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Q-borderenergetic graphs

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Abstract

A graph G is said to be borderenergetic (L-borderenergetic, respectively) if its energy (Laplacian energy, respectively) equals the energy (Laplacian energy, respectively) of the complete graph K_n. We extend this concept to signless Laplacian energy of a graph. A graph G is called Q-borderenergetic if its signless Laplacian energy is same as that of the complete graph K_n, i.e., $QE(G) = QE(K_n) = 2n - 2$. In this paper, we construct some infinite family of graphs satisfying $QE(G) = LE(G) = 2n - 2$, this happens to give a positive answer to the open problem mentioned by Nair Abreu et al. in Nair Abreu et al. (2011), that is whether there are connected non-bipartite, non-regular graphs satisfying $QE(G) = LE(G)$. We also obtain some bounds on the order and size of Q-borderenergetic graphs. Finally, we use a computer search to find out all Q-borderenergetic graphs on no more than 10 vertices, the number of such graphs is 39.

1. Introduction

Throughout this paper we consider simple undirected and connected graphs only. Let G be such a graph on n vertices, the energy of G [1–3], denoted by $E(G)$, is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. If a graph G on n vertices has the same energy as the complete graph K_n, i.e., $E(G) = E(K_n) = 2n - 2$, then G is said to be borderenergetic, see [4]. For more details on borderenergetic, we refer to [5–8].

Let A be adjacency matrix and D diagonal matrix of vertex degrees of G, respectively, then $L = D - A$ and $Q = D + A$ are called the Laplacian matrix and the signless Laplacian matrix of G, respectively. The Laplacian energy of a graph G [9] is defined as $LE(G) = \sum_{i=1}^{n} |\lambda_i - \overline{d}|$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. For its basic properties, see [10–15]. More recently, Fernando Tura [16] proposed the concept of L-borderenergetic graphs, which means $LE(G) = LE(K_n) = 2n - 2$,

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and gave several classes of L-borderenergetic graphs. For more results on L-borderenergetic graphs, see [17–19]. In this paper, we extend this concept to signless Laplacian energy of a graph.

For signless Laplacian spectral properties see [20]. The signless Laplacian energy of a graph G is defined as $QE(G) = \sum_{i=1}^{n} |q_i - \overline{d}|$, where $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ are the signless Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G. We refer to a graph G as Q-borderenergetic if its signless Laplacian energy equals that of the complete graph K_n, i.e., $QE(G) = QE(K_n) = 2n - 2$. We construct an infinite family of graphs satisfying $QE(G) = LE(G) = 2n - 2$, see Theorem 3. It is interesting that this family of graphs also gives a positive answer to the open problem mentioned by Nair Abreu et al. in Ref. [21], that is whether there are connected non-bipartite, non-regular graphs satisfying $QE(G) = LE(G)$. We also construct another two infinite families of Q-borderenergetic graphs. Then, we obtain some bounds on the order and size of Q-borderenergetic graphs. Finally, we use a computer search to find out all Q-borderenergetic graphs on no more than 10 vertices, the number of such graphs is 39.

2. Main results

If G is a regular graph of degree k, noting that $D = kI$, $Q = kI + A$, and $L = kI - A$, it follows that $E(G) = LE(G) = QE(G)$, also note that the graph $\overline{pC_4 \bigcup qC_6 \bigcup rC_3}$ is regular of degree $n - 3$, from [4] we can easily get the following Theorem.

Theorem 1. *Let p, q, and r are non-negative integers with $p + q = 2$, then $\overline{pC_4 \bigcup qC_6 \bigcup rC_3}$ is Q-borderenergetic.*

It is well-known that for bipartite graphs the Laplacian spectrum coincides with the signless Laplacian spectrum, obviously, in this case, $QE(G) = LE(G)$. From Theorem 10 in [17] we know that there is not any L-borderenergetic tree, thus there is not any Q-borderenergetic tree. While there does exist bipartite Q-borderenergetic graph, see graph G_3^{10} in Fig. 2, for example.

Let $G_1 \triangledown G_2$ denote the join of graphs G_1 and G_2, obtained from the union of G_1 and G_2 by joining every vertex of G_1 with every vertex of G_2. The following lemma is from [16].

Lemma 2. *Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. Let L_1 and L_2 be the Laplacian matrices for G_1 and G_2, respectively, and let L be the Laplacian matrix for $G_1 \triangledown G_2$. If $0 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n_1}$ and $0 = \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n_2}$ are the eigenvalues of L_1 and L_2, respectively, then the eigenvalues of L are $\{0, n_2 + \alpha_2, n_2 + \alpha_3, \ldots, n_2 + \alpha_{n_1}, n_1 + \beta_2, n_1 + \beta_3, \ldots, n_1 + \beta_{n_2}, n_1 + n_2\}$.*

Theorem 3. *For each integer $p \geq 1$, $K_1 \triangledown (K_3 \cup pK_2)$ is L-borderenergetic and Q-borderenergetic.*

Proof. On the one hand, by Lemma 2 and direct calculation, the Laplacian spectrum of $K_1 \triangledown (K_3 \cup pK_2)$ is $\{0, 1^{(p)}, 3^{(p)}, 4^{(2)}, 2p + 4\}$. Noting that its average degree is 3, it can be easily verified that $LE(K_1 \triangledown (K_3 \cup pK_2)) = 4p + 6 = 2n - 2$, where $n = 2p + 4$ is the order of $K_1 \triangledown (K_3 \cup pK_2)$.

On the other hand, the signless Laplacian matrix of $K_1 \triangledown (K_3 \cup pK_2)$ with suitable labeling has the form

$$Q(K_1 \triangledown (K_3 \cup pK_2)) = \begin{pmatrix} 2p+3 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & 3 & 1 & 0 & & & \vdots \\ 1 & 1 & 1 & 3 & 0 & & & \\ 1 & 0 & 0 & 0 & 2 & 1 & & \vdots \\ 1 & \vdots & & & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & & & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

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Thus, its signless Laplacian characteristic polynomial is $|xI - Q(K_1 \bigtriangledown (K_3 \cup pK_2))|$

$$= \begin{vmatrix} x-(2p+3) & -1 & -1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & x-3 & -1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & -1 & x-3 & -1 & 0 & & & \vdots \\ -1 & -1 & -1 & x-3 & 0 & & & \\ -1 & 0 & 0 & 0 & x-2 & -1 & & \vdots \\ -1 & \vdots & & & -1 & x-2 & \ddots & 0 \\ \vdots & \vdots & & & & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & & \cdots & 0 & -1 & x-2 \end{vmatrix}.$$

Performing $C_1 + \frac{1}{x-5}(C_2 + C_3 + C_4)$, where C_i is the ith column of the above determinant, we obtain $|xI - Q(K_1 \bigtriangledown (K_3 \cup pK_2))|$

$$= \begin{vmatrix} x-(2p+3)+\dfrac{-3}{x-5} & -1 & -1 & -1 & -1 & -1 & \cdots & -1 \\ 0 & x-3 & -1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & x-3 & -1 & 0 & & & \vdots \\ 0 & -1 & -1 & x-3 & 0 & & & \\ -1 & 0 & 0 & 0 & x-2 & -1 & & \vdots \\ -1 & \vdots & & & -1 & x-2 & \ddots & 0 \\ \vdots & \vdots & & & & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & & \cdots & 0 & -1 & x-2 \end{vmatrix}.$$

Again performing $C_1 + \frac{1}{x-3}(C_5 + C_6), \ldots, C_1 + \frac{1}{x-3}(C_{n-1} + C_n)$, then directly expanding along the first column, we have

$$\begin{aligned} &|xI - Q(K_1 \bigtriangledown (K_3 \cup pK_2))| \\ &= (x - (2p+3) + \frac{-3}{x-5} + \frac{-2p}{x-3})(x-5)(x-2)^2(x-3)^p(x-1)^p \\ &= (x^2 - (2p+9)x + 10p + 18)(x-2)^3(x-3)^{p-1}(x-1)^p. \end{aligned}$$

So its signless Laplacian spectrum is $\{1^{(p)}, 2^{(3)}, 3^{(p-1)}, p + \frac{9}{2} \pm \frac{1}{2}\sqrt{4p^2 - 4p + 9}\}$. Hence, we have

$$\begin{aligned} &QE(K_1 \bigtriangledown (K_3 \cup pK_2)) \\ &= 2p + 3 + 0 \cdot (p-1) + |p + \frac{9}{2} + \frac{1}{2}\sqrt{4p^2 - 4p + 9} - 3| + |p + \frac{9}{2} - \frac{1}{2}\sqrt{4p^2 - 4p + 9} - 3| \\ &= 4p + 6 \\ &= 2n - 2. \end{aligned}$$

This completes the proof. □

It is not difficult to verify that $K_1 \bigtriangledown (K_3 \cup pK_2)$ is connected non-bipartite and non-regular, therefore, Theorem 3 also gives a positive answer to the problem mentioned by Nair Abreu et al. in Ref. [21], that is whether there are connected non-bipartite, non-regular graphs satisfying $QE(G) = LE(G)$.

Note that graph $K_1 \bigtriangledown (K_3 \cup pK_2)$ can be seen as constructed by connecting one vertex of K_4 with both ends of each of p copies of K_2. If we do the same thing on two or three vertices of K_4, which has the form as graph H_1 and H_2 in Fig. 1, respectively, we obtain another two families of Q-borderenergetic graphs.

Theorem 4. (1) *For each integer $p \geq 1$, let H_1 be a graph constructed by connecting two vertices of K_4 with both ends of each of p copies of K_2, respectively. Then H_1 is a Q-borderenergetic graph of order $4p + 4$.*
(2) *For each integer $p \geq 1$, let H_2 be a graph constructed by connecting three vertices of K_4 with both ends of each of p copies of K_2, respectively. Then H_2 is a Q-borderenergetic graph of order $6p + 4$.*

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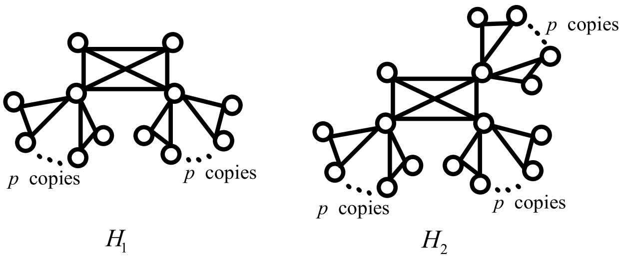

Fig. 1. Two families of Q-borderenergetic graphs.

Proof. (1) According to the form of signless Laplacian characteristic polynomial $|xI - Q(H_1)|$, by direct calculation similar to the proof of Theorem 3, we get that $|xI - Q(H_1)| = (x^2 - (2p+9)x + 8p + 18)(x - 2p - 3)(x-2)^3(x-3)^{2p-2}(x-1)^{2p}$. So the signless Laplacian spectrum of H_1 is $\{1^{(2p)}, 2^{(3)}, 3^{(2p-2)}, 2p+3, p+\frac{9}{2} \pm \frac{1}{2}\sqrt{4p^2+4p+9}\}$, and noting that the average degree of H_1 is 3, consequently, by simple calculation, its signless Laplacian energy is $QE(H_1) = 8p + 6 = 2n - 2$, where $n = 4p + 4$.
(2) Similarly, by direct calculation we obtain the signless Laplacian characteristic polynomial of H_2, $|xI - Q(H_2)| = (x - (2p+6))(x - (2p+3))^2(x-2)^3(x-3)^{3p-2}(x-1)^{3p}$. Hence the signless Laplacian spectrum of H_2 is $\{1^{(3p)}, 2^{(3)}, 3^{(3p-2)}, (2p+3)^{(2)}, 2p+6\}$, also note that the average degree of H_2 is 3, it can be verified that $QE(H_2) = 12p + 6 = 2n - 2$, where $n = 6p + 4$. This completes the proof. □

Next, we present a lower bound on the size and a bound on the order of Q-borderenergetic graphs, respectively.

Lemma 5 ([22]). *Let G be a connected graph of order $n \geq 3$ with m edges and having first Zagreb index $M_1(G) = \sum_{i=1}^{n} d_i^2$. Then $QE(G) \geq 2(\frac{M_1(G)}{m} - \frac{2m}{n})$.*

Lemma 6 ([22]). *Let G be a connected graph of order n with m edges and having maximum degree Δ. Then $QE(G) \leq 2(2m + 1 - \Delta - \frac{2m}{n})$ with equality if and only if $G \cong K_{1,n-1}$.*

Theorem 7. *If G is a Q-borderenergetic graph of order n with m edges, then*

$$m > \frac{1}{4}(n - n^2 + \sqrt{n^2(n-1)^2 + 8M_1(G)n}).$$

Proof. By Lemma 5, we have

$$2(n-1) \geq 2(\frac{M_1(G)}{m} - \frac{2m}{n})$$

by simplification, we obtain

$$2m^2 + (n^2 - n)m - M_1(G)n \geq 0$$

solving this inequality of m, it is easy to get

$$m \geq \frac{1}{4}(n - n^2 + \sqrt{n^2(n-1)^2 + 8M_1(G)n}). \quad (1)$$

Next, we should point out that the equality in Eq. (1) does not hold. This can be seen from the proof of Lemma 5 (Theorem 3.3 in [22]). In order to make the equality in Lemma 5 hold, on the one hand, graph G must have only one signless Laplacian eigenvalue greater than or equal to average degree $\frac{2m}{n}$, that is, there should hold $q_2 < \frac{2m}{n}$ for G. On the other hand, for the line graph $\ell(G)$ of G, its largest adjacency eigenvalue should satisfy $\mu_1(\ell(G)) = \frac{2m(\ell(G))}{n(\ell(G))}$, that is, the line graph $\ell(G)$ of G is regular, (see page 55, theorem 3.2.1 in [23]). Therefore, G should be a regular graph or semi-regular bipartite graph, (see page 8, proposition 1.2.2 in [23]).

For a regular graph G, if $G \ncong K_n$, G has at least two non-adjacent vertices with equal degree $\Delta_1 = \Delta_2$, where Δ_1, Δ_2 are the largest and the second largest degree of G. According to the proof of theorem 3.1 in [24], if a graph G has

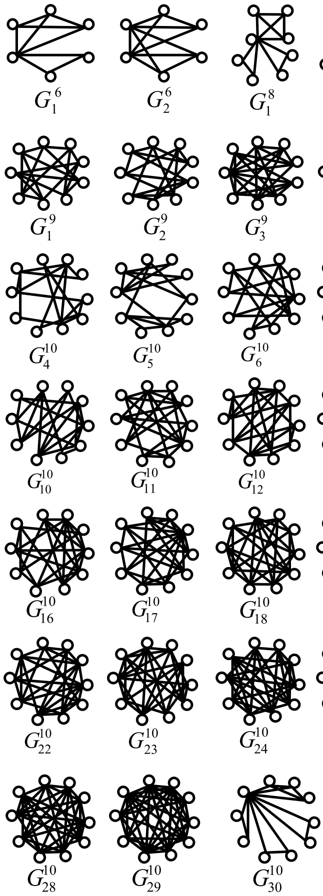

Fig. 2. Q-borderenergetic graphs on no more than 10 vertices.

two non-adjacent vertices with the largest and the second largest degree Δ_1 and Δ_2, respectively, then $q_2 \geq \Delta_2$. Thus the second largest signless Laplacian eigenvalue of G satisfying $q_2 \geq \Delta_2 = \frac{2m}{n}$.

For a semi-regular bipartite graph G, if $G \ncong K_{1,n-1}$, (where $K_{1,n-1}$ is star graph, which is not Q-borderenergetic), G has at least two non-adjacent vertices with equal degree $\Delta_1 = \Delta_2 \geq \frac{2m}{n}$, therefore, similarly, we have $q_2 \geq \Delta_2 \geq \frac{2m}{n}$.

From the above, we know that there is no Q-borderenergetic graph such that the equality in Eq. (1) holds. □

Theorem 8. *If G is a Q-borderenergetic graph of order n with m edges, then*

$$\frac{1}{2}((2m+2-\Delta) - \sqrt{(2m+2-\Delta)^2 - 8m}) < n < \frac{1}{2}((2m+2-\Delta) + \sqrt{(2m+2-\Delta)^2 - 8m}).$$

Proof. By Lemma 6, we have

$$2(n-1) \leq 2(2m+1-\Delta-\frac{2m}{n})$$

by simplification, we obtain

$$n^2 - (2m+2-\Delta)n + 2m \leq 0$$

solving this inequality of n, we get

$$\frac{1}{2}((2m+2-\Delta) - \sqrt{(2m+2-\Delta)^2 - 8m}) \leq n \leq \frac{1}{2}((2m+2-\Delta) + \sqrt{(2m+2-\Delta)^2 - 8m}). \tag{2}$$

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Further, the equality in Lemma 6 holds if and only if $G \cong K_{1,n-1}$, which is not Q-borderenergetic. Therefore, there is no Q-borderenergetic graph such that the equalities in Eq. (2) hold. □

At last, we perform a computer-aided search for Q-borderenergetic graphs on $n \le 10$ vertices and obtain that there are totally 39 non-complete Q-borderenergetic graphs, among which there are 2 on $n = 6$ vertices, 4 on $n = 8$ vertices, 3 on $n = 9$ vertices and 30 on $n = 10$ vertices. We list the result in the following theorem.

Theorem 9. *There are totally 39 non-complete Q-borderenergetic graphs on $n \le 10$ vertices, which are listed in Fig. 2 in Appendix, their signless Laplacian spectra are listed in the following table, where q_1, q_2, q_3 in $SQ(G_1^{10})$ are three roots of cubic equation $x^3 - 18x^2 + 101x - 176 = 0$.*

$SQ(G_1^6) = \{1,2,2,2,4,7\}$	$SQ(G_{12}^{10}) = \{2,2,3,4,4,5,5,6,7,10\}$
$SQ(G_2^6) = \{2,2,2,4,4,8\}$	$SQ(G_{13}^{10}) = \{1,3,3,4,5,5,6,6,6,11\}$
$SQ(G_1^8) = \{1,1,2,2,2,3,\frac{1}{2}(13 \pm \sqrt{17})\}$	$SQ(G_{14}^{10}) = \{2,3,3,3,5,5,5,7,\frac{1}{2}(17 \pm \sqrt{17})\}$
$SQ(G_2^8) = \{1,1,2,2,2,5,\frac{1}{2}(11 \pm \sqrt{17})\}$	$SQ(G_{15}^{10}) = \{2,2,3,4,5,5,6,7,8 \pm 2\sqrt{2}\}$
$SQ(G_3^8) = \{2,2,4,4,5,5,6,10\}$	$SQ(G_{16}^{10}) = \{2,2,4,4,5,6,6,6,\frac{1}{2}(17 \pm \sqrt{33})\}$
$SQ(G_4^8) = \{2,4,4,4,4,6,6,10\}$	$SQ(G_{17}^{10}) = \{2,3,3,5,5,6,6,6,6,12\}$
$SQ(G_1^9) = \{2,2,2,3,3,5,5,6,8\}$	$SQ(G_{18}^{10}) = \{2,4,4,4,5,6,7,7,\frac{1}{2}(17 \pm \sqrt{33})\}$
$SQ(G_2^9) = \{2,2,2,2,5,5,5,5,8\}$	$SQ(G_{19}^{10}) = \{3,3,4,4,5,6,6,8,\frac{1}{2}(17 \pm \sqrt{33})\}$
$SQ(G_3^9) = \{3,3,3,5,5,5,\frac{1}{2}(17 \pm \sqrt{33})\}$	$SQ(G_{20}^{10}) = \{3,3,4,5,5,6,6,6,8,12\}$
$SQ(G_1^{10}) = \{1,1,1,2,2,2,3,q_1, q_2, q_3\}$	$SQ(G_{21}^{10}) = \{3,4,5,6,6,6,4 \pm \sqrt{3}, 10 \pm \sqrt{7}\}$
$SQ(G_2^{10}) = \{1,1,1,2,2,2,3,5,5,8\}$	$SQ(G_{22}^{10}, G_{23}^{10}) = \{2,4,4,5,5,6,6,7,7,12\}$
$SQ(G_3^{10}) = \{0,1,2,2,2,4,5,5,\frac{1}{2}(11 \pm \sqrt{17})\}$	$SQ(G_{24}^{10}) = \{3,4,4,4,6,6,7,7,7,12\}$
$SQ(G_4^{10}, G_5^{10}) = \{1,1,1,2,2,4,4,6,\frac{1}{2}(11 \pm \sqrt{17})\}$	$SQ(G_{25}^{10}) = \{3,4,4,5,5,6,6,7,8,12\}$
$SQ(G_6^{10}) = \{1,2,2,2,2,4,4,5,6,8\}$	$SQ(G_{26}^{10}) = \{6,7,7,7,\frac{1}{2}(21 \pm \sqrt{41})$ $\frac{1}{2}(9 \pm \sqrt{5}), \frac{1}{2}(9 \pm \sqrt{5})\}$
$SQ(G_7^{10}) = \{1,2,2,2,3,4,5,6,\frac{1}{2}(13 \pm \sqrt{17})\}$	$SQ(G_{27}^{10}) = \{4,4,4,6,6,7,7,7,\frac{1}{2}(21 \pm \sqrt{33})\}$
$SQ(G_8^{10}) = \{1,2,2,3,3,4,4,5,7,9\}$	$SQ(G_{28}^{10}) = \{4,5,5,6,6,7,7,8,8,14\}$
$SQ(G_9^{10}) = \{2,2,2,2,3,4,4,5,8 \pm \sqrt{2}\}$	$SQ(G_{29}^{10}) = \{5,5,6,7,7,8,8,8,8,16\}$
$SQ(G_{10}^{10}) = \{2,5,5,6,7 \pm \sqrt{5},$ $\frac{1}{2}(5 \pm \sqrt{5}), \frac{1}{2}(5 \pm \sqrt{5})\}$	$SQ(G_{30}^{10}) = \{1,1,1,2,2,2,3,3,\frac{1}{2}(15 \pm \sqrt{33})\}$
$SQ(G_{11}^{10}) = \{2,2,2,4,4,5,5,7,\frac{1}{2}(15 \pm \sqrt{33})\}$	

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Appendix. Fig. 2

See Fig. 2.

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