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Some classes of trees with maximum number of holes two

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Abstract

An $L(2,1)$-coloring of a simple connected graph G is an assignment of non-negative integers to the vertices of G such that adjacent vertices color difference is at least two, and vertices that are at distance two from each other get different colors. The maximum color assigned in an $L(2,1)$-coloring is called span of that coloring. The span of a graph G denoted by $\lambda(G)$ is the smallest span taken over all $L(2,1)$-colorings of G. A hole is an unused color within the range of colors used by the coloring. An $L(2,1)$-coloring f is said to be irreducible if no other $L(2,1)$-coloring can be produced by decreasing a color of f. The maximum number of holes of a graph G, denoted by $H_\lambda(G)$, is the maximum number of holes taken over all irreducible $L(2,1)$-colorings with span $\lambda(G)$. Laskar and Eyabi (Christpher, 2009) conjectured that if T is a tree, then $H_\lambda(T) = 2$ if and only if $T = P_n, n > 4$. We show that this conjecture does not hold by providing a counterexample. Also, we give some classes of trees with maximum number of holes two.

Keywords: $L(2,1)$-coloring; Span of a graph; Irreducible coloring; Maximum number of holes

1. Introduction

The Channel assignment problem is the problem of assigning frequencies to transmitters without interference. One of the variations of Channel assignment problem is $L(2,1)$-coloring of graphs. An $L(2,1)$-coloring of a graph G, introduced by Griggs and Yeh [1] is an assignment f from the vertex set of G to $\{0, 1, 2, \ldots, k\}$ such that $|f(u) - f(v)| \geqslant 2$ if $d(u,v) = 1$, and $|f(u) - f(v)| \geqslant 1$ if $d(u,v) = 2$, where $d(u,v)$ denotes the distance between the vertices u and v. The span of f is the largest integer assigned by f. The $L(2,1)$-span or span $\lambda(G)$ of a graph G is the smallest span of f taken over all $L(2,1)$-colorings of G. In the introductory paper, they have proved that $\lambda(P_n) = 4$ for $n \geqslant 5$ and they have shown that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$ for any tree T with maximum degree Δ. We refer a tree is Type-I if $\lambda(T) = \Delta + 1$, otherwise Type-II. In a graph G with maximum degree Δ, we refer a vertex v as a major vertex if its degree is Δ, otherwise it is a minor vertex. A Δ-path segment is a path

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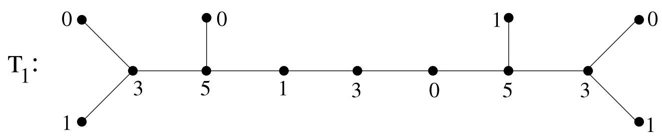


Fig. 2.1. Irreducible L (2, 1)-span coloring of T_1 with two holes.

between two major vertices. Wang [2] has proved that if a tree T does not contain Δ-path segments of length 1, 2 and 4, then T is Type-I. Zhai et al. [3] improved the above condition as if T does not contains Δ-path segment of length 2 and 4, then $\lambda(T) = \Delta + 1$. Mandal and Panigrahi [4] have found that $\lambda(T) = \Delta + 1$ if T has at most one Δ-path segment of length either 2 or 4 and all other Δ-path segments are of length at least 7. Wood and Jacob [5] have given a complete characterization of the $L(2, 1)$-span of trees up to twenty vertices. Fishburn et al. [6] have introduced the concept of irreducibility of $L(2, 1)$-coloring. An $L(2, 1)$-coloring f of a graph G is said to be irreducible if there is no $L(2, 1)$-coloring g of G such that $g(v) \leqslant f(v)$ for all vertices v in G and $g(u) < f(u)$ for at least one vertex u of G. An integer l between 0 and the span of an $L(2, 1)$-coloring f is said to be a hole if there is no vertex v such that $f(v) = l$. The maximum number of holes over all irreducible span colorings of a graph G is denoted by $H_\lambda(G)$. Laskar and Eyabi [7] have determined the maximum number of holes for paths, cycles, stars and complete bipartite graphs as 2, 2, 1 and 1 respectively. Also, they showed that $H_\lambda(T) \leq 1$ for any Type-I tree T and $H_\lambda(T) \leq 2$ if T is Type-II tree. Further, they conjectured as below.

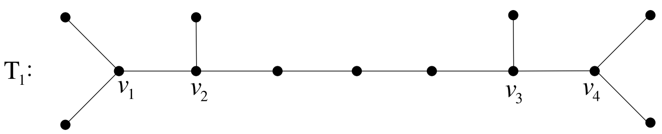
Conjecture 1.1 (*[7]*). *For any tree T, $H_\lambda(T) = 2$ if and only if T is a path P_n, $n > 4$.*

In this paper, we give a counterexample for Conjecture 1.1 by giving a two hole irreducible span coloring for a Type-II tree which is not a path. Also, we consider some Type-II trees given by Wood and Jacob [5] and for each tree, we construct infinitely many Type-II trees with maximum number of holes two.

2. Counterexample

In this section, we give a counterexample to Conjecture 1.1. Wood and Jacob [5] have proved that a tree T with maximum degree $\Delta = 3$ and containing a subtree T_1 with four major vertices v_1, v_2, v_3 and v_4 of T such that v_1v_2, $v_3v_4 \in E(T)$, $d(v_2, v_3) = 4$ and $d(v_1, v_4) = 6$ is Type-II. For this subtree T_1, we define an irreducible span coloring with two holes which disproves Conjecture 1.1. For the sake of completeness, we give the proof given by Wood and Jacob [5] to show T is Type-II.

Theorem 2.1 (*[5]*). *A tree T with maximum degree $\Delta = 3$ and containing a subtree T_1 as below, is Type-II.*



Proof. Suppose T_1 is Type-I, that is $\lambda(T_1) = 4$. Let f be a span coloring of T_1. In any span coloring of a Type-I tree, any major vertex receives either 0 or $\Delta + 1$. Without loss of generality, let $f(v_1) = 4$ and $f(v_2) = 0$. If $f(v_3) = 4$ and $f(v_4) = 0$, then there is no possibility for coloring the vertices between v_2 and v_3. Suppose that $f(v_3) = 0$ and $f(v_4) = 4$. Since $f(v_1) = 4$, $f(v_2) = 0$ and $f(v_3) = 0$, the only possibility for coloring the vertices between v_2 and v_3 is $\langle 3\ 1\ 4 \rangle$. which is not possible as $f(v_4) = 4$. Therefore T_1 is Type-II and hence T. ■

The following example gives a counterexample for Conjecture 1.1.

Example 2.2. It is clear that the coloring of T_1 given in Fig. 2.1 is an irreducible L (2, 1)-span coloring with two holes.

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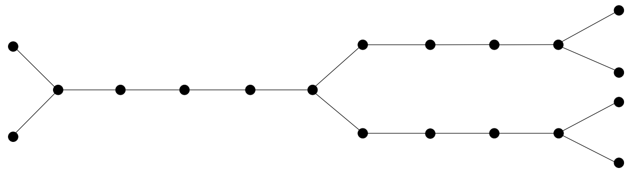
3. Some classes of Type-II trees with maximum number of holes two

Wood and Jacob [5] have given some sufficient conditions for a tree to be Type-II. We consider some of their sufficient conditions as below.

Theorem 3.1 (*[5]*). *A tree T with maximum degree Δ and containing any of the following subtrees is Type-II.*

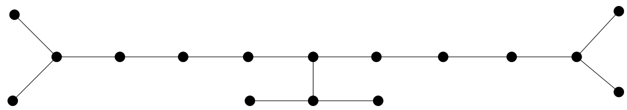
1. $\Delta = 3$,

T_2:



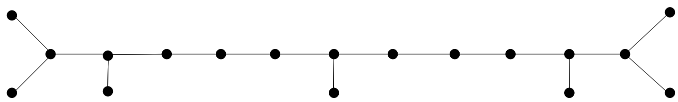
2. $\Delta = 3$,

T_3:



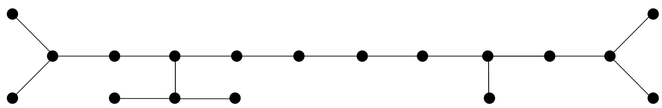
3. $\Delta = 3$,

T_4:



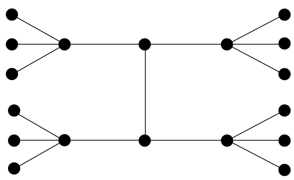
4. $\Delta = 3$,

T_5:



5. $\Delta = 4$,

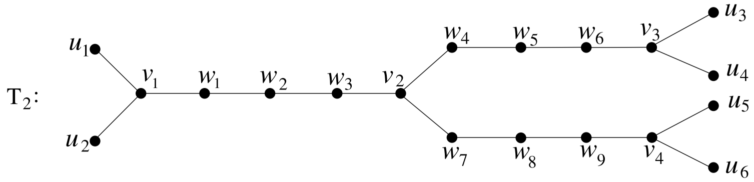
T_6:



Now, we show that $H_\lambda(T_i) = 1$, $2 \leqslant i \leqslant 6$. Later, in this section, we construct some classes of Type-II trees from T_i, $2 \leqslant i \leqslant 6$ with maximum number of holes two. In the following theorem, we prove that there is no irreducible $L(2, 1)$-span coloring with two holes for T_i, $2 \leqslant i \leqslant 6$.

Theorem 3.2. *For the trees T_i, $2 \leqslant i \leqslant 6$, $H_\lambda(T_i) \leqslant 1$.*

Proof. First, we consider T_2 with the following labeling and prove $H_\lambda(T_2) \leqslant 1$.

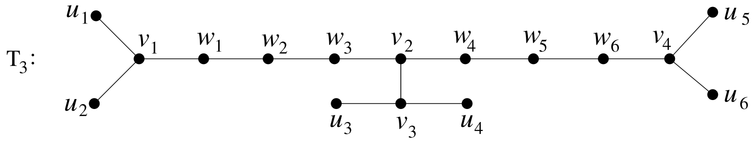


Suppose that T_2 has an irreducible $L(2,1)$-span coloring f with two holes h, h'. Since 0, the span of f and any two consecutive colors cannot be holes, the possibilities for $\{h, h'\}$ are $\{1,3\}$, $\{1,4\}$ and $\{2,4\}$. If $\{h, h'\} = \{1,3\}$, then any major vertex receives only either 0 or 2. If $f(v_1) = 0$, then neighbors of v_1 receives 2, 4 and 5. Since at least one of the pendant vertices u_1 and u_2 receives the color 4 or 5 which is reducible to 3, $f(v_1) \neq 0$. So, $f(v_1)$, $f(v_3)$ and $f(v_4)$ must be 2. If $f(w_1) = 0$, then $f(w_3)$ and $f(v_2)$ must be 2 and 0 respectively (as $f(w_2)$ is either 4 or 5). With this partial coloring there is no possibility for coloring the path $w_4w_5w_6$ (only two colors 4 and 5 are available). If $f(w_1)$ is either 4 or 5. Then $f(w_2)$ and $f(v_2)$ must be 0 and 2 respectively. Since either w_4 or w_7 receives 0, there is no possible coloring for either w_5w_6 or w_8w_9. Therefore, there is no irreducible span coloring of T_2 with holes 1 and 3.

If $\{h, h'\} = \{1,4\}$, then any major vertex receives only 0 or 5. If $f(v_1) = 5$, then one of the colors assigned to u_1 or u_2 reduces to 1. So, $f(v_1)$, $f(v_3)$ and $f(v_4)$ must be 0. If u_1 or u_2 receives 5 then it reduces to a hole 4. Therefore u_1 and u_2 receive the colors 2 and 3, and w_1 receives the color 5. Since $f(w_1) = 5$ and $f(w_2) = 2$ or 3, w_3 and v_2 receive 0 and 5 respectively. With this partial coloring there is no possibility for coloring the path $w_4w_5w_6$ (only two colors 2 and 3 are available). Therefore, there is no irreducible span coloring of T_2 with 1 and 4 as holes.

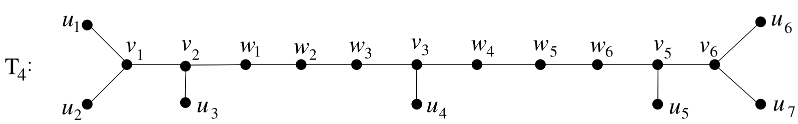
If $\{h, h'\} = \{2,4\}$, then any major vertex receives only 3 or 5. If $f(v_1) = 3$ and $f(w_1)$ is 0 or 1, then $f(w_2) = 5$ and $f(v_2) = 3$. In this case color 5 received by w_2 reduces to a hole 4. If $f(v_1) = 3$ and $f(w_1) = 5$, then $f(w_3)$ and $f(v_2)$ must be 3 and 5 respectively. With this partial coloring the possibilities for coloring the path $w_4w_5w_6v_3$ are $\langle 0\ 3\ 1\ 5\rangle$ and $\langle 1\ 3\ 0\ 5\rangle$. In both the cases, one of the pendant vertices u_3 or u_4 receives 3, which reduces to a hole 2. If $f(v_1) = 5$, then $f(w_1)$ must be 3 otherwise u_1 or u_2 receives 3 which reduces to a hole 2. So, $f(w_3)$ and $f(v_2)$ must be 5 and 3 respectively. With this partial coloring the possibilities for coloring the path $w_4w_5w_6$ are $\langle 0\ 5\ 1\rangle$ and $\langle 1\ 5\ 0\rangle$. In both the cases, the color 5 reduces to a hole 4. Therefore $H_\lambda(T_2) \leqslant 1$.

Now, we consider T_3 with labeling as below.



Suppose that T_3 has an irreducible $L(2,1)$-span coloring f with two holes h, h'. The possibilities for $\{h, h'\}$ are $\{1,3\}$, $\{1,4\}$ and $\{2,4\}$. If $\{h, h'\} = \{1,3\}$, then as in T_2, $f(v_1)$, $f(v_3)$ and $f(v_4)$ must be 2. Since v_2 is adjacent to v_3, $f(v_2) = 0$. With this partial coloring there is no possibility for coloring the path $w_1w_2w_3$. If $\{h, h'\} = \{1,4\}$, then $f(v_1)$, $f(v_3)$ and $f(v_4)$ must be 0. As v_2 and v_3 are adjacent, v_2 receives 5. With this partial coloring there is no possibility for coloring the path $w_1w_2w_3$. If $\{h, h'\} = \{2,4\}$, then any major vertex receives only 3 or 5. If $f(v_3) = 3$ and $f(v_2) = 5$, the possibilities for coloring the path $v_1w_1w_2w_3$ are $\langle 5\ 0\ 3\ 1\rangle$ and $\langle 5\ 1\ 3\ 0\rangle$. In both the cases, u_1 or u_3 receives 3 which reduces to 2. If $f(v_3) = 5$ and $f(v_2) = 3$, then the possibilities to color the path $w_1w_2w_3$ are $\langle 0\ 5\ 1\rangle$ and $\langle 1\ 5\ 0\rangle$. In both the cases, the color 5 reduces to a hole 4. Therefore, there is no irreducible $L(2,1)$-span coloring for T_3 with two holes.

Next, we consider T_4 along with the following labeling.



Suppose that T_4 has an irreducible $L(2,1)$-span coloring f with two holes h, h'. Then $\{h, h'\}$ is $\{1,3\}$ or $\{1,4\}$ or $\{2,4\}$. If $\{h, h'\} = \{1,3\}$, then any major vertex receives only either 0 or 2. Since one of the vertices v_1 or v_2 must

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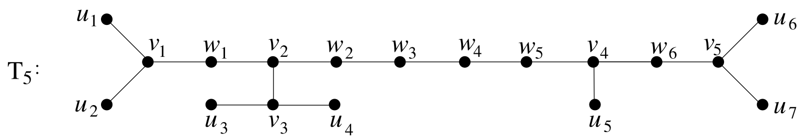
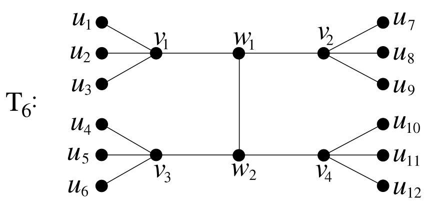
receive 0, a pendant vertex adjacent to it receives 4 or 5 which reduces to a hole 3. If $\{h, h'\} = \{1, 4\}$, then any major vertex receives only either 0 or 5. Since one of the vertices v_1 or v_2 must receive 5, a pendant vertex adjacent to it receives 2 or 3 which reduces to a hole 1. If $\{h, h'\} = \{2, 4\}$, then any major vertex receives only 3 or 5. If $f(v_1) = 3$ and $f(v_2) = 5$, then $f(w_2) = 3$ which implies $f(v_3) = 5$. Since the coloring is irreducible, $f(u_4)$ cannot be 3. With this partial coloring, there is no possibility for coloring the path $w_4w_5w_6$. If $f(v_1) = 5$ and $f(v_2) = 3$ then $f(w_2) = 5$ which reduces to 4. Therefore $H_\lambda(T_4) \leqslant 1$.

Now, consider T_5 with labeling as below.



Suppose that T_5 has an irreducible $L(2, 1)$-span coloring f with two holes h, h'. The possibilities for $\{h, h'\}$ are $\{1, 3\}$, $\{1, 4\}$ and $\{2, 4\}$. If $\{h, h'\} = \{1, 3\}$, then as in T_2, $f(v_1)$, $f(v_3)$ and $f(v_5)$ must be 2, and $f(v_2) = f(v_4) = 0$. With this partial coloring there is no possibility for coloring the path $w_2w_3w_4w_5$. If $\{h, h'\} = \{1, 4\}$, then as in T_2, $f(v_1)$, $f(v_3)$ and $f(v_5)$ must be 0. Then one of the pendant vertices u_1 and u_2 receives the color 5 which is reducible to 4. If $\{h, h'\} = \{2, 4\}$, then any major vertex receives only 3 or 5. If v_1 receives 5, then $f(v_2) = 3$ and one of the pendant vertices u_1 and u_2 receives the color 3 which is reducible to 2. Therefore $f(v_1)$ and $f(v_5)$ must be 3 and hence $f(v_2) = 5$, $f(v_3) = 3$ and $f(v_4) = 5$. With this partial coloring there is no possibility to color the path $w_2w_3w_4w_5$. Therefore $H_\lambda(T_5) \leqslant 1$.

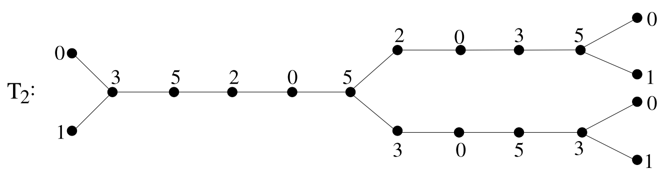
Finally, we consider the tree T_6 along with labeling as below.

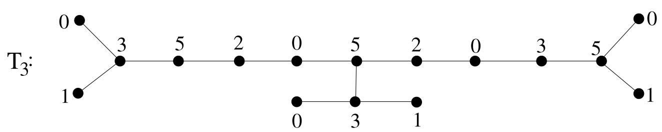


Suppose T_6 has an irreducible $L(2, 1)$-span coloring f with holes h, h'. The possibilities for $\{h, h'\}$ are $\{1, 3\}$, $\{1, 5\}$ and $\{3, 5\}$. If $\{h, h'\} = \{1, 3\}$, then any major vertex receives 0 or 2 only. Without loss of generality, we assume $f(v_1) = 0$ and $f(v_2) = 2$. Then one of the pendant vertices adjacent to v_1 must receive 4 which reduces to 3. Similarly one can see that other two cases are not possible. Therefore $H_\lambda(T_6) \leqslant 1$. ■

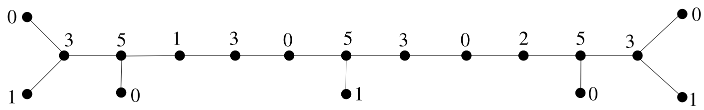
Theorem 3.3. *For the trees T_i, $2 \leqslant i \leqslant 6$, $H_\lambda(T_i) = 1$.*

Proof. From Theorem 3.2, we have $H_\lambda(T_i) \leqslant 1$, $2 \leqslant i \leqslant 6$. It is easy to see that the colorings of T_i, $2 \leqslant i \leqslant 6$ given below are irreducible $L(2, 1)$-span colorings with one hole. Therefore $H_\lambda(T_i) = 1$, $2 \leqslant i \leqslant 6$.

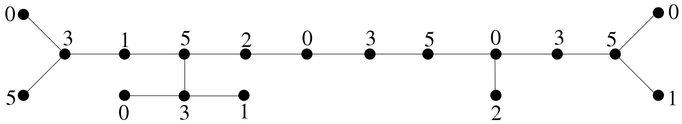




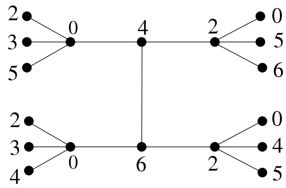
T_4:



T_5:



T_6:



■

3.1. *Construction of Type-II trees with maximum number of holes two*

We start this subsection with a lemma that gives a Type-II tree with maximum number of holes two from a Type-II tree with a two hole $L(2,1)$-span coloring without changing the span. Later, we apply this lemma on trees T_i, $1 \leqslant i \leqslant 6$ to construct infinitely many trees with maximum number of holes two.

Lemma 3.4. *If T is a Type-II tree and T has a two hole reducible span coloring, then there exists a tree T' such that T is a subtree of T', $\lambda\left(T'\right) = \lambda(T)$ and $H_\lambda\left(T'\right) = 2$.*

Proof. Let T be a Type-II tree with maximum degree Δ and let f be a reducible $L(2,1)$-span coloring of T with two holes h and h'. Without loss of generality $h < h'$. Now, we give a procedure to construct T' from T.

Step-I: Whenever a vertex color reduction is possible to a color other than hole, we reduce the color. Finally, f is an $L(2,1)$-span coloring of T with no vertex color can be reduced to a color other than hole.

Step-II: Suppose that a color received by a minor vertex u reduces to h'. Let k be the order of the set $S_u = \{c : 0 \leqslant c < h', c \neq h$ and c is not the color assigned to any neighbor of u$\}$. Now we attach k new pendant vertices and we assign them the k colors from S_u. Apply this procedure to all the minor vertices whose color reduces to h'.

Step-III: We follow the procedure of Step-II for all the minor vertices in the tree obtained from Step-II by replacing h' by h. Let T' obtained finally.
Next, we show that the coloring of T' is an irreducible span coloring. From Step-I, Step-II and Step-III, it is clear that none of the minor vertex colors is reducible. Now, we prove that any major vertex color is not reducible to a hole. Suppose that a color l assigned to a major vertex u reduces to a hole l'. It is clear that $l \neq 0, 1$. If $2 \leqslant l \leqslant \Delta + 1$, then none of the colors $l'-1, l', l'+1, l-1, l$ and $l+1$ can be given to the neighbors of u. Since in any case the colors $l'-1, l', l'+1, l-1, l$ and $l+1$ are at least 4, it is not possible to color the Δ neighbors of u. If $l = \Delta + 2$, then $l' < \Delta + 1$ is not possible as above. So, $l = \Delta + 2$ and $l' = \Delta + 1$. Since $0, 1, 2, \ldots, \Delta - 1$ are used to color the Δ neighbors of u, the other hole must be Δ which is not possible as Δ and $\Delta + 1$ are consecutive. ■

Theorem 3.5. *There are infinitely many Type-II trees with $\Delta = 3$ and maximum number of holes two.*

Proof. Since the trees T_i, $2 \leqslant i \leqslant 5$ have reducible $L(2,1)$-span colorings with two holes, we apply Lemma 3.4 and get the following trees T_i', $2 \leqslant i \leqslant 5$ with irreducible span colorings f_i, $2 \leqslant i \leqslant 5$ with two holes.

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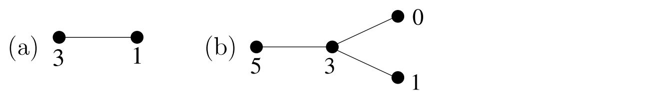
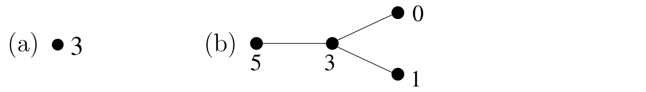
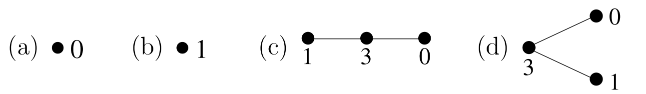
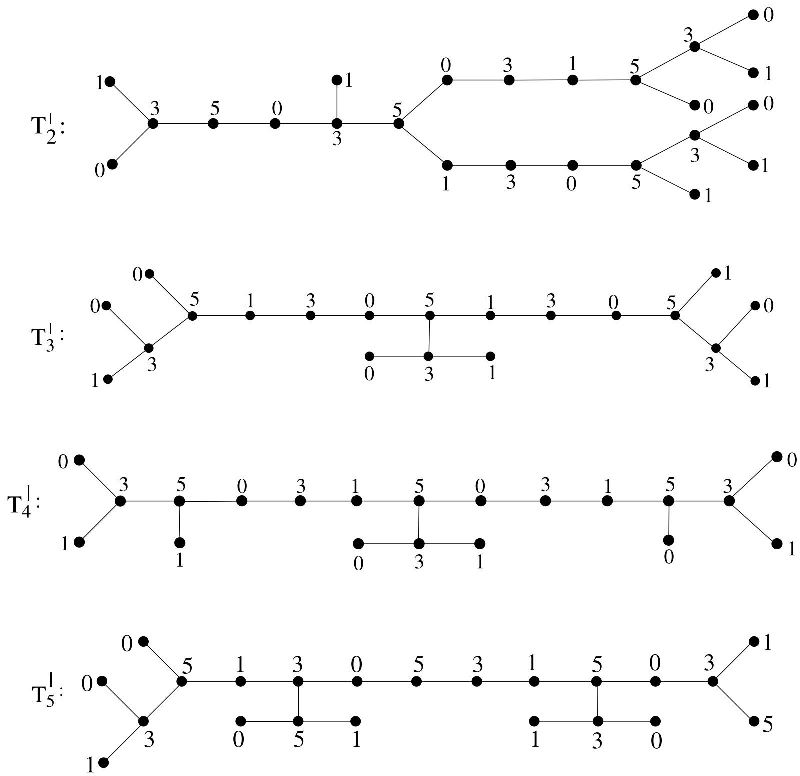
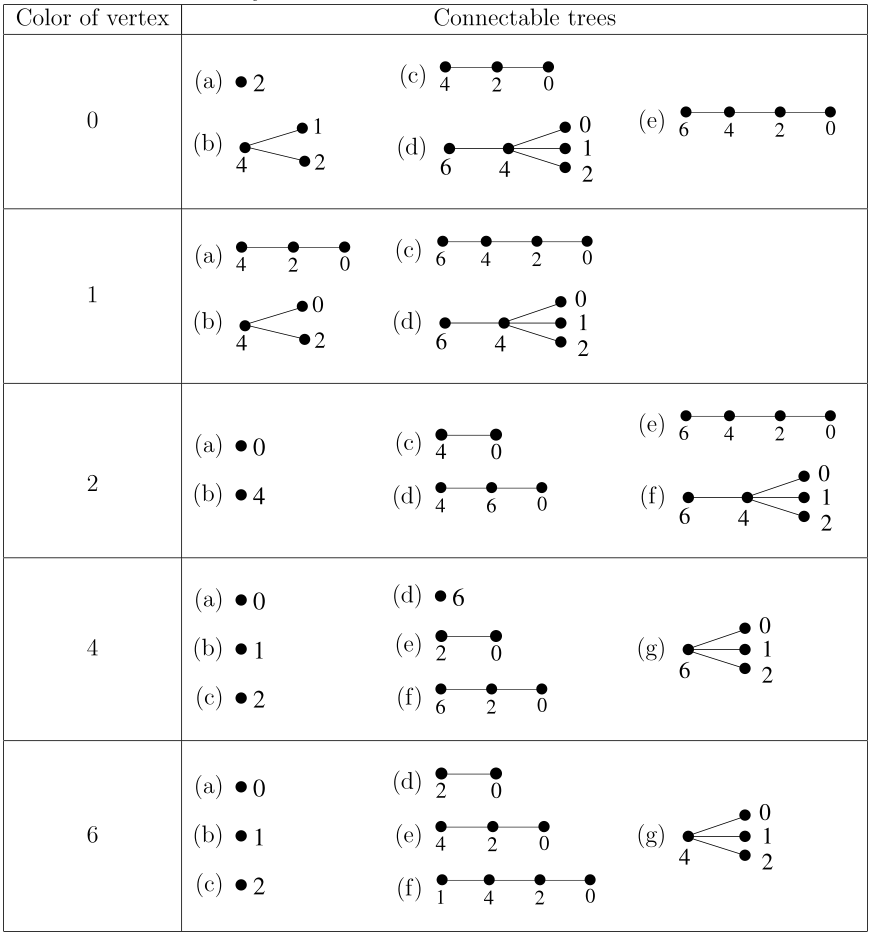
Table 3.1
Trees connectable to the trees T_i', $1 \leqslant i \leqslant 5$.

Color of vertex	Connectable trees
0	
1	
3	
5	



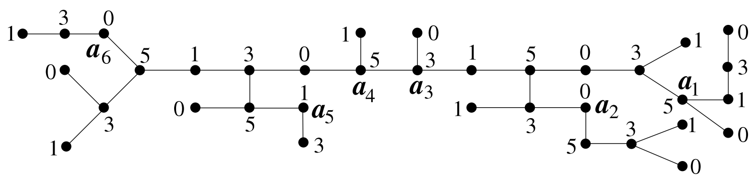
Let f_1 be the irreducible $L(2, 1)$-span coloring of T_1 given in Example 2.2 and let $T_1' = T_1$. Now, we construct infinitely many Type-II trees with $\Delta = 3$ and maximum number of holes two from T_i', $1 \leqslant i \leqslant 5$. When we say connecting two trees, we mean adding an edge between them. Let c be a color of a vertex u in T_i', $1 \leqslant i \leqslant 5$. Depending on the colors of neighbors of u, we connect the trees (one at a time) given in Table 3.1 corresponding to the color c with the first vertex to T_i' at u. In the case of connecting a single vertex, first we connect the smallest colored vertex to maintain irreducibility. It is easy to see that after every step the tree obtained is Type-II with maximum

Table 3.2
Trees connectable to the tree T_6'.



number of holes two and $\Delta = 3$. Also, since at every step, connecting of trees to the pendant vertices is possible, we get infinitely many trees. ■

Example 3.6. In this example, we give an illustration of Theorem 3.5 for the tree T_5'. The vertex a_1 has the color 5 and its neighbor's color is 3 in T_5'. From Theorem 3.5, there are only two possibilities (a) and (c) corresponding to color 5 in Table 3.1 out of which (a) is connected first. Later, out of the two possibilities, (b) and (c) for the vertex a_1, (c) is connected. Similarly, some trees are connected for the vertices a_i, $1 \leqslant i \leqslant 6$.



Theorem 3.7. *There are infinitely many Type-II trees with $\Delta = 4$ and maximum number of holes two.*

Proof. We apply Lemma 3.4 on T_6 to get figure T_6' with coloring f_6 as below. Rest of the proof is similar to that of Theorem 3.5 and using Table 3.2. ■

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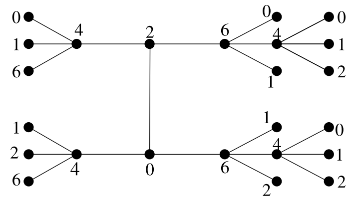
T_6^1 :



Remark. Tables 3.1 and 3.2 are obtained using the concept in the proof of Lemma 3.4. It is easy to see that connecting a tree (not a tree obtained by connecting some trees from the table) to T_i', $1 \leqslant i \leqslant 6$ other than the trees listed in the tables, produces a reducible $L(2, 1)$-span coloring for the resultant tree. Therefore, the class of trees generated from the tables is complete with respect to f_i, $1 \leqslant i \leqslant 6$. Changing two hole coloring of T_i', $1 \leqslant i \leqslant 6$ produces different class of Type-II trees with maximum number of holes two.

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