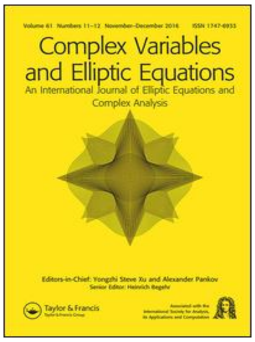

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An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gcov20

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To cite this article: Prashanta Garain (2020): Existence and nonexistence results for anisotropic *p*-Laplace equation with singular nonlinearities, Complex Variables and Elliptic Equations, DOI: 10.1080/17476933.2020.1801655

To link to this article: https://doi.org/10.1080/17476933.2020.1801655

Published online: 09 Aug 2020.

COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
https://doi.org/10.1080/17476933.2020.1801655

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Existence and nonexistence results for anisotropic *p*-Laplace equation with singular nonlinearities

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ABSTRACT

Let $p_i \geq 2$ and consider the following anisotropic p-Laplace equation

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = g(x)f(u), \quad u > 0 \text{ in } \Omega.$$

Under suitable hypothesis on the weight function g we present an existence result for $f(u) = e^{\frac{1}{u}}$ in a bounded smooth domain Ω and nonexistence results for $f(u) = -e^{\frac{1}{u}}$ or $-(u^{-\delta} + u^{-\gamma})$, $\delta, \gamma > 0$ with $\Omega = \mathbb{R}^N$ respectively.

ARTICLE HISTORY
Received 6 September 2019
Accepted 23 July 2020

COMMUNICATED BY
T. Bartsch

KEYWORDS
Anisotropic p-Laplacian; existence; nonexistence; stable solution

AMS SUBJECT CLASSIFICATIONS
35A01; 35B53; 35J75

1. Introduction

In this article we are interested in the question of existence of a weak solution to the following anisotropic p-Laplace equation

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = g(x)\, e^{\frac{1}{u}}, \quad u > 0 \text{ in } \Omega \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 3$ and $g \in L^1(\Omega)$ is nonnegative which is not identically zero.

Alongside we present nonexistence results concerning stable solutions to the following equation

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = g(x)f(u) \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N \tag{2}$$

where $f(u)$ is either $-(u^{-\delta} + u^{-\gamma})$ with $\delta, \gamma > 0$ or $-e^{\frac{1}{u}}$. The weight function $g \in L^1_{\text{loc}}(\mathbb{R}^N)$ is such that $g \geq c > 0$ for some constant c.

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Throughout the article, we assume that $p_i \geq 2$. If $p_i = 2$ for all i and $g \equiv 1$ Equation (2) becomes the Laplace equation

$$-\Delta u = f(u) \quad \text{in } \Omega. \tag{3}$$

Observe that the nonlinearities in our consideration is singular in the sense that it blows up near the origin. Starting from the pioneering work of Crandall et al. [1] where the existence of a unique positive classical solution for $f(u) = u^{-\delta}$ with any $\delta > 0$ has been proved for the problem (3) with zero Dirichlet boundary value. Lazer-McKenna [2] observed that the above classical solution is a weak solution in $H_0^1(\Omega)$ iff $0 < \delta < 3$. Boccardo-Orsina [3] investigated the case of any $\delta > 0$ concerning the existence of a weak solution in $H^1_{\mathrm{loc}}(\Omega)$. Moreover, Canino-Degiovanni [4] and Canino-Sciunzi [5] investigated the question of existence and uniqueness of solution for singular Laplace equations. Canino et al. [6] generalized the problem (3) to the following singular p-Laplace equation

$$-\Delta_p u = \frac{f(x)}{u^{\delta}} \text{in}\Omega, \quad u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \tag{4}$$

to obtain existence and uniqueness of weak solution for any $\delta > 0$ under suitable hypothesis on f. For more details concerning singular problems reader can look at [7–9] and the references therein.

Farina [10] settled the question of nonexistence of stable solution for the Equation (3) with $f(u) = e^u$. There is a huge literature in this direction for various type of nonlinearity $f(u)$, reader can look at the nice surveys [11, 12]. For $f(u) = -u^{-\delta}$ with $\delta > 0$ Ma-Wei [13] proved that the Equation (3) does not admit any $C^1(\mathbb{R}^N)$ stable solution provided

$$2 \leq N < 2 + \frac{4}{1+\delta}\left(\delta + \sqrt{\delta^2 + \delta}\right).$$

Moreover many other qualitative properties of solutions has been obtained there. Consider the weighted p-Laplace equation

$$-\operatorname{div}\left(w(x)|\nabla u|^{p-2}\nabla u\right) = g(x)f(u) \quad \text{in } \mathbb{R}^N. \tag{5}$$

For $w = g = 1$, Guo-Mei [14] showed nonexistence results in $C^1(\mathbb{R}^N)$ for (5), provided $2 \leq p < N < \frac{p(p+3)}{p-1}$ and $\delta > q_c$ where

$$q_c = \frac{(p-1)[(1-p)N^2 + (p^2+2p)N - p^2] - 2p^2\sqrt{(p-1)(N-1)}}{(N-p)[(p-1)N - p(p+3)]}.$$

By considering a more general weight $g \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $|g| \geq C|x|^a$ for large $|x|$, Chen et al. [15] proved nonexistence results for the Equation (5), provided $w = 1$ and $2 \leq p < N < \frac{p(p+3)+4a}{p-1}$ and $\delta > q_c$ where

$$q_c = \frac{2(N+a)(p+a) - (N-p)[(p-1)(N+a) - p - a] - \beta}{(N-p)[(p-1)N - p(p+3)]},$$

for

$$\beta = 2(p+a)\sqrt{(p+a)\left(N + a + \frac{N-p}{p-1}\right)}.$$

Recently this has been extended for a general weight function w in [16, 17].

Our main motive in this article is to investigate such results in the framework of the anisotropic p-Laplace operator, which is non-homogeneous. Such operators appear in many physical phenomena, for example, it reflects anisotropic physical properties of some reinforced materials [18], appears in image processing [19], to study the dynamics of fluids in anisotropic media when the conductivities of the media are different in each direction [20]. The first part of this article is devoted to the existence of a weak solution for the anisotropic problem (1). Some recent works on singular anisotropic problems can be found in [21, 22]. The singularity $e^{\frac{1}{u}}$ is more singular in nature compared to $u^{-\delta}$ which protects one to obtain the uniform boundedness of u_n as in [3]. We overcome this difficulty using the domain approximation method following [23]. In the second part we provide nonexistence results of stable solutions for the anisotropic p-Laplace equation (2) with the mixed singularities $-(u^{-\delta}+u^{-\gamma})$ and $-e^{\frac{1}{u}}$. We employ the idea introduced in [10] to establish our main results stated in Section 2 for which Caccioppoli type estimates (see Section 5) will be the main ingredient. The main difficulty to obtain such estimates arises due to the nonhomogenity of the anisotropic p-Laplace operator which we overcome by choosing suitable test functions in the stability condition.

2. Preliminaries

In this section, we present some basic results in the anisotropic Sobolev space.

Anisotropic Sobolev Space: Let $p_i \geq 2$ for all i, then for any domain D define the anisotropic Sobolev space by

$$W^{1,p_i}(D) = \left\{ v \in W^{1,1}(D) : \frac{\partial v}{\partial x_i} \in L^{p_i}(D) \right\}$$

and

$$W_0^{1,p_i}(D) = W^{1,p_i}(D) \cap W_0^{1,1}(D)$$

endowed with the norm

$$\|v\|_{W_0^{1,p_i}(D)} = \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(D)}.$$

The space $W_{\mathrm{loc}}^{1,p_i}(D)$ is defined analogously.

We denote by $\bar{p}$ to be the harmonic mean of $p_1, p_2, \ldots, p_N$ defined by

$$\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$$

and

$$\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}.$$

The following Sobolev embedding theorem can be found in [24–26].

Theorem 2.1: *For any bounded domain Ω, the inclusion map*

$$W_0^{1,p_i}(\Omega) \to L^r(\Omega)$$

is continuous for every $r \in [1, \bar{p}^*]$ if $\bar{p} < N$ and for every $r \geq 1$ if $\bar{p} \geq N$. Moreover, there exists a positive constant C depending only on Ω such that for every $v \in W_0^{1,p_i}(\Omega)$

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(\Omega)}, \quad \forall\, r \in [1, \bar{p}^*].$$

Weak Solution: We say that $u \in W_{\text{loc}}^{1,p_i}(\Omega)$ is a weak solution of the problem (1) if $u > 0$ a.e. in Ω and for every $\phi \in C_c^1(\Omega)$

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = \int_{\Omega} g(x)\, \mathrm{e}^{\frac{1}{u}} \phi \, \mathrm{d}x. \tag{6}$$

Stable Solution: We say that $u \in W_{\text{loc}}^{1,p_i}(\mathbb{R}^N)$ is a stable solution of the problem (2), if $u > 0$ a.e. in Ω such that both $g(x)f(u), g(x)f'(u) \in L_{\text{loc}}^1(\mathbb{R}^N)$ and for all $\varphi \in C_c^1(\mathbb{R}^N)$,

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = \int_{\mathbb{R}^N} g(x) f(u) \varphi \, \mathrm{d}x \tag{7}$$

and

$$\int_{\mathbb{R}^N} g(x) f'(u) \varphi^2 \, \mathrm{d}x \leq \sum_{i=1}^{N} (p_i - 1) \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \mathrm{d}x. \tag{8}$$

For a general theory of anisotropic Sobolev space, we refer the reader to [24, 25, 27, 28].

Assumption and notation for the nonexistence results: We denote by $\Omega = \mathbb{R}^N$ for $N \geq 1$ and assume $2 < p_1 \leq p_2 \leq \cdots \leq p_N$.

We will make use of the following truncated functions later. For $k \in \mathbb{N}, \alpha > p_N - 1$ and $t \geq 0$, define

$$a_k(t) = \begin{cases} \dfrac{(1-\alpha)}{2} k^{\frac{\alpha+1}{2}} \left(t + \dfrac{1+\alpha}{k(1-\alpha)} \right), & \text{if } 0 \leq t < \dfrac{1}{k}, \\ t^{\frac{1-\alpha}{2}}, & \text{if } t \geq \dfrac{1}{k}, \end{cases}$$

and

$$b_k(t) = \begin{cases} -\alpha k^{\alpha+1} \left(t - \dfrac{1+\alpha}{k\alpha} \right), & \text{if } 0 \leq t < \dfrac{1}{k}, \\ t^{-\alpha}, & \text{if } t \geq \dfrac{1}{k}. \end{cases}$$

Then it can be easily verified that both a_k and b_k are positive $C^1[0, \infty)$ decreasing functions. Moreover, a_k and b_k satisfies the following properties:

(a)

$$a_k(t)^2 \geq t\, b_k(t), \quad \forall\, t \geq 0.$$

(b)

$$a_k(t)^{p_i} |a_k'(t)|^{2-p_i} + b_k(t)^{p_i} |b_k'(t)|^{1-p_i} \leq C\, |t|^{p_i - \alpha - 1},$$

for some positive constant $C(p_1, p_2, \dots, p_N, \alpha)$.

(c)

$$a_k'(t)^2 = \frac{(\alpha-1)^2}{4\alpha} |b_k'(t)|, \quad \forall\, t \geq 0.$$

The following notations will be used for the nonexistence results.

Notation: The Equation (2) will be denoted by $(2)_s$ and $(2)_e$ for $f(u) = -u^{-\delta} - u^{-\gamma}$ and $f(u) = -\mathrm{e}^{\frac{1}{u}}$ respectively. Without loss of generality we assume $0 < \delta \leq \gamma$.

We denote by $B_r(0)$ to be the ball centred at 0 with radius $r > 0$.
We denote by $u_i = \frac{\partial u}{\partial x_i}$ for all $i = 1, 2, \dots, N$ and $q = \frac{\sum_{i=1}^N p_i}{N}$.
Denote by

$$l_1 = \frac{p_N - q}{2}, \quad l_2 = \frac{2\delta}{N(q-1)} - \frac{q-1}{2} \quad \text{and} \quad l_3 = \frac{2}{MN(q-1)} - \frac{q-1}{2}.$$

We denote by

$$A = \left(\frac{N(q-1)(p_N-1)}{4}, \infty \right),$$
$$B = \left(0, \frac{4}{N(q-1)(p_N-1)} \right), \quad C = \left(0, \frac{4}{N(N-1)(q-1)} \right).$$

Define

$$I = \bigcap_{i=1}^{N} I_i$$

where

$$I_i = \left(\frac{N^2(q-1)(p_i-1)}{p_i\big(N(q-1)+4\big) - N^2(q-1)}, \infty \right),$$

provided $p_i(N(q-1)+4) - N^2(q-1) > 0$ for all $i = 1, 2, \dots, N$ and $J = B \cap C$.

We assume $\delta \in A$ and $M \in J$. Observe that $\delta \in A$ implies $l_2 > l_1$ and $l_2 > 0$. Also $M \in J$ implies $l_3 > l_1$ and $l_3 > 0$.

If C depends on ϵ we denote by C_ϵ and if C depends on $r_1, r_2, \dots, r_m$ we denote it by $C(r_1, r_2, \dots, r_m)$.

Throughout this article $\psi_R \in C_c^1(\mathbb{R}^N)$ is a test function such that

$$0 \le \psi_R \le 1 \text{ in } \mathbb{R}^N, \quad \psi_R = 1 \text{ in } B_R(0),$$
$$\psi_R = 0 \quad \text{in } \mathbb{R}^N \setminus B_{2R}(0)$$

with

$$|\nabla \psi_R| \le \frac{C}{R}$$

for some constant $C > 0$ (independent of R).

3. Main results

The main results of this article reads as follows:

Theorem 3.1: *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $N \ge 3$ and $p_N \ge \cdots p_2 \ge p_1 \ge 2$. Then the problem* (1) *admits a weak solution u in $W^{1,p_i}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ such that $(u - \epsilon)^+ \in W_0^{1,p_i}(\Omega)$ for every $\epsilon > 0$, provided*

(a) *$g \in L^m(\Omega)$ for some $m > \frac{\bar{p}^*}{\bar{p}^* - \bar{p}}$ if $\bar{p} < N$ where $\bar{p}^* \ge p_N$.*
(b) *$g \in L^m(\Omega)$ for some $m > \frac{r}{r - p_N}$ if $\bar{p} \ge N$ where $r > p_N$.*

Theorem 3.2: *Let $u \in W^{1,p_i}_{\text{loc}}(\Omega)$ be such that $0 < u \le 1$ a.e. in Ω. Assume that $1 \le \delta < \gamma$ be such that $\delta \in A \cap I$. Then u is not a stable solution to the problem $(2)_s$.*

Theorem 3.3: *Let $u \in W^{1,p_i}_{\text{loc}}(\Omega)$ be such that $u \ge 1$ a.e. in Ω. Assume that $0 < \delta < \gamma$ be such that $\delta \in A$ and $\gamma \in I \cap [1, \infty)$. Then u is not a stable solution to the problem $(2)_s$.*

Theorem 3.4: *Let $u \in W^{1,p_i}_{\text{loc}}(\Omega)$ be such that $u > 0$ a.e. in Ω. Assume that $1 \le \delta = \gamma \in A \cap I$. Then u is not a stable solution to the problem $(2)_s$.*

Theorem 3.5: *Let $u \in W^{1,p_i}_{\text{loc}}(\Omega)$ be such that $0 < u \le M$ a.e in Ω, provided $M \in J$. Then u is not a stable solution to the Equation $(2)_e$.*

We present the proof of the above theorems in the following two sections.

4. Proof of existence results

For $n \in \mathbb{N}$, define $g_n(x) = \min\{g(x), n\}$ and consider the following approximated problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(|u_i|^{p_i - 2} u_i) = g_n(x)\, \mathrm{e}^{\frac{1}{(u + \frac{1}{n})}} \quad \text{in } \Omega. \tag{9}$$

Lemma 4.1: *Let*

(1) *$g \in L^m(\Omega)$ for some $m > \frac{\bar{p}^*}{\bar{p}^* - \bar{p}}$ if $\bar{p} < N$ or*

(2) *$g \in L^m(\Omega)$ for some $m > \frac{r}{r-p_N}$ if $\bar{p} \geq N$ where $r > p_N$.*

Then for every $n \in \mathbb{N}$ the problem (9) has a positive solution $u_n \in W_0^{1,p_i}(\Omega)$. Moreover, one has

(i) *$\|u_n\|_{L^\infty(\Omega)} \leq C$ for some constant C independent of n.*
(ii) *$u_{n+1} \geq u_n$ and each u_n is unique.*
(iii) *there exists a positive constant $c_\omega > 0$ such that for every $\omega \subset\subset \Omega$ we have $u_n \geq c_\omega > 0$.*

Proof: *Existence:* Let $v \in L^r(\Omega)$ for some $r \geq 1$. Then the problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(|u_i|^{p_i-2}u_i) = g_n(x)\, \mathrm{e}^{\frac{1}{(|v|+\frac{1}{n})}}$$

has a unique solution $u = A(v) \in W_0^{1,p_i}(\Omega)$ since the r.h.s belongs to $L^\infty(\Omega)$, see [25]. Choosing $u = A(v)$ as a test function and using Theorem 2.1 together with Hölder's inequality we obtain

$$\|u\|_{L^r(\Omega)} \leq C_N$$

for some constant C_N independent of u. Now arguing as in Lemma 2.1 of [21] gives the existence of u_n.

(i) (1) Let $\bar{p} < N$ and $g \in L^m(\Omega)$ for some $m > \frac{\bar{p}^*}{\bar{p}^*-\bar{p}}$. Choosing $G_k(u_n) = (u_n - k)^+$ for $k > 1$ as a test function in (9) we get

$$\|G_k(u_n)\|_{W_0^{1,p_i}(\Omega)} \leq e\left(\int_\Omega g|G_k(u_n)|\,\mathrm{d}x\right)^{\frac{1}{p_i}}.$$

Using Theorem 2.1 with $r = \bar{p}^*$ and Hölder's inequality we get

$$\|G_k(u_n)\|_{L^{\bar{p}^*}(\Omega)} \leq c\left(\int_{A(k)} |g|^{\bar{p}^{*'}}\,\mathrm{d}x\right)^{\frac{\bar{p}^*-1}{\bar{p}^*(\bar{p}-1)}}.$$

Now for $1 < k < h$ denote by $A(h) = \{x \in \Omega : u(x) > h\}$, we get

$$\begin{aligned}(h-k)^{p_i}|A(h)|^{\frac{p_i}{\bar{p}^*}} &\leq \left(\int_{A(k)} |G_k(u_n)|^{\bar{p}^*}\right)^{\frac{p_i}{\bar{p}^*}} \\ &\leq c\left(\int_{A(k)} |g|^{\bar{p}^{*'}}\,\mathrm{d}x\right)^{\frac{p_i(\bar{p}^*-1)}{\bar{p}^*(\bar{p}-1)}}.\end{aligned}$$

Now using Hölder's inequality with exponents $q = \frac{m}{\bar{p}^{*'}}$ and $q' = \frac{q}{q-1}$ we get

$$(h-k)^{p_i}|A(h)|^{\frac{p_i}{\bar{p}^*}} \leq c\|g\|_{L^m(\Omega)}^{\frac{p_i}{\bar{p}-1}}|A(k)|^{\frac{p_i(\bar{p}^*-1)(m-\bar{p}^{*'})}{\bar{p}^*m(\bar{p}-1)}}.$$

Therefore we have

$$|A(h)| \leq \frac{c\|g\|_{L^m(\Omega)}^{\frac{\bar{p}^*}{\bar{p}-1}}}{(h-k)^{\bar{p}^*}} |A(k)|^{\beta},$$

where $\beta = \frac{(\bar{p}^*-1)(m-\bar{p}^{*'})}{m(\bar{p}-1)} > 1$ since $m > \frac{\bar{p}^*}{\bar{p}^*-\bar{p}}$. By Stampacchia's result [29] we get $\|u_n\|_{L^\infty(\Omega)} \leq C$ where C is independent of n.

(2) Choosing $G_k(u_n) = (u_n - k)^+$ as a test function in (9) and using Hölder's inequality we get

$$\|G_k(u_n)\|_{W_0^{1,p_i}(\Omega)} \leq e\|g\|_{L^{r'}(A(k))}^{\frac{1}{p_i-1}}.$$

Using Hölder's inequality with exponents $\frac{m}{r'}$ and $\frac{m}{m-r'}$ we get

$$\|G_k(u_n)\|_{W_0^{1,p_i}(\Omega)} \leq c\|g\|_{L^m(\Omega)}^{\frac{1}{p_i-1}} |A(k)|^{\frac{(m-r')}{mr'(p_i-1)}}.$$

Now for $1 < k < h$ we have

$$\begin{aligned}
&(h-k)^{p_i}|A(h)|^{\frac{p_i}{r}} \\
&\leq \left(\int_{A(h)} (u-k)^r \, dx\right)^{\frac{p_i}{r}} \\
&\leq \left(\int_{A(k)} (u-k)^r \, dx\right)^{\frac{p_i}{r}} \\
&\leq \sum_{i=1}^{N} \int_\Omega |\partial_i G_k(u_n)|^{p_i} \, dx \\
&\leq c\|g\|_{L^m(\Omega)}^{p_i'} |A(k)|^{\frac{p_i(m-r')}{mr'(p_i-1)}}.
\end{aligned}$$

Therefore we have

$$|A(h)| \leq c\frac{\|g\|^{\frac{r}{p_i-1}} |A(k)|^{\gamma}}{(h-k)^r},$$

where $\gamma = \frac{r(m-r')}{mr'(p_i-1)} > 1$ since $m > \frac{r}{r-p_N}$. By Stampacchia's result [29] we get $\|u_n\|_{L^\infty(\Omega)} \leq C$ where C is independent of n.

(ii) Let u_n and u_{n+1} satisfies the Equations (9). Then for every $\phi \in W_0^{1,p_i}(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega |(u_n)_i|^{p_i-2}(u_n)_i\phi_i \, dx = \int_\Omega g_n \, e^{\frac{1}{(u_n+\frac{1}{n})}} \phi \, dx \tag{10}$$

and

$$\sum_{i=1}^{N} \int_\Omega |(u_{n+1})_i|^{p_i-2}(u_{n+1})_i\phi_i \, dx = \int_\Omega g_{n+1} \, e^{\frac{1}{(u_{n+1}+\frac{1}{n+1})}} \phi \, dx. \tag{11}$$

Choosing $\phi = (u_n - u_{n+1})^+$ as a test function and subtracting (10) and (11) we have

$$\sum_{i=1}^{N} \int_{\Omega} \left(|(u_n)_i|^{p_i-2}(u_n)_i - |(u_{n+1})_i|^{p_i-2}(u_{n+1})_i\right)(u_n - u_{n+1})_i^+ \, \mathrm{d}x$$
$$\leq \int_{\Omega} g_{n+1}(x) \left\{ \mathrm{e}^{\frac{1}{(u_n+\frac{1}{n})}} - \mathrm{e}^{\frac{1}{(u_{n+1}+\frac{1}{n+1})}} \right\} (u_n - u_{n+1})_i^+ \, \mathrm{d}x \leq 0.$$

Using the algebraic inequality (Lemma A.0.5 of [30]) we get for any $p_i \geq 2$

$$\|(u_n - u_{n+1})^+\|_{W_0^{1,p_i}(\Omega)} = 0.$$

Therefore (i) holds. The uniqueness follows similarly as in the monotonicity.

(iii) Observe that $u_1 \in L^\infty(\Omega)$ by using (i). Hence

$$\sum_{i=1}^{N} \int_{\Omega} |(u_1)_i|^{p_i-2}(u_1)_i \phi_i \, \mathrm{d}x = g_1 \, \mathrm{e}^{\frac{1}{(u_1+1)}} \geq g_1 \, \mathrm{e}^{\frac{1}{\|u_1\|_\infty+1}}.$$

Since g is nonnegative and not identically zero, by the strong maximum principle (Theorem 3.18 of [24]) we get the property (iii).

■

Proof of Theorem 3.1: Let $\bar{p} < N$ such that $\bar{p}^* \geq p_N$ and $\Omega = \bigcup_k \Omega_k$ where $\Omega_k \subset\subset \Omega_{k+1}$ for each k. Let $\gamma_k = \inf_{\Omega_k} u_n > 0$. Choosing $\phi = (u_n - \gamma_1)^+$ as a test function in (9), using Lemma 4.1 and Theorem 2.1 we get

$$\sum_{i=1}^{N} \int_{\{u_n > \gamma_1\}} |(u_n)_i|^{p_i} \, \mathrm{d}x$$
$$= \int_{\{u_n > \gamma_1\}} g_n \, \mathrm{e}^{\frac{1}{(u_n+\frac{1}{n})}} (u_n - \gamma_1)^+ \, \mathrm{d}x$$
$$\leq c\|g\|_{L^m(\Omega)} \|(u_n - \gamma_1)^+\|_{W_0^{1,p_i}(\Omega)}$$

where c is a constant independent of n. Using Lemma 4.1 and the fact

$$\|u_n\|_{W^{1,p_i}(\Omega_1)} \leq \|u_n\|_{W^{1,p_i}(\{u_n > \gamma_1\})}$$

we get the sequence $\{u_n\}$ is uniformly bounded in $W^{1,p_i}(\Omega_1)$ and as a consequence of Theorem 2.1 it has a subsequence $\{u^1_{n_k}\}$ converges weakly in $W^{1,p_i}(\Omega_1)$ and strongly in $L^{p_i}(\Omega_1)$ and almost everywhere in Ω_1 to $u_{\Omega_1} \in W^{1,p_i}(\Omega_1)$, say.

Proceeding in the same way for any k, we obtain a subsequence $\{u^k_{n_l}\}$ of $\{u_n\}$ such that $u^k_{n_l}$ converges weakly in $W^{1,p_i}(\Omega_k)$, strongly in $L^{p_i}(\Omega_k)$ and almost everywhere to $u_{\Omega_k} \in W^{1,p_i}(\Omega_k)$. We may assume $u^{k+1}_{n_l}$ is a subsequence of $u^k_{n_l}$ for every k, and that $n^k_k \to \infty$ as $k \to \infty$. Therefore $u_{\Omega_{k+1}} = u_{\Omega_k}$ on Ω_k. Define $u = u_{\Omega_1}$ and $u = u_{\Omega_{k+1}}$ on $\Omega_{k+1} \setminus \Omega_k$ for each k. Therefore by our construction the diagonal subsequence $\{u_{n_k}\} := \{u^k_{n_k}\}$ converges weakly to u in $W^{1,p_i}_{\mathrm{loc}}(\Omega_k)$, strongly in $L^{p_i}(\Omega_k)$ and almost everywhere in Ω. Now we claim that $\{u_{n_k}\}$ converges strongly to u in $W^{1,p_i}_{\mathrm{loc}}(\Omega_k)$. Let $\Omega' \subset\subset \Omega$. Let $\phi \in C^\infty_c(\Omega)$ such that $0 \le \phi \le 1$ in Ω, $\phi = 1$ on Ω' and let $k_1 \ge 1$ such that $\operatorname{supp}\phi \subset \Omega_{k_1}$. For every $k, m \ge 1$ we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega'}\left(|(u_{n_k})_i|^{p_i-2}(u_{n_k})_i - |(u_{n_m})_i|^{p_i-2}(u_{n_m})_i\right)(u_{n_k}-u_{n_m})_i\,\mathrm{d}x\\
&\quad\le \sum_{i=1}^{N}\int_{\Omega}\left(|(u_{n_k})_i|^{p_i-2}(u_{n_k})_i - |(u_{n_m})_i|^{p_i-2}(u_{n_m})_i\right)\left(\phi(u_{n_k}-u_{n_m})\right)_i\,\mathrm{d}x\\
&\qquad - \sum_{i=1}^{N}\int_{\Omega_{k_1}}\left\{\left(|(u_{n_k})_i|^{p_i-2}(u_{n_k})_i - |(u_{n_m})_i|^{p_i-2}(u_{n_m})_i\right).\phi_i\right\}(u_{n_k}-u_{n_m})\,\mathrm{d}x\\
&\quad := A - B.
\end{aligned}
$$

Now the fact that u_{n_k} is uniformly bounded in $W^{1,p_i}(\Omega_{k_1})$ and converges strongly in $L^{p_i}(\Omega_{k_1})$ implies $B \to 0$ as $k, m \to \infty$. Choosing $\psi = \phi(u_{n_k} - u_{n_m})$ and either $n = n_k$ or $n = n_m$ we get for $l = k, m$

$$
\begin{aligned}
&\left|(u_{n_l})_i|^{p_i-2}(u_{n_l})_i\left(\phi(u_{n_k}-u_{n_m})\right)_i\,\mathrm{d}x\right|\\
&\quad\le \int_{\Omega_{k_1}} g_n(x)\,\mathrm{e}^{\frac{1}{(u_{n_l}+\frac{1}{n_l})}}|u_{n_k}-u_{n_m}|\,\mathrm{d}x.
\end{aligned}
$$

Now Lemma 4.1, $g \in L^m(\Omega)$ and the strong convergence of u_{n_k} gives $A \to 0$ as $k, m \to \infty$. Now the algebraic inequality (Lemma A.0.5 of [30]) gives

$$
\sum_{i=1}^{N}\int_{\Omega'}|(u_{n_k})_i - (u_{n_m})_i|^{p_i}\,\mathrm{d}x \to 0
$$

as $k, m \to \infty$. Therefore for any $\phi \in C^1_c(\Omega)$ we have

$$
\sum_{i=1}^{N}\int_{\Omega}|(u_{n_k})_i|^{p_i-2}(u_{n_k})_i\phi_i\,\mathrm{d}x = \sum_{i=1}^{N}\int_{\Omega}|u_i|^{p_i-2}u_i\phi_i\,\mathrm{d}x.
$$

Lemma 4.1 and the fact $u_{n_k} \ge c_{\operatorname{supp}\phi} > 0$ gives

$$
\left|\int_{\Omega} g_{n_k}(x)\,\mathrm{e}^{\frac{1}{(u_{n_k}+\frac{1}{n_k})}}\phi\,\mathrm{d}x\right| \le \mathrm{e}^{\frac{1}{c_{\operatorname{supp}\phi}}}\|\phi\|_{L^\infty(\Omega)}\|g\|_{L^1(\Omega)}.
$$

By Lebesgue dominated theorem we obtain

$$
\int_{\Omega} g_{n_k}(x)\,\mathrm{e}^{\frac{1}{(u_{n_k}+\frac{1}{n_k})}}\phi\,\mathrm{d}x = \int_{\Omega} g(x)\,\mathrm{e}^{\frac{1}{u}}\phi\,\mathrm{d}x.
$$

Hence $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ is a weak solution of the problem (7). Now observe that $(u_{n_k} - \epsilon)^+$ in bounded in $W^{1,p_i}_0(\Omega)$ and it has a subsequence converges to v weakly in $W^{1,p_i}_0(\Omega)$. Since

u_{n_k} converges almost everywhere to u, we have $v = (u-\epsilon)^+ \in W_0^{1,p_i}(\Omega)$. The case $\bar{p} \geq N$ follows similarly using Theorem 2.1. ■

5. Proof of nonexistence results

To prove our main results we establish the following a priori estimate on the stable solution to the problem (2).

5.1. A priori estimate

Lemma 5.1: *Let $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ be a positive stable solution to either of the Equation $(2)_s$ or $(2)_e$ and $\alpha > p_N - 1$ be fixed. Then for every $\epsilon \in (0,\alpha)$, there exists a positive constant $C = C_\epsilon(p_1, p_2, \ldots, p_N, q, \alpha)$ such that for any nonnegative $\psi \in C_c^1(\Omega)$, one has*

$$\begin{aligned}\int_\Omega g(x)uf'(u)b_k(u)\psi^q\,\mathrm{d}x &\\ \leq C\sum_{i=1}^N \int_\Omega u^{p_i-\alpha-1}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x &\\ - \frac{(\alpha-1)^2(N(q-1)+\epsilon)}{4\alpha(1-\epsilon)}\int_\Omega g(x)f(u)b_k(u)\psi^q\,\mathrm{d}x. &\end{aligned} \tag{12}$$

As a corollary of Lemma 5.1 we obtain the following Caccioppoli type estimates.

Corollary 5.2: *Let $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ be a positive stable solution to the problem $(2)_s$. Then the following holds:*

(1) *Assume that $0 < u \leq 1$ a.e. in Ω and $1 \leq \delta < \gamma$ be such that $\delta \in A \cap I$. Then for any $\beta \in (l_1, l_2)$, there exists a constant $C = C(p_1, p_2, \ldots, p_N, q, N, \beta)$ such that for every $\psi \in C_c^1(\Omega)$ with $0 \leq \psi \leq 1$ in Ω, we have*

$$\int_\Omega g(x)\left(\frac{\psi}{u}\right)^{2\beta+\delta+q-1}\mathrm{d}x \leq C\sum_{i=1}^N\int_\Omega |\psi_i|^{p_i\theta_i'}\,\mathrm{d}x \tag{13}$$

where

$$\theta_i = \frac{2\beta+\delta+q-1}{2\beta+q-p_i}, \quad \theta_i' = \frac{2\beta+\delta+q-1}{\delta+p_i-1}.$$

(2) *Assume that $u \geq 1$ a.e. in Ω and $0 < \delta < \gamma$ be such that $\delta \in A$ and $\gamma \in I \cap [1,\infty)$. Then for any $\beta \in (l_1, l_2)$, there exists a constant $C = C(p_1, p_2, \ldots, p_N, q, N, \beta)$ such that for every $\psi \in C_c^1(\Omega)$ with $0 \leq \psi \leq 1$ in Ω, we have*

$$\int_\Omega g(x)\left(\frac{\psi}{u}\right)^{2\beta+\gamma+q-1}\mathrm{d}x \leq C\sum_{i=1}^N\int_\Omega |\psi_i|^{p_i\zeta_i'}\,\mathrm{d}x \tag{14}$$

where

$$\zeta_i = \frac{2\beta+\gamma+q-1}{2\beta+q-p_i}, \quad \zeta_i' = \frac{2\beta+\gamma+q-1}{\gamma+p_i-1}.$$

(3) *Assume that $u > 0$ a.e. in Ω and $1 \le \delta = \gamma \in A \cap I$. Then for any $\beta \in (l_1, l_2)$, there exists a constant $C = C(p_1, p_2, \ldots, p_N, q, N, \beta)$ such that for every $\psi \in C_c^1(\Omega)$ with $0 \le \psi \le 1$ in Ω, we have*

$$\int_\Omega g(x) \left(\frac{\psi}{u}\right)^{2\beta+\delta+q-1} \mathrm{d}x \le C \sum_{i=1}^N \int_\Omega |\psi_i|^{p_i\theta_i'} \mathrm{d}x \tag{15}$$

where

$$\theta_i = \frac{2\beta+\delta+q-1}{2\beta+q-p_i}, \quad \theta_i' = \frac{2\beta+\delta+q-1}{\delta+p_i-1}.$$

Corollary 5.3: *Let $u \in W_{\mathrm{loc}}^{1,p_i}(\Omega)$ be a positive stable solution to the problem $(2)_e$ such that $0 < u \le M$ a.e. in Ω for some positive constant M. Then for any $\beta \in (l_1, l_3)$ there exists a constant $C = C(p_1, p_2, \ldots, p_N, q, N, \beta)$ such that for every $\psi \in C_c^1(\Omega)$ with $0 \le \psi \le 1$ in Ω, we have*

$$\int_\Omega g(x) \left(\frac{\psi}{u}\right)^{2\beta+q} \mathrm{d}x \le C \sum_{i=1}^N \int_\Omega |\psi_i|^{2\beta+q} \mathrm{d}x. \tag{16}$$

Proof of Lemma 5.1: Let $u \in W_{\mathrm{loc}}^{1,p_i}(\Omega)$ be a positive stable solution to the Equation (2) and $\psi \in C_c^1(\Omega)$ be nonnegative in Ω. Then u satisfies both the equations (7) and (8). We prove the lemma into the following two steps.

Step 1. Choosing $\phi = b_k(u)\psi^q$ as a test function in (7), we have

$$\sum_{i=1}^N \int_\Omega |b_k'(u)||u_i|^{p_i}\psi^q \,\mathrm{d}x$$
$$\le q \sum_{i=1}^N \int_\Omega \psi^{q-1} b_k(u)|u_i|^{p_i-2}u_i\psi_i \,\mathrm{d}x - \int_\Omega g(x)f(u)b_k(u)\psi^q \,\mathrm{d}x. \tag{17}$$

Using Young's inequality with $\epsilon \in (0,1)$, we obtain

$$q \sum_{i=1}^N \int_\Omega \psi^{q-1} b_k(u)|u_i|^{p_i-2}u_i\psi_i \,\mathrm{d}x$$
$$\le \epsilon \sum_{i=1}^N \int_\Omega |b_k'(u)||u_i|^{p_i}\psi^q \,\mathrm{d}x + C \sum_{i=1}^N \int_\Omega b_k(u)^{p_i}|b_k'(u)|^{1-p_i}|\psi_i|^{p_i}\psi^{q-p_i} \,\mathrm{d}x,$$

for some positive constant depending $C = C_\epsilon(p_1, p_2, \ldots, p_N, q)$.

Therefore for $\epsilon \in (0,1)$, we obtain

$$(1-\epsilon) \sum_{i=1}^N |b_k'(u)||u_i|^{p_i}\psi^q \,\mathrm{d}x$$
$$\le C \sum_{i=1}^N \int_\Omega b_k(u)^{p_i}|b_k'(u)|^{1-p_i}|\psi_i|^{p_i}\psi^{q-p_i} \,\mathrm{d}x - \int_\Omega g(x)f(u)b_k(u)\psi^q \,\mathrm{d}x. \tag{18}$$

Step 2. Choosing $\phi = a_k(u)\psi^{\frac{q}{2}}$ in the inequality (8), we obtain

$$\int_\Omega g(x)f'(u)a_k(u)^2\psi^q \,\mathrm{d}x \le \sum_{i=1}^N (p_i-1)\left(X_i + \frac{q^2}{4}Y_i + qZ_i\right), \tag{19}$$

where

$$X_i = \int_\Omega |a_k'(u)|^2|u_i|^{p_i}\psi^q \,\mathrm{d}x, \quad Y_i = \int_\Omega \psi^{q-2}a_k(u)^2|u_i|^{p_i-2}|\psi_i|^2 \,\mathrm{d}x,$$

and

$$Z_i = \int_\Omega |a_k'(u)|a_k(u)\psi^{q-1}|u_i|^{p_i-1}|\psi_i| \,\mathrm{d}x.$$

Using (c) noting that

$$X_i = \frac{(\alpha-1)^2}{4\alpha}\int_\Omega |b_k'(u)||u_i|^{p_i}\psi^q \,\mathrm{d}x,$$

from the estimate (18), we obtain

$$\begin{aligned}\sum_{i=1}^N X_i &= \frac{(\alpha-1)^2}{4\alpha}\sum_{i=1}^N\int_\Omega |b_k'(u)||u_i|^{p_i}\psi^q \,\mathrm{d}x\\ &\le \frac{(\alpha-1)^2}{4\alpha(1-\epsilon)}\left\{C\sum_{i=1}^N\int_\Omega b_k(u)^{p_i}|b_k'(u)|^{1-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x\right.\\ &\quad\left. - \int_\Omega g(x)f(u)b_k(u)\psi^q \,\mathrm{d}x\right\}.\end{aligned}$$

Moreover, using Young's inequality we have

$$\begin{aligned}&(p_i-1)\frac{q^2}{4}Y_i\\ &\quad= (p_i-1)\frac{q^2}{4}\int_\Omega \psi^{q-2}a_k(u)^2|u_i|^{p_i-2}|\psi_i|^2\,\mathrm{d}x\\ &\quad= (p_i-1)\frac{q^2}{4}\int_\Omega \left(|u_i|^{p_i-2}|a_k'(u)|^{\frac{2(p_i-2)}{p_i}}\psi^{\frac{q(p_i-2)}{p_i}}\right)\\ &\qquad\times\left(a_k(u)^2|a_k'(u)|^{\frac{2(2-p_i)}{p_i}}|\psi_i|^2\psi^{\frac{2(q-p_i)}{p_i}}\right)\mathrm{d}x\\ &\quad\le \frac{\epsilon}{2N}X_i + \frac{C}{2}\int_\Omega a_k(u)^{p_i}|a_k'(u)|^{2-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x,\end{aligned}$$

and

$$
\begin{aligned}
&(p_i-1)qZ_i \\
&\quad = (p_i-1)q\int_\Omega |a_k'(u)|a_k(u)\psi^{q-1}|u_i|^{p_i-1}|\psi_i|\,\mathrm{d}x \\
&\quad = (p_i-1)q\int_\Omega \left(|u_i|^{p_i-1}|a_k'(u)|^{\frac{2}{p_i'}}\psi^{\frac{q}{p_i'}}\right)\left(a_k(u)|a_k'(u)|^{\frac{2-p_i}{p_i}}|\psi|^{p_i}\psi^{q-p_i}\right)\mathrm{d}x \\
&\quad \le \frac{\epsilon}{2N}X_i + \frac{C}{2}\int_\Omega a_k(u)^{p_i}|a_k'(u)|^{2-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x
\end{aligned}
$$

for some positive constant $C = C_\epsilon(p_1, p_2, \dots, p_N, q, N)$.

Using the above estimates in (19) together with (a) and (b) we obtain

$$
\begin{aligned}
&\int_\Omega g(x)uf'(u)b_k(u)\psi^q\,\mathrm{d}x \\
&\quad \le \int_\Omega g(x)f'(u)a_k(u)^2\psi^q\,\mathrm{d}x \\
&\quad \le \sum_{i=1}^N \left(p_i - 1 + \frac{\epsilon}{N}\right)X_i + C\sum_{i=1}^N\int_\Omega a_k(u)^{p_i}|a_k'(u)|^{2-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x \\
&\quad \le \left(p_1 - 1 + \frac{\epsilon}{N}\right)\sum_{i=1}^N X_i + \left(p_2 - 1 + \frac{\epsilon}{N}\right)\sum_{i=1}^N X_i + \cdots + \left(p_N - 1 + \frac{\epsilon}{N}\right)\sum_{i=1}^N X_i \\
&\qquad + C\sum_{i=1}^N\int_\Omega a_k(u)^{p_i}|a_k'(u)|^{2-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x \\
&\quad = \big(N(q-1)+\epsilon\big)\sum_{i=1}^N X_i + C\sum_{i=1}^N\int_\Omega a_k(u)^{p_i}|a_k'(u)|^{2-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x \\
&\quad \le \frac{(\alpha-1)^2\big(N(q-1)+\epsilon\big)}{4\alpha(1-\epsilon)}\left\{C\sum_{i=1}^N\int_\Omega b_k(u)^{p_i}|b_k'(u)|^{1-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x\right. \\
&\qquad \left. - \int_\Omega g(x)f(u)b_k(u)\psi^q\,\mathrm{d}x\right\} + C\sum_{i=1}^N\int_\Omega a_k(u)^{p_i}|a_k'(u)|^{2-p_i}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x \\
&\quad \le C\sum_{i=1}^N\int_\Omega\left\{b_k(u)^{p_i}|b_k'(u)|^{1-p_i} + a_k(u)^{p_i}|a_k'(u)|^{2-p_i}\right\}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x \\
&\qquad - \frac{(\alpha-1)^2(N(q-1)+\epsilon)}{4\alpha(1-\epsilon)}\int_\Omega g(x)f(u)b_k(u)\psi^q\,\mathrm{d}x \\
&\quad \le C\sum_{i=1}^N\int_\Omega u^{p_i-\alpha-1}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x \\
&\qquad - \frac{(\alpha-1)^2(N(q-1)+\epsilon)}{4\alpha(1-\epsilon)}\int_\Omega g(x)f(u)b_k(u)\psi^q\,\mathrm{d}x
\end{aligned}
$$

for some positive constant $C = C_\epsilon(p_1, \dots, p_N, q, N, \alpha)$. ■

Proof of Corollary 5.2: Let $u \in W^{1,p_i}_{\text{loc}}(\Omega)$ be a positive stable solution to the problem $(2)_s$. Observe that the fact $\beta > l_1$ implies $\alpha = 2\beta + q - 1 > p_N - 1$. Then by Lemma 5.1, using the fact $0 < \delta \le \gamma$ and $f(u) = -u^{-\delta} - u^{-\gamma}$ in the inequality (12), for some $C = C_\epsilon(p_1, \dots, p_N, q, N, \alpha)$ we obtain

$$\alpha_\epsilon \int_\Omega g(x)(u^{-\delta} + u^{-\gamma}) b_k(u)\psi^q \, dx \le C \sum_{i=1}^N \int_\Omega u^{p_i - \alpha - 1} |\psi_i|^{p_i} \psi^{q-p_i} \, dx,$$

where $\alpha_\epsilon = \delta - \frac{(\alpha-1)^2(N(q-1)+\epsilon)}{4\alpha(1-\epsilon)}$. Observe that

$$\lim_{\epsilon \to 0} \alpha_\epsilon = \delta - \frac{N(q-1)(\alpha-1)^2}{4\alpha} > 0, \quad \forall\, \beta \in (l_1, l_2).$$

Hence we can fix $\beta \in (l_1, l_2)$ and choose $\epsilon \in (0,1)$ such that $\alpha_\epsilon > 0$. As a consequence we have

$$\int_\Omega g(x)(u^{-\delta} + u^{-\gamma}) b_k(u)\psi^q \, dx \le C \sum_{i=1}^N \int_\Omega u^{p_i - 2\beta - q} |\psi_i|^{p_i} \psi^{q-p_i} \, dx \tag{20}$$

for some positive constant $C = C(p_1, \dots, p_N, q, N, \alpha)$.

(1) Since $\delta < \gamma$ and $0 < u \le 1$ a.e. in Ω, for any $\beta \in (l_1, l_2)$ the inequality (20) becomes

$$\int_\Omega g(x) u^{-\delta} b_k(u)\psi^q \, dx \le C \sum_{i=1}^N \int_\Omega |u|^{p_i - 2\beta - q} |\psi_i|^{p_i} \psi^{q-p_i} \, dx.$$

By the monotone convergence theorem we obtain

$$\int_\Omega g(x) u^{-2\beta - \delta - q + 1} \psi^q \, dx \le C \sum_{i=1}^N \int_\Omega |u|^{p_i - 2\beta - q} |\psi_i|^{p_i} \psi^{q-p_i} \, dx.$$

Replacing ψ by $\psi^{\frac{2\beta+\delta+q-1}{q}}$ and using the Young's inequality for $\epsilon \in (0,1)$ with the exponents $\theta_i = \frac{2\beta+\delta+q-1}{2\beta+q-p_i}$, $\theta_i{}' = \frac{2\beta+\delta+q-1}{\delta+p_i-1}$ in the above inequality we obtain

$$\begin{aligned}
&\int_\Omega g(x) \left(\frac{\psi}{u}\right)^{2\beta+\delta+q-1} dx \\
&\le C \sum_{i=1}^N \int_\Omega u^{p_i - 2\beta - q} \psi^{2\beta+\delta+q-p_i-1} |\psi_i|^{p_i} \, dx \\
&= C \sum_{i=1}^N \int_\Omega \left(\left(\frac{\psi}{u}\right)^{2\beta+q-p_i} \right) \left(\psi^{\delta-1} |\psi_i|^{p_i}\right) dx \\
&\le \epsilon \int_\Omega g(x) \left(\frac{\psi}{u}\right)^{2\beta+\delta+q-1} dx + C \sum_{i=1}^N \int_\Omega g^{-\frac{\theta_i'}{\theta_i}} \psi^{(\delta-1)\theta_i{}'} |\psi_i|^{p_i \theta_i'} \, dx.
\end{aligned}$$

Using $\delta \geq 1$ and choosing $0 \leq \psi \leq 1$ in Ω together with the fact $g \geq c$ we obtain

$$\int_\Omega g(x)\left(\frac{\psi}{u}\right)^{2\beta+\delta+q-1} \mathrm{d}x \leq C\sum_{i=1}^N \int_\Omega |\psi_i|^{p_i\theta_i'}\,\mathrm{d}x,$$

for some positive constant $C = C(p_1,\dots,p_N,q,N,\beta)$.

(2) Since $\delta < \gamma$ and $u \geq 1$ a.e. in Ω, for any $\beta \in (l_1,l_2)$ the inequality (20) becomes

$$\int_\Omega g(x)u^{-\gamma}b_k(u)\psi^q\,\mathrm{d}x \leq C\sum_{i=1}^N \int_\Omega |u|^{p_i-2\beta-q}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x.$$

By the monotone convergence theorem we obtain

$$\int_\Omega g(x)u^{-2\beta-\gamma-q+1}\psi^q\,\mathrm{d}x \leq C\sum_{i=1}^N \int_\Omega |u|^{p_i-2\beta-q}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x.$$

Replacing ψ by $\psi^{\frac{2\beta+\gamma+q-1}{q}}$ and using the Young's inequality for $\epsilon \in (0,1)$ with the exponents $\zeta_i = \frac{2\beta+\gamma+q-1}{2\beta+q-p_i}$, $\zeta_i' = \frac{2\beta+\gamma+q-1}{\gamma+p_i-1}$ in the above inequality we obtain

$$\begin{aligned}
&\int_\Omega g(x)\left(\frac{\psi}{u}\right)^{2\beta+\gamma+q-1}\mathrm{d}x\\
&\leq C\sum_{i=1}^N \int_\Omega u^{p_i-2\beta-q}\psi^{2\beta+\gamma+q-p_i-1}|\psi_i|^{p_i}\,\mathrm{d}x\\
&= C\sum_{i=1}^N \int_\Omega \left(\left(\frac{\psi}{u}\right)^{2\beta+q-p_i}\right)\left(\psi^{\gamma-1}|\psi_i|^{p_i}\right)\mathrm{d}x\\
&\leq \epsilon\int_\Omega g(x)\left(\frac{\psi}{u}\right)^{2\beta+\gamma+q-1}\mathrm{d}x + C\sum_{i=1}^N \int_\Omega g^{-\frac{\zeta_i'}{\zeta_i}}\psi^{(\gamma-1)\zeta_i'}|\psi_i|^{p_i\zeta_i'}\,\mathrm{d}x.
\end{aligned}$$

Using $\gamma \geq 1$ and choosing $0 \leq \psi \leq 1$ in Ω together with the fact $g \geq c$ we obtain

$$\int_\Omega g(x)\left(\frac{\psi}{u}\right)^{2\beta+\gamma+q-1} \mathrm{d}x \leq C\sum_{i=1}^N \int_\Omega |\psi_i|^{p_i\zeta_i'}\,\mathrm{d}x,$$

for some positive constant $C = C(p_1,\dots,p_N,q,N,\beta)$.

(3) Since $\delta = \gamma \geq 1$ and $u > 0$ a.e. in Ω, for any $\beta \in (l_1,l_2)$ the inequality (20) becomes

$$\int_\Omega g(x)u^{-\delta}b_k(u)\psi^q\,\mathrm{d}x \leq C\sum_{i=1}^N \int_\Omega |u|^{p_i-2\beta-q}|\psi_i|^{p_i}\psi^{q-p_i}\,\mathrm{d}x.$$

Now proceeding similarly as in Case (1) we obtain the required estimate.

■

Proof of Corollary 5.3: Assume $M \in J$ and let $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ be such that $0 < u \leq M$ a.e. in Ω is a positive stable solution of the Equation $(2)_e$. Let $\beta \in (l_1, l_3)$ and define $\alpha = 2\beta + q - 1$. Observe that the fact $\beta > l_1$ implies $\alpha > p_N - 1$. Therefore we can apply Lemma 5.1 to choose $f(u) = -\mathrm{e}^{\frac{1}{u}}$ and use the assumption $0 < u \leq M$ a.e. in Ω in the estimate (12) and obtain

$$\alpha_\epsilon \int_\Omega g(x)\, \mathrm{e}^{\frac{1}{u}} b_k(u) \psi^q \,\mathrm{d}x \leq C \sum_{i=1}^{N} \int_\Omega u^{p_i - \alpha - 1} |\psi_i|^{p_i} \psi^{q-p_i} \,\mathrm{d}x,$$

for some positive constant $C = C_\epsilon(p_1, \dots, p_N, q, N, \alpha)$ where $\alpha_\epsilon = \frac{1}{M} - \frac{(\alpha-1)^2(N(q-1)+\epsilon)}{4\alpha(1-\epsilon)}$. Observe that

$$\lim_{\epsilon \to 0} \alpha_\epsilon = \frac{1}{M} - \frac{N(q-1)(\alpha-1)^2}{4\alpha} > 0, \quad \forall\, \beta \in (l_1, l_3).$$

Hence we can fix $\beta \in (l_1, l_3)$ and choose $\epsilon \in (0,1)$ such that $\alpha_\epsilon > 0$. Using $\mathrm{e}^x > x$ for $x > 0$, in the above estimate we obtain

$$\int_\Omega g(x) \frac{1}{u} b_k(u) \psi^q \,\mathrm{d}x \leq \int_\Omega g(x)\, \mathrm{e}^{\frac{1}{u}} b_k(u) \psi^q \,\mathrm{d}x \leq C \sum_{i=1}^{N} \int_\Omega u^{p_i - 2\beta - q} |\psi_i|^{p_i} \psi^{q-p_i} \,\mathrm{d}x,$$

for some positive constant $C = C(\beta, p_1, \dots, p_N, q, N)$. By the monotone convergence theorem we obtain

$$\int_\Omega g(x) u^{-2\beta - q} \psi^q \,\mathrm{d}x \leq C \sum_{i=1}^{N} \int_\Omega u^{p_i - 2\beta - q} |\psi_i|^{p_i} \psi^{q-p_i} \,\mathrm{d}x.$$

Replacing ψ by $\psi^{\frac{2\beta+q}{q}}$ and using the Young's inequality for $\epsilon \in (0,1)$ with exponents $\gamma_i = \frac{2\beta+q}{2\beta+q-p_i}$, $\gamma_i' = \frac{2\beta+q}{p_i}$ in the above inequality we obtain

$$\begin{aligned}
&\int_\Omega g(x) \left(\frac{\psi}{u}\right)^{2\beta+q} \mathrm{d}x \\
&\leq C \sum_{i=1}^{N} \int_\Omega \left(\frac{\psi}{u}\right)^{2\beta+q-p_i} |\psi_i|^{p_i} \,\mathrm{d}x \\
&\leq \epsilon \int_\Omega g(x) \left(\frac{\psi}{u}\right)^{2\beta+q} \mathrm{d}x + C \sum_{i=1}^{N} \int_\Omega g^{-\frac{\gamma_i'}{\gamma_i}} |\psi_i|^{2\beta+q} \,\mathrm{d}x.
\end{aligned}$$

Therefore, using the fact that $g \geq c$, we have

$$\int_\Omega g(x) \left(\frac{\psi}{u}\right)^{2\beta+q} \mathrm{d}x \leq C \sum_{i=1}^{N} \int_\Omega |\psi_i|^{2\beta+q} \,\mathrm{d}x,$$

for some positive constant $C = C(\beta, p_1, \dots, p_N, q, N)$. ∎

5.2. Proof of the main results

Proof of Theorem 3.2: Let $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ be a stable solution of the Equation $(2)_s$ such that $0 < u \leq 1$ a.e. in Ω. Then by Corollary 5.2 we have

$$\int_{\Omega} g(x) \left(\frac{\psi}{u}\right)^{2\beta+\delta+q-1} \mathrm{d}x \leq C \sum_{i=1}^{N} \int_{\Omega} |\psi_i|^{p_i\theta_i'} \mathrm{d}x.$$

Choosing $\psi = \psi_R$ in the above inequality we obtain

$$\int_{B_R(0)} g(x) \left(\frac{1}{u}\right)^{2\beta+\delta+q-1} \mathrm{d}x \leq C \sum_{i=1}^{N} R^{N-p_i\theta_i'}, \tag{21}$$

for some positive constant C independent of R. Observe that,

$$\lim_{\beta \to l_2} (N - p_i\theta_i') = N - \frac{p_i(2l_2 + \delta + q - 1)}{\delta + p_i - 1} < 0$$

which follows from the assumption $\delta \in I$, since

$$\delta > \frac{N^2(q-1)(p_i-1)}{p_i(N(q-1)+4) - N^2(q-1)} \quad \text{for all } i = 1, 2, \ldots, N.$$

As a consequence, we can choose $\beta \in (l_1, l_2)$, such that $N - p_i\theta_i' < 0$ for all i. Therefore, letting $R \to \infty$ in (21), we obtain

$$\int_{\Omega} g(x) \left(\frac{1}{u}\right)^{2\beta+\delta+q-1} \mathrm{d}x = 0,$$

which is a contradiction. ■

Proof of Theorem 3.3: Let $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ be a stable solution of the Equation $(2)_s$ such that $u \geq 1$ a.e. in Ω. Then by Corollary 5.2 we have

$$\int_{\Omega} g(x) \left(\frac{\psi}{u}\right)^{2\beta+\gamma+q-1} \mathrm{d}x \leq C \sum_{i=1}^{N} \int_{\Omega} |\psi_i|^{p_i\zeta_i'} \mathrm{d}x.$$

Choosing $\psi = \psi_R$ in the above inequality we obtain

$$\int_{B_R(0)} g(x) \left(\frac{1}{u}\right)^{2\beta+\gamma+q-1} \mathrm{d}x \leq C \sum_{i=1}^{N} R^{N-p_i\zeta_i'}, \tag{22}$$

for some positive constant C independent of R. Observe that,

$$\lim_{\beta \to l_2} (N - p_i\zeta_i') = N - \frac{p_i(2l_2 + \gamma + q - 1)}{\delta + p_i - 1} < 0$$

which follows from the assumption $\gamma \in I$, since $\gamma > \frac{N^2(q-1)(p_i-1)}{p_i(N(q-1)+4)-N^2(q-1)}$ for all $i = 1, 2, \ldots, N$. As a consequence, we can choose $\beta \in (l_1, l_2)$, such that $N - p_i\zeta_i' < 0$ for all i.

Therefore, letting $R \to \infty$ in (22), we obtain

$$\int_{\Omega} g(x) \left(\frac{1}{u}\right)^{2\beta+\gamma+q-1} \mathrm{d}x = 0,$$

which is a contradiction. ■

Proof of Theorem 3.4: Let $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ be a positive stable solution of the Equation $(2)_s$. Then by Corollary 5.2 we have

$$\int_{\Omega} g(x) \left(\frac{\psi}{u}\right)^{2\beta+\delta+q-1} \mathrm{d}x \leq C \sum_{i=1}^{N} \int_{\Omega} |\psi_i|^{p_i\theta_i'} \,\mathrm{d}x.$$

Now proceeding similarly as in Theorem 3.2 we obtain

$$\int_{\Omega} g(x) \left(\frac{1}{u}\right)^{2\beta+\delta+q-1} \mathrm{d}x = 0,$$

which is a contradiction. ■

Proof of Theorem 3.5: Let $u \in W^{1,p_i}_{\mathrm{loc}}(\Omega)$ be a stable solution to the problem $(2)_e$ such that $0 < u \leq M$ a.e. in Ω. Then by Corollary 5.3 we have

$$\int_{\Omega} g(x) \left(\frac{\psi}{u}\right)^{2\beta+q} \mathrm{d}x \leq C \sum_{i=1}^{N} \int_{\Omega} |\psi_i|^{2\beta+q} \,\mathrm{d}x.$$

Choosing $\psi = \psi_R$ in the above inequality we obtain

$$\int_{B_R(0)} g(x) \left(\frac{1}{u}\right)^{2\beta+q} \mathrm{d}x \leq CR^{N-2\beta-q}, \tag{23}$$

where C is a positive constant independent of R. Observe that, since $M \in J$ we have $0 < M < \frac{4}{N(N-1)(q-1)}$ which implies $N < 2l_3 + q$ and hence

$$\lim_{\beta \to l_3} (N - 2\beta - q) = N - 2l_3 - q < 0.$$

As a consequence, we can choose $\beta \in (l_1, l_3)$ such that $N - 2\beta - q < 0$.

Therefore, letting $R \to \infty$ in (23), we obtain

$$\int_{\Omega} g(x) \left(\frac{1}{u}\right)^{2\beta+q} \mathrm{d}x = 0,$$

which is a contradiction. Hence the Theorem follows. ■

Acknowledgments

The author would like to thank the anonymous referee for his/her valuable comments and suggestions. The author thanks Professor Adi Adimurthi for some fruitful discussion on the topic. The author thanks TIFR CAM, Bangalore for the financial support.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The author thanks Tata Institute of Fundamental Research (TIFR), Centre for Applicable Mathematics (CAM), Bangalore for the financial support.

References

[1] Crandall MG, Rabinowitz PH, Tartar L. On a Dirichlet problem with a singular nonlinearity. Comm Partial Differential Equations. 1977;2(2):193–222.
[2] Lazer AC, McKenna PJ. On a singular nonlinear elliptic boundary-value problem. Proc Amer Math Soc. 1991;111(3):721–730.
[3] Boccardo L, Orsina L. Semilinear elliptic equations with singular nonlinearities. Calc Var Partial Differential Equations. 2010;37(3–4):363–380.
[4] Canino A, Degiovanni M. A variational approach to a class of singular semilinear elliptic equations. J Convex Anal. 2004;11(1):147–162.
[5] Canino A, Sciunzi B. A uniqueness result for some singular semilinear elliptic equations. Commun Contemp Math. 2016;18(6):1550084, 9.
[6] Canino A, Sciunzi B, Trombetta A. Existence and uniqueness for p-Laplace equations involving singular nonlinearities. NoDEA Nonlinear Differential Equations Appl. 2016;23(2):Art. 8, 18.
[7] Arcoya D, Moreno-Mérida L. Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity. Nonlinear Anal. 2014;95:281–291.
[8] Badra M, Bal K, Giacomoni J. A singular parabolic equation: existence, stabilization. J Differential Equations. 2012;252(9):5042–5075.
[9] Ghergu M, Rădulescu VD. Singular elliptic problems: bifurcation and asymptotic analysis. Oxford: The Clarendon Press, Oxford University Press; 2008. (Oxford lecture series in mathematics and its applications: vol. 37).
[10] Farina A. Stable solutions of $-\Delta u = e^u$ on $\mathbb{R}^N$. C R Math Acad Sci Paris. 2007;345(2):63–66.
[11] Dupaigne L, Farina A. Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$. J Eur Math Soc (JEMS). 2010;12(4):855–882.
[12] Dupaigne L. Stable solutions of elliptic partial differential equations. Boca Raton (FL): Chapman & Hall/CRC; 2011. (Chapman & Hall/CRC monographs and surveys in pure and applied mathematics: vol. 143).
[13] Ma L, Wei JC. Properties of positive solutions to an elliptic equation with negative exponent. J Funct Anal. 2008;254(4):1058–1087.
[14] Guo ZM, Mei LF. Liouville type results for a p-Laplace equation with negative exponent. Acta Math Sin (Engl Ser). 2016;32(12):1515–1540.
[15] Chen C, Song H, Yang H. Liouville type theorems for stable solutions of p-Laplace equation in $\mathbb{R}^N$. Nonlinear Anal. 2017;160:44–52.
[16] Bal K, Garain P. Nonexistence results for weighted p-Laplace equations with singular nonlinearities. Electron J Differential Equations. 2019;2019:pages Paper No. 95, 12.
[17] Le P. Nonexistence of stable solutions to p-Laplace equations with exponential nonlinearities. Electron J Differential Equations. 2016;2016:pages Paper No. 326, 5.
[18] Lions J-L. Quelques méthodes de résolution des problèmes aux limites non linéaires. Paris: Dunod; Gauthier-Villars; 1969.
[19] Weickert J. Anisotropic diffusion in image processing. Stuttgart: B. G. Teubner; 1998. (European consortium for mathematics in industry).
[20] Antontsev SN, Díaz JI, Shmarev S. Applications to nonlinear PDEs and fluid mechanics. Energy methods for free boundary problems. Boston (MA): Birkhäuser Boston, Inc.; 2002. (Progress in nonlinear differential equations and their applications: vol. 48).

[21] Leggat AR, El-Hadi Miri S. Anisotropic problem with singular nonlinearity. Complex Var Elliptic Equ. 2016;61(4):496–509.
[22] El-Hadi Miri S. On an anisotropic problem with singular nonlinearity having variable exponent. Ric Mat. 2017;66(2):415–424.
[23] Perera K, Silva EAB. On singular p-Laplacian problems. Differential Integral Equations. 2007;20(1):105–120.
[24] Di Castro A. Elliptic problems for some anisotropic operators [PhD thesis]. Rome: University of Rome "Sapienza", a. y.; 2008/2009.
[25] Di Castro A. Existence and regularity results for anisotropic elliptic problems. Adv Nonlinear Stud. 2009;9(2):367–393.
[26] Kruzhkov SN, Kolodiĭ $\bar{I}$M. On the theory of anisotropic Sobolev spaces. Uspekhi Mat Nauk. 1983;38(2(230)):207–208.
[27] El Manouni S. Note on an anisotropic p-Laplacian equation in $\mathbb{R}^n$. Electron J Qual Theory Differ Equ. 2010;2010:pages No. 73, 9.
[28] Trudinger NS. An imbedding theorem for $H_0(G, \Omega)$ spaces. Studia Math. 1974;50:17–30.
[29] Kinderlehrer D, Stampacchia G. An introduction to variational inequalities and their applications. New York-London: Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers]; 1980. (Pure and applied mathematics: vol. 88).
[30] Peral. I. Multiplicity of solutions for the p-Laplacian, Lecture Notes at the Second School on Nonlinear Functional Analysis and Applications to Differential Equations at ICTP, Trieste, 1997.