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Degree associated edge reconstruction number of split graphs with biregular independent set is one

N. Kalai Mathi and S. Monikandan

Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamil Nadu, India

ABSTRACT

A degree associated edge card of a graph G is an edge deleted subgraph of G with which the degree of the deleted edge is given. The degree associated edge reconstruction number of a graph G (or dern(G)) is the size of the smallest collection of the degree associated edge cards of G that uniquely determines G. A split graph G is a graph in which the vertices can be partitioned into an independent set and a clique. We prove that the dern of all split graphs with biregular independent set is one.

KEYWORDS

Reconstruction; split graphs; card; deck

AMS SUBJECT CLASSIFICATION (2010)

Primary 05C60; Secondary 05C07

1. Introduction

All graphs considered are nonempty, simple and finite. We shall mostly follow the graph theoretic terminology of [5]. A *vertex-deleted subgraph* or *card* $G - v$ of a graph (digraph) G is the unlabeled graph (digraph) obtained from G by deleting the vertex v and all edges (arcs) incident with v. The *deck* of a graph (digraph) G is the collection of all cards of G. Following the formulation in [2], a graph (digraph) G is reconstructible if it can be uniquely determined from its deck. The well-known Reconstruction Conjecture (RC), due to Kelly [9] and Ulam [23], asserts that every graph with at least three vertices is reconstructible. Several classes of graphs and many parameters of graphs are proved to be reconstructible. Nevertheless, the full conjecture remains open. Surveys of results on the RC and related problems include [4, 12, 17]. In their paper, Harary and Plantholt [7] have defined the *reconstruction number* of a graph G, denoted by $rn(G)$, to be the minimum number of cards which can only belong to the deck of G and not to the deck of any other graph H, $H \ncong G$, these cards thus uniquely identifying G. Reconstruction number is known for only few classes of graphs [2].

An extension of the RC to digraphs is the *Digraph Reconstruction Conjecture* (DRC), proposed by Harary [6], which asserts that every digraph with at least seven vertices is reconstructible. The DRC was disproved by Stockmeyer [22] by exhibiting several infinite families of counter-examples. In 1981, Ramachandran [18] studied degree associated reconstruction for digraphs and proposed a new conjecture. It was proved [18] that the digraphs in all these counterexamples to the DRC obey the new conjecture.

The ordered triple (a, b, c) where a, b and c are respectively the number of unpaired outarcs, unpaired inarcs and symmetric pair of arcs incident with v in a digraph D is called the *degree triple of v*. The degree associated card or dacard of a digraph (graph) is a pair (d, C) consisting of a card C and the degree triple (degree) d of the deleted vertex. The *dadeck* of a digraph D is the multiset of all the dacards of D. A digraph is said to be *N-reconstructible* if it can be uniquely determined from its dadeck. The *new digraph reconstruction conjecture* [18] (NDRC) asserts that all digraphs are N-reconstructible. Ramachandran [19, 20] then studied the degree associated reconstruction number of graphs and digraphs in 2000. The *degree* (*degree triple*) *associated reconstruction number* of a graph (digraph) D is the size of the smallest collection of dacards of D that uniquely determines D. Articles [1, 3] and [11] are recent papers on this parameter.

The *edge card, edge deck, edge reconstructible graphs* and *edge reconstruction number* are defined similarly with edge deletions instead of vertex deletions. The *edge reconstruction conjecture*, proposed by Harary [6], states that all graphs with at least 4 edges are edge reconstructible. The ordered pair $(d(e), G - e)$ is called a *degree associated edge card* or *da-ecard* of the graph G, where $d(e)$ (called the *degree* of e) is the number of edges adjacent to e in G. The *edeck* (*da-edeck*) of a graph G is the collection of all ecards (da-ecards) of G. For an edge reconstructible graph G, the *edge reconstruction number* of G is defined to be the size of the smallest subcollection of the edeck of G which is not contained in the edeck of any other graph H, $H \ncong G$. The edge reconstruction number is known for only few classes of graphs [13, 14]. For an edge reconstructible graph G from its

CONTACT S. Monikandan monikandans@gmail.com Department of Mathematics, Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, 627012, Tamil Nadu, India.

da-edeck, the *degree associated edge reconstruction number* of a graph G, denoted by $dern(G)$, is the size of the smallest subcollection of the da-edeck of G which is not contained in the da-edeck of any other graph H, $H \not\cong G$. Articles [10, 15] and [16] are recent papers on this parameter.

A *split graph* G is a graph in which the vertices can be partitioned into an independent set (say X) and a clique (say Y). Throughout this paper, we use G, X and Y in the sense of this definition. In their paper, Ramachandran and Monikandan [21] proved a reduction on the RC that the family $\mathscr{F}$ of all 2-connected graphs G with $diam(G) = 2$ or $diam(G) = diam(\bar{G}) = 3$ is reconstructible if and only if all graphs are reconstructible. So, any result proving or determining the possibility of the (edge) reconstructibility of a subclass of $\mathscr{F}$ is of interest. In this paper, as all connected split graphs without end vertices belong to $\mathscr{F}$, we shall determine the *dern* of split graphs. We prove that the *dern* of all split graphs with biregular independent set is one.

2. Dern of split graphs

For a da-ecard (d, C), let $dsum(C) = \{degu + degv : u$ and v are nonadjacent vertices in C and $degu + degv$ is possibly equal to $d\}$. For the sake of clarity of the proof, even though the elements in $dsum(C)$ have no specific ordering, we denote the first element, second element and so on in $dsum(C)$ by $d_1, d_2, \ldots,$ respectively. For $U, W \subseteq V(G)$, the set of edges of G that join a vertex in U to a vertex in W is denoted by $E(U, W)$. By $N_X(U)$, we mean the set of all vertices in X that are adjacent to a vertex in a subset U of Y. Let $|X| = m > 0$ and $|Y| = n > 0$. For $i = 1, 2$, let X_{r_i} denote the set of vertices in X of degree r_i. For $i = 0, 1, \ldots, m$, let Y_i denote the set of vertices in Y adjacent to exactly i vertices in X; therefore, the degree of a vertex $v \in Y_i$ is $n - 1 + i$ in G.

Let $k_1, k_2, \ldots, k_t$ be integers with $1 \leq k_1 < k_2 < \ldots < k_t \leq m$ such that $Y_{k_i} \neq \phi$ for all $i = 1, 2, \ldots, t$ and $Y = Y_0 \cup (\cup_{i=1}^{t} Y_{k_i})$. *For expediency, we shall write sometimes a da-ecard as* $(d_1 + d_2 - 2, G - e)$, *which indicates that the deleted edge e is joined to a* d_1*-vertex and a* d_2*-vertex in G* (Figure 1).

Lemma 1. *The dern of a graph G is 1 if G has a da-ecard (d, C) containing only one pair of nonadjacent vertices whose degree sum is d.*

Proof. Such a da-ecard has a unique extension and it is isomorphic to G. □

Lemma 2. *If G is a split graph such that every vertex in X is of degree zero in G, then* $dern(G) = 1$.

Proof. Now G is $K_n \cup \bar{K}_m$ and all its da-ecards are $(2n - 4, G - e)$. Since $|E(G)| \geq 4$ and $E(X, Y) = \phi$, it follows that $n \geq 4$. Since $G - e$ contains exactly one pair of nonadjacent vertices with degree sum $2n - 4$, every extension $H(2n - 4, G - e)$ obtained by adding a new edge that joins the two nonadjacent vertices of degree sum $2n - 4$ is isomorphic to G and hence $dern(G) = 1$. □

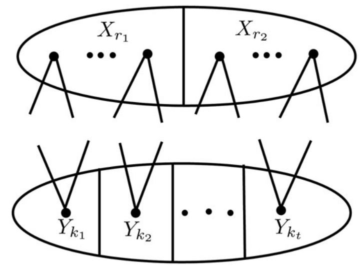

Figure 1. The split graph G.

Theorem 3. *Let G be a split graph such that every vertex in X is of degree* r_1 *or* r_2. *Then* $dern(G) = 1$ *if every vertex in Y is adjacent to at least one vertex in X.*

Proof. Now $Y = \cup_{i=1}^{t} Y_{k_i}$ since $Y_0 = \phi$. We proceed by two cases depending upon the cardinality of Y_{k_t}. By Lemma 2, we can take that $0 < r_1 < r_2$.

Case 1. $|Y_{k_t}| = 1$

Since $k_1 \geq 1$, we have $k_t \geq 2$ and we proceed by two cases depending upon the values of k_1.

Case 1.1. $k_1 = 1$

We proceed by three subcases depending upon the values of r_2.

Case 1.1.1. $r_2 = n$

If $r_1 = n - 1$, then, since $k_1 = 1$, it follows that $|X_n| = 1$. Also, since $r_1 = n - 1$, we have $|Y_1| = 1$ and $Y = Y_1 \cup Y_m$. Clearly $|Y_m| = 1$, $n = 2$ and $r_1 = 1$. Consider a da-ecard $(2 + (m + 1) - 2, G - e)$, where $e \in (Y_1, Y_m)$. Then $dsum(G - e) = \{1 + 1,\ 1 + 2,\ 1 + m\}$. If d_1 were equal to $m + 1$, then m would be one, giving a contradiction. If d_2 is equal to $m + 1$, then $m = 2$ and the two 1-vertices of $G - e$ have the same neighbourhood in $G - e$. Hence the extension $H(3, G - e)$ is isomorphic to G and hence $dern(G) = 1$. Finally, if d_3 is equal to $m + 1$, then, since there is only one such pair of non adjacent vertices in $G - e$ $(m > 2)$, $dern(G) = 1$ by Lemma 1.

Now suppose that $r_1 \leq n - 2$. Since $|Y_{k_t}| = 1$ and $k_1 = 1, k_t \geq 2$ and $|X_n| = 1$. Consider a da-ecard $(n + (n - 1 + k_t) - 2, G - e)$, where $e \in E(Y_1, Y_{k_t})$. Then $dsum(G - e) = \{r_1 + r_1,\ r_1 + n,\ r_1 + (n - 1),\ r_1 + (n - 1 + k_t - 1),\ r_1 + (n - 1 + k_1),\ r_1 + (n - 1 + k_2), \ldots,\ r_1 + (n - 1 + k_{t-1}),\ (n - 1) + (n - 1 + k_t - 1)\}$. If d_1, d_2 or d_3 were equal to $d(e)$, then k_t would be strictly less than 2, giving a contradiction. If d_4 were equal to $d(e)$, then r_1 would be $n - 1$, again a contradiction. If d_i were equal to $d(e)$ for $i = 5, 6, \ldots, t + 3$, then k_t would be at most k_j, $j = 1, 2, \ldots, t - 1$, again a contradiction. But for the last element d_{t+4} in $dsum(G - e)$, since there is only one such pair of non adjacent vertices in $G - e$, it follows that $dern(G) = 1$ by Lemma 1.

Case 1.1.2. $r_2 = n - 1$

First, suppose that $k_t \neq k_{t-1} + 1$. Now we consider a da-ecard $(n + (n - 1 + k_t) - 2, G - e)$, where $e \in E(Y_1, Y_{k_t})$. Then $dsum(G - e) = \{r_1 + r_1,\ r_1 + (n - 1),\ (n - 1) + (n - 1),\ r_1 + (n - 1 + k_t - 1),\ (r_1) + (n - 1 + k_1),\ (r_1) + (n -$

$1+k_2), \ldots, \ (r_1)+(n-1+k_{t-1}), \ (n-1)+(n-1+k_1), \ (n-1)+(n-1+k_2), \ \ldots, (n-1)+(n-1+k_{t-1}), \ (n-1)+(n-1+k_t-1)\}$. If $d_1, \ d_2$ or d_3 were equal to $d(e)$, then k_t would be at most one, a contradiction. If d_4 were equal to $d(e)$, then r_1 would be $n-1$, again a contradiction. If d_i were equal to $d(e)$ for $i=5,6,\ldots,t+3$, then k_t would be at most $k_j, \ j=1,2,\ldots,t-1$, contradicting. Similarly, if d_i was equal to $d(e)$ for $i=t+4,t+5,\ldots,2t+2,$, then k_t would be equal to $k_j+1, \ j=1,2,\ldots,t-1$, again a contradiction. Finally, if the last degree sum d_{2t+3} in $dsum(G-e)$ is equal to $d(e)$, then either there is only one such pair of nonadjacent vertices in $G-e$ or the two $(n-1)$-vertices of $G-e$ have the same neighbourhood in $G-e$. Hence the extension $H(2n+k_t-3)$ is isomorphic to G and $dern(G)=1$.

Next, assume that $k_t=k_{t-1}+1$ and $k_t\geq 3$. Consider a da-ecard $((n-1+k_{t-1})+(n-1+k_t)-2, G-e)$, where $e\in E(Y_{k_{t-1}}, Y_{k_t})$. Then $dsum(G-e)=\{r_1+r_1, \ r_1+(n-1), \ r_1+(n-1+k_{t-1}-1), \ r_1+(n-1+k_t-1), \ (n-1)+(n-1), \ (n-1)+(n-1+k_{t-1}-1), \ (n-1)+(n-1+k_t-1), \ r_1+(n-1+k_1), \ r_1+(n-1+k_2), \ \ldots, r_1+(n-1+k_{t-1}), (n-1)+(n-1+k_1), (n-1)+(n-1+k_2), \ldots, (n-1)+(n-1+k_{t-1}), \ (n-1+k_{t-1}-1)+(n-1+k_t-1)\}$. If d_i was equal to $d(e)$ for $i=1,2,\ldots,7$, then k_t would be at most two, a contradiction. If d_i was equal to $d(e)$ for $i=8,9,\ldots,t+6$, then $2k_t$ would be at most $k_j+2, \ j=1,2,\ldots,t-1$, again a contradiction. If d_i was equal to $d(e)$ for $i=t+7,t+8,\ldots,2t+5$, then $2k_t$ would be equal to $k_j+3, \ j=1,2,\ldots,t-1$, again a contradiction to $|Y_{k_t}|=1, \ k_t\geq 3$ and $k_t>k_j$. Finally, if the last degree sum in $dsum(G-e)$ is equal to $d(e)$, then $dern(G)=1$ by Lemma 1, since there is only one such pair of nonadjacent vertices in $G-e$.

Finally, we consider the case that $k_t=k_{t-1}+1$ and $k_t=2$. Clearly, the graph G has the following properties:

(i) $n\geq 3$;
(ii) $Y=Y_1\cup Y_2$, where $|Y_2|=1$;
(iii) $|X_{n-1}|=1$ (because $|Y_2|=1$ and $|X_{r_1}|\geq 1$); and
(iv) $|X_{r_1}|=1$ or 2 (as otherwise, $|X_{r_1}|\geq 3$ and so $|Y_2|\geq 2$ or $k_t\geq 3$, a contradiction).

If $|X_{r_1}|=1$, then clearly $r_1=2$. Therefore $r_2=n-1\geq 3$ and hence $n\geq 4$. Now consider a da-ecard $(n+(n+1)-2, G-e)$, where $e\in E(Y_1,Y_2)$. Clearly $dsum(G-e)=\{2+(n-1), \ 2+n, \ (n-1)+(n-1), \ (n-1)+n\}$. If d_1 or d_2 was equal to $(n+(n+1)-2)$, then n would be at most 3, a contradiction. If d_3 was equal to $d(e)$, then we would have $-1=0$, contradicting. But the last degree sum in $dsum(G-e)$, since there is only one such pair non adjacent vertices in $G-e$, it follows that $dern(G)=1$ by Lemma 1. *Otherwise*, $|X_{r_1}|=2$. Now $r_1=1$ and the two vertices in X_1 are adjacent either to the same vertex or to two different vertices in Y. Consider a da-ecard $(n+(n+1)-2, G-e)$, where $e\in E(Y_1,Y_2)$. Clearly $dsum(G-e)=\{1+1, \ 1+(n-1), \ 1+n, \ (n-1)+n\}$. If d_1, d_2 or d_3 was equal to $d(e)$, then n would be at most 2, a contradiction. For the last degree sum in $dsum(G-e)$, either only one such pair of nonadjacent vertices in $G-e$ or the two $(n-1)$-vertices of $G-e$ have the same neighbourhood and so the extension $H(2n-1)$ is isomorphic to G. Hence $dern(G)=1$.

Case 1.1.3. $r_2\leq n-2$

Consider the da-ecard $(n+(n-1+k_t)-2, G-e)$, where $e\in E(Y_1, Y_{k_t})$. Clearly, $dsum(G-e)=\{r_1+r_1, \ r_1+r_2, \ r_2+r_2, \ r_1+(n-1), r_2+(n-1), r_1+(n-1+k_t-1), \ r_2+(n-1+k_t-1), \ r_1+(n-1+k_1), \ r_1+(n-1+k_2), \ \ldots, r_1+(n-1+k_{t-1}), \ r_2+(n-1+k_1), \ r_2+(n-1+k_2), \ \ldots, \ r_2+(n-1+k_{t-1}), \ (n-1)+(n-1+k_t-1)\}$. If d_i was equal to $d(e)$ for $i=1,2,3,4,5$, then, since $r_1\leq n-3$ and $r_2\leq n-2$, we have k_t would be at most zero, a contradiction. If d_i was equal to $d(e)$ for $i=6,7$, then both the values of r_1 and r_2 would be $n-1$, a contradiction. If d_i was equal to $d(e)$ for $i=8,9,\ldots,t+6$, then k_t would be at most $k_j-1, \ j=1,2,\ldots,t-1$, again a contradiction. Similarly, for each of $d_{t+7}, \ d_{t+8}, \ \ldots, \ d_{2t+5}$, the value of k_t would be at most $k_j, \ j=1,2,\ldots,t-1$, again a contradiction. But for the last degree sum in $dsum(G-e)$, since there is only one such pair of nonadjacent vertices in $G-e$, it follows that $dern(G)=1$ by Lemma 1.

Case 1.2. $k_1>1$

Since $k_1\geq 2$, we have $k_t\geq 3$ and $k_1+k_t\geq 5$. Consider the da-ecard $((n-1+k_1)+(n-1+k_t)-2, G-e)$, where $e\in E(Y_{k_1}, Y_{k_t})$. Clearly, $dsum(G-e)=\{r_1+r_1, \ r_1+r_2, \ r_2+r_2, \ r_1+(n-1+k_1-1), \ r_2+(n-1+k_1-1), \ r_1+(n-1+k_t-1), \ r_2+(n-1+k_t-1), \ r_1+(n-1+k_1), \ r_1+(n-1+k_2), \ \ldots, r_1+(n-1+k_{t-1}), \ r_2+(n-1+k_1), \ r_2+(n-1+k_2), \ \ldots, r_2+(n-1+k_{t-1}), \ (n-1+k_1-1)+(n-1+k_t-1)\}$. If d_i was equal to $d(e)$ for $i=1,2,3$, then, since $r_1\leq n-1$ and $r_2\leq n$, the value of k_1+k_t would be at most four, again a contradiction. If d_i was equal to $d(e)$ for $i=4,5$, then the value of k_t would be at most one, a contradiction. If d_i was equal to $d(e)$ for $i=6,7$, then the value of k_1 would be at most one, a contradiction. If d_i was equal to $d(e)$ for $i=8,9,\ldots,t+6$, then, the value of k_t would be at most k_j-k_1+2, since $k_j<k_t, \ j=1,2,\ldots,t-1, \ k_1\leq 1$, again a contradiction. Similarly, for each of the degree sum $d_{t+8}, \ d_{t+9}, \ \ldots, \ d_{2t+6},$, we would have $k_t\leq k_j-k_1+3, \ j=1,2,\ldots,t-1$. If $k_1\geq 3$ then k_t would be less than or equal to $k_j, \ j=1,2,\ldots,t-1$, a contradiction. Otherwise, $k_1=2$. Now $k_t\leq k_j+1$ and $k_t=k_j+1, \ j=1,2,\ldots,t-1$, which implies $r_2=n$. This is not possible, because no nonadjacent pair of r_2-vertex and $(n-1+k_j)$-vertex exists $(j=1,2,\ldots,t-1)$. For the last degree sum in $dsum(G-e)$, since there is only one such pair of non adjacent vertices in $G-e$, it follows that $dern(G)=1$ by Lemma 1.

Case 2. $|Y_{k_t}|\geq 1$

We proceed by three cases depending upon the values of k_t.

Case 2.1. $k_t=1$

Clearly $Y=Y_1$ and $r_2\leq n-1$ (as otherwise, X_{r_1} would be empty, contradicting). Hence we proceed by three subcases depending on the possible values of r_2.

Case 2.1.1. $r_2=n-1$

Clearly $|X_{r_1}| = 1, r_1 = 1, |X_{n-1}| = 1$ and $n \geq 3$. For this case, the da-ecard we consider is $(1 + n - 2, G - e)$, where $e \in E(X_1, Y_1)$. Clearly $dsum(G - e) = \{0 + (n - 1),\ 0 + (n),\ (n - 1) + (n - 1)\}$. If d_1 is equal to $d(e)$, then, since two $(n - 1)$- vertices of $G - e$ have the same neighbourhood in $G - e$, the extension $H(1 + n - 2)$ is isomorphic to G and hence $dern(G) = 1$. If d_2 was equal to $d(e)$, then we would have $0 = -1$, a contradiction. Similarly, If d_3 was equal to $d(e)$, then n would be one, again a contradiction.

Case 2.1.2. $r_2 = n - 2$

Since $k_t = 1$ and $|X_{r_1}| \neq \phi$, we have $|X_{n-2}| = 1$ and $|X_{r_1}| = 1$ or 2.

If $|X_{r_1}| = 1$, then $r_1 = 2$ and $n \geq 5$. For this case, the da-ecard we consider is $(2 + n - 2, G - e)$, where $e \in E(X_2, Y_1)$. Clearly $dsum(G - e) = \{1 + n,\ 1 + (n - 2),\ (n - 2) + n,\ (n - 2) + (n - 1),\ 1 + (n - 1)\}$. If d_1 or d_2 was equal to $d(e)$, then we would have $0 = 1$ or $0 = -1$, a contradiction. If d_3 or d_4 was equal to $d(e)$, then the value of n would be 2 or 3, again a contradiction. If d_5 is equal to $d(e)$, then, since there is only one such pair of nonadjacent vertices in $G - e$, it follows that $dern(G) = 1$, by Lemma 1. Now suppose that $|X_{r_1}| = 2$. Clearly $r_1 = 1$ and $n \geq 4$. We consider a da-ecard $(n + n - 2, G - e)$, where $e = u_1u_2 \in E(Y_1, Y_1)$ such that both u_1 and u_2 are adjacent to none of the $(n - 2)$-vertices in X. Clearly $dsum(G - e) = \{1 + 1,\ 1 + (n - 2),\ 1 + (n - 1),\ 1 + n,\ (n - 2) + (n - 1),\ (n - 1) + (n - 1)\}$. If d_i was equal to $d(e)$ for $i = 1, 2, 3, 4,$ then n would be at most three, a contradiction. If d_5 was equal to $d(e)$, then we would have $-1 = 0$, again a contradiction. But, for $d_6 = d(e)$, since there is only one such pair of non adjacent vertices in $G - e$, we have $dern(G) = 1$.

Case 2.1.3. $r_2 \leq n - 3$

We consider a da-ecard $(n + n - 2, G - e)$, where $e \in E(Y_1, Y_1)$. Then $dsum(G - e) = \{r_1 + r_1,\ r_1 + r_2,\ r_2 + r_2,\ r_1 + (n - 1),\ r_1 + n,\ r_2 + (n - 1),\ r_2 + n,\ (n - 1) + (n - 1)\}$. If d_i was equal to $(n + n - 2)$ for $i = 1, 2, \ldots, 7,$ then, since $r_1 \leq n - 4$ and $r_2 \leq n - 3$, the value of r_1 would be $n - 1$ or $n - 2$ and the value of r_2 would be $n - 1$, $n - 2$ or at least $n + 2$, contradicting. For $d_7 = d(e)$, since there is only one such pair of nonadjacent vertices in $G - e$, it follows that $dern(G) = 1$.

Case 2.2. $k_t = 2$

We proceed three cases depending upon the values of r_2.

Case 2.2.1. $r_2 = n$

Since $k_t = 2$, we have $|X_{r_2}| = |X_n| = 1$.

If $r_1 = n - 1$ and $n \geq 3$, then $|X_{n-1}| = 1$, $m = 2$, $|Y_1| = 1$ and $Y = Y_1 \cup Y_2$. Now consider a da-ecard $((n + 1) + (n + 1) - 2, G - e)$, where $e \in E(Y_2, Y_2)$. Clearly $dsum(G - e) = \{(n - 1) + n,\ n + n\}$. If d_1 was equal to $d(e)$, then we would have $-1 = 0$, a contradiction. If $d_2 = d(e)$, then $dern(G) = 1$, since there is only one such pair of nonadjacent vertices in $G - e$.

Next, suppose that $r_1 = n - 1$ and $n = 2$. Then $|X_{n-1}| \leq 2$ and $m = 2$ or 3. If $m = 2$, then $Y = Y_1 \cup Y_2, |Y_1| = 1$ and $|Y_2| = 1$, a contradiction. If $m = 3$, then we consider a da-ecard $(3 + 3 - 2, G - e)$, where $e \in E(Y_2, Y_2)$. Clearly $dsum(G - e) = \{1 + 1, 1 + 2, 2 + 2\}$ and the only possible degree sum is d_3. But in this case $dern(G) = 1$, since there is only one such pair of non adjacent vertices in $G - e$.

Finally, we consider the case that $r_1 \leq n - 2$. Now we consider a da-ecard $((n + 1) + (n + 1) - 2, G - e)$, where $e \in E(Y_2, Y_2)$. Then $dsum(G - e) = \{r_1 + r_1,\ r_1 + n,\ r_1 + (n + 1),\ n + n\}$. If d_i was equal to $d(e)$ for $i = 1, 2, 3,$ then r_1 would be $n - 1$ or n, a contradiction. If $d_4 = d(e)$, then $dern(G) = 1$, since there is only one such pair of nonadjacent vertices in $G - e$.

Case 2.2.2. $r_2 = n - 1$

Now $n \geq 3$ and $|X_{n-1}| \leq 2$ (as otherwise, $|X_{r_1}| = \phi$ or $k_t \geq 3$, contradicting). Now we consider a da-ecard $((n + 1) + (n + 1) - 2, G - e)$, where the edge e is chosen in $E(Y_2, Y_2)$ as below:

i. If $|X_{n-1}| = 1$ and if there exists exactly one vertex, say u, of degree $n+1$ and also it is adjacent to no $(n - 1)$-vertex, then we choose an edge incident to u as e; choose any edge in $E(Y_2, Y_2)$ as e otherwise.
ii. If $|X_{n-1}| = 2$ and if there exists exactly one vertex, say u, of degree $n+1$ and also it is adjacent to no $(n - 1)$-vertex, then we choose an edge incident to u as e; if there exists exactly one vertex, say v, of degree $n+1$ and also it is adjacent to only one $(n - 1)$-vertex, then we choose an edge incident to u as e; if there exists two vertices, say u_1 and u_2 of degree $n+1$ and they are adjacent to only one $(n - 1)$-vertex, then we take the edge e to be u_1u_2; choose any edge in $E(Y_2, Y_2)$ as e otherwise.

Clearly $dsum(G - e) = \{r_1 + r_1,\ r_1 + (n - 1),\ r_1 + n,\ r_1 + (n + 1),\ (n - 1) + n,\ n + n\}$. If d_i was equal to $d(e)$ for $i = 1, 2, 3, 4$ then r_1 would be at least $n - 1$, contradicting. If d_5 was equal to $d(e)$, then we would have $-1 = 0$, again a contradiction. Finally, for $d_6 = d(e)$, we have $dern(G) = 1$, since there is only one such pair of nonadjacent vertices in $G - e$.

Case 2.2.2. $r_2 \leq n - 2$

We consider a da-ecard $((n + 1) + (n + 1) - 2, G - e)$, where $e \in E(Y_2, Y_2)$. Then $dsum(G - e) = \{r_1 + r_1,\ r_1 + n,\ r_1 + (n + 1),\ r_1 + r_2,\ r_2 + r_2,\ r_2 + n,\ r_2 + (n + 1),\ n + n\}$. If d_i was equal to $d(e)$ for $i = 1, 2, 3,$ then r_1 would be $n - 1$ or n, giving a contradiction. If d_i was equal to $d(e)$ for $i = 4, 5, 6, 7,$ then r_2 would be $n - 1$ or n or at least $n + 3$, again a contradiction. Finally, for $d_8 = d(e)$, we have $dern(G) = 1$, since there is only one such pair of nonadjacent vertices in $G - e$.

Case 2.3. $k_t \geq 3$

Now consider a da-ecard $((n - 1 + k_t) + (n - 1 + k_t) - 2, G - e)$, where $e \in E(Y_{k_t}, Y_{k_t})$. Clearly, $dsum(G - e) = \{r_1 + r_1,\ r_1 + r_2,\ r_2 + r_2,\ r_1 + (n - 1 + k_t - 1),\ r_2 + (n - 1 + k_t - 1),\ r_1 + (n - 1 + k_1),\ r_1 + (n - 1 + k_2),\ \ldots, r_1 + (n - 1 + k_t),\ r_2 + (n - 1 + k_1),\ r_2 + (n - 1 + k_2),\ \ldots, r_2 + (n - 1 + k_t),\ (n - 1 + k_t - 1) + (n - 1 + k_t - 1)\}$. If d_i was equal to $d(e)$ for $i = 1, 2, 3,$ then, since $r_1 \leq n - 1, r_2 \leq n$, the value of k_t would be at most two, a contradiction. If d_i

was equal to $d(e)$ for $i = 4, 5$, then both r_1 and r_2 would be at least $n + 1$, giving a contradiction. If d_i was equal to $d(e)$ for $i = 6, 7, \ldots, t + 5$, then either r_1 would be at least n (when $k_j = k_t$) or the value of k_t would be at most $\frac{k_j+2}{2}$ (when $k_j \neq k_t$), $j = 1, 2, \ldots, t$, giving a contradiction. If d_i was equal to $d(e)$ for $i = t + 6, t + 7, \ldots, 2t + 5$, then either k_t would be 3 (when $k_t = k_j$) and r_2 would be n, giving a contradiction to the fact that no pair of vertices of degree r_2 and $(n - 1 + k_j)$, $j = 1, 2, \ldots, t$ is nonadjacent, or the value of k_t would be at most $\frac{k_j+3}{2}$ (when $k_t \neq k_j$, $j = 1, 2, \ldots, t$), again a contradiction. For the last degree sum in $dsum(G - e)$, we have $dern(G) = 1$ since there is only one such pair of nonadjacent vertices in $G - e$, which completes the proof. □

Theorem 4. *Let G be a split graph such that every vertex in X is of degree r_1 or r_2. Then dern(G) = 1 if there is a vertex in Y nonadjacent to any vertex in X.*

Proof. Now $Y_0 \neq \phi$ and $r_2 \leq n - 1$. If $r_2 = n - 1$, then let y_0 be the unique vertex in Y_0. Now the set $X \cup \{y_0\}$ and $Y - \{y_0\}$ will become an independent set and a clique of G, respectively, such that it satisfies the hypothesis of Theorem 3 and hence $dern(G) = 1$. So, we can take that $r_2 \leq n - 2$ and that $0 < r_1 < r_2$ in view of Lemma 2. We proceed by three cases depending upon the values of r_2.

Case 1. $r_2 = n - 2$

Clearly $r_1 \leq n - 3$ and $|Y_0| = 1$ or 2. We proceed by two cases depending upon the values of $|Y_{k_t}|$.

Case 1.1. $|Y_{k_t}| = 1$

Consider the da-ecard $((n - 1 + k_1) + (n - 1 + k_t) - 2, G - e)$, where $e \in E(Y_{k_1}, Y_{k_t})$. Clearly, $dsum(G - e) = \{r_1 + r_1, r_1 + (n - 2), r_1 + (n - 1), (n - 2) + (n - 2), (n-2) + (n - 1), r_1 + (n - 1 + k_1 - 1), r_1 + (n - 1 + k_t - 1), (n - 2) + (n - 1 + k_1 - 1), (n - 2) + (n - 1 + k_t - 1), r_1 + (n - 1 + k_1), r_1 + (n - 1 + k_2), \ldots, r_1 + (n - 1 + k_{t-1}), (n-2) + (n - 1 + k_1), (n - 2) + (n - 1 + k_2), \ldots, (n - 2) + (n-1 + k_{t-1}), (n - 1 + k_1 - 1) + (n - 1 + k_t - 1)\}$. If d_i was equal to $d(e)$ for $i = 1, 2, \ldots, 5$, then, since $r_1 \leq n - 3$, the value of $k_1 + k_t$ would be at most one, a contradiction. If d_i was equal to $d(e)$ for $i = 6, 7, 8, 9$, then, since $r_1 \leq n - 3$, the values of k_1 and k_t would be at most -1, again a contradiction. If d_i was equal to $d(e)$ for $i = 10, 11, \ldots, t + 8$, then k_t would be at most $k_j - k_1$ and at most k_j, $j = 1, 2, \ldots, t - 1$ (as $k_1 \geq 1$), again a contradiction. Similarly, if d_i was equal to $d(e)$ for $i = t + 9, t + 10, \ldots, 2t + 7$, then the value of k_t would be equal to $k_j - k_1 + 1$ and so k_t would be at most k_j, $j = 1, 2, \ldots, t - 1$, a contradiction. For the last degree sum in $dsum(G - e)$, we have $dern(G) = 1$ since there is only one such pair of nonadjacent vertices in $G - e$.

Case 1.2. $|Y_{k_t}| \geq 2$

Consider the da-ecard $((n - 1 + k_t) + (n - 1 + k_t) - 2, G - e)$, where $e \in E(Y_{k_t}, Y_{k_t})$. Clearly, $dsum(G - e) = \{r_1 + (n - 1 + k_1), r_1 + (n - 1 + k_2), \ldots, r_1 + (n - 1 + k_t), (n - 2) + (n - 1 + k_1), (n - 2) + (n - 1 + k_2), \ldots, (n - 2) + (n - 1 + k_t), (n - 1 + k_t - 1) + (n - 1 + k_t - 1), r_1 + r_1, r_1 + (n - 2), r_1 + (n - 1), r_1 + (n - 1 + k_t - 1), (n - 2) + (n - 2), (n - 2) + (n - 1), (n - 2) + (n - 1 + k_t - 1)\}$. If d_i was equal to $d(e)$ for $i = 1, 2, \ldots, t$, then either k_t would be at most zero (when $k_j = k_t$) or k_t would be at most $k_j/2$ (when $k_j \neq k_t$), which is a contradiction to $k_j < k_t$, $j = 1, 2, \ldots, t$. If d_i was equal to $d(e)$ for $i = t + 1, t + 2, \ldots, 2t$, then we will get either a contradiction or $dern(G) = 1$ as follows: If $k_j \neq k_t$, then $k_j = 2k_t - 1$, giving a contradiction to $k_j < k_t$, $j = 1, 2, \ldots, t$. Otherwise, that is $k_j = k_t$, $j = 1, 2, \ldots, t$. Then $k_t = 1$. Since $r_2 = n - 2$ and $|X_{r_1}| \neq \phi$, we have $|Y_0| = 1$. Now, consider the graph G', obtained from G, whose independent set $X' = X \cup \{y_0\}$ and clique $Y' = Y - \{y_0\}$, where $Y_0 = \{y_0\}$. Then $|X'_{r_1}| = |X'_1| = 1$ and $|X'_{r_2}| = |X'_{n-2}| = 1$. Consider the da-ecard $(1 + n' - 2, G - e)$, where $e \in E'(X'_1, Y'_1)$, where $n' = n - 1$. Clearly, $dsum(G' - e) = \{0 + n' - 2, \ 0 + n', \ 0 + n' - 1, \ n' - 2 + n' - 1\}$ and $n' \geq 4$. If each of the first two elements in $dsum(G' - e)$ was equal to $(1 + n' - 2)$, then we would have $-2 = -1$ or $-1 = 0$. If the third element in $dsum(G' - e)$ is equal to $(1 + n' - 2)$, then, since the two $(n' - 1)$-vertices of G – e have the same neighbourhood in $G' - e$, the extension $H'(n' - 1)$ is isomorphic to G' and hence $dern(G) = dern(G') = 1$. If the last element in $dsum(G' - e)$ was equal to $(1 + n' - 2)$, then n' would be 2, again a contradiction. If d_{2t+1} is equal to $d(e) = (n - 1 + k_t) + (n - 1 + k_t) - 2$, then $dern(G) = 1$, since there is only one such pair of non adjacent vertices in $G - e$. If d_i was equal to $d(e)$ for $i = 2t + 2, 2t + 3, \ldots, 2t + 8$, then since $r_1 \leq n - 3$, the value of k_t would be at most zero, a contradiction.

Case 2. $r_2 \leq n - 3$

We proceed by two cases depending upon the value of $|Y_{k_t}|$.

Case 2.1. $|Y_{k_t}| = 1$

Consider the da-ecard $((n - 1 + k_1) + (n - 1 + k_t) - 2, G - e)$, where $e \in E(Y_{k_1}, Y_{k_t})$. Clearly, $dsum(G - e) = \{r_1 + r_1, \ r_1 + r_2, \ r_1 + (n - 1), \ r_2 + r_2, \ r_2 + (n - 1), \ r_1 + (n - 1 + k_1 - 1), r_1 + (n - 1 + k_t - 1), \ r_2 + (n - 1 + k_1 - 1), \ r_2 + (n - 1 + k_t - 1), \ r_1 + (n - 1 + k_1), \ r_1 + (n - 1 + k_2), \ldots, r_1 + (n - 1 + k_{t-1}), r_2 + (n - 1 + k_1), \ r_2 + (n - 1 + k_2), \ldots, r_2 + (n - 1 + k_{t-1}), \ (n - 1 + k_1 - 1) + (n - 1 + k_t - 1)\}$. If d_i was equal to $d(e)$ for $i = 1, 2, \ldots, 5$, then, since $r_1 \leq n - 4$ and $r_2 \leq n - 3$, the value of $k_1 + k_t$ would be at most zero, a contradiction. If d_i was equal to $d(e)$ for $i = 6, 7, 8, 9$, then, since $r_1 \leq n - 4$ and $r_2 \leq n - 3$, both the values of k_1 and k_t would be at most -1, a contradiction. If d_i was equal to $d(e)$ for $i = 10, 11, \ldots, t + 8$, then, since $r_1 \leq n - 4$, the value of k_t would be at most $k_j - k_1 - 1$, which is at most k_j, $j = 1, 2, \ldots, t - 1$ (since $k_1 \geq 1$), again a contradiction. Similarly, If d_i was equal to $d(e)$ for $i = t + 9, t + 10, \ldots, 2t + 7$, then k_t would be at most $k_j - k_1$, which is at most $k_j - 1$, $j = 1, 2, \ldots, t - 1$ (since $k_1 \geq 1$), a contradiction. For the last degree sum in $dsum(G - e)$, we have $dern(G) = 1$ since there is only one such pair of nonadjacent vertices in $G - e$.

Case 2.2. $|Y_{k_t}| \geq 2$

Consider the da-ecard $((n - 1 + k_t) + (n - 1 + k_t) - 2, G - e)$, where $e \in E(Y_{k_t}, Y_{k_t})$. Clearly, $dsum(G - e) =$

$\{r_1 + (n-1+k_1),\ r_1 + (n-1+k_2), \ldots,\ r_1 + (n-1+k_t),\ r_2 + (n-1+k_1),\ r_2 + (n-1+k_2), \ldots,\ r_2 + (n-1+k_t),\ (n-1+k_t-1) + (n-1+k_t-1),\ r_1 + r_1,\ r_1 + r_2,\ r_1 + (n-1),\ r_1 + (n-1+k_t-1),\ r_2 + r_2,\ r_2 + (n-1),\ r_2 + (n-1+k_t-1)\}$. s If d_i was equal to $d(e)$ for $i = 1, 2, \ldots, t$, then either k_t would be at most −1 (if $k_j = k_t$), or k_t would be at most $\frac{k_j-1}{2}$ (if $k_j \neq k_t$), $j = 1, 2, \ldots, t$, giving a contradiction. Similarly, if d_i was equal to $d(e)$ for $i = t+1, t+2, \ldots, 2t$, then k_t would be at most zero (if $k_j = k_t$) or k_t would be at most $\frac{k_j}{2}$ (if $k_j \neq k_t$), giving a contradiction to $k_j < k_t$, $j = 1, 2, \ldots, t$. If d_{2t+1} is equal to $d(e)$, then, since there is only one such pair of non adjacent vertices in $G - e$, it follows that $dern(G) = 1$. Finally, if d_i was equal to $d(e)$ for $i = 2t+2, 2t+3, \ldots, 2t+8$, then, since $r_1 \leq n-4$ and $r_2 \leq n-3$, the value of k_t would be at most zero, giving a contradiction and completing the proof. □

3. Conclusion

It seems that the value of *dern* of split graphs not covered under this paper and [8] is also likely to be one or two. In most of the cases of Theorems 3 and 4, we have determined $dern(G)$, by using the da-ecards obtained by deleting edges lying in the partite set Y that is complete. If one can able to prove this result by using the da-ecards obtained by deleting edges joining a vertex in X to a vertex in Y, then it may lead to a way to find the *dern* of bipartite graphs, which remains open in both reconstruction and edge reconstruction problems [4]. Degree associated (edge) reconstruction number might be a strong tool for providing evidence to support or reject the Edge Reconstruction Conjecture that remains open.

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