (when we do so, we say that G-layers are oriented *according* to D_1, and H-layers are oriented according to D_2).

We orient the Cartesian edges of G$\boxtimes H$ by the following rules.

(A) For every $y \in B_2$ the edges of G_y are oriented according to D_1, and for every $y \in A_2$ the edges of G_y are oriented according to $-D_1$.

(B) For every $x \in A_1$ the edges of H_x are oriented according to D_2, and for every $x \in B_1$ the edges of H_x are oriented according to $-D_2$.

To define the orientations of direct edges of $G \boxtimes H$ assume $x_1 \to x_2$ in D_1 and $y_1 \to y_2$ in D_2, and apply the following rules.

(G1) $(x_1, y_1) \to (x_2, y_2)$ and $(x_2, y_1) \to (x_1, y_2)$, if $(x_1, y_1) \in (A_1 \times B_2) \cup (B_1 \times A_2)$.

(G2) $(x_2, y_2) \to (x_1, y_1)$ and $(x_1, y_2) \to (x_2, y_1)$, if $(x_1, y_1) \in (A_1 \times A_2) \cup (B_1 \times B_2)$.

Call the obtained digraph D. The orientation is defined in such a way that the "neighboring diagonals" are directed in opposite directions (see Figure 1).

The diameter of the obtained digraph is $\frac{1}{2}\max\{m, n\} + 1$ (we skip the proof of this claim). Note that there are exactly two white vertices in Figure 1. Call them x and y and note that $d(x, y) = 2$ and $d(y, x) = 4 = \frac{1}{2}\max\{6, 4\} + 1$. Note that this orientation is near-optimal because $\mathrm{diam}(C_m \boxtimes C_n) = \frac{1}{2}\max\{m, n\}$, if m and n are even.

PROPOSITION 2.1. *For every even $m, n \geq 4$, $\mathrm{diam}_{\min}(C_m\boxtimes C_n) \leq \frac{1}{2}\max\{m, n\}+1$.*

Rules (A), (B), (G1) and (G2) can be applied to any product $G \boxtimes H$ with bipartite factors G and H, and the resulting digraph will be well-defined. However the resulting digraph might not be strong because there might be some vertices that have only in-neighbors or only out-neighbors (if both factors have a vertex of degree one).

To obtain a strong orientation of $G \boxtimes H$ when G and H have vertices of degree one, and in particular when G and H are trees, additional rules (C), (D), (E) and (F) are introduced in the following section. These rules deal with orientations of direct edges of $G \boxtimes H$ that are incident to vertices of degree 3 in $G \boxtimes H$.

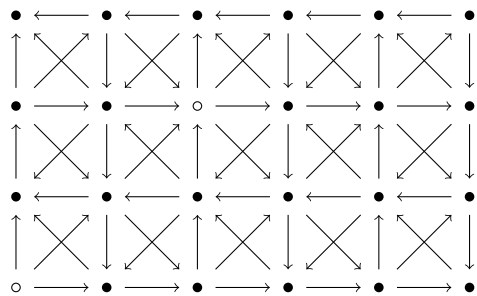

Figure 1: The orientation of $P_6 \boxtimes P_4 \subseteq C_6 \boxtimes C_4$.

3. The diameter of strong products of trees. Let T be a tree and $r \in V(T)$ be the root of T. For $x, y \in V(T)$ we write $x < y$ if x lies on the path between y and r.

Let T_1 and T_2 be trees, and let r_1 and r_2 be their roots, respectively (the roots may be chosen arbitrarily). Let $A_i \cup B_i$ be the bipartition of T_i, and assume that $r_i \in A_i$ for $i \in \{1, 2\}$.

Let D_1 be the orientation of T_1 such that every edge is oriented away from the root r_1. More precisely, if xy is an edge in T_1 and $x < y$ then we orient xy as $x \to y$. Let D_2 be the orientation of T_2 such that every edge is oriented away from the root r_2.

With these settings we are ready to define an orientation of $T_1 \boxtimes T_2$. We orient the Cartesian edges of $T_1 \boxtimes T_2$ according to rules (A) and (B), where $G = T_1$ and $H = T_2$. To define the orientations of direct edges of $T_1 \boxtimes T_2$ assume $x_1 \to x_2$ in D_1 and $y_1 \to y_2$ in D_2, and apply the following rules (note that the objective of rules (C) to (F) is that all vertices of $G \boxtimes H$ have at least one in-neighbor and at least one out-neighbor).

(C) If $x_1 = r_1$, and $y_2 \in A_2$ is a leaf, then orient $(x_1, y_1) \to (x_2, y_2)$ and $(x_1, y_2) \to (x_2, y_1)$.

(D) If $x_2 \in A_1$ is a leaf, $y_1 = r_2$, and y_2 is not a leaf, then orient $(x_1, y_1) \to (x_2, y_2)$ and $(x_1, y_2) \to (x_2, y_1)$.

(E) If $x_2 \in A_1$ is a leaf, $y_1 = r_2$, and y_2 is a leaf, then orient $(x_1, y_2) \to (x_2, y_1)$ and $(x_2, y_2) \to (x_1, y_1)$.

(F) If $x_2 \in A_1$ is a leaf, $y_2 \in B_2$ is a leaf, and $y_1 \neq r_2$, then orient $(x_2, y_1) \to (x_1, y_2)$ and $(x_2, y_2) \to (x_1, y_1)$.

(G) Otherwise (if assumptions of (C), (D), (E) and (F) are false) then apply rules (G_1) and (G_2).

If T_1 and T_2 are rooted paths, then the orientation of $T_1 \boxtimes T_2$, obtained by rules (A) to (G), is shown in Figure 2.

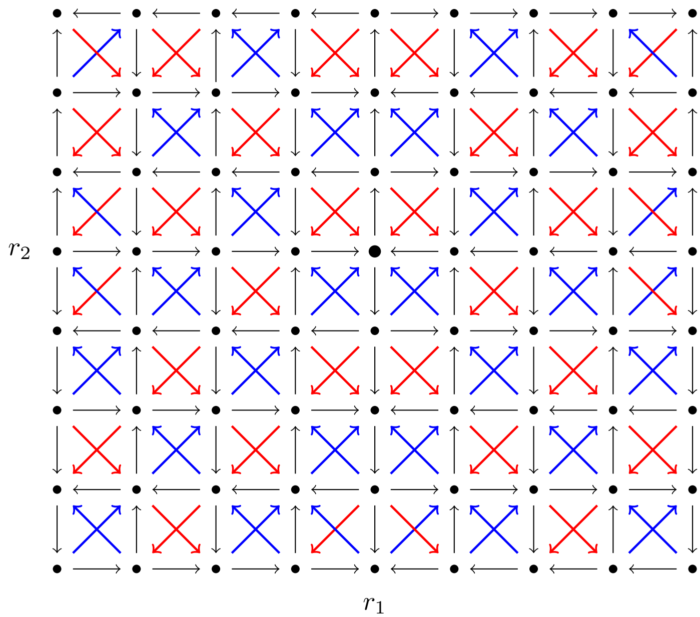

Figure 2: Orientation of $P_9 \boxtimes P_8$ obtained from rules (A) to (G).

When T_1 and T_2 are arbitrary trees, the orientation of $T_1 \boxtimes T_2$ obtained by rules (A) to (G) produces a digraph with a "small" diameter. The diameter of this digraph is given by the following theorem, which is our main result.

THEOREM 3.1. *For any trees T_1 and T_2 we have*

$$\mathrm{diam}_{\min}(T_1 \boxtimes T_2) \leq \max\{\mathrm{diam}(T_1), \mathrm{diam}(T_2)\} + 15.$$

The proof of the above theorem is given in Section 5. It follows from the theorem that strong products of trees admit near-optimal orientations, and we made no attempt to optimize the constant 15 (in fact, we think that the constant 15 can be reduced, if a very detailed case analysis is applied). We now apply the bound of Theorem 3.1 to obtain a bound for $\mathrm{diam}_{\min}(G \boxtimes H)$ when G and H are arbitrary graphs.

COROLLARY 3.2. *For any connected graphs G and H*, $\mathrm{diam}_{\min}(G \boxtimes H) \leq 2\,\mathrm{rad}(G \boxtimes H) + 15$.

Proof. Let T_1 and T_2 be shortest path trees in G and H with respect to their central vertices, respectively. Then we have

$$\begin{aligned}\mathrm{diam}_{\min}(G \boxtimes H) &\leq \mathrm{diam}_{\min}(T_1 \boxtimes T_2) \leq \max\{\mathrm{diam}(T_1), \mathrm{diam}(T_2)\} + 15\\ &\leq 2\max\{\mathrm{rad}(T_1), \mathrm{rad}(T_2)\} + 15 = 2\max\{\mathrm{rad}(G), \mathrm{rad}(H)\} + 15\\ &= 2\,\mathrm{rad}(G \boxtimes H) + 15. \qquad \square\end{aligned}$$

The above result bounds $\mathrm{diam}_{\min}(G \boxtimes H)$ in terms of $\mathrm{rad}(G \boxtimes H)$. We ask the following question.

QUESTION 3.3. *Does there exist a constant k, such that for every connected graphs G and H, $\mathrm{diam}_{\min}(G \boxtimes H) \leq \mathrm{diam}(G \boxtimes H) + k$?*

If the answer to the above question is positive, then strong products of connected graphs admit near-optimal orientations.

4. Short directed paths between neighbouring vertices. In this section we state several local properties of the orientation D of $T_1 \boxtimes T_2$ obtained by rules (A) to (G), as they are given in Sections 2 and 3. The proofs of all results given here can be found in [4]. However, we do provide the proof of Lemma 4.4, which is the most important result of this section. In the sequal we assume that T_1 and T_2 are trees with roots r_1 and r_2. The roots may be arbitrarily chosen, and we assume that D_1 and D_2 are digraphs obtained by orienting all edges of T_1 and T_2 away from their respective roots.

LEMMA 4.1. *Let T_1 and T_2 be trees and D the orientation of $T_1 \boxtimes T_2$ according to rules (A) to (G). If $x_1, x_2 \in V(T_1)$ are not leaves in T_1 and $x_1x_2 \in E(T_1)$, and $y_1, y_2 \in V(T_2)$ are not leaves in T_2 and $y_1y_2 \in E(T_2)$, then we have the following orientations of direct edges (see Figure 3):*

(a) If $(x_1, y_1) \in (A_1 \times A_2) \cup (B_1 \times B_2)$ and $x_2 \to x_1$, or $(x_1, y_1) \in (A_1 \times B_2) \cup (B_1 \times A_2)$ and $x_1 \to x_2$, then $(x_1, y_1) \to (x_2, y_2)$ and $(x_2, y_1) \to (x_1, y_2)$.

(b) If $(x_1, y_1) \in (A_1 \times A_2) \cup (B_1 \times B_2)$ and $x_1 \to x_2$, or $(x_1, y_1) \in (A_1 \times B_2) \cup (B_1 \times A_2)$ and $x_2 \to x_1$, then $(x_1, y_2) \to (x_2, y_1)$ and $(x_2, y_2) \to (x_1, y_1)$.

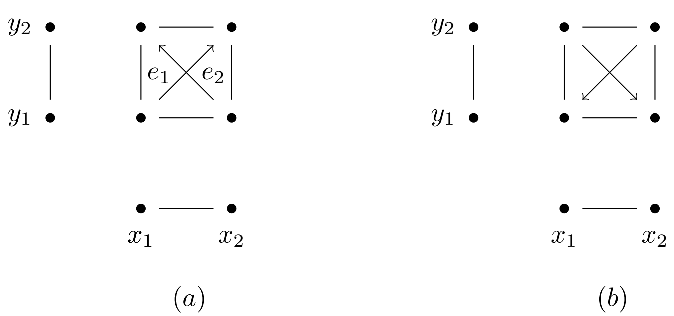

Figure 3: The orientation of direct edges of $\{x_1, x_2\} \times \{y_1, y_2\}$.

Let P be the path $x = x_0, x_1, \ldots, x_n = y$ between x and y in a rooted tree T. The *root of the path* P is the vertex x_k (where $0 \leq k \leq n$) such that $x_k < x_i$ for every $i \neq k$. The root of the path P is the vertex of P that is nearest to the root of the tree.

Lemma 4.2. *Let T_1 and T_2 be trees and D the orientation of $T_1 \boxtimes T_2$ according to rules (A) to (G). Let x_1, x_2, x_3 be a path in T_1, and let y_1 and y_2 be adjacent vertices in T_2. If x_2 is not the root of the path x_1, x_2, x_3, then the Cartesian edges of the subgraph induced by $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$ are oriented as shown in Figure 4 (cases (a) to (d)).*

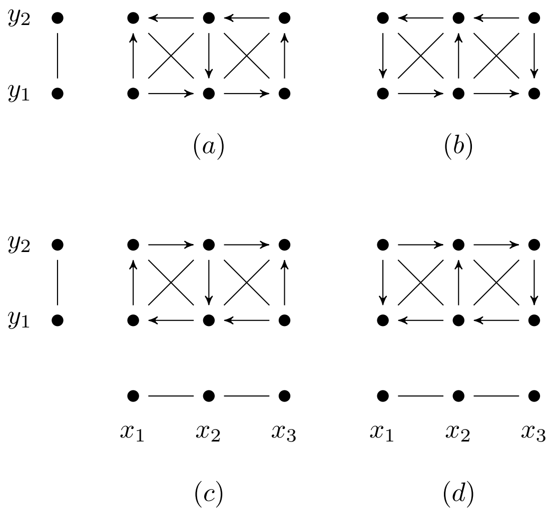

Figure 4: The orientation of Cartesian edges of $\{x_1, x_2, x_3\} \times \{y_1, y_2\}$.

If $x_1 \to x_2 \to x_3 \to x_4 \to x_1$ in D, then we say that x_1, x_2, x_3 and x_4 *induce a directed 4-cycle*. Observe that in all cases of Lemma 4.2, if $\{x_1, x_2\} \times \{y_1, y_2\}$ doesn't induce a directed 4-cycle, then $\{x_2, x_3\} \times \{y_1, y_2\}$ induces a directed 4-cycle.

The following lemma is analogous to Lemma 4.2.

Lemma 4.3. *Let T_1 and T_2 be trees and D the orientation of $T_1 \boxtimes T_2$ according to rules (A) to (G). Let x_1 and x_2 be adjacent vertices in T_1, and let y_1, y_2, y_3 be a path in T_2. If y_2 is not the root of the path y_1, y_2, y_3, then the Cartesian edges of the subgraph induced by $\{x_1, x_2\} \times \{y_1, y_2, y_3\}$ are oriented as shown in Figure 5 (cases (a) to (d)).*

Lemma 4.4. *For any trees T_1 and T_2 let D be the orientation of $T_1 \boxtimes T_2$ according to rules (A) to (G). Let $x_1, x_2 \in V(T_1)$ be adjacent vertices in T_1 and $y_1, y_2 \in V(T_2)$ adjacent vertices in T_2. Then there exists a path of length at most 4 from (x_1, y_1) to (x_2, y_2) in D.*

Proof. We may assume that $(x_2, y_2) \to (x_1, y_1)$ in D, for otherwise the statement of the lemma is true.

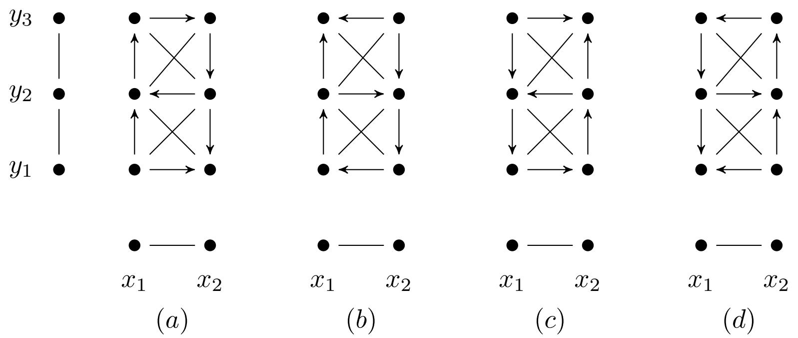

Figure 5: The orientation of Cartesian edges of $\{x_1, x_2\} \times \{y_1, y_2, y_3\}$.

If $\{x_1, x_2\} \times \{y_1, y_2\}$ induces a directed 4-cycle, then there is a path of length 2 from (x_1, y_1) to (x_2, y_2).

Suppose that $(x_1, y_1) \to (x_2, y_1)$ and $(x_1, y_1) \to (x_1, y_2)$ in D. If x_2 is not a leaf and $x_2 \neq r_1$, then there is a vertex $x_3 \in V(T_1)$ adjacent to x_2, such that x_2 is not the root of the path x_1, x_2, x_3. Therefore $\{x_2, x_3\} \times \{y_1, y_2\}$ induces a directed 4-cycle (by Lemma 4.2), and so there is a directed path from (x_2, y_1) to (x_2, y_2) of length at most 3. Since $(x_1, y_1) \to (x_2, y_1)$ we have a path of length at most 4 from (x_1, y_1) to (x_2, y_2).

If y_2 is not a leaf or $y_2 \neq r_2$, the proof is similar, therefore we can assume that both x_2 and y_2 are either a leaf or the root in D_1 and D_2, respectively. We distinguish the following cases:

(a) Suppose that x_2 is a leaf in D_1 and y_2 is a leaf in D_2. Then $x_1 \to x_2$ in D_1 and $y_1 \to y_2$ in D_2. Hence $x_1 \in A_1$ and $y_1 \in B_2$ and the orientation of the edge with endvertices (x_1, y_1) and (x_2, y_2) is obtained by the rule (G1) (if $x_1 \neq r_1$) or the rule (C) (if $x_1 = r_1$). In either case we have $(x_1, y_1) \to (x_2, y_2)$, a contradiction.

(b) Suppose that x_2 is a leaf in D_1 and $y_2 = r_2$. Since x_2 is a leaf we have $x_1 \to x_2$ in D_1 and since $y_2 = r_2$ we have $y_2 \to y_1$ in D_2. Therefore $x_1 \in B_1$ and $y_1 \in B_2$. The orientation of the edge with endvertices (x_1, y_1) and (x_2, y_2) is obtained by the rule (D) (if y_1 is not a leaf) or the rule (E) (if y_1 is a leaf). In either case we have $(x_1, y_1) \to (x_2, y_2)$, a contradiction.

(c) Suppose that $x_2 = r_1$ and y_2 is a leaf in D_2. Then $y_1 \to y_2$ in D_2 and therefore $x_2 \in B_1$, a contradiction (since $x_2 = r_1 \in A_1$).

(d) Suppose that $x_2 = r_1$ and $y_2 = r_2$. In this case we have $x_2 \to x_1$ in D_1 and therefore $y_2 \in B_2$, a contradiction (since $y_2 = r_2 \in A_2$).

Suppose that $(x_2, y_1) \to (x_1, y_1)$ and $(x_1, y_2) \to (x_1, y_1)$ in D. If x_1 is not a leaf and $x_1 \neq r_1$, or if y_1 is not a leaf and $y_1 \neq r_2$, then there is a vertex $x_0 \in V(T_1)$ such that $\{x_0, x_1\} \times \{y_1, y_2\}$ induces a directed 4-cycle, or there is a vertex $y_0 \in V(T_2)$

such that $\{x_1, x_2\} \times \{y_0, y_1\}$ induces a directed 4-cycle. Since $(x_1, y_2) \to (x_2, y_2)$ and $(x_2, y_1) \to (x_2, y_2)$ we get (in either case) a directed path of length at most 4 from (x_1, y_1) to (x_2, y_2).

(a) Suppose that x_1 is a leaf in D_1 and y_1 is a leaf in D_2. Then we have $x_2 \to x_1$ in D_1 and $y_2 \to y_1$ in D_2. Hence $x_1 \in A_1$ and $y_1 \in B_2$. By the rule (E) (if $y_2 = r_2$) or the rule (F) (if $y_2 \neq r_2$) we get the edge $(x_1, y_1) \to (x_2, y_2)$, a contradiction.

(b) Suppose that x_1 is a leaf in D_1 and $y_1 = r_2$. Then we have $x_2 \to x_1$ in D_1 and therefore $y_1 \in B_2$ (because $(x_2, y_1) \to (x_1, y_1)$ in D). This is a contradiction, since $y_1 = r_2 \in A_2$.

(c) Suppose that $x_1 = r_1$ and y_1 is a leaf in D_2. Then $x_1 \to x_2$ in D_1 and $y_2 \in B_2$. Since $x_1 \in A_1$ we get, by the rule (C), the edge $(x_1, y_1) \to (x_2, y_2)$, a contradiction.

(d) Suppose that $x_1 = r_1$ and $y_1 = r_2$. Since $y_1 \to y_2$ in D_1 we get $x_1 \in B_1$. This is a contradiction, since $x_1 = r_1 \in A_1$. □

Lemma 4.5. *For any trees T_1 and T_2 let D be the orientation of $T_1 \boxtimes T_2$ according to rules (A) to (G). Let $x_1, x_2 \in V(T_1)$ be adjacent vertices in T_1 and $y_1 \in V(T_2)$. Then there exists a path of length at most 4 from (x_1, y_1) to (x_2, y_1) in D.*

Lemma 4.6. *For any trees T_1 and T_2 let D be the orientation of $T_1 \boxtimes T_2$ according to rules (A) to (G). Let $y_1, y_2 \in V(T_2)$ be adjacent vertices in T_2 and $x_1 \in V(T_1)$. Then there exists a path of length at most 5 from (x_1, y_1) to (x_1, y_2) in D.*

5. Proof of the main theorem. In this section we prove Theorem 3.1.

Choose a root r_i in T_i, and let D_i be the orientation of T_i, such that every edge is oriented away from r_i, for $i \in \{1, 2\}$ (any vertex of T_i may be chosen as the root of T_i). We orient the edges of $T_1 \boxtimes T_2$ according to rules (A) to (G), and call the obtained digraph D. Let $(x, y), (x', y') \in V(D)$. We claim that there is a directed path P from (x, y) to (x', y') in D such that the length of P is at most $\max\{\mathrm{diam}(T_1), \mathrm{diam}(T_2)\} + 15$. Let

$$x = x_0, x_1, \ldots, x_m = x'$$

be the path between x and x' in T_1, and let

$$y = y_0, y_1, \ldots, y_n = y'$$

be the path between y and y' in T_2. Denote these two paths by P_1 and P_2, respectively. Let $\ell = \min\{m, n\}$.

A. (x, y) *and* (x', y') *are contained in the same G-layer*

Suppose that $y = y'$ and that x_i is the root of P_1 (here we are refering to the root of the path P_1). If $m = 1$ then, by Lemma 4.5, there exists a path of length at most 4 from (x_0, y) to (x_1, y), therefore we may assume that $m > 1$. Let v be any neighbour of y in T_2. If $y \in A_2$ and $i \neq m-1$ then

$$(x_0, y) \to \ldots \to (x_i, y) \xrightarrow{4} (x_{i+1}, v) \to \cdots \to (x_{m-1}, v) \xrightarrow{4} (x_m, y)$$

is a path of length $m+6$ in D (for paths of length 4 above we applied Lemma 4.6).

If $i = m-1$ then

$$(x_0, y) \to \ldots \to (x_{m-1}, y) \xrightarrow{4} (x_m, y)$$

is a path of length $m+3$ in D (for the path of length 4 we applied Lemma 4.5).

If $y \in B_2$ then

$$(x_0, y) \xrightarrow{4} (x_1, v) \to \ldots \to (x_{i-1}, v) \xrightarrow{4} (x_i, y) \to (x_{i+1}, y) \to \cdots \to (x_m, y)$$

is a path of length $m+6$ in D.

B. *(x, y) and (x', y') are contained in the same H-layer*
If $x = x'$ we prove analogously as in case A that there is a path from (x, y_0) to (x, y_n) of length at most $n+6$ in D.

C. *(x, y) and (x', y') are not contained in the same G-layer or H-layer*
Suppose that $x \neq x'$ and $y \neq y'$. Let x_i be the root of P_1 and $m, n \geq 3$. Note that $x_1, x_2, \ldots, x_{\ell-1}$ and $y_1, y_2, \ldots, y_{\ell-1}$ are not leaves, therefore we may apply Lemma 4.1 to find the orientations of direct edges with endvertices in $\{x_1, x_2, \ldots, x_{\ell-1}\} \times \{y_1, y_2, \ldots, y_{\ell-1}\}$.

(a). Suppose that $(x_0, y_0) \in (A_1 \times A_2) \cup (B_1 \times B_2)$.

(i) Suppose that $1 \leq i \leq \ell-2$. By Lemma 4.4 we have $(x_0, y_0) \xrightarrow{4} (x_1, y_1)$. Since $x_i \to x_{i-1} \to \ldots \to x_1$ we have, by Lemma 4.1, $(x_1, y_1) \to (x_2, y_2) \to \ldots \to (x_i, y_i)$ in D. Hence,

$$(x_0, y_0) \xrightarrow{4} (x_1, y_1) \to (x_2, y_2) \to \ldots \to (x_i, y_i)$$

is a path from (x_0, y_0) to (x_i, y_i) in D.

We claim that $(x_i, y_i) \to (x_i, y_{i+1})$, $(x_i, y_i) \to (x_i, y_{i-1})$ or $(x_i, y_i) \to (x_{i+1}, y_i)$ in D.

If $y_{i-1} \to y_i \to y_{i+1}$ or $y_{i+1} \to y_i \to y_{i-1}$ in D_2, then we have, by the rule (B), either $(x_i, y_{i-1}) \to (x_i, y_i) \to (x_i, y_{i+1})$ or $(x_i, y_{i+1}) \to (x_i, y_i) \to (x_i, y_{i-1})$. In this case the claim is true.

Suppose that $y_{i-1} \leftarrow y_i \to y_{i+1}$ in D_2. If $(x_i, y_i) \in (A_1 \times A_2)$, then we have (since $x_i \in A_1$) $(x_i, y_i) \to (x_i, y_{i+1})$ and $(x_i, y_i) \to (x_i, y_{i-1})$. If $(x_i, y_i) \in (B_1 \times B_2)$, then we have (since $y_i \in B_2$ and $x_i \to x_{i+1}$ in D_1) $(x_i, y_i) \to (x_{i+1}, y_i)$. This proves the claim.

Suppose that $(x_i, y_i) \to (x_i, y_{i+1})$ in D. Since $(x_i, y_{i+1}) \in (A_1 \times B_2) \cup (B_1 \times A_2)$ and $x_i \to x_{i+1} \to \ldots \to x_{\ell-2}$ we have, by Lemma 4.1, the path

$$(x_i, y_{i+1}) \to (x_{i+1}, y_{i+2}) \to \ldots \to (x_{\ell-2}, y_{\ell-1}).$$

To obtain the orientation of the edge $(x_{\ell-2}, y_{\ell-1})(x_{\ell-1}, y_\ell)$ one of the rules (C), (G1) or (G2) is applied (since $x_{\ell-1}$ is not a leaf, rules (D), (E) and (F) do not apply). In either case we have $(x_{\ell-2}, y_{\ell-1}) \to (x_{\ell-1}, y_\ell)$.

Altogether we have the path

$$(x_0, y_0) \xrightarrow{4} (x_1, y_1) \to \ldots \to (x_i, y_i) \to (x_i, y_{i+1}) \to \ldots \to (x_{\ell-1}, y_\ell)$$

of length $\ell + 3$.

If $m \geq n$ we use case A. of this theorem to find that there is the path from $(x_{\ell-1}, y_\ell)$ to (x_m, y_n) of length at most $m - \ell + 7$. When we combine all of the above paths we obtain a path from (x_0, y_0) to (x_m, y_n) of length at most $m + 10$.

If $n > m$ then $(x_{\ell-1}, y_\ell) \xrightarrow{4} (x_\ell, y_{\ell+1})$, by Lemma 4.4. As in case B. of this theorem there is a path from $(x_\ell, y_{\ell+1})$ to (x_m, y_n) of length at most $n - \ell + 5$. In this case, by combining all of the above paths, we get a path from (x_0, y_0) to (x_m, y_n) of length at most $n + 12$.

Suppose that $(x_i, y_i) \to (x_i, y_{i-1})$ or $(x_i, y_i) \to (x_{i+1}, y_i)$ in D. If y_{i-1} is not a leaf then we have, by Lemma 4.1, the edge $(x_i, y_{i-1}) \to (x_{i+1}, y_i)$. If y_{i-1} is a leaf (note that this is possible only if $i = 1$) then we have again $(x_i, y_{i-1}) \to (x_{i+1}, y_i)$ by the rule (G2) (it is easy to see that rules (C), (D), (E), (F) and (G1) do not apply in this case). Since $(x_{i+1}, y_i) \in (A_1 \times B_2) \cup (B_1 \times A_2)$ and $x_{i+1} \to x_{i+2} \to \ldots \to x_{\ell-2}$ we have, by Lemma 4.1, the path

$$(x_{i+1}, y_i) \to (x_{i+2}, y_{i+1}) \to \ldots \to (x_{\ell-1}, y_{\ell-2}).$$

When we combine this path with

$$(x_0, y_0) \xrightarrow{4} (x_1, y_1) \to \ldots \to (x_i, y_i) \to (x_i, y_{i-1}) \to (x_{i+1}, y_i)$$

we get a path from (x_0, y_0) to $(x_{\ell-1}, y_{\ell-2})$ of length $\ell + 3$. To construct a path from (x_0, y_0) to (x_m, y_n) we use the claim below, and obtain a path of length at most $\max\{m, n\} + 11$.

Claim 1: There is a path from $(x_{\ell-1}, y_{\ell-2})$ to (x_m, y_n) of length at most $\max\{m, n\} - \ell + 8$.

To obtain the orientation of the edge $(x_{\ell-1}, y_{\ell-2})(x_\ell, y_{\ell-1})$ one of the rules (D), (G1) or (G2) is applied (since $y_{\ell-1}$ is not a leaf, rules (C), (E) and (F) do not apply). In either case we have $(x_{\ell-1}, y_{\ell-2}) \to (x_\ell, y_{\ell-1})$.

If $m > n$ then $(x_\ell, y_{\ell-1}) \xrightarrow{4} (x_{\ell+1}, y_\ell)$, by Lemma 4.4. By case A. of this theorem there is a path from $(x_{\ell+1}, y_\ell)$ to (x_m, y_n) of length at most $m - \ell + 5$.

If $n \geq m$ then there is a path from $(x_\ell, y_{\ell-1})$ to (x_m, y_n) of length at most $n - \ell + 7$ (case B. of this theorem). When we combine this path with $(x_{\ell-1}, y_{\ell-2}) \to (x_\ell, y_{\ell-1})$ we get a path of length at most $n - \ell + 8$. This proves the claim.

(ii) Suppose that $i \geq \ell - 1$. In this case we have

$$(x_0, y_0) \xrightarrow{4} (x_1, y_1) \rightarrow (x_2, y_2) \rightarrow \ldots \rightarrow (x_{\ell-1}, y_{\ell-1}).$$

By Lemma 4.4 we have $(x_{\ell-1}, y_{\ell-1}) \xrightarrow{4} (x_\ell, y_\ell)$. If $m \geq n$ we use case A. of this theorem, otherwise we use case B. of this theorem, to find a path from (x_ℓ, y_ℓ) to (x_m, y_n) of length at most $\max\{m, n\} - \ell + 6$. It follows that there is a path from (x_0, y_0) to (x_m, y_n) of length at most $\max\{m, n\} + 12$.

(iii) Suppose that $i = 0$. Since there is either $(x_0, y_0) \rightarrow (x_1, y_0)$ or $(x_1, y_1) \rightarrow (x_2, y_1)$ and since $(x_0, y_0) \xrightarrow{4} (x_1, y_1)$ and $(x_1, y_0) \xrightarrow{4} (x_2, y_1)$, we have a path of length 5 from (x_0, y_0) to (x_2, y_1). By Lemma 4.1 we have

$$(x_2, y_1) \rightarrow (x_3, y_2) \rightarrow \ldots \rightarrow (x_{\ell-1}, y_{\ell-2}).$$

By Claim 1 we have a path from $(x_{\ell-1}, y_{\ell-2})$ to (x_m, y_n) of length at most $\max\{m, n\} - \ell + 8$. If we combine all of these paths we obtain a path from (x_0, y_0) to (x_m, y_n) of length at most $\max\{m, n\} + 10$.

To finish the proof of case (a) it remains to construct a path from (x_0, y_0) to (x_m, y_n) when $m < 3$ or $n < 3$. Without loss of generality we can assume $m < 3$ and $m \leq n$. If $m = 2$ we have $(x_0, y_0) \xrightarrow{4} (x_1, y_1) \xrightarrow{4} (x_2, y_2)$. By case B. we have a path from (x_2, y_2) to (x_2, y_n) of length at most $n + 4$ and therefore there is a path from (x_0, y_0) to (x_m, y_n) of length at most $n + 12$. For $m = 1$ the proof is similar.

(b). Suppose that $(x_0, y_0) \in (A_1 \times B_2) \cup (B_1 \times A_2)$. By Lemma 4.5 we have $(x_0, y_0) \xrightarrow{4} (x_1, y_0)$. Since $(x_1, y_0) \in (A_1 \times A_2) \cup (B_1 \times B_2)$ this case reduces to case (a). By case (a) we have a path from (x_1, y_0) to (x_m, y_n) of length at most $\max\{m - 1, n\} + 12$, and therefore (when we use $(x_0, y_0) \xrightarrow{4} (x_1, y_0)$) we have a path from (x_0, y_0) to (x_m, y_n) of length at most $\max\{m, n\} + 15$. This completes the proof of Theorem 3.1. □

REFERENCES

1. J. BANG-JENSEN AND G. GUTIN, *Digraphs: Theory, Algorithms and Applications,* Springer, London, 2000.
2. V. CHVÁTAL AND C. THOMASSEN, Distances in orientations of graphs, *J. Combin. Theory Ser. B* **24** (1978), 61–75.
3. R. HAMMACK, W. IMRICH, AND S. KLAVŽAR, *Handbook of product graphs,* Second edition, CRC Press, Boca Raton, FL, 2011.
4. I. HRASTNIK LADINEK AND S. ŠPACAPAN, The diameter of strong orientations of strong products of graphs, arXiv:1909.12022.
5. K.M. KOH AND E.G. TAY, On optimal orientations of Cartesian products of even cycles and paths, *Networks* **30** (1997), 1–7.
6. ———————, Optimal orientations of products of paths and cycles, *Discrete Appl. Math.* **78** (1997), 163–174.

7. ______________, On optimal orientations of Cartesian products of graphs (I), *Discrete Math.* **190** (1998), 115–136.

8. ______________, On optimal orientations of Cartesian products of graphs (II): complete graphs and even cycles, *Discrete Math.* **211** (2000), 75–102.

9. ______________, On optimal orientations of Cartesian products of trees, *Graphs Combin.* **17** (2001), 79–97.

10. M. Knor, R. Škrekovski, and A. Tepeh, Orientations of graphs with maximum Wiener index, *Discrete Appl. Math.* **211** (2016), 121–129.

11. ______________, Some remarks on the Wiener index of digraphs, *Appl. Math. Comput.* **273** (2016), 631–636.

12. T. Paj-Erker, Optimal orientations of strong products of paths, *ADAM* **2** (2019), #P1.04

13. I. Peterin and P. Žigert Pleteršek, Wiener index of strong product of graphs, *Opuscula Math.* **38**(1) (2018), 81–94.

14. J. Plesník, On the sum of all distances in a graph or digraph, *J. Graph Theory* **8** (1984) 1–21.

15. H.E. Robbins, A theorem on graphs with an application to a problem on traffic control, *Am. Math. Month.* **46** (1939), 281–283.

16. F.S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs I: Large grids, *SIAM J. Discrete Math.* **1**(2) (1988), 199–222.

17. ______________, On the optimal strongly connected orientations of city street graphs II: Two east-west avenues or north-south streets, *Networks* **19** (1989), 221–233.

18. ______________, On the optimal strongly connected orientations of city street graphs III: Three east-west avenues or north-south streets, *Networks* **22** (1992), 109–143.

19. Y. Shitov, Counterexamples to Hedetniemi's conjecture, arXiv:1905.02167v2.

20. S. Špacapan, The diameter of strong orientations of Cartesian products of graphs, *Discrete Appl. Math.* **247** (2018), 116–121.

Received 6 December, 2019 and in revised form 21 March, 2020.