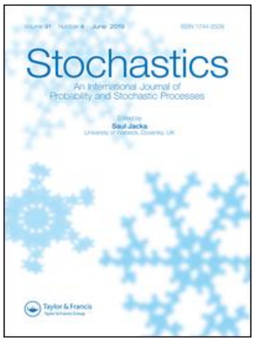

Taylor & Francis
Taylor & Francis Group

Stochastics

An International Journal of Probability and Stochastic Processes

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gssr20

General methods for bounding multidimensional ruin probabilities in regime-switching models

Lesław Gajek & Marcin Rudź

STOCHASTICS
https://doi.org/10.1080/17442508.2020.1801684

General methods for bounding multidimensional ruin probabilities in regime-switching models

Lesław Gajek and Marcin Rudź

Institute of Mathematics, Lodz University of Technology, Łódź, Poland

ABSTRACT
We present a universal methodology for bounding multidimensional ultimate ruin probabilities $\boldsymbol{\Psi}$ in regime-switching models. Some new lower and upper bounds on $\boldsymbol{\Psi}$ are given. The considered methods are applicable to several discrete- and continuous time risk models. As an example, we construct a variety of new two-sided operator bounds which converge to $\boldsymbol{\Psi}$ with an exponential rate. Several numerical examples are also provided.

ARTICLE HISTORY
Received 22 October 2019
Accepted 22 July 2020

KEYWORDS
Multidimensional risk operators; regime-switching models; Markov chains; multidimensional ruin probabilities; two-sided bounds

1. Introduction

The problem of ruin of an insurer has been a point of interest for many probabilists. In the classical sense, it can be tackled by the methods of renewal theory as it was shown, e.g. in Chapters XIII-XIV of Feller [6]. For other ideas and the detailed references to risk theory, we refer the reader to Asmussen and Albrecher [2], Kyprianou [17], Rolski *et al.* [24], among others.

In the present paper, we investigate the classical ruin problem in a more general regime-switching framework in which a Markov chain is assumed to switch the amount and/or waiting time distributions of claims. Regime-switching models have received considerable attention recently, see, e.g. Q. Liu *et al.* [20], Jacka and Ocejo [15], Momeya [22], Xu *et al.* [28], R.H. Liu [19], G. Wang *et al.* [27], Landriault *et al.* [18], Chen *et al.* [4], Guillou *et al.* [13], Lu [21], Asmussen [1], Gajek and Rudź [9–12] and the references therein for an overview of selected developments and applications of Markov-modulated models and related problems.

We will focus our attention on bounding multidimensional ultimate ruin probabilities in the following regime-switching Sparre Andersen model. By a claim we understand an individual claim, or a total claim after a given number of claims, or a total claim from a given period of time, respectively.

Let a random variable X_m denote the amount of the mth claim, T_1 – the moment when the first claim appears and T_m – the time between the $(m-1)$th claim and the mth one. Clearly, $A_n = T_1 + \cdots + T_n$, $n \geqslant 1$, is the moment when the nth claim appears with $A_0 = 0$. Let $\{I_0, I_1, I_2, \ldots\}$ be a time-homogeneous Markov chain with a finite state space

CONTACT Marcin Rudź marcin.rudz@p.lodz.pl

$S = \{1, 2, \dots, s\}$ such that the probabilities $p_i = \mathbb{P}(I_0 = i)$ are positive and $p_{ij} = \mathbb{P}(I_{m+1} = j | I_m = i)$ are non-negative for all $i, j \in S$. The jump from I_{m-1} to I_m can change the distribution of T_m and/or X_m at the moment A_m only (cf. the discussion in Gajek and Rudź [9, p. 237]), so one can interpret $\{I_0, I_1, I_2, \dots\}$ as 'switches'. We assume that a random variable $C_m = c(I_{m-1})$, where c is a known positive function defined on S, denote the insurance premium rate during the time interval $[A_{m-1}, A_m)$. The conditional distribution of X_1 (respectively T_1), given the initial state i and the state j at the moment A_1, will be denoted by F^{ij} (respectively G^{ij}, see Section 2 for details).

Let $u \geqslant 0$ denote the insurer's surplus at the moment A_0. Let $\mathcal{R}$ be the set of all measurable functions defined on $[0, \infty)$ and taking values in the interval [0,1] almost everywhere. The symbol $\mathcal{R}^s$ denotes the Cartesian product $\{(\rho_1, \dots, \rho_s) : \rho_i \in \mathcal{R} \text{ for every } i \in S\}$ and its elements will be written in bold. We call $\mathbf{L} : \mathcal{R}^s \to \mathcal{R}^s$ the generalized Volterra risk operator if $\mathbf{L}\boldsymbol{\rho}(u) = (\mathrm{L}_1\boldsymbol{\rho}(u), \dots, \mathrm{L}_s\boldsymbol{\rho}(u))$, where

$$\mathrm{L}_i\boldsymbol{\rho}(u) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \rho_j(u + c(i)t - x)\, \mathrm{d}F^{ij}(x)\, \mathrm{d}G^{ij}(t)$$

$$+ \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{u+c(i)t}^\infty \mathrm{d}F^{ij}(x)\, \mathrm{d}G^{ij}(t) \qquad (1)$$

for all $i \in S$, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_s) \in \mathcal{R}^s$ and $u \geqslant 0$, see Taylor [26] for a special case of the operator (1) in the classical non-switching Cramér–Lundberg model. Let us denote $M^i(r) = \sum_{j=1}^s p_{ij} \int_0^\infty \int_0^\infty \exp(-r(c(i)t - x))\, \mathrm{d}F^{ij}(x)\, \mathrm{d}G^{ij}(t)$ for all $i \in S$ and $r \in \mathbb{R}$. Assume that there exists (see Lemma 2.4 for details) the adjustment vector $(r^1, \dots, r^s)$ with positive coordinates which satisfy the equations $M^i(r^i) = 1$, $i \in S$. Set $\boldsymbol{\Psi}(u) = (\Psi^1(u), \dots, \Psi^s(u))$, where $\Psi^i(u) = \Psi(u, i)$ is the conditional ultimate ruin probability, given the initial state i of the Markov chain, considered as a function of u (see Section 2 for details). Let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_s) \in \mathcal{R}^s$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_s) \in \mathcal{R}^s$. To simplify the notation, we will write $\boldsymbol{\rho} \preceq \boldsymbol{\xi}$ if and only if $\rho_i(u) \leqslant \xi_i(u)$ for all $i \in S$ and $u \geqslant 0$.

The paper is arranged as follows. In Section 2, we briefly sketch some useful facts about the risk operator $\mathbf{L}$, the adjustment vector $(r^1, \dots, r^s)$ and the vector $(M^1, \dots, M^s)$ of generalized moment generating functions. In Section 3, we present the main results of the paper. We prove that any function $\boldsymbol{\rho} \in \mathcal{R}^s$ that satisfies

$$\mathbf{L}\boldsymbol{\rho} \preceq \boldsymbol{\rho}$$

is an upper bound for $\boldsymbol{\Psi}$ (see Theorem 3.1 for details). Surprisingly enough,

$$\boldsymbol{\rho} \preceq \mathbf{L}\boldsymbol{\rho}$$

does not imply that $\boldsymbol{\rho} \preceq \boldsymbol{\Psi}$ and a sufficient condition for bounding $\boldsymbol{\Psi}$ from below is more complex (see Theorem 3.2 and Remark 3.3 for details). An important reason lies in the fact that the vector $\boldsymbol{\Psi}$ is not the unique fixed point of $\mathbf{L}$. Indeed, the unit function $\mathbf{1}$ in $\mathcal{R}^s$ or, in general, any convex combination of $\boldsymbol{\Psi}$ and $\mathbf{1}$ constitutes a fixed point of $\mathbf{L}$ as well. Thereby, the risk operator $\mathbf{L}$ has in general infinitely many fixed points.

Theorem 3.1 and Corollary 3.4 provide some new useful and effective methods for bounding $\boldsymbol{\Psi}$ from above and below. As an example, we will use it to prove the following two-sided bound for $\mathbf{\Psi}$:

$$A_* \mathbf{Q}_0 \preceq \mathbf{\Psi} \preceq A^* \mathbf{R}_0,$$

where $r_* = \min\{r^i : i \in S\}$, $r^* = \max\{r^i : i \in S\}$, $R_0^i(u) = \exp(-r_* u)$, $Q_0^i(u) = \exp(-r^* u)$, $\mathbf{R}_0(u) = (R_0^1(u), \ldots, R_0^s(u))$, $\mathbf{Q}_0(u) = (Q_0^1(u), \ldots, Q_0^s(u))$, $i \in S$, $u \geqslant 0$, and the constants A_* and A^* are properly chosen (see Section 3 and Theorem 3.5 for details).

Set $D_0^i(u) = A_* Q_0^i(u)$, $U_0^i(u) = A^* R_0^i(u)$ and $M^*(r) = \max\{M^i(r) : i \in S\}$. Let $\mathbf{D}_n(u) = (D_n^1(u), \ldots, D_n^s(u))$ and $\mathbf{U}_n(u) = (U_n^1(u), \ldots, U_n^s(u))$ be the nth iteration of $\mathbf{L}$ on $\mathbf{D}_0(u) = (D_0^1(u), \ldots, D_0^s(u))$ and $\mathbf{U}_0(u) = (U_0^1(u), \ldots, U_0^s(u))$, respectively. The following new two-sided bounds for the vector $\mathbf{\Psi}$:

$$\mathbf{D}_n \preceq \mathbf{\Psi} \preceq \mathbf{U}_n$$

hold for any $n \in \mathbb{N}$ (see Theorem 3.6 for details). Moreover, for any $i \in S$, the sequences $\{D_n^i\}_{n \in \mathbb{N}}$ and $\{U_n^i\}_{n \in \mathbb{N}}$ converge monotonically, as $n \to \infty$, to Ψ^i with the exponential rate of convergence:

$$\left| U_n^i(u) - D_n^i(u) \right| \leqslant A^* \, \mathrm{e}^{-ru} [M^*(r)]^n, \quad i \in S, \; n \in \mathbb{N}, \; r \in (0, r_*), \; u \geqslant 0 \qquad (2)$$

(cf. Theorem 3.6 and Corollary 3.7). Inequality (2) may provide bounds more general as well as sharper than some existing results (see Remark 3.8 for details).

In Example 4.1, we use Theorem 3.1 to obtain other upper bounds for multidimensional ultimate ruin probabilities $\mathbf{\Psi}$. Inequality (20) improves as well as generalizes some known results (see Example 4.1 for details).

A distinct mathematical methodology for obtaining the unique fixed point of the risk operator, based on Banach Contraction Principle, can be found in Gajek and Rudź [10]. Some stochastic developments of Banach-type fixed point theorems are discussed in Saipara *et al.* [25]. Some asymptotic results for ruin probabilities can be found in Cheng and Yu [5], J. Peng and D. Wang [23], Guo *et al.* [14], Yang and Li [29], Konstantinides and Li [16], Cai and Dickson [3], among others.

2. An overview of basic properties of the risk operator, the adjustment vector and the vector of generalized moment generating functions

To make the paper self-contained, we briefly outline the model and some basic properties concerning the risk operator $\mathbf{L}$, the adjustment vector $(r^1, \ldots, r^s)$ and the vector $(M^1, \ldots, M^s)$ of generalized moment generating functions.

Assume first that all stochastic objects considered in the paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As usual, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the real line, respectively. We will use the following notation: $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_+^0 = [0, \infty)$ and $\overline{\mathbb{R}}_+ = (0, \infty]$. The random variables C_m, T_m and X_m, $m \in \mathbb{N}$, are assumed to be positive almost surely and their distributions – to have no singular parts. Let $U(n, u)$ denote the insurer's surplus at the moment A_n. The surplus process $\{U(n, \cdot)\}_{n \in \mathbb{N}}$ is defined then as $U(n, u) = u - \sum_{m=1}^n Z_m$, where $Z_m = X_m - c(I_{m-1}) T_m$, $m \in \mathbb{N}$. A more thorough interpretation of the above model, including the detailed simulation scheme for it, can be found in Gajek and Rudź [11]. Selected special cases of this regime-switching model are given in Gajek and Rudź [10, p. 46] with the detailed references, including Markov additive processes.

As usual, the time of ruin $\tau = \tau(u) = \inf\{n \in \mathbb{N} : U(n,u) < 0\}$ is the first moment when the insurer's surplus becomes negative with $\tau = \infty$ if $U(n,u) \geqslant 0$ for every $n \in \mathbb{N}$. For the convenience of the reader, we write $\mathbb{P}^i(B) = \mathbb{P}(B|I_0 = i), \mathbb{P}^{ij}(B) = \mathbb{P}(B|I_0 = i, I_1 = j)$ if $p_{ij} > 0$, or 0 otherwise and, consequently, $F^{ij}(x) = \mathbb{P}^{ij}(X_1 \leqslant x)$, $G^{ij}(t) = \mathbb{P}^{ij}(T_1 \leqslant t)$ and $\Psi^i(u) = \Psi(u,i) = \mathbb{P}^i(\tau(u) < \infty)$ for all $B \in \mathcal{F}$, $i,j \in S$ and $t,x \in \mathbb{R}_+$. Moreover, $\Psi_n^i(u) = \Psi_n(u,i)$ denotes $\mathbb{P}^i(\tau(u) \leqslant n)$, i.e. the conditional probability of ruin at or before the nth claim, $n \in \mathbb{N}$, given the initial state i, considered as a function of u, with $\Psi_0^i(u)$ equal to zero for all $i \in S$ and $u \geqslant 0$. The ruin probabilities $\Psi^1(u), \ldots, \Psi^s(u)$ (respectively $\Psi_n^1(u), \ldots, \Psi_n^s(u)$) form a vector $\boldsymbol{\Psi}(u)$ (respectively $\boldsymbol{\Psi}_n(u)$).

In Lemma 2.1 below, we will assume that the following condition holds.

Condition C1: For all $i,j \in S$, $m \in \{2,3,\ldots\}$ and $t,x \in \mathbb{R}_+$, $\mathbb{P}^{ij}(T_1 \leqslant t, X_1 \leqslant x) = F^{ij}(x)G^{ij}(t)$ and the conditional distribution of the random variables $Z_2, \ldots, Z_m$, given $(I_0 = i, I_1 = j, T_1 = t, X_1 = x)$, is the same as the conditional distribution of the random variables $Z_1, \ldots, Z_{m-1}$, given $I_0 = j$.

An important property is the upcoming relationship between $\boldsymbol{\Psi}_{n+1}$, $\boldsymbol{\Psi}_1$ and $\mathbf{L}$:

Lemma 2.1: *Let Condition* C1 *hold. Then*

$$\boldsymbol{\Psi}_{n+1}(u) = \mathbf{L}\boldsymbol{\Psi}_n(u) = \mathbf{L}^n\boldsymbol{\Psi}_1(u) \tag{3}$$

for all $n \in \{0,1,\ldots\}$ and $u \geqslant 0$.

Here and throughout the paper, $\mathbf{L}^0\boldsymbol{\Psi}_1 = \boldsymbol{\Psi}_1$ and $\mathbf{L}^1\boldsymbol{\Psi}_1 = \mathbf{L}\boldsymbol{\Psi}_1$. For a simulation scheme in which Condition C1 is satisfied, see Gajek and Rudź [11]. The proofs of Lemmas 2.1, 2.4, 2.6 and Remark 2.3 can be found in Gajek and Rudź [10].

Let us define a linear operator $\ell_i : \mathcal{R}^s \to \mathcal{R}$, $i \in S$, by

$$\ell_i\boldsymbol{\rho}(u) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \rho_j(u + c(i)t - x)\,\mathrm{d}F^{ij}(x)\,\mathrm{d}G^{ij}(t), \quad u \geqslant 0,$$

and denote the vector $(\ell_1\boldsymbol{\rho}, \ldots, \ell_s\boldsymbol{\rho})$ by $\boldsymbol{\ell}\boldsymbol{\rho}$. Clearly, $\boldsymbol{\ell}^n\boldsymbol{\rho}(u) = (\ell_1\boldsymbol{\ell}^{n-1}\boldsymbol{\rho}(u), \ldots, \ell_s\boldsymbol{\ell}^{n-1}\boldsymbol{\rho}(u))$ for all $n \in \mathbb{N}$, $\boldsymbol{\rho} \in \mathcal{R}^s$ and $u \geqslant 0$, with the understanding that $\boldsymbol{\ell}^0\boldsymbol{\rho}(u) = \boldsymbol{\rho}(u)$ and $\boldsymbol{\ell}^1\boldsymbol{\rho}(u) = \boldsymbol{\ell}\boldsymbol{\rho}(u)$.

Write $B_n = \{\tau \leqslant n\}$ for each $n \in \mathbb{N}$. Clearly, the sequence $\{B_n\}_{n\in\mathbb{N}}$ of events is non-decreasing and $\{\tau < \infty\} = \bigcup_{n=1}^\infty B_n$. Thus, the continuity and the monotonicity of conditional probability, given the initial state i of the Markov chain, imply that $\{\Psi_n^i\}_{n\in\mathbb{N}}$ converges monotonically from below, as $n \to \infty$, to Ψ^i.

From Lemma 2.1 and the above-mentioned convergence, it follows that the vector $\boldsymbol{\Psi}$ is a fixed point of the multidimensional risk operator $\mathbf{L}$.

Corollary 2.2: *Let Condition* C1 *hold. Then* $\mathbf{L}\boldsymbol{\Psi}(u) = \boldsymbol{\Psi}(u)$, *for every* $u \geqslant 0$.

Proof: By the definition (1) and Lemma 2.1,

$$\mathbf{L}\boldsymbol{\rho}(u) = \boldsymbol{\ell}\boldsymbol{\rho}(u) + \boldsymbol{\Psi}_1(u) \tag{4}$$

for all $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_s) \in \mathcal{R}^s$ and $u \geqslant 0$. Since $\{\Psi_n^i\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, to Ψ^i, Equation (3) and the Lebesgue dominated convergence theorem imply that

$$\begin{aligned}\Psi^i(u) &= \lim_{n\to\infty} \Psi_{n+1}^i(u) = \lim_{n\to\infty} \ell_i \boldsymbol{\Psi}_n(u) + \Psi_1^i(u)\\ &= \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \lim_{n\to\infty} \Psi_n^j(u + c(i)t - x)\, \mathrm{d}F^{ij}(x)\, \mathrm{d}G^{ij}(t) + \Psi_1^i(u) = \mathrm{L}_i \boldsymbol{\Psi}(u).\end{aligned}$$

Since $i \in S$ and $u \geqslant 0$ are arbitrary, the assertion follows. ■

Clearly, the following remark also holds.

Remark 2.3: The function $M^i : \mathbb{R} \to \overline{\mathbb{R}}_+$, defined by

$$M^i(r) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_0^\infty \mathrm{e}^{-r(c(i)t-x)}\, \mathrm{d}F^{ij}(x)\, \mathrm{d}G^{ij}(t),$$

is convex on $\mathbb{R}$ for any $i \in S$.

Let V^i, $i \in S$, denote the set $\{r \geqslant 0 : M^i(r) < \infty\}$. In Lemma 2.4 below, we will assume that the following condition holds.

Condition C2: All the sets $V^1, \ldots, V^s$ are right-open and for every $i \in S$

$$\sum_{j=1}^{s} p_{ij} \int_0^\infty x\, \mathrm{d}F^{ij}(x) < c(i) \sum_{j=1}^{s} p_{ij} \int_0^\infty t\, \mathrm{d}G^{ij}(t)$$

and $P^{ij_0}(X_1 > c(i)T_1) > 0$ for some $j_0 \in S$.

The next lemma provides a sufficient condition for the existence of the adjustment vector. It generalizes the notion of the adjustment coefficient which is commonly used (cf. Rolski *et al.* [24]) in ruin theory.

Lemma 2.4: *Let Condition* C2 *hold. Then there exists the adjustment vector* $(r^1, \ldots, r^s)$.

The following corollary will be used in the proof of Theorem 3.5 and in Example 4.1.

Corollary 2.5: *Let Condition* C2 *hold and* $i \in S$. *Then* $M^i(\epsilon r^i) \leqslant 1$ *for* $\epsilon \in [0, 1]$ *and* $M^i(\epsilon r^i) \geqslant 1$ *for* $\epsilon \geqslant 1$.

Proof: From Remark 2.3 and the equalities $M^i(0) = 1 = M^i(r^i)$, $i \in S$, it holds that $M^i(\epsilon r^i) \leqslant 1$ for $\epsilon \in [0, 1]$ and $M^i(\epsilon r^i) \geqslant 1$ for $\epsilon \geqslant 1$. ■

Let us recall that $\boldsymbol{\rho} \preceq \boldsymbol{\xi}$ if and only if $\rho_i(u) \leqslant \xi_i(u)$ for all $i \in S$ and $u \geqslant 0$. To prove Lemma 2.7, Theorems 3.1, 3.2, 3.6 and Corollary 3.7, we will use the monotonicity of the risk operator **L**.

Lemma 2.6: *Assume that $\boldsymbol{\rho}, \boldsymbol{\xi} \in \mathcal{R}^s$ satisfy $\boldsymbol{\rho} \preceq \boldsymbol{\xi}$. Then $\mathbf{L}\boldsymbol{\rho} \preceq \mathbf{L}\boldsymbol{\xi}$.*

Under Condition C2, let us consider the adjustment vector $(r^1, \ldots, r^s)$ and recall that $r_* = \min\{r^i : i \in S\}$, $R_0^i(u) = \exp(-r_* u)$ and $\mathbf{R}_0(u) = (R_0^1(u), \ldots, R_0^s(u))$, where $i \in S$ and $u \geqslant 0$. Define iteratively a sequence $\{R_n^i(\cdot)\}_{n\in\mathbb{N}}$ by $R_n^i(u) = \mathrm{L}_i \mathbf{R}_{n-1}(u)$, where $\mathbf{R}_n(u) = (R_n^1(u), \ldots, R_n^s(u))$ for all $i \in S, n \in \mathbb{N}$ and $u \geqslant 0$. We also recall that $M^*(r) = \max\{M^i(r) : i \in S\}$.

The following lemma plays an important role in the proofs of Corollary 3.4 and Theorem 3.6.

Lemma 2.7: *Under Conditions* C1 *and* C2,

(i) $|R_n^i(u) - \Psi_n^i(u)| \leqslant \mathrm{e}^{-ru}[M^*(r)]^n, \; n \in \mathbb{N}, \; r \in (0, r_*)$,
(ii) $\lim_{n\to\infty} R_n^i(u) = \Psi^i(u)$,

for all $i \in S$ and $u \geqslant 0$.

Proof: Clearly, the assumed existence of the adjustment vector implies that r_* is well-defined. Since, by Equation (3),

$$R_n^i(u) - \Psi_n^i(u) = \ell_i \boldsymbol{\ell}^{n-1} \mathbf{R}_0(u), \tag{5}$$

we claim that

$$\ell_i \boldsymbol{\ell}^{n-1} \mathbf{R}_0(u) \leqslant \mathrm{e}^{-ru}[M^*(r)]^n, \quad i \in S, \; r \in (0, r_*), \; u \geqslant 0. \tag{6}$$

We will show first that it holds for $n = 1$. Indeed,

$$\ell_i \boldsymbol{\ell}^0 \mathbf{R}_0(u) = \ell_i \mathbf{R}_0(u) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \mathrm{e}^{-r_*(u+c(i)t-x)} \, \mathrm{d}F^{ij}(x) \, \mathrm{d}G^{ij}(t)$$

$$\leqslant \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \mathrm{e}^{-r(u+c(i)t-x)} \, \mathrm{d}F^{ij}(x) \, \mathrm{d}G^{ij}(t) \leqslant \mathrm{e}^{-ru} M^i(r) \leqslant \mathrm{e}^{-ru} M^*(r).$$

Let us assume now that Inequality (6) holds for some $n \in \mathbb{N}$. We will show that it holds for $n+1$ as well. Indeed, note that

$$\begin{aligned}
\ell_i \boldsymbol{\ell}^n \mathbf{R}_0(u) &= \ell_i(\ell_1 \boldsymbol{\ell}^{n-1} \mathbf{R}_0(u), \ldots, \ell_s \boldsymbol{\ell}^{n-1} \mathbf{R}_0(u)) \\
&= \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \ell_j \boldsymbol{\ell}^{n-1} \mathbf{R}_0(u + c(i)t - x) \, \mathrm{d}F^{ij}(x) \, \mathrm{d}G^{ij}(t) \\
&\leqslant [M^*(r)]^n \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \mathrm{e}^{-r(u+c(i)t-x)} \, \mathrm{d}F^{ij}(x) \, \mathrm{d}G^{ij}(t) \\
&\leqslant [M^*(r)]^n \, \mathrm{e}^{-ru} M^i(r) \leqslant \mathrm{e}^{-ru}[M^*(r)]^{n+1}.
\end{aligned}$$

By the principle of mathematical induction, Inequality (6) holds for every $n \in \mathbb{N}$.

It follows from a counterpart of the Cramér–Lundberg bound (cf. Inequality (21)) that $\mathbf{\Psi} \preceq \mathbf{R}_0$. Therefore, by Corollary 2.2 and Lemma 2.6, $\mathbf{\Psi} \preceq \mathbf{R}_n$ for every $n \in \mathbb{N}$. On the other hand, $\mathbf{\Psi}_n \preceq \mathbf{\Psi}$, $n \in \mathbb{N}$, which results from the monotone convergence of $\{\Psi_n^i\}_{n\in\mathbb{N}}$. Thus, the assertion (i) holds because of Equation (5) and Inequality (6).

Now, the assertion (ii) follows from (i), since $M^*(r) < 1$ for every $r \in (0, r_*)$ and $\lim_{n\to\infty} \Psi_n^i(u) = \Psi^i(u)$ for all $i \in S$ and $u \geqslant 0$. ■

3. Main results

The first result gives a convenient method for bounding multidimensional ultimate ruin probabilities $\mathbf{\Psi}$ from above.

Theorem 3.1: *Let Condition* C1 *hold and* $\boldsymbol{\rho}$ *be any function from* $\mathcal{R}^s$ *that satisfies*

$$\mathbf{L}\boldsymbol{\rho} \preceq \boldsymbol{\rho}. \tag{7}$$

Then

$$\mathbf{\Psi} \preceq \boldsymbol{\rho}.$$

Proof: Since $\boldsymbol{\rho} \in \mathcal{R}^s$, Equation (4) implies that $\mathbf{\Psi}_1 \preceq \mathbf{L}\boldsymbol{\rho}$ and, consequently, by the assumption (7), $\mathbf{\Psi}_1 \preceq \boldsymbol{\rho}$. Applying Lemma 2.6 and Equation (3) gives $\mathbf{\Psi}_2 = \mathbf{L}\mathbf{\Psi}_1 \preceq \mathbf{L}\boldsymbol{\rho}$. Thus, the assumption (7) yields $\mathbf{\Psi}_2 \preceq \boldsymbol{\rho}$. By iterating the above steps, one can verify that $\mathbf{\Psi}_n \preceq \boldsymbol{\rho}$ for every $n \in \mathbb{N}$. Letting $n \to \infty$ completes the proof because $\lim_{n\to\infty} \Psi_n^i(u) = \Psi^i(u)$ for all $i \in S$ and $u \geqslant 0$ (see Section 2 for details). ■

Theorem 3.1 can be successfully applied to obtain a variety of upper bounds for the vector $\mathbf{\Psi}$. Selected several of them are considered in Theorems 3.5 and 3.6 and listed in Example 4.1.

Our next theorem provides a general recipe to bound the vector $\mathbf{\Psi}$ from below. It is not an immediate counterpart of Theorem 3.1 because the symmetry is not complete here. The reason is that one has to separate (see Remark 3.3 below) 'inadmissible' fixed points of $\mathbf{L}$ (such as the unit function $\mathbf{1}$ in $\mathcal{R}^s$ or, in general, any convex combination of $\mathbf{\Psi}$ and $\mathbf{1}$, as it was mentioned in Section 1).

We say that the sequence $\{\boldsymbol{\zeta}_n\}_{n\geqslant 0}$ of elements $\boldsymbol{\zeta}_n = (\zeta_n^1, \dots, \zeta_n^s)$ from $\mathcal{R}^s$ converges to $\boldsymbol{\zeta} = (\zeta^1, \dots, \zeta^s) \in \mathcal{R}^s$ if $\lim_{n\to\infty} \zeta_n^i(u) = \zeta^i(u)$ for all $i \in S$ and $u \geqslant 0$.

Theorem 3.2: *Assume that there exists* $\boldsymbol{\xi}_0 \in \mathcal{R}^s$ *such that the sequence* $\boldsymbol{\xi}_n = \mathbf{L}\boldsymbol{\xi}_{n-1}$ *converges to* $\mathbf{\Psi}$. *Let* $\boldsymbol{\rho}$ *be any function from* $\mathcal{R}^s$ *that satisfies*

$$\boldsymbol{\rho} \preceq \mathbf{L}\boldsymbol{\rho} \quad \textit{and} \quad \boldsymbol{\rho} \preceq \boldsymbol{\xi}_0. \tag{8}$$

Then

$$\boldsymbol{\rho} \preceq \mathbf{\Psi}.$$

Before we get to the proof, let us look at the following counterexample, which shows that the assumption $\boldsymbol{\rho} \preceq \mathbf{L}\boldsymbol{\rho}$ does not suffice to construct a lower bound for $\mathbf{\Psi}$.

Remark 3.3: Assume that $\rho_i(u) = 1$ for all $i \in S$ and $u \geqslant 0$. Clearly, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_s) \in \mathcal{R}^s$ and $\mathrm{L}_i\boldsymbol{\rho}(u) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \mathrm{d}F^{ij}(x)\,\mathrm{d}G^{ij}(t) + \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{u+c(i)t}^\infty \mathrm{d}F^{ij}(x)$ $\mathrm{d}G^{ij}(t) = \rho_i(u)$. Let $\boldsymbol{\xi}_0$ be an arbitrary function from $\mathcal{R}^s$ such that $\boldsymbol{\rho} \preceq \boldsymbol{\xi}_0$. In the considered case, $\xi_1^i(u) = \mathrm{L}_i\boldsymbol{\xi}_0(u) \geqslant 1$, $i \in S$, and, consequently, $\xi_n^i(u) \geqslant 1$ for every $n \in \mathbb{N}$. Although the assumption $\boldsymbol{\rho} \preceq \mathbf{L}\boldsymbol{\rho}$ is satisfied, $\boldsymbol{\rho}$ is not a lower bound for ruin probabilities $\boldsymbol{\Psi}$ unless $\boldsymbol{\Psi} = \mathbf{1}$.

We will prove now Theorem 3.2.

Proof: By the assumptions (8) and Lemma 2.6, $\mathbf{L}\boldsymbol{\rho} \preceq \mathbf{L}\boldsymbol{\xi}_0 = \boldsymbol{\xi}_1$ and, consequently, $\boldsymbol{\rho} \preceq \boldsymbol{\xi}_1$. By iterating the above steps, one may notice that $\boldsymbol{\rho} \preceq \boldsymbol{\xi}_n$ for every $n \in \mathbb{N}$. Letting $n \to \infty$ completes the proof because, by the assumption, $\{\xi_n^i\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, to Ψ^i, $i \in S$. ■

The question arises whether there exists a non-trivial sequence $\{\xi_n^i(\cdot)\}_{n\geqslant 0}$ satisfying the assumptions of Theorem 3.2. Representative examples, i.e. $\xi_n^i = R_n^i$ or $\xi_n^i = U_n^i$, are given in Corollary 3.4 and Theorem 3.6, respectively.

Corollary 3.4: *Let Conditions* C1 *and* C2 *hold. Let* $\boldsymbol{\rho}$ *be any function from* $\mathcal{R}^s$ *that satisfies*

$$\boldsymbol{\rho} \preceq \mathbf{L}\boldsymbol{\rho} \quad \textit{and} \quad \boldsymbol{\rho} \preceq \mathbf{R}_0. \tag{9}$$

Then

$$\boldsymbol{\rho} \preceq \boldsymbol{\Psi}.$$

Proof: By Lemma 2.7, the sequence $\{R_n^i\}_{n\in\mathbb{N}}$ converges, as $n \to \infty$, to Ψ^i for every $i \in S$. Hence, the assertion follows from Theorem 3.2. ■

We will show now how to apply Theorem 3.1 and Corollary 3.4 in order to construct new two-sided bounds for vector-valued ruin probabilities $\boldsymbol{\Psi}$. First of all, we will deal with a multidimensional Lundberg–Taylor type bound (cf. Taylor [26]) which will subsequently be used to obtain a variety of sharper operator bounds for $\boldsymbol{\Psi}$.

Let us denote:

$$A^i(r,u) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{u+c(i)t}^\infty \mathrm{d}F^{ij}(x)\,\mathrm{d}G^{ij}(t) \Big/ \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{u+c(i)t}^\infty \mathrm{e}^{-r(u+c(i)t-x)}\mathrm{d}F^{ij}(x)\,\mathrm{d}G^{ij}(t),$$

where $i \in S$ and $r > 0$. The following convention will be used throughout the paper: if the support of any F^{ij} is bounded from above in the previous quotient, then we restrict our consideration to all these $u \geqslant 0$ for which the denominator is positive. Then $0 \leqslant A^i(r,u) \leqslant 1$. Under Condition C2, let us recall that $r_* = \min\{r^i : i \in S\}$, $r^* = \max\{r^i : i \in S\}$ and set

$$A_* = \inf_{i\in S} \inf_{u\geqslant 0} \{A^i(r^*,u)\} \quad \text{and} \quad A^* = \sup_{i\in S} \sup_{u\geqslant 0} \{A^i(r_*,u)\},$$

where the inner *inf* and sup in A_* and A^*, respectively, runs over all non-negative u such that the denominator of $A^i(r^*,u)$ (respectively $A^i(r_*,u)$) is positive. Let us also recall that

$Q_0^i(u) = \exp(-r^* u)$, $\mathbf{Q}_0(u) = (Q_0^1(u), \ldots, Q_0^s(u))$, $i \in S$, $u \geqslant 0$, and

$$\Psi_1^i(u) = \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{u+c(i)t}^\infty \mathrm{d}F^{ij}(x)\,\mathrm{d}G^{ij}(t), \quad i \in S,\ u \geqslant 0, \tag{10}$$

by the definition (1) and Equation (3).

Theorem 3.1 and Corollary 3.4 can be applied to prove the following two-sided bound for multidimensional ruin probabilities $\boldsymbol{\Psi}$.

Theorem 3.5: *Let Conditions* C1 *and* C2 *hold. Then*

$$A_* \mathbf{Q}_0 \preceq \boldsymbol{\Psi} \preceq A^* \mathbf{R}_0. \tag{11}$$

Proof: Since $A^i(r_*, u) \leqslant A^*$, the formula (10) implies that

$$\Psi_1^i(u) \leqslant A^* R_0^i(u) \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{u+c(i)t}^\infty \mathrm{e}^{-r_*(c(i)t-x)}\,\mathrm{d}F^{ij}(x)\,\mathrm{d}G^{ij}(t),$$

$i \in S$, $u \geqslant 0$. Therefore, by the definition (1), the formula (10) and the inequality $M^i(r_*) \leqslant 1$ resulting from Corollary 2.5,

$$\begin{aligned}\mathrm{L}_i(A^*\mathbf{R}_0)(u) &= A^* R_0^i(u) \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \mathrm{e}^{-r_*(c(i)t-x)}\,\mathrm{d}F^{ij}(x)\,\mathrm{d}G^{ij}(t) + \Psi_1^i(u) \\ &\leqslant A^* R_0^i(u) M^i(r_*) \leqslant A^* R_0^i(u), \quad i \in S,\ u \geqslant 0,\end{aligned} \tag{12}$$

or, equivalently, $\mathbf{L}(A^*\mathbf{R}_0) \preceq A^*\mathbf{R}_0$. Clearly, the assumption (7) is satisfied for the vector $\boldsymbol{\rho} = A^*\mathbf{R}_0$ and, consequently,

$$\boldsymbol{\Psi} \preceq A^* \mathbf{R}_0, \tag{13}$$

by Theorem 3.1.

To apply Corollary 3.4 with $\boldsymbol{\rho} = A_*\mathbf{Q}_0$, one has to bound it by $\mathbf{R}_0$ first. We have

$$A_* \mathbf{Q}_0 \preceq \mathbf{Q}_0 \preceq \mathbf{R}_0, \tag{14}$$

because of $A_* \leqslant 1$. Since $M^i(r^*) \geqslant 1$, by Corollary 2.5, and $A^i(r^*, u) \geqslant A_*$, similar arguments show that

$$\mathrm{L}_i(A_*\mathbf{Q}_0)(u) \geqslant A_* Q_0^i(u) M^i(r^*) \geqslant A_* Q_0^i(u), \quad i \in S,\ u \geqslant 0,$$

or, equivalently,

$$A_* \mathbf{Q}_0 \preceq \mathbf{L}(A_* \mathbf{Q}_0). \tag{15}$$

By the properties (14) and (15), the assumptions (9) are satisfied for the vector $\boldsymbol{\rho} = A_*\mathbf{Q}_0$. Therefore, by Corollary 3.4,

$$A_* \mathbf{Q}_0 \preceq \boldsymbol{\Psi}. \tag{16}$$

Finally, the assertion (11) follows from the properties (13) and (16). The proof is complete. ∎

Write $D_0^i(u) = A_* Q_0^i(u)$, $U_0^i(u) = A^* R_0^i(u)$, $\mathbf{D}_0(u) = (D_0^1(u), \ldots, D_0^s(u))$ and $\mathbf{U}_0(u) = (U_0^1(u), \ldots, U_0^s(u))$, where $i \in S$ and $u \geqslant 0$. For every $i \in S$, define iteratively sequences $\{D_n^i(\cdot)\}_{n\in\mathbb{N}}$ and $\{U_n^i(\cdot)\}_{n\in\mathbb{N}}$ by $D_n^i(u) = \mathrm{L}_i\mathbf{D}_{n-1}(u)$ and $U_n^i(u) = \mathrm{L}_i\mathbf{U}_{n-1}(u)$, respectively, where $\mathbf{D}_n(u) = (D_n^1(u), \ldots, D_n^s(u))$ and $\mathbf{U}_n(u) = (U_n^1(u), \ldots, U_n^s(u))$, $u \geqslant 0$. We also recall that $M^*(r) = \max\{M^i(r) : i \in S\}$ and $\ell_i\boldsymbol{\rho}(u) = \sum_{j=1}^s p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \rho_j(u + c(i)t - x)\, \mathrm{d}F^{ij}(x)\, \mathrm{d}G^{ij}(t)$ for all $i \in S$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_s) \in \mathcal{R}^s$ and $u \geqslant 0$ (see Section 2 for details).

The next theorem provides new two-sided operator bounds converging to the vector $\boldsymbol{\Psi}$ with the rate which is exponentially bounded from above.

Theorem 3.6: *Let Conditions* C1 *and* C2 *hold. Then*

$$\mathbf{D}_n \preceq \boldsymbol{\Psi} \preceq \mathbf{U}_n \tag{17}$$

and

$$\left|U_n^i(u) - D_n^i(u)\right| \leqslant A^*\, \mathrm{e}^{-ru}[M^*(r)]^n \tag{18}$$

for all $i \in S$, $n \in \mathbb{N}$, $r \in (0, r_*)$ *and* $u \geqslant 0$.

Proof: By Theorem 3.5, $\mathbf{D}_0 \preceq \boldsymbol{\Psi} \preceq \mathbf{U}_0$ and, consequently, by Lemma 2.6, $\mathbf{L}\mathbf{D}_0 \preceq \mathbf{L}\boldsymbol{\Psi} \preceq \mathbf{L}\mathbf{U}_0$. Applying Corollary 2.2 and the definitions of $\mathbf{D}_1$ and $\mathbf{U}_1$ gives $\mathbf{D}_1 \preceq \boldsymbol{\Psi} \preceq \mathbf{U}_1$. By iterating the above steps, one can verify that $\mathbf{D}_n \preceq \boldsymbol{\Psi} \preceq \mathbf{U}_n$ for every $n \in \mathbb{N}$, so the assertion (17) holds.

To prove Inequality (18), we will first show that

$$\left|U_n^i(u) - D_n^i(u)\right| \leqslant \ell_i \boldsymbol{\ell}^{n-1}\mathbf{U}_0(u) \tag{19}$$

holds for all $i \in S$, $n \in \mathbb{N}$ and $u \geqslant 0$. Indeed, by Equation (4) and the definitions of $\mathbf{D}_1$ and $\mathbf{U}_1$, $U_1^i(u) - D_1^i(u) = \ell_i\mathbf{U}_0(u) - \ell_i\mathbf{D}_0(u) \leqslant \ell_i\mathbf{U}_0(u) = \ell_i\boldsymbol{\ell}^0\mathbf{U}_0(u)$ for all $i \in S$ and $u \geqslant 0$. Thus, Inequality (19) is valid for $n = 1$. Under the assumption that it holds for some $n \in \mathbb{N}$, we get

$$\begin{aligned} &U_{n+1}^i(u) - D_{n+1}^i(u) \\ &\quad \leqslant \sum_{j=1}^s p_{ij} \int_0^\infty \int_{(0,u+c(i)t]} \ell_j \boldsymbol{\ell}^{n-1}\mathbf{U}_0(u + c(i)t - x)\, \mathrm{d}F^{ij}(x)\, \mathrm{d}G^{ij}(t) = \ell_i\boldsymbol{\ell}^n\mathbf{U}_0(u) \end{aligned}$$

for all $i \in S$ and $u \geqslant 0$. By the principle of mathematical induction, Inequality (19) holds for every $n \in \mathbb{N}$ and, consequently, by the definition of $\mathbf{U}_0$ and Inequality (6),

$$|U_n^i(u) - D_n^i(u)| \leqslant A^*\ell_i\boldsymbol{\ell}^{n-1}\mathbf{R}_0(u) \leqslant A^*\, \mathrm{e}^{-ru}[M^*(r)]^n$$

for all $i \in S$, $n \in \mathbb{N}$, $r \in (0, r_*)$ and $u \geqslant 0$. Hence, the assertion (18) holds and the proof is complete. ■

The sequence $\{\boldsymbol{\zeta}_n\}_{n\geqslant 0}$ of elements $\boldsymbol{\zeta}_n = (\zeta_n^1, \ldots, \zeta_n^s)$ from $\mathcal{R}^s$ is said to be non-decreasing (respectively non-increasing) if $\boldsymbol{\zeta}_n \preceq \boldsymbol{\zeta}_{n+1}$ (respectively $\boldsymbol{\zeta}_{n+1} \preceq \boldsymbol{\zeta}_n$) for every $n \in \mathbb{N}$.

Property (17) implies the following corollary.

Corollary 3.7: *Let Conditions* C1 *and* C2 *hold. Then*

(i) $\{\mathbf{D}_n\}_{n\geqslant 0}$ *is a non-decreasing sequence which converges, as* $n \to \infty$, *to* $\boldsymbol{\Psi}$,
(ii) $\{\mathbf{U}_n\}_{n\geqslant 0}$ *is a non-increasing sequence which converges, as* $n \to \infty$, *to* $\boldsymbol{\Psi}$.

Proof: Since $A^* \leqslant 1$ and $A_* \geqslant 0$, $\mathbf{U}_0 \preceq \mathbf{R}_0$, $\boldsymbol{\Psi}_0 \preceq \mathbf{D}_0$ and, consequently, by Lemma 2.6, $\mathbf{U}_n \preceq \mathbf{R}_n$ and $\boldsymbol{\Psi}_n \preceq \mathbf{D}_n$ for every $n \in \mathbb{N}$. Therefore, by the property (17),

$$\lim_{n\to\infty} D_n^i(u) = \Psi^i(u) = \lim_{n\to\infty} U_n^i(u)$$

because both the sequences $\{\Psi_n^i\}_{n\geqslant 0}$ and $\{R_n^i\}_{n\geqslant 0}$ converge, as $n \to \infty$, to Ψ^i (see Section 2 for details). Moreover, the properties (12), (15) and Lemma 2.6 imply that $\mathbf{U}_{n+1} \preceq \mathbf{U}_n$ and $\mathbf{D}_n \preceq \mathbf{D}_{n+1}$ for every $n \in \mathbb{N}$. The proof is complete. ■

Remark 3.8: Theorem 3.6 improves as well as generalizes the following result to the regime-switching framework. Assume that there exists the one-dimensional adjustment coefficient r_0, i.e. a positive solution of $M(r) = 1$, where $M(r) = \mathbb{E}\exp(r(X_1 - \gamma))$ and γ is the total amount of premiums received in each period for the non-switching discrete time risk model. This can be obtained when $s = 1$, $X_1, X_2, \ldots$ are a.s. non-negative i.i.d. random variables sharing a distribution function F and $\mathbb{P}(T_k = \eta) = 1,\ k \in \mathbb{N}$, so that $\gamma = c\eta$ for some positive constants c and η. Then superscripts i can be skipped and the following inequality has been known:

$$|U_n(u) - D_n(u)| \leqslant \frac{r_0}{r_0 - r} \sup_{d\geqslant\gamma} A(d)\,\mathrm{e}^{-ru}[M(r)]^n,$$

where $A(d) = \int_d^\infty \mathrm{d}F(x)/\int_d^\infty \exp(-r_0(d-x))\,\mathrm{d}F(x)$ (see Gajek and Rudź [8] for details). Since $1 < r_0/(r_0 - r)$ for $r \in (0, r_0)$, Theorem 3.6 may provide bounds more general as well as sharper (even in the case of one-dimensional models) than some existing results (cf. Gajek [7] or Gajek and Rudź [8]).

Remark 3.9: The two-sided bounds resulting from Corollary 3.7 might even be asymptotically exact under the assumption that the adjustment coefficient exists. As an application, let us consider the following family of claim distributions. Under the notation of Remark 3.8, set

$$F(x) = \begin{cases} 0, & x < \alpha \\ P_2, & x = \alpha \\ 1 - P_1\exp(-\beta(x-\alpha)), & x > \alpha, \end{cases}$$

where $\alpha \in \mathbb{R}_+^0$, $\beta \in \mathbb{R}_+$, $P_2 \in [0,1)$ and $P_1 = 1 - P_2$. Assume that $\gamma \in (\alpha + P_1/\beta, +\infty)$ satisfies for some $r_0 \in (0,\beta)$ the following equation $P_2 + \beta P_1/(\beta - r_0) = \exp(r_0(\gamma - \alpha))$. Both the limits $\lim_{n\to\infty} D_n(u)$ and $\lim_{n\to\infty} U_n(u)$ are equal to $(\beta - r_0)\exp(-r_0 u)/\beta$, which is the exact ultimate ruin probability $\Psi(u) = \mathbb{P}(\tau(u) < \infty)$ in the considered case (cf. Gajek and Rudź [8]). In particular, the property holds for exponential distributions with a probability mass P_2 at zero (when $\alpha = 0$) or exponential distributions shifted by α to the right (when $P_2 = 0$), including the case of exponential distributions (when $P_2 = \alpha = 0$).

4. Examples

In this section, we highlight some other applications of our methodology. The next result concerns several upper bounds for the vector $\boldsymbol{\Psi}$, which can be proven using Theorem 3.1.

Example 4.1: Let Conditions C1 and C2 hold. Then

$$\Psi^i(u) \leqslant \inf_{r \in (0, r_*]} \{\mathrm{e}^{-ru} M^i(r)\} \tag{20}$$

for all $i \in S$ and $u \geqslant 0$.

The proof of Inequality (20) relies on applying Theorem 3.1 to the following vector-valued function $\boldsymbol{\rho}_r = (\rho_{r,1}, \dots, \rho_{r,s})$, where $\rho_{r,i}(u) = \exp(-ru)M^i(r)$ for all $i \in S$, $r \in (0, r_*]$ and $u \geqslant 0$. For these i and r, $M^i(r) \leqslant 1$, by Corollary 2.5. Therefore, $\boldsymbol{\rho}_r \in \mathcal{R}^s$ and, by the definition (1) and the formula (10),

$$\begin{aligned} \mathrm{L}_i \boldsymbol{\rho}_r(u) &= \sum_{j=1}^{s} M^j(r) p_{ij} \int_0^\infty \int_{(0, u+c(i)t]} \mathrm{e}^{-r(u+c(i)t-x)} \, \mathrm{d}F^{ij}(x) \, \mathrm{d}G^{ij}(t) + \Psi_1^i(u) \\ &\leqslant \sum_{j=1}^{s} p_{ij} \int_0^\infty \int_0^\infty \mathrm{e}^{-r(u+c(i)t-x)} \, \mathrm{d}F^{ij}(x) \, \mathrm{d}G^{ij}(t) = \rho_{r,i}(u) \end{aligned}$$

for all $i \in S$, $r \in (0, r_*]$ and $u \geqslant 0$. Thus, $\mathbf{L}\boldsymbol{\rho}_r \preceq \boldsymbol{\rho}_r$ and, by Theorem 3.1, $\boldsymbol{\Psi} \preceq \boldsymbol{\rho}_r$. Since $r \in (0, r_*]$ is arbitrary, Inequality (20) follows. It can also be proven using an inductive approach discussed in Gajek and Rudź [11].

The upper bound (20) generalizes and improves (cf. Inequalities (22) below) the following results. In particular:

(i) if there exists the one-dimensional adjustment coefficient r_0 for the non-switching discrete time risk model, then the ultimate ruin probability Ψ can be bounded as follows

$$\Psi(u) \leqslant \inf_{r \in (0, r_0]} \{\mathrm{e}^{-ru} M(r)\}, \quad u \geqslant 0,$$

(see Gajek [7] for details),

(ii)

$$\Psi^i(u) \leqslant \mathrm{e}^{-r_* u} M^i(r_*), \quad i \in S, \ u \geqslant 0,$$

(see Gajek and Rudź [10] for the detailed methodology based on Banach Contraction Principle),

(iii)

$$\Psi^i(u) \leqslant \mathrm{e}^{-r_* u}, \quad i \in S, \ u \geqslant 0, \tag{21}$$

which is a counterpart of the well-known Cramér–Lundberg bound.

Since

$$\inf_{r \in (0, r_*]} \{\mathrm{e}^{-ru} M^i(r)\} \leqslant \mathrm{e}^{-r_* u} M^i(r_*) \leqslant \mathrm{e}^{-r_* u}, \quad i \in S, \ u \geqslant 0, \tag{22}$$

our methodology may provide bounds sharper than the ones obtained by some different mathematical approaches.

The infimum in Inequality (20) runs over all positive r that are not greater than r_*. There are known bounds for ruin probabilities with the infimum being taken conversely, i.e. for $r \geqslant r_*$. In the following example, we compare Inequality (20) with such a result. It turns out that, depending on the parameters of the model, the infinite-horizon formula (20) can sometimes better approximate Ψ_n^i than a generalized Gerber-type inequality designed especially for the valuation of finite-horizon ruin probabilities Ψ_n^i.

To be more precise, we compare Inequality (20) with the result (23) from [9, Theorem 3.1, p. 239].

Example 4.2: Let us consider a discrete time regime-switching model obtained when $\mathbb{P}(T_k = \eta) = 1$ for every $k \in \mathbb{N}$, so that the total amount of premiums in the kth period of length η equals $\gamma_k = \gamma(I_{k-1}) = c(I_{k-1})\eta$ for some positive real η and a known positive function c which is defined on S. Then a.s. non-negative random variables $X_1, X_2, \ldots$ denote the total sums of the claims during the consecutive periods.

Assume that: $S = \{1, 2\}$, the amount of premiums during the first period, γ_1, given the state i in the beginning, equals $\gamma(1) = 3.15$ or $\gamma(2) = 4.15$. Set $p_{11} = 0.94, p_{12} = 0.06, p_{21} = 0.9, p_{22} = 0.1$, $F^{11}(x) = F^{21}(x) = (1 - \exp(-\beta_1 x))\mathbb{I}_{(0,\infty)}(x)$ and $F^{12}(x) = F^{22}(x) = (1 - \exp(-\beta_2 x))\mathbb{I}_{(0,\infty)}(x)$, where $\beta_1 = 1$, $\beta_2 = 0.6$ and $\mathbb{I}_{(0,\infty)}(x) = 1$ if $x > 0$ and 0 otherwise. In the considered case,

$$\begin{aligned} M^i(r) &= p_{i1} \int_{[0,\infty)} \mathrm{e}^{-r(\gamma(i)-x)}\, \mathrm{d}F^{i1}(x) + p_{i2} \int_{[0,\infty)} \mathrm{e}^{-r(\gamma(i)-x)}\, \mathrm{d}F^{i2}(x) \\ &= \mathrm{e}^{-r\gamma(i)} \big(p_{i1}\beta_1/(\beta_1 - r) + p_{i2}\beta_2/(\beta_2 - r)\big) \end{aligned}$$

for $i \in \{1, 2\}$ and $r < \beta_2$. Then $M^*(r) = M^1(r)$ and $r_* = r^1 \approx 0.591$.

We will compare Inequality (20) with a generalized Gerber-type bound

$$\Psi_n^i(u) \leqslant \inf_{r \geqslant r_*} \{\mathrm{e}^{-ru} M^i(r) [M^*(r)]^{n-1}\}, \quad i \in S,\ n \in \mathbb{N},\ u \geqslant 0. \tag{23}$$

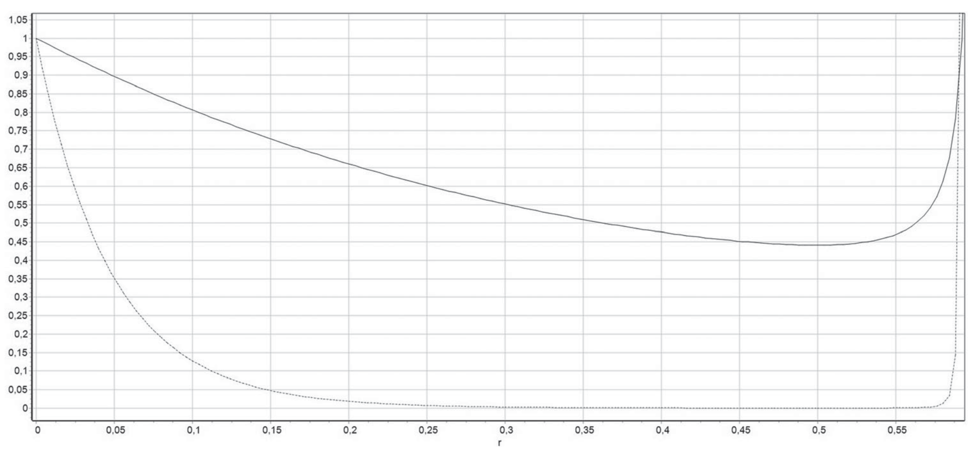

Figure 1. $\exp(-ru)M^1(r)$ (the solid line) and $\exp(-ru)M^1(r)[M^*(r)]^{n-1}$ (the dotted line), as functions of r, for the initial surplus $u = 0.1$ and $n = 10$.

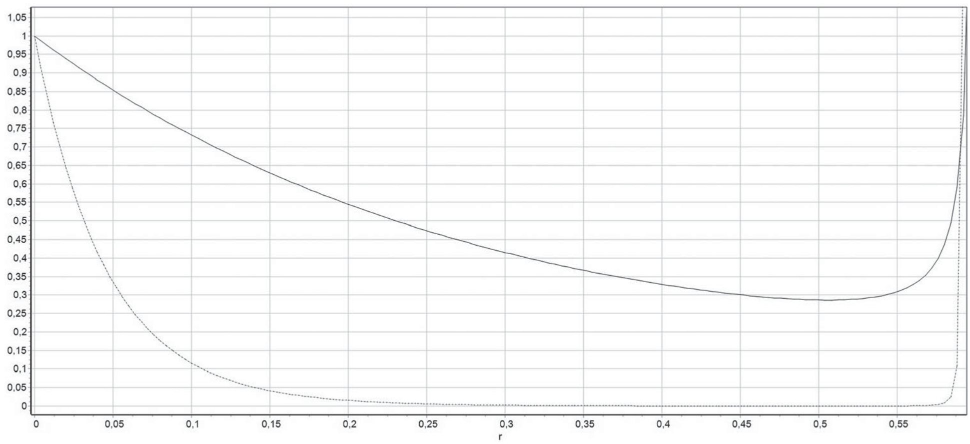

Figure 2. $\exp(-ru)M^2(r)$ (the solid line) and $\exp(-ru)M^2(r)[M^*(r)]^{n-1}$ (the dotted line), as functions of r, for the initial surplus $u = 0.1$ and $n = 10$.

Inequality (23) can be more accurate than other upper bounds for finite-horizon ruin probabilities (see Gajek and Rudź [9,12] for details). Nevertheless, the bound (20) can be even sharper than Inequality (23) in some situations. For instance, set $n = 10$ and $u = 0.1$. The graphs of $\exp(-ru)M^i(r)$ and $\exp(-ru)M^i(r)[M^*(r)]^{n-1}$, as functions of r, are presented in Figures 1 and 2. Although

$$\inf_{r\in\mathbb{R}_+} \{e^{-ru}M^i(r)[M^*(r)]^{n-1}\} < \inf_{r\in\mathbb{R}_+} \{e^{-ru}M^i(r)\}$$

for both $i = 1$ and $i = 2$, the parameters of the model have been selected in such a way that the infima over $(0, r_*]$ are substantially smaller than over $[r_*, \infty)$. To be more precise, we have

$$\inf_{r\in(0,r_*]} \{e^{-ru}M^1(r)\} = 0.441 < 0.943 = \inf_{r\geqslant r_*} \{e^{-ru}M^1(r)[M^*(r)]^{n-1}\}$$

and

$$\inf_{r\in(0,r_*]} \{e^{-ru}M^2(r)\} = 0.286 < 0.738 = \inf_{r\geqslant r_*} \{e^{-ru}M^2(r)[M^*(r)]^{n-1}\},$$

so Inequality (20) can give more accurate bounds of $\Psi^1_{10}(0.1)$ and $\Psi^2_{10}(0.1)$ than the finite-horizon upper bound (23).

Acknowledgments

The authors thank the reviewers and the editors for helpful comments.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Marcin Rudź http://orcid.org/0000-0003-0644-6655

References

[1] S. Asmussen, *Risk theory in a Markovian environment*, Scand. Actuar. J. 1989(2) (1989), pp. 69–100.

[2] S. Asmussen and H. Albrecher, *Ruin Probabilities*, 2nd ed., World Scientific Publishing Co. Pte. Ltd., Singapore, 2010.

[3] J. Cai and D.C.M. Dickson, *Ruin probabilities with a Markov chain interest model*, Insurance Math. Econom. 35 (2004), pp. 513–525.

[4] X. Chen, T. Xiao, and X. Yang, *A Markov-modulated jump-diffusion risk model with randomized observation periods and threshold dividend strategy*, Insurance Math. Econom. 54 (2014), pp. 76–83.

[5] D. Cheng and C. Yu, *Uniform asymptotics for the ruin probabilities in a bidimensional renewal risk model with strongly subexponential claims*, Stochastics 91(5) (2019), pp. 643–656.

[6] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. I, 3rd ed., John Wiley & Sons, Inc., New York 1968.

[7] L. Gajek, *On the deficit distribution when ruin occurs-discrete time model*, Insurance Math. Econom. 36 (2005), pp. 13–24.

[8] L. Gajek and M. Rudź, *Sharp approximations of ruin probabilities in the discrete time models*, Scand. Actuar. J. 2013(5) (2013), pp. 352–382.

[9] L. Gajek and M. Rudź, *A generalization of Gerber's inequality for ruin probabilities in risk-switching models*, Statist. Probab. Lett. 129 (2017), pp. 236–240.

[10] L. Gajek and M. Rudź, *Banach Contraction Principle and ruin probabilities in regime-switching models*, Insurance Math. Econom. 80 (2018), pp. 45–53.

[11] L. Gajek and M. Rudź, *Deficit distributions at ruin in a regime-switching Sparre Andersen model*, J. Appl. Anal. 24(1) (2018), pp. 99–107.

[12] L. Gajek and M. Rudź, *Finite-Horizon Ruin Probabilities in a Risk-Switching Sparre Andersen Model*, Methodol. Comput. Appl. Probab. (2018). Available at https://doi.org/10.1007/s11009-018-9627-2.

[13] A. Guillou, S. Loisel, and G. Stupfler, *Estimation of the parameters of a Markov-modulated loss process in insurance*, Insurance Math. Econom. 53 (2013), pp. 388–404.

[14] F. Guo, D. Wang, and H. Yang, *Asymptotic results for ruin probability in a two-dimensional risk model with stochastic investment returns*, J. Comput. Appl. Math. 325 (2017), pp. 198–221.

[15] S. Jacka and A. Ocejo, *On the regularity of American options with regime-switching uncertainty*, Stoch. Proc. Appl. 128(3) (2018), pp. 803–818.

[16] D.G. Konstantinides and J. Li, *Asymptotic ruin probabilities for a multidimensional renewal risk model with multivariate regularly varying claims*, Insurance Math. Econom. 69 (2016), pp. 38–44.

[17] A.E. Kyprianou, *Gerber-Shiu Risk Theory*, EAA Series, Springer, Cham, 2013.

[18] D. Landriault, B. Li, and S. Li, *Analysis of a drawdown-based regime-switching Lévy insurance model*, Insurance Math. Econom. 60 (2015), pp. 98–107.

[19] R.H. Liu, *Optimal stopping of switching diffusions with state dependent switching rates*, Stochastics 88(4) (2016), pp. 586–605.

[20] Q. Liu, D. Jiang, T. Hayat, and A. Alsaedi, *Dynamical behavior of a hybrid switching SIS epidemic model with vaccination and Lévy jumps*, Stoch. Anal. Appl. 37(3) (2019), pp. 388–411.

[21] Y. Lu, *On the severity of ruin in a Markov-modulated risk model*, Scand. Actuar. J. 2006(4) (2006), pp. 183–202.

[22] R. Momeya, *Viscosity solutions and the pricing of European-style options in a Markov-modulated exponential Lévy model*, Stochastics 90(8) (2018), pp. 1238–1275.

[23] J. Peng and D. Wang, *Uniform asymptotics for ruin probabilities in a dependent renewal risk model with stochastic return on investments*, Stochastics 90(3) (2018), pp. 432–471.

[24] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels, *Stochastic Processes for Insurance and Finance*, John Wiley & Sons Ltd, Chichester, 1999.

[25] P. Saipara, P. Kumam, and Y.J. Cho, *Random fixed point theorems for Hardy-Rogers self-random operators with applications to random integral equations*, Stochastics 90(2) (2018), pp. 297–311.
[26] G.C. Taylor, *Use of differential and integral inequalities to bound ruin and queuing probabilities*, Scand. Actuar. J. 1976(4) (1976), pp. 197–208.
[27] G. Wang, G. Wang, and H. Yang, *On a multi-dimensional risk model with regime switching*, Insurance Math. Econom. 68 (2016), pp. 73–83.
[28] L. Xu, L. Zhang, and D. Yao, *Optimal investment and reinsurance for an insurer under Markov-modulated financial market*, Insurance Math. Econom. 74 (2017), pp. 7–19.
[29] H. Yang and J. Li, *Asymptotic ruin probabilities for a bidimensional renewal risk model*, Stochastics 89(5) (2017), pp. 687–708.