Fault diagnosis and fault-tolerant control for system with fast time-varying delay

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Xiao-Ye Xi[a], Jian-Fei Zhao[a], Ting-Zhang Liu[a] and Li-Min Yan[b]

[a]Shanghai Key Laboratory of Power Station Automation Technology, Department of Automation, Shanghai University, Shanghai, People's Republic of China; [b]Microelectronics R&D Center, Shanghai University, Shanghai, People's Republic of China

ABSTRACT

This paper proposes a fault diagnosis and fault-tolerant control method for a system with a fast time-varying delay and time-varying parameters. A fault observer is designed to estimate faults, and an improved fast adaptive fault estimation (FAFE) algorithm is developed to reduce the relevant constraints in the general form of this algorithm. With newly introduced relaxation matrices, this study estimates faults in a system exhibiting a fast time-varying delay. Based on the estimated faults, an output feedback controller is designed to accommodate the faults. The fault-tolerant control is realized using the introduced relaxation matrices. An algorithm is derived to solve for the observer and controller. Finally, the theory and method are validated using a real example of a helicopter system.

1. Introduction

With the increasing complexity of control systems in modern industrial processes and the increasing requirements of reliable systems [1,2], the diagnosis and accommodation of faults for dynamic systems have become more important.

The diagnosis and accommodation of faults for systems with a time delay is an important and challenging task [3–5] because many realistic systems exhibit a time delay [6]. Furthermore, because of the uncertain parameters in most practical systems, the design becomes more difficult [7,8]. Only few studies have been conducted in the area of fault estimation for time-delay systems. A sliding mode observer has been designed to estimate faults; however, in this method, the system demand is so high that its applicability is poor [9]. Moreover, it is only suitable for systems with a constant or a slow time-varying delay. In another study [10], a fault estimation filter method was used to estimate the faults in a near spacecraft. However, this method is not suitable for all types of faults because of its strict constraints. An observer-based iterative learning method was used to estimate the fault in a spacecraft; however, the design process of this method is so complicated that its generality is poor, and this method must also satisfy difficult conditions [11]. An adaptive observer was designed to estimate faults in a system with a time-varying delay using the common adaptive fault estimation (CAFE) algorithm. This method is simple and effective while being less restrictive on the system and faults. However, the method is unsuitable for systems with a fast time-varying delay [12]. A fast adaptive fault estimation (FAFE) algorithm has been developed to estimate faults in a system with a fast time-varying delay by introducing relaxation matrices. However, in the FAFE algorithm, the corresponding constraints need to be calculated via an approximate method. Moreover, no method has been provided for fault-tolerant control of the system [13].

Studies on the fault-tolerant control for systems with a time-varying delay are limited [14]. A fault-tolerant controller for a linear system without time delay has been designed based on online fault estimates [15]. In another study, a fault-tolerant controller for a linear system with a slow time-varying delay has been designed via the adaptive control method based on online fault estimates [16]. A fault accommodation method for a system with a slow time-varying delay has been proposed based on adaptive sliding mode control [17]. The above methods can achieve satisfactory results for systems with slow time-varying delays or without any time delay but are unsuitable for systems with a fast time-varying delay. Studies on the fault-tolerant control for systems with a fast time-varying delay are lacking [18].

The main objective of this study was to develop a method for the fault diagnosis and fault-tolerant control of systems with a fast time-varying delay. Estimating and accommodating faults for such systems with loose constraints and improved applicability is a challenging task.

CONTACT Jian-Fei Zhao ✉ mdklvjkkj@163.com 🏢 Shanghai Key Laboratory of Power Station Automation Technology, Department of Automation, Shanghai University, Shanghai 200072, People's Republic of China

In this study, a fault observer is designed based on the improved FAFE algorithm for systems with a fast time-varying delay and time-varying parameters. The faults are estimated by introducing relaxation matrices. Based on this, an output feedback fault-tolerant controller is designed. The relaxation matrices are introduced to realize fault-tolerant control. Because the observer and controller cannot be solved directly, this paper derives a corresponding iterative algorithm to solve the problem. Finally, the accuracy of the method is proven through numerical and real examples.

The contributions of this study are as follows. (1) The system reported in this paper has a fast time-varying delay and time-varying parameters; we estimate the faults in this system, despite the challenging nature of the task. (2) After the fault estimation, we propose a method for fault accommodation.

2. System description and preliminaries

A system with a fast time-varying delay can be represented as

$$
\begin{cases}
\dot{x}(t) = (A + D_1 F(t) E_1) x(t) \\
\quad + (A_d + D_1 F(t) E_d) x(t - d(t)) \\
\quad + (B + D_1 F(t) E_2) u(t) + (G + D_1 F(t) E_3) \\
\quad \times f(t) + (W + D_1 F(t) E_4) w(t) \\
y(t) = C x(t)
\end{cases}
\tag{1}
$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^p$ are the state, control input, and output vectors, respectively. $f(t) \in R^q$ indicates the fault in the actuator of the system, satisfying the condition $\dot{f}(t) \in L_2[0, \infty)$; $d(t)$ is the time-varying delay in the system, which satisfies $0 < d(t) \le h$ and $\dot{d}(t) \le \tau$; and $w(t) \in R^s$ is the noise in the system, which satisfies $w(t) \in L_2[0, \infty)$. In addition, A, A_d, B, G, W, C, D_1, E_1, E_d, E_2, E_3, and E_4 are known real constant matrices with appropriate dimensions, $F(t)$ represents the matrix associated with the time-varying parameters of the system, satisfying the condition $F^T(t) F(t) \le I$, and I is a unit matrix.

Prior to diagnosing the faults and developing a fault-tolerant control for the system represented in (1), we assume the following:

Assumption 2.1: *B is a full column rank matrix, and C is a full row rank matrix.*

Assumption 2.2: *(A, B) is controllable, and (A, C) is observable.*

Assumption 2.3: *$rank(B, G) = rank(B)$ and $rank(E_2, E_3) = rank(E_2)$.*

Remark 2.1: Assumptions 2.1 and 2.2 are reasonable for the control system and are necessary conditions for

fault estimation and accommodation. The fault studied in this paper occurs in actuators. According to a previous study [19], assumption 2.3 is reasonable and is equivalent to considering a matrix $G^* \in R^{m \times n}$, implying that $BG^* + G = 0$ and $E_2 G^* + E_3 = 0$ hold.

Lemma 2.1: *[20]: Y is assumed to be a symmetric matrix, where D and E are matrices of appropriate dimensions. Accordingly, the inequality*

$$
Y + DFE + E^T F^T D^T < 0
$$

will hold for all matrices satisfying $F^T F \le I$, if and only if the inequality

$$
Y + \varepsilon DD^T + \varepsilon^{-1} E^T E < 0
$$

holds for a positive scalar ε.

3. Design of a fault estimation observer

The following observer is designed to estimate faults:

$$
\begin{cases}
\dot{\hat{x}}(t) = (A + D_1 F(t) E_1) \hat{x}(t) + (A_d + D_1 F(t) E_d) \\
\quad \times \hat{x}(t - d(t)) + (B + D_1 F(t) E_2) u(t) \\
\quad + (G + D_1 F(t) E_3) \hat{f}(t) - L(\hat{y}(t) - y(t)) \\
\quad - H(\hat{y}(t - d(t)) - y(t - d(t))) \\
\hat{y}(t) = C \hat{x}(t)
\end{cases}
\tag{2}
$$

where $\hat{x}(t) \in \hat{R}n$ is the observer state vector, $\hat{y}(t) \in R^p$ is the observer output vector, $\hat{f}(t) \in R^q$ is the fault estimates, and L and H are the gain matrices of the observer to be designed.

We used the improved FAFE algorithm to estimate the fault:

$$
\dot{\hat{f}}(t) = -\Gamma \left(K_1 \dot{\tilde{y}}(t) + K_2 \tilde{y}(t) \right)
\tag{3}
$$

where $\Gamma = \Gamma^T > 0$ is the adaptive learning factor, and K_1 and K_2 are matrices with unknown parameters to be solved.

From the improved FAFE algorithm (3), the fault estimation signal can be obtained.

$$
\hat{f}(t) = -\Gamma \left(K_1 \tilde{y}(t) + K_2 \int_0^t \tilde{y}(s) ds \right)
\tag{4}
$$

Remark 3.1: The CAFE algorithm is expressed as

$$
\dot{\hat{f}}(t) = -\Gamma K \tilde{y}(t)
\tag{5}
$$

and the FAFE algorithm is expressed as

$$
\dot{\hat{f}}(t) = -\Gamma K \left(\dot{\tilde{y}}(t) + \tilde{y}(t) \right)
\tag{6}
$$

Equations (5) and (6) show that the FAFE is the improved version of the CAFE. Because the FAFE includes an additional $\dot{\tilde{y}}(t)$ term, the performance of the

FAFE is better than that of the CAFE. From (4) and (6), we see that in the general FAFE algorithm, $K = K_1 = K_2$. It needs to satisfy the equation $(G + D_1F(t)E_3)^T P = KC$. Because of the uncertain parameter matrix $F(t)$, this equation is difficult to solve. The improved FAFE reported in this paper does not require the constraints of this equation; thus, we compensate for the shortcomings of the general FAFE algorithm. In (3), Γ is an adaptive learning factor; its role is only to improve the fault estimation performance and has nothing to do with the performance indicators, namely the robustness and stability, of the observer. Therefore, we do not consider Γ (we take $\Gamma = I$ at first, where I is a unit matrix) when the observer is designed using the H_∞ performance indicators. Once the parameters affecting the performance indicators are determined, we assign Γ a specific value to improve the fault estimation performance.

We define

$$\tilde{x} = \hat{x} - x, \quad \tilde{y} = \hat{y} - y, \quad \text{and} \quad \tilde{f} = \hat{f} - f$$

where $\tilde{x}$ is the state estimation error, $\tilde{y}$ is the output estimation error, and $\tilde{f}$ is the fault estimation error.

Furthermore, we define

$$A' = A + D_1F(t)E_1, \; A_d' = A_d + D_1F(t)E_d,$$

$$B' = B + D_1F(t)E_2, \; G' = G + D_1F(t)E_3, \; \text{and}$$

$$W' = W + D_1F(t)E_4$$

Thus, we can obtain the following extension system:

$$\begin{cases} \dot{\bar{x}}(t) = (\bar{A} - \bar{L}\bar{C} + \bar{D}_1F(t)\bar{E}_1)\bar{x}(t) \\ \quad + (\bar{A}_d - \bar{H}\bar{C} + \bar{D}_1F(t)\bar{E}_d)\bar{x}(t - d(t)) \\ \quad - (\bar{W} + \bar{D}F(t)\bar{E}_4)\mu(t) \\ \tilde{y}(t) = \bar{C}\tilde{x}(t) \end{cases} \quad (7)$$

where

$$\bar{A} = \begin{pmatrix} A & G \\ -K_1CA - K_2C & -K_1CG \end{pmatrix}, \; \bar{L} = \begin{pmatrix} L \\ -K_1CL \end{pmatrix},$$

$$\bar{C} = (C0), \; \bar{D}_1 = \begin{pmatrix} D_1 \\ -K_1CD_1 \end{pmatrix}, \; \bar{E}_1 = (E_1E_3),$$

$$\bar{H} = \begin{pmatrix} H \\ -K_1CH \end{pmatrix}, \; \bar{E}_d = (E_d 0), \; \bar{W} = \begin{pmatrix} W & 0 \\ -K_1CW & I \end{pmatrix},$$

$$\bar{E}_4 = (E_40), \; \bar{x}(t) = (\tilde{x}^T(t)\tilde{f}^T(t))^T, \; \text{and}$$

$$\mu(t) = \left(w^T(t)f^{\dot{T}}(t)\right)^T.$$

To reduce the steady state error in the system, we define the following H_∞ performance indicator:

$$J = \int_0^\infty \frac{1}{\gamma}\tilde{y}^T(t)\tilde{y}(t) - \gamma\mu^T(t)\mu(t)\mathrm{d}t \quad (8)$$

Theorem 3.1: *For the given H_∞ performance indicators $\gamma > 0$ and constants $h > 0$, if there exist symmetric positive definite matrices $P_1, P_2, Z_1, Q_1, \Pi_1,$ and $\Pi_2 \in R^{n\times n}$ and matrices $K_1, K_2 \in R^{q\times p}, N_1, N_2 \in R^{n\times n}, L,$ and $H \in R^{n\times p}$, such that (9) holds,*

$$\begin{pmatrix} \Phi_0 & hN & h\Phi_1^T & \Phi_2 \\ * & -hZ_1 & 0 & 0 \\ * & * & -hZ_1^{-1} & 0 \\ * & * & * & -\Pi_1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} \Phi_3^T & \Phi_2 & \Phi_4^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Pi_1^{-1} & 0 & 0 \\ * & -\Pi_2 & 0 \\ * & * & -\Pi_2^{-1} \end{pmatrix} < 0 \quad (9)$$

where

$$\Phi_0 = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & -P_1W' & 0 \\ * & \varphi_4 & \varphi_5 & 0 & 0 \\ * & * & \varphi_6 & P_2K_1CW' & -P_2 \\ * & * & * & -\gamma I & 0 \\ * & * & * & * & -\gamma I \end{pmatrix},$$

$$\varphi_1 = P_1(A' - LC) + (A' - LC)^TP_1 + Q_1 \\ + N_1^T + N_1 + \gamma^{-1}C^TC,$$

$$\varphi_1 = P_1(A' - LC) + (A' - LC)^TP_1 \\ + Q_1 + N_1^T + N_1 + \gamma^{-1}C^TC,$$

$$\varphi_1 = P_1(A' - LC) + (A' - LC)^TP_1 + Q_1 + N_1^T + N_1 \\ + \gamma^{-1}C^TC, \varphi_2 = P_1(A_d' - HC) + N_2^T - N_1,$$

$$\varphi_3 = P_1G' - A'^TC^TK_1^TP_2 \\ - C^TK_2^TP_2, \varphi_4 = -(1 - \tau)Q_1 - N_2^T - N_2,$$

$$\varphi_5 = -C^TK_2^TP_2, \varphi_6 = -P_2K_1CG' \\ - G'^TC^TK_1^TP_2, \text{and} N = (N_1^TN_2^T000)^T$$

the improved FAFE algorithm (4) can make the state and fault estimation errors robust and stable, and the extension system (7) will satisfy the H_∞ performance $\|\tilde{y}(t)\|_2 < \gamma\|\mu(t)\|_2$.

Proof: The following Lyapunov function is selected:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \quad (10)$$

where

$$V_1(t) = \bar{x}^T(t) P \bar{x}(t) \tag{11}$$

$$V_2(t) = \int_{t-d(t)}^{t} \bar{x}^T(t) Q \bar{x}(t) dt, \text{ and} \tag{12}$$

$$V_3(t) = \int_{-h}^{0} \int_{t+\theta}^{t} \bar{x}^{\dot{T}}(s) Z \dot{\bar{x}}(s) ds d\theta \tag{13}$$

where $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$, $Q = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix}$.

Taking the derivatives of $V_1(t)$, $V_2(t)$, and $V_3(t)$ with respect to time, we obtain:

$$
\begin{aligned}
\dot{V}_1 &= 2\tilde{x}^T(t) P_1 \dot{\tilde{x}}(t) + 2\tilde{f}^T(t) P_2 \dot{\tilde{f}}(t) \\
&= 2\tilde{x}^T(t) P_1 ((A' - LC)\tilde{x}(t) \\
&\quad + (A_d' - HC)\tilde{x}(t - d(t)) \\
&\quad - W'w(t)) + 2\tilde{f}^T(t) P_2(-K_1 C((A' - LC)\tilde{x}(t) \\
&\quad + (A_d' - HC)\tilde{x}(t - d(t)) - W'w(t)) \\
&\quad - K_2 C\tilde{x}(t) - \dot{f}(t)) \\
&\leq 2\tilde{x}^T(t) P_1(A' - LC)\tilde{x}(t) + 2\tilde{x}^T(t) \\
&\quad \times P_1(A_d' - HC)\tilde{x}(t - d(t)) \\
&\quad + 2\tilde{x}^T(t) P_1 G'\tilde{f}(t) - 2\tilde{x}^T(t) P_1 W'w(t) \\
&\quad - 2\tilde{f}^T(t) P_2(K_1 CA' + K_2 C)\tilde{x}(t) \\
&\quad - 2\tilde{f}^T(t) P_2 K_1 CA_d'\tilde{x}(t - d(t)) \\
&\quad - 2\tilde{f}^T(t) P_2 K_1 CG'\tilde{f}(t) + 2\tilde{f}^T(t) P_2 K_1 CW'w(t) \\
&\quad - 2\tilde{f}^T(t) P_2 \dot{f}(t) + \tilde{f}^T(t) P_2 K_1 C \Pi_1^{-1} \\
&\quad \times (P_2 K_1 C)^T \tilde{f}(t) + \tilde{x}^T(t) C^T L^T \Pi_1 LC\tilde{x}(t) \\
&\quad + \tilde{f}^T(t) P_2 K_1 C \Pi_2^{-1} (P_2 K_1 C)^T \tilde{f}(t) \\
&\quad + \tilde{x}^T(t - d(t)) C^T H^T \Pi_2 HC\tilde{x}(t - d(t)) \tag{14}
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_2 &= \tilde{x}^T(t) Q_1 \tilde{x}(t) - (1 - \dot{d}(t)) \tilde{x}^T(t) Q_1 \tilde{x}(t) \\
&\leq \tilde{x}^T(t) Q_1 \tilde{x}(t) - (1 - \tau) \tilde{x}^T(t) Q_1 \tilde{x}(t), \quad \text{and} \tag{15}
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_3 &\leq h\dot{\tilde{x}}^T(t) Z_1 \dot{\tilde{x}}(t) - \int_{t-d(t)}^{t} \dot{\tilde{x}}^T(s) Z_1 \dot{\tilde{x}}(s) ds \\
&= h\eta^T(t) \Phi_1^T Z_1 \Phi_1 \eta(t) + 2\eta^T(t) N K_0^T \eta(t) \\
&\quad + h\eta^T(t) N Z_1^{-1} N^T \eta(t) \\
&\quad - \int_{t-d(t)}^{t} \left(\dot{\tilde{x}}^T(s) Z_1 + \eta^T(s) N \right) Z_1^{-1} \\
&\quad \times \left(Z_1 \dot{\tilde{x}}(s) + N^T \eta(s) \right) ds \\
&\leq h\eta^T(t) \Phi_1^T Z_1 \Phi_1 \eta(t) + 2\eta^T(t) N K_0^T \eta(t) \\
&\quad + h\eta^T(t) N Z_1^{-1} N^T \eta(t) \tag{16}
\end{aligned}
$$

where

$$\eta(t) = \left(\tilde{x}^T(t) \; \tilde{x}^T(t - d(t)) \; \tilde{f}^T(t) \quad w^T(t) \quad \dot{f}^T(t) \right)^T$$

$$K_0 = (1\,1\,0\,0\,0)^T,$$

$$\Phi_1 = ((A' - LC)(A_d' - HC)\,G' - W'\,0),$$

$$\Phi_2 = (0\,0\,(P_2 K_1 C)^T\,0\,0)^T, \text{ and } \Phi_3 = (LC\,0\,0\,0\,0),$$

$$\Phi_4 = (0\,HC\,0\,0\,0) \qquad \blacksquare$$

Remark 3.2: Because $\tilde{x}(t) - \tilde{x}(t - d(t)) = \int_{t-d(t)}^{t} \dot{\tilde{x}}(s) ds$,

we can obtain

$$
\begin{aligned}
&(\tilde{x}^T(t) N_1 + \tilde{x}^T(t - d(t)) N_2) \\
&\cdot \left(\tilde{x}(t) - \tilde{x}(t - d(t)) - \int_{t-d(t)}^{t} \dot{\tilde{x}}(s) ds \right) = 0 \tag{17}
\end{aligned}
$$

Based on (17), we can solve (16); thus, the relaxation matrices N_1 and N_2 are introduced. If the relaxation matrices are not introduced, φ_4 of Theorem 3.1 will be $-(1 - \tau)Q_1 > 0$. In this case, the matrix inequality of Theorem 3.1 will be unsolved when $\tau > 1$. Because the relaxation matrices N_2 and N_2^T appear in φ_4, the matrix inequality can be solved. Thus, faults can be estimated.

From (8), the following can be obtained:

$$
\begin{aligned}
J &\leq \int_0^{\infty} \frac{1}{\gamma} \tilde{y}^T(t) \tilde{y}(t) - \gamma \mu^T(t)\mu(t) + \dot{V}(t) dt \\
&= \int_0^{\infty} \eta^T(t)(\Phi_0 + h\Phi_1^T Z_1 \Phi_1 + hN Z_1^{-1} N^T \\
&\quad + \Phi_2^T \Pi_1^{-1} \Phi_2 + \Phi_3^T \Pi_1 \Phi_3 \\
&\quad + \Phi_2^T \Pi_2^{-1} \Phi_2 + \Phi_4^T \Pi_2 \Phi_4)\eta(t) dt \tag{18}
\end{aligned}
$$

According to Schur complements, when (7) holds, we have

$$
\begin{aligned}
&\Phi_0 + h\Phi_1^T Z_1 \Phi_1 + hN Z_1^{-1} N^T + \Phi_2^T \Pi_1^{-1} \Phi_2 \\
&\quad + \Phi_3^T \Pi_1 \Phi_3 + \Phi_2^T \Pi_2^{-1} \Phi_2 + \Phi_4^T \Pi_2 \Phi_4 < 0 \tag{19}
\end{aligned}
$$

Equation (19) can be expressed as

$$\frac{1}{\gamma} \tilde{y}^T(t) \tilde{y}(t) - \gamma \mu^T(t)\mu(t) + \dot{V}(t) < 0$$

Therefore, according to [21], the state and fault estimation errors are robust and stable. From (18), (19), and considering 0 the initial state, extension system (7) satisfies the H_∞ performance $||\tilde{y}(t)||_2 < \gamma ||\mu(t)||_2$.

It is difficult to solve (9) of Theorem 3.1; therefore, we transformed it into a matrix inequality, which can be solved using the iterative method.

Note that

$$
\begin{aligned}
\Phi_0 &= \Phi_{01} + \Phi_{02} \\
&= \Phi_{01} + \Phi_{021} F(t)\Phi_{022} + \Phi_{022}^T F^T(t)\Phi_{021}^T \tag{20}
\end{aligned}
$$

where

$$\Phi_{01} = \begin{pmatrix} \varphi_{11} & \varphi_{21} & \varphi_{31} & -P_1 W & 0 \\ * & \varphi_4 & \varphi_5 & 0 & 0 \\ * & * & \varphi_{61} & P_2 K_1 CW & -P_2 \\ * & * & * & -\gamma I & 0 \\ * & * & * & * & -\gamma I \end{pmatrix}$$

$$\Phi_{022} = (E_1 \; E_d \; E_3 \; -E_4 \; 0)$$

$$\Phi_{021} = ((P_1 D_1)^T \; 0 \; 0 \; -(P_2 K_1 C D_1)^T \; 0)^T$$

$$\Phi_1 = \Phi_{11} + D_1 F(t) \Phi_{022}$$

$$\Phi_{11} = ((A - LC) \; (A_d - HC) \; G \; -W \; 0),$$

$$\varphi_{11} = P_1(A - LC) + (A - LC)^T P_1 + Q_1 \\ + N_1{}^T + N_1 + \gamma^{-1} C^T C$$

$$\varphi_{21} = P_1(A_d - HC) + N_2{}^T - N_1$$

$$\varphi_{31} = P_1 G - A^T C^T K_1{}^T P_2 - C^T K_2{}^T P_2, \text{ and}$$

$$\varphi_{61} = -P_2 \Gamma K_1 CG - G^T C^T K_1{}^T P_2.$$

Thus, inequality (9) is equivalent to

$$\begin{pmatrix} \Phi_{01} & hN & h\Phi_{11}{}^T P_1 & \Phi_2 \\ * & -hZ_1 & 0 & 0 \\ * & * & -hP_1 Z_1{}^{-1} P_1 & 0 \\ * & * & * & -\Pi_1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{matrix} \Phi_3{}^T & \Phi_2 & \Phi_4{}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Pi_1{}^{-1} & 0 & 0 \\ * & -\Pi_2 & 0 \\ * & * & -\Pi_2{}^{-1} \end{matrix}$$

$$+ \begin{pmatrix} \Phi_{021} \\ 0 \\ hP_1 D_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} F(t) \begin{pmatrix} \Phi_{022}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$

$$+ \begin{pmatrix} \Phi_{022}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} F^T(t) \begin{pmatrix} \Phi_{021} \\ 0 \\ hP_1 D_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T < 0 \quad (21)$$

According to lemma 1, (21) is equivalent to considering $\varepsilon > 0$. Therefore, the following inequality holds:

$$\begin{pmatrix} \Phi_{01} & hN & h\Phi_{11}{}^T P_1 & \Phi_2 \\ * & -hZ_1 & 0 & 0 \\ * & * & -hP_1 Z_1{}^{-1} P_1 & 0 \\ * & * & * & -\Pi_1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{matrix} \Phi_3{}^T & \Phi_2 & \Phi_4{}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Pi_1{}^{-1} & 0 & 0 \\ * & -\Pi_2 & 0 \\ * & * & -\Pi_2{}^{-1} \end{matrix}$$

$$+ \varepsilon_1 \begin{pmatrix} \Phi_{021} \\ 0 \\ hP_1 D_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \Phi_{021} \\ 0 \\ hP_1 D_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$

$$+ \varepsilon_1^{-1} \begin{pmatrix} \Phi_{022}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \Phi_{022}^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T < 0 \quad (22)$$

According to the Schur complements and lemma 1, we define $Y_L = P_1 L$, $Y_H = P_1 H$, $P_{K1} = P_2 K_1$, and $P_{K2} =$

$P_2 K_2$ and obtain

$$
\begin{pmatrix}
\Phi_{01} & hN & h\Phi^T{}_{11P} & \Phi_2 & \Phi_{3P}{}^T \\
* & -hZ_1 & 0 & 0 & 0 \\
* & * & -hP_1Z_1^{-1}P_1 & 0 & 0 \\
* & * & * & -\Pi_1 & 0 \\
* & * & * & * & -P_1\Pi_1^{-1}P_1 \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}
$$

$$
\begin{pmatrix}
\Phi_2 & \Phi_{4P}{}^T & \Phi_{021} & \Phi_{022}{}^T \\
0 & 0 & 0 & 0 \\
0 & 0 & hP_1D_1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\Pi_2 & 0 & 0 & 0 \\
* & -P_1\Pi_2^{-1}P_1 & 0 & 0 \\
* & * & -\varepsilon_1^{-1}I & 0 \\
* & * & * & -\varepsilon_1 I
\end{pmatrix} < 0 \quad (23)
$$

where

$$
\Phi_{01} = \begin{pmatrix}
\varphi_{11} & \varphi_{21} & \varphi_{31} & -P_1W & 0 \\
* & \varphi_4 & \varphi_5 & 0 & 0 \\
* & * & \varphi_{61} & P_{K1}CW & -P_2 \\
* & * & * & -\gamma I & 0 \\
* & * & * & * & -\gamma I
\end{pmatrix},
$$

$\Phi_{022} = (E_1 \; E_d \; E_3 \; -E_4 \; 0)$,

$\Phi_{021} = ((P_1D_1)^T \; 0 \; 0 \; -(P_{K1}CD_1)^T \; 0)^T$,

$\Phi_{11p} = ((P_1A - Y_LC)\;(P_1A_d - Y_HC)\;P_1G\;-P_1W\;0)$

$\Phi_2 = (0\;0\;(P_{K1}C)^T\;0\;0)^T$, $\Phi_{3P} = (Y_LC\;0\;0\;0\;0)$,

$\Phi_{4P} = (0\;Y_HC\;0\;0\;0)$, $\varphi_{61} = -P_{K1}CG - G^TC^TP_{K1}^T$

$\varphi_{11} = (P_1A - Y_LC) + (P_1A - Y_LC)^T + Q_1$
$\qquad + N_1{}^T + N_1 + \gamma^{-1}C^TC$, and

$\varphi_{21} = (P_1A_d - Y_HC) + N_2{}^T - N_1$,

$\varphi_{31} = P_1G - A^TC^TP_{K1}^T - C^TP_{K2}^T$.

Therefore, we can write the following Theorem 3.2.

Theorem 3.2: *For given H_∞ performance indicators $\gamma > 0$ and constants $h > 0$, if there exist symmetric positive definite matrices P_1, P_2, Z_1, Q_1, Π_1, and $\Pi_2 \in R^{n\times n}$; matrices K_1, $K_2 \in R^{q\times p}$, N_1, $N_2 \in R^{n\times n}$, Y_L,*

$Y_H \in R^{n\times p}$, P_{K1}, and $P_{K2} \in R^{n\times n}$; and arithmetic number ε_1 such that (23) holds, then the improved FAFE algorithm (4) makes the state and fault estimation errors robust and stable. The extension system (7) satisfies the H_∞ performance $||\tilde{y}(t)||_2 < \gamma ||\mu(t)||_2$.

The inequality in (23) is not linear; nevertheless, it can be solved using the iterative method. We define $P_1Z_1^{-1}P_1 \geq S$, $P_1\Pi_1^{-1}P_1 \geq S_1$, and $P_1\Pi_2^{-1}P_1 \geq S_2$ and express (23) as

$$
\begin{pmatrix}
\Phi_{01} & hN & h\Phi^T{}_{11P} & \Phi_2 & \Phi_{3P}{}^T \\
* & -hZ_1 & 0 & 0 & 0 \\
* & * & -hS & 0 & 0 \\
* & * & * & -\Pi_1 & 0 \\
* & * & * & * & -S_1 \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}
$$

$$
\begin{pmatrix}
\Phi_2 & \Phi_{4P}{}^T & \Phi_{021} & \Phi_{022}{}^T \\
0 & 0 & 0 & 0 \\
0 & 0 & hP_1D_1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\Pi_2 & 0 & 0 & 0 \\
* & -S_2 & 0 & 0 \\
* & * & -\varepsilon_1^{-1}I & 0 \\
* & * & * & -\varepsilon_1 I
\end{pmatrix} < 0
$$

Thus $P_1^{-1}Z_1P_1^{-1} \leq S^{-1}$, $P_1^{-1}\Pi_1P_1^{-1} \leq S_1{}^{-1}$, and $P_1^{-1}\Pi_2P_1^{-1} \leq S_2{}^{-1}$, and according to Schur complements, we have

$$
\begin{pmatrix} S^{-1} & P_1^{-1} \\ P_1^{-1} & Z_1^{-1} \end{pmatrix} \geq 0 \tag{24}
$$

$$
\begin{pmatrix} S_1^{-1} & P_1^{-1} \\ P_1^{-1} & \Pi_1^{-1} \end{pmatrix} \geq 0, \text{ and} \tag{25}
$$

$$
\begin{pmatrix} S_2^{-1} & P_1^{-1} \\ P_1^{-1} & \Pi_2^{-1} \end{pmatrix} \geq 0 \tag{26}
$$

The new invertible matrices $U_1 = S^{-1}$, $U_2 = P_1^{-1}$, $U_3 = Z_1^{-1}$, $J_1 = S_1^{-1}$, $J_2 = \Pi_1^{-1}$, $R_2 = S_2^{-1}$, and $J_3 = \Pi_2^{-1}$ are introduced. We now have

$$
\begin{pmatrix} U_1 & U_2 \\ U_2 & U_3 \end{pmatrix} \geq 0 \tag{27}
$$

$$
\begin{pmatrix} J_1 & U_2 \\ U_2 & J_2 \end{pmatrix} \geq 0 \quad \text{and} \tag{28}
$$

$$\begin{pmatrix} R_2 & U_2 \\ U_2 & J_3 \end{pmatrix} \geq 0 \tag{29}$$

We define $\varepsilon_1^{-1} = \delta_1$. Therefore, the conditions of Theorem 3.2 can be transformed into the following optimization problem:

$$Min\ Trace\begin{pmatrix} P_1 U_2 + Z_1 U_3 + S U_1 + S_1 J_1 \\ +\Pi_1 J_2 + S_2 R_2 + \Pi_2 J_3 + \varepsilon_1 \delta_1 \end{pmatrix}$$

s.t. (23), (27), (28), (29)

$$\begin{pmatrix} P_1 & I \\ I & U_2 \end{pmatrix} \geq 0,\ \begin{pmatrix} Z_1 & I \\ I & U_3 \end{pmatrix} \geq 0$$

$$\begin{pmatrix} S & I \\ I & U_1 \end{pmatrix} \geq 0,\ \begin{pmatrix} S_1 & I \\ I & J_1 \end{pmatrix} \geq 0,\ \begin{pmatrix} \varepsilon_1 & 1 \\ 1 & \delta_1 \end{pmatrix} \geq 0,$$

$$\begin{pmatrix} \Pi_1 & I \\ I & J_2 \end{pmatrix} \geq 0,\ \begin{pmatrix} S_2 & I \\ I & R_2 \end{pmatrix} \geq 0,\ and\ \begin{pmatrix} \Pi_2 & I \\ I & J_3 \end{pmatrix} \geq 0 \tag{30}$$

The steps involved in obtaining the solution are as follows.

First, we set the error threshold $\in$ and the maximum number of iterations k^*.

Step 1: Solve (23), (27), (28), (29), and (30) to obtain feasible solutions $P_1^{\ 1}, U_1^{\ 1}, Z_1^{\ 1}, U_3^{\ 1}, S^1, U_1^{\ 1}, S_1^{\ 1}, J_1^{\ 1}, \Pi_1^{\ 1}, J_3^{\ 1}, S_2^{\ 1}, R_2^{\ 1}, \Pi_2^{\ 1}, J_3^{\ 1}, \varepsilon_1^{\ 1},$ and $\delta_1^{\ 1}$. Calculate the objective function value Tr^0, define $Tr^1 = 2Tr^0$, and let $k = 1$.

Step 2: With $k = k + 1$, obtain the optimal solutions $P_1^{\ k}, U_1^{\ k}, Z_1^{\ k}, U_3^{\ k}, S^k, U_1^{\ k}, S_1^{\ k}, J_1^{\ k}, \Pi_1^{\ k}, J_3^{\ k}, S_2^{\ k}, R_2^{\ k}, \Pi_2^{\ k}, J_3^{\ k}, \varepsilon_1^{\ k},$ and $\delta_1^{\ k}$, which satisfy (23), (27), (28), (29), and (30), and minimize the objective function

$$Trace\begin{pmatrix} P_1^{\ k}U_2^{\ k-1}+P_1^{\ k-1}U_2^{\ k}+Z_1^{\ k}U_3^{\ k-1}+Z_1^{\ k-1}U_3^{\ k} \\ +S^kU_1^{\ k-1}+S^{k-1}U_1^{\ k}+S_1^{\ k}J_1^{\ k-1}+S_1^{\ k-1}J_1^{\ k} \\ +\Pi_1^{\ k}J_2^{\ k-1}+\Pi_1^{\ k-1}J_2^{\ k}+S_2^{\ k}R_2^{\ k-1}+S_2^{\ k-1}R_2^{\ k} \\ +\Pi_2^{\ k}J_3^{\ k-1}+\Pi_2^{\ k-1}J_3^{\ k}+\varepsilon_1^{\ k}\delta_1^{\ k-1}+\varepsilon_1^{\ k-1}\delta_1^{\ k} \end{pmatrix}.$$ Calculate the minimized value Tr^k and determine $\Delta Tr^k = |Tr^{k-1} - Tr^k|$.

Step 3: If $\Delta Tr < \in$, terminate the process. Otherwise, proceed to the next step.

Step 4: If $k > k^*$, terminate the process. Otherwise, proceed to step 2.

4. Design of H_∞ fault-tolerant controller

We design the following feedback fault-tolerant controller based on the fault estimates:

$$\begin{cases} \dot{\xi}(t) = A_c\zeta(t) + B_c y(t) \\ u(t) = C_c\zeta(t) + D_c y(t) + G^*\hat{f}(t) + r(t) \end{cases} \tag{31}$$

where $\xi(t) \in R^n$ is the controller state vector, $r(t) \in R^m$ is the reference input, and $A_c, B_c, C_c,$ and D_c are the parameter matrices of the controller to be designed.

Taking $r(t) = 0$, we obtain the following system based on (1).

$$\begin{cases} \dot{\zeta}(t) = (\tilde{A} + \tilde{D}_1 F(t)\tilde{E}_1)\zeta(t) \\ \qquad + (\tilde{A}_d + \tilde{D}_1 F(t)\tilde{E}_d)\zeta(t - d(t)) \\ \qquad + (\tilde{W} + \tilde{D}_1 F(t)\tilde{E}_3)v(t) \\ y(t) = \tilde{C}\zeta(t) \end{cases} \tag{32}$$

where

$$\tilde{A} = \begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix},\ \tilde{D}_1 = \begin{pmatrix} D_1 \\ 0 \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} C & 0 \end{pmatrix},\ \tilde{E}_1 = \begin{pmatrix} E_1 + E_2D_cC & E_2C_c \end{pmatrix},$$

$$\tilde{E}_d = \begin{pmatrix} E_d & 0 \end{pmatrix},\ \tilde{A}_d = \begin{pmatrix} A_d & 0 \\ 0 & 0 \end{pmatrix},$$

$$\tilde{W} = \begin{pmatrix} W & -G \\ 0 & 0 \end{pmatrix},\ \tilde{E}_3 = \begin{pmatrix} E_4 & -E_3 \end{pmatrix},$$

$$\zeta(t) = (x^T(t)\ \xi^T(t))^T,\ and\ v(t) = (w^T(t)\ \tilde{f}^T(t))^T.$$

Remark 4.1: According to the output feedback controller (31) and the system state equation, i.e. (1), the following can be obtained:

$$\dot{x}(t) = (A + D_1F(t)E_1)x(t) + (B + D_1F(t)E_2)D_cCx(t)$$
$$\qquad + (A_d + D_1F(t)E_d)x(t - d(t))$$
$$\qquad + (B + D_1F(t)E_2)C_c\xi(t)$$
$$\qquad + (BG^* + G + D_1F(t)(E_2G^* + E_3))\hat{f}(t)$$
$$\qquad - (G + D_1F(t)E_3)\tilde{f}(t) + (W + D_1F(t)E_4)w(t)$$

According to assumption 2.3,

$$\dot{x}(t) = (A + D_1F(t)E_1)x(t) + (B + D_1F(t)E_2)D_cCx(t)$$
$$\qquad + (A_d + D_1F(t)E_d)x(t - d(t))$$
$$\qquad + (B + D_1F(t)E_2)C_c\xi(t)$$
$$\qquad - (G + D_1F(t)E_3)\tilde{f}(t) + (W + D_1F(t)E_4)w(t).$$

We can now obtain the extension system (32).

Theorem 4.1: For given H_∞ performance indicators $\gamma_2 > 0$ and constants $h > 0$, if there exist symmetric positive definite matrices $P_c \in R^{2n \times 2n}, Z_{c1},$ and $Q_{c1} \in R^{n \times n}$; matrices $N_{c1}, N_{c2} \in R^{n \times n}, A_c \in R^{n \times n}, B_c \in R^{n \times p}, C_c \in R^{m \times n},$ and $D_c \in R^{m \times p}$; and arithmetic number ε_2 such

that (33) holds,

$$\begin{pmatrix} \varphi_{c11} & \varphi_{c21} & P_c\tilde{W} & hN_{c10} \\ * & \varphi_{c3} & 0 & hN_{c2} \\ * & * & -\gamma_2 I & 0 \\ * & * & * & -hZ_{c1} \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\left.\begin{matrix} h\tilde{A}_1^T & P_c\tilde{D}_1 & \tilde{E}_1^T & C^T \\ hA_d{}^T & 0 & E_d^T & 0 \\ h\widetilde{W}_1{}^T & 0 & \tilde{E}_3^T & 0 \\ 0 & 0 & 0 & 0 \\ -hZ_{c1}{}^{-1} & hD_1 & 0 & 0 \\ * & -\varepsilon_2^{-1}I & 0 & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\gamma_2 I \end{matrix}\right) < 0 \quad (33)$$

where

$$\varphi_{c11} = P_c\tilde{A} + \tilde{A}^T P_c + Q_c + N_{c100}{}^T + N_{c100},$$

$$\varphi_{c21} = P_c\tilde{A}_{d0} - N_{c10} + N_{c20}{}^T, \quad Q_c = \begin{pmatrix} Q_{c1} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\widetilde{A_{d0}} = \begin{pmatrix} A_d \\ 0 \end{pmatrix}, \quad \varphi_{c3} = -(1-\tau)Q_{c1} - N_{c2} - N_{c2}^T,$$

$$N_{c10} = (N_{c1}{}^T\ 0)^T, \quad N_{c20} = (N_{c2}\ 0), \text{ and}$$

$$N_{c100} = \begin{pmatrix} N_{c1} & 0 \\ 0 & 0 \end{pmatrix},$$

the extension system (32) satisfies the H_∞ performance $\|y(t)\|_2 < \gamma_2\|v(t)\|_2$.

Proof: We define the Lyapunov function of the system represented in (32) as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \quad (34)$$

where

$$V_1(t) = \zeta^T(t)P_c\zeta(t) \quad (35)$$

$$V_2(t) = \int_{t-d(t)}^{t} \zeta^T(s)Q_c\zeta(s)ds, \text{ and} \quad (36)$$

$$V_3(t) = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_{c1}\dot{x}(s)ds d\theta \quad (37)$$

Taking the derivatives of $V_1(t)$, $V_2(t)$, and $V_3(t)$ with respect to time, we have

$$\dot{V}_1 = 2\zeta^T(t)P_c(\tilde{A}'\zeta(t) + \widetilde{A_d}'\zeta(t-d(t)) + \tilde{W}'v(t))$$

$$= 2\zeta^T(t)P_c(\tilde{A}'\zeta(t) + \widetilde{A_{d0}}'x(t-d(t)) + \tilde{W}'v(t)) \quad (38)$$

where $\widetilde{A_{d0}}' = (A_d'^T 0)^T$.

$$\dot{V}_2 \leq x^T(t)Q_{c1}x(t) - (1-\tau)x^T(t-d(t)) \\ \times Q_{c1}x(t-d(t)), \text{ and} \quad (39)$$

$$\dot{V}_3 \leq h\dot{x}^T(t)Z_{c1}\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s)Z_{c1}\dot{x}(s)ds$$

$$+ 2(x^T(t)N_{c1} + x^T(t-d(t))N_{c2})$$

$$\cdot \left(x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds\right)$$

$$= h\dot{x}^T(t)Z_{c1}\dot{x}(t) - \int_{t-d(t)}^{t} \dot{x}^T(s)Z_{c1}\dot{x}(s)ds$$

$$+ 2\eta_c^T(t)N_c K_0^T \eta_c^T(t) - 2\eta_c^T(t)N_c \int_{t-d(t)}^{t} \dot{x}(s)ds$$

$$\leq h\eta_c^T(t)\Phi_{c1}{}^T Z_{c1}\Phi_{c1}\eta_c(t) + 2\eta_c^T(t)N_c K_0^T \eta_c(t)$$

$$+ h\eta_c^T(t)N_c Z_{c1}{}^{-1}N_c^T \eta_c(t)$$

$$- \int_{t-d(t)}^{t} \left(\dot{x}^T(s)Z_{c1} + \eta_c^T(t)N_c\right)Z_{c1}{}^{-1}$$

$$\times \left(Z_{c1}\dot{x}(s) + N_c^T \eta_c(t)\right)ds$$

$$\leq h\eta_c^T(t)\Phi_{c1}{}^T Z_{c1}\Phi_{c1}\eta_c(t) + 2\eta_c^T(t)N_c K_0^T \eta_c(t)$$

$$+ h\eta_c^T(t)N_c Z_{c1}{}^{-1}N_c^T \eta_c(t) \quad (40)$$

where

$$\eta_c(t) = (x^T(t)\xi^T(t)x^T(t-d(t))v^T(t))^T,$$

$$K_0 = (1\ 0\ -1\ 0)^T, \quad \Phi_{c1} = ((A' + B'D_c C)$$

$$\times B'C_c\ A_d'\ \widetilde{W_1}'), \text{ and}$$

$$N_c = (N_{c1}{}^T\ 0\ N_{c2}{}^T\ 0)^T.$$

■

Remark 4.2: Because $x(t) - x(t-d(t)) = \int_{t-d(t)}^{t} \dot{x}(s)ds$, we have

$$(x^T(t)N_{c1} + x^T(t-d(t))N_{c2})$$

$$\cdot \left(x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds\right) = 0 \quad (41)$$

Based on (41), we can solve (40). Thus, the relaxation matrices N_{c1} and N_{c2} are introduced to realize a fault-tolerant control for the system with a fast time-varying delay with a rate of change $\tau > 1$. If the relaxation matrices are not introduced, φ_{c3} of Theorem 4.1 will be $-(1-\tau)Q_{c1} > 0$, with no solution to the matrix inequality in Theorem 4.1.

We select the following performance indicator

$$J = \int_0^\infty \frac{1}{\gamma_2}y^T(t)y(t) - \gamma_2 v^T(t)v(t)dt \quad (42)$$

and obtain

$$J \leq \int_0^\infty \frac{1}{\gamma_2} y^T(t)y(t) - \gamma_2 v^T(t)v(t) + \dot{V}(t)dt$$

$$= \int_0^\infty \eta_c^T(t)(\Phi_{c0} + hN_cZ_{c1}^{-1}N_c^T$$

$$+ h\Phi_{c1}^T Z_{c1}\Phi_{c1})\eta_c(t)dt < 0 \qquad (43)$$

where

$$\Phi_{c0} = \begin{pmatrix} \varphi_{c1} & \varphi_{c2} & P_c\widetilde{W}' \\ * & \varphi_{c3} & 0 \\ * & * & -\gamma_2 I \end{pmatrix}, \varphi_{c1} = P_c\tilde{A}' + \tilde{A}'^T P_c$$

$$+ Q_c + N_{c100}{}^T + N_{c100} + \frac{1}{\gamma_2}\tilde{C}^T\tilde{C}, \text{ and}$$

$$\varphi_{c2} = P_c\widetilde{A_{d0}}' - N_{c10} + N_{c20}{}^T.$$

According to the Schur complements, we can conclude that

$$\Phi_{c0} + hN_cZ_{c1}^{-1}N_c^T + h\Phi_{c1}^T Z_{c1}\Phi_{c1} < 0$$

is equivalent to

$$\begin{pmatrix} \Phi_{c0} & hN_c & h\Phi_{c1}{}^T \\ * & -hZ_{c1} & 0 \\ * & * & -hZ_{c1}^{-1} \end{pmatrix} < 0 \qquad (44)$$

therefore,

$$\begin{pmatrix} \varphi_{c1} & \varphi_{c2} & P_c\widetilde{W}' & hN_{c10} & h\widetilde{A_1}'^T \\ * & \varphi_{c3} & 0 & hN_{c2} & hA_d'^T \\ * & * & -\gamma_2 I & 0 & h\widetilde{W_1}'^T \\ * & * & * & -hZ_{c1} & 0 \\ * & * & * & * & -hZ_{c1}^{-1} \end{pmatrix} < 0 \quad (45)$$

The inequality in (43) can be expressed as

$$\begin{pmatrix} \varphi_{c1} & \varphi_{c2} & P_c\tilde{W} & hN_{c10} & h\tilde{A}_1^T \\ * & \varphi_{c3} & 0 & hN_{c2} & hA_d^T \\ * & * & -\gamma_2 I & 0 & h\tilde{W}_1^T \\ * & * & * & -hZ_{c1} & 0 \\ * & * & * & * & -hZ_{c1}^{-1} \end{pmatrix}$$

$$+ \begin{pmatrix} P_c\widetilde{D_1} \\ 0 \\ 0 \\ 0 \\ hD_1 \end{pmatrix} F(t) \begin{pmatrix} \tilde{E}_1^T \\ E_d^T \\ \tilde{E}_3^T \\ 0 \\ 0 \end{pmatrix}^T$$

$$+ \begin{pmatrix} \tilde{E}_1^T \\ E_d^T \\ \tilde{E}_3^T \\ 0 \\ 0 \end{pmatrix} F^T(t) \begin{pmatrix} P_c\widetilde{D_1} \\ 0 \\ 0 \\ 0 \\ hD_1 \end{pmatrix}^T + \begin{pmatrix} \tilde{C}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \tilde{C} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} < 0$$

$$(46)$$

According to lemma 1, (46) will hold if and only if there is a constant $\varepsilon_2 > 0$ that can hold (47).

$$\begin{pmatrix} \varphi_{c1} & \varphi_{c2} & P_c\tilde{W} & hN_{c10} & h\tilde{A}_1^T \\ * & \varphi_{c3} & 0 & hN_{c2} & hA_d{}^T \\ * & * & -\gamma_2 I & 0 & h\tilde{W}_1^T \\ * & * & * & -hZ_{c1} & 0 \\ * & * & * & * & -hZ_{c1}^{-1} \end{pmatrix}$$

$$+ \varepsilon_2 \begin{pmatrix} P_c\tilde{D}_1 \\ 0 \\ 0 \\ 0 \\ hD_1 \end{pmatrix} \begin{pmatrix} P_c\tilde{D}_1 \\ 0 \\ 0 \\ 0 \\ hD_1 \end{pmatrix}^T$$

$$+ \varepsilon_2^{-1} \begin{pmatrix} \tilde{E}_1^T \\ E_d{}^T \\ \tilde{E}_3^T \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \tilde{E}_1^T \\ E_d{}^T \\ \tilde{E}_3^T \\ 0 \\ 0 \end{pmatrix}^T + \begin{pmatrix} \tilde{C}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \tilde{C}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T < 0$$

$$(47)$$

According to the Schur complements, the inequality

$$\Phi_{c0} + hN_cZ_{c1}^{-1}N_c^T + h\Phi_{c1}^T Z_{c1}\Phi_{c1} < 0 \qquad (48)$$

will hold when (33) holds.

Equation (48) can be expressed as

$$\frac{1}{\gamma_2} y^T(t)y(t) - \gamma_2 v^T(t)v(t) + \dot{V}(t) < 0 \qquad (49)$$

Hence, according to [21], the state and fault estimation errors are robust and stable. Using (49) and considering 0 the initial state, we find that the extension system (32) satisfies the H_∞ performance $||y(t)||_2 < \gamma_2 ||v(t)||_2$.

The matrix inequality (33) in Theorem 4.1 is not linear, but can be transformed into one and then solved. We take matrix P_c and its inverse matrix P_c^{-1} in the following forms:

$$P_c = \begin{pmatrix} Y_1 & R_{c1} \\ R_{c1}{}^T & T_1 \end{pmatrix} \text{ and } P_c^{-1} = \begin{pmatrix} Y_2 & R_{c2} \\ R_{c2}{}^T & T_2 \end{pmatrix},$$

where Y_1 and $Y_2 \in R^{n \times n}$ are symmetric positive definite matrices, and $R_{c1}, R_{c2}, T_1,$ and T_2 are matrices with appropriate dimensions satisfying $Y_1Y_2 + R_{c1}R_{c2}^T = I$.

Given that

$$P_c \begin{pmatrix} Y_2 & I \\ R_{c2}{}^T & 0 \end{pmatrix} = \begin{pmatrix} I & Y_1 \\ 0 & R_{c1}{}^T \end{pmatrix} \qquad (50)$$

we define

$$F_1 = \begin{pmatrix} Y_2 & I \\ R_{c2}{}^T & 0 \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} I & Y_1 \\ 0 & R_{c1}{}^T \end{pmatrix} \text{ and}$$

$$\hat{A} = Y_1(A + B\hat{D}C)Y_2 + R_{c1}B_cCY_2 + Y_1BC_cR_{c2}^T$$

$$+ R_{c1}^T A_c R_{c2}^T, \hat{B} = Y_1B\hat{D} + R_{c1}B_c, \hat{C} = \hat{D}CY_2$$

$$+ C_c R_{c2}^T \text{ and } \hat{D} = D_c.$$

We can obtain

$$F_1{}^{\mathrm{T}} P_c \tilde{A} F_1 = \begin{pmatrix} AY_2 + B\hat{C} & A + B\hat{D}C \\ \hat{A} & Y_1 A + \hat{B}C \end{pmatrix},$$

$$F_1{}^{\mathrm{T}} \tilde{A}_1^{\mathrm{T}} = \begin{pmatrix} Y_2 A^{\mathrm{T}} + \hat{C}^{\mathrm{T}} B^{\mathrm{T}} \\ A^{\mathrm{T}} + C^{\mathrm{T}} \hat{D}^{\mathrm{T}} B^{\mathrm{T}} \end{pmatrix},$$

and $F_1{}^{\mathrm{T}} \tilde{E}_1^{\mathrm{T}} = \begin{pmatrix} Y_2 E_1^{\mathrm{T}} + \hat{C}^{\mathrm{T}} E_2^{\mathrm{T}} \\ E_1^{\mathrm{T}} + C^{\mathrm{T}} \hat{D}^{\mathrm{T}} E_2^{\mathrm{T}} \end{pmatrix}.$

We multiply both sides of (33) with $diag(F_1^{\mathrm{T}}, I, ..I)$ on the left and with $diag(F_1, I, ..I)$ on the right. Now, according to the Schur complements, (33) can be transformed into a matrix inequality that can be solved to prove Theorem 4.2.

Remark 4.3: Once both sides of (33) are multiplied with $diag(F_1^{\mathrm{T}}, I, ..I)$ on the left and $diag(F_1, I, ..I)$ on the right, the matrices $Y_2(N_{c2}^{\mathrm{T}} - N_{c1})$ and $hY_2 N_{c1}$, along with their transpose matrices, will appear in the transformed matrix inequality. According to the Schur complements and (51), we can obtain (52).

$$= \begin{pmatrix} Y_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ (N_2 - N_1^T) \\ 0 \\ hN_{c1}^T \end{pmatrix}^T + \begin{pmatrix} 0 \\ (N_{c2} - N_{c1}^T) \\ 0 \\ hN_{c1}^T \end{pmatrix} \begin{pmatrix} Y_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$

$$\leq \begin{pmatrix} Y_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Pi_{c1}^{-1} \begin{pmatrix} Y_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T + \begin{pmatrix} 0 \\ (N_{c2} - N_{c1}^T) \\ 0 \\ hN_{c1}^T \end{pmatrix}$$

$$\times \Pi_{c1} \begin{pmatrix} 0 \\ (N_{c2} - N_{c1}^T) \\ 0 \\ hN_{c1}^T \end{pmatrix}^T \tag{51}$$

where Π_{c1} is a symmetric positive definite matrix.

Theorem 4.2: *For given H_∞ performance indicators $\gamma_2 > 0$ and constants $h > 0$, if there exist symmetric positive definite matrices Y_1, Π_{c1}, Z_{c1}, Q_{c1}, and $Y_2 \in R^{n \times n}$; matrices N_{c1}, $N_{c2} \in R^{n \times n}$, $A_c \in R^{n \times n}$, $B_c \in R^{n \times p}$, $C_c \in R^{m \times n}$, and $D_c \in R^{m \times p}$; and arithmetic number ε_2 such that (48) holds, the extension system (32) satisfies the H_∞ performance $||y(t)||_2 < \gamma_2 ||v(t)||_2$.*

$$\begin{pmatrix} 0 & Y_2(N_{c2}^T - N_{c1}) & 0 & hY_2 N_{c1} \\ (N_{c2} - N_{c1}^T)Y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ hN_{c1}^T Y_2 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccccccc} AY_2 + B\hat{C} + Y_2 A^T + \hat{C}^T B^T & A + B\hat{D}C + \hat{A}^T & A_d & W & -G & 0 \\ * & Y_1 A + \hat{B}C + A^T Y_1 + C^T\hat{B}^T & Y_1 A_d + N_{c2}{}^T - N_{c1} & Y_1 W & -Y_1 G & hN_{c1} \\ * & * & -(1-\tau)Q_{c1} - N_{c2}^T - N_{c2} & 0 & 0 & hN_{c2} \\ * & * & * & -\gamma_2 I & 0 & 0 \\ * & * & * & * & -\gamma_2 I & 0 \\ * & * & * & * & * & -hZ_{c1} \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{array}\right.$$

$$\left.\begin{array}{ccccccc} h(Y_2 A^T + \hat{C}^T B^T) & D_1 & Y_2 E_1^T + \hat{C}^T E_2^T & Y_2 C^T & Y_2 & Y_2 & 0 & Y_2 \\ h(A^T + C^T\hat{D}^T B^T) & Y_1 D_1 & E_1^T + C^T\hat{D}^T E_2{}^T & C^T & I & I & 0 & 0 \\ hA_d{}^T & 0 & E_d{}^T & 0 & 0 & 0 & N_{c2} - N_{c1}^T & 0 \\ hW^T & 0 & E_4^T & 0 & 0 & 0 & 0 & 0 \\ -hG^T & 0 & -E_3^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & hN_{c1}^T & 0 \\ -hZ_{c1}{}^{-1} & hD_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\varepsilon_2{}^{-1}I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_{c1}{}^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & -\frac{1}{2}N_{c1}{}^{-1} & 0 & 0 \\ * & * & * & * & * & * & -\Pi_{c1} & 0 \\ * & * & * & * & * & * & * & -\Pi_{c1}{}^{-1} \end{array}\right) < 0$$

$$\tag{52}$$

The parameter matrices of the fault-tolerant controller are

$$D_c = \hat{D},$$

$$C_c = (\hat{C} - \hat{D}CY_2)R_{c2}^{-T},$$

$$B_c = R_{c1}^{(-1)}(\hat{B} - Y_1 B\hat{D}), \text{ and}$$

$$A_c = R_{c1}^{-1}(\hat{A} - Y_1(A + B\hat{D}C)Y_2)R_{c2}^{-T} - B_c CY_2 R_{c2}^{-T}$$
$$- R_{c1}^{-1} Y_1 B C_c$$

The matrix inequality (51) in Theorem 4.1 is not linear but can be solved using the iterative method. We take $Z_{c1}^{-1} = S_{c1}$, $\Pi_{c1}^{-1} = \Pi_{c2}$, $Q_{c1}^{-1} = U_{c1}$, $N_{c1}^{-1} = M_{c1}$, and $\varepsilon_2^{-1} = \delta_2$. Theorem 4.2 can now be transformed into the following optimization problem.

$$\text{Min Trace} \begin{pmatrix} Z_{c1}S_{c1} + \Pi_{c1}\Pi_{c2} + Q_{c1}U_{c1} \\ + N_{c1}M_{c1} + \varepsilon_2\delta_2 \end{pmatrix}$$

S.t. (52),

$$\begin{pmatrix} Z_{c1} & I \\ I & S_{c1} \end{pmatrix} \geq 0, \begin{pmatrix} \Pi_{c1} & I \\ I & \Pi_{c2} \end{pmatrix} \geq 0, \begin{pmatrix} Q_{c1} & I \\ I & U_{c1} \end{pmatrix} \geq 0,$$

$$\text{and } \begin{pmatrix} N_{c1} & I \\ I & M_{c1} \end{pmatrix} \geq 0 \tag{53}$$

The following are the steps involved in obtaining the solution:

First, we set the error threshold $\in_c$ and the maximum number of iterations k_c^*.

Step 1: Solve (52) and (53) to obtain feasible solutions Z_{c1}^1, S_{c1}^1, Π_{c1}^1, Π_{c2}^1, Q_{c1}^1, U_{c1}^1, N_{c1}^1, M_{c1}^1, ε_2^1, and δ_2^1. Calculate the objective function value Tr_c^0, define $Tr_c^1 = 2Tr_c^0$, and let $k_c = 1$.

Step 2: With $k = k + 1$, calculate the optimal solutions Z_{c1}^k, S_{c1}^k, Π_{c1}^k, Π_{c2}^k, Q_{c1}^k, U_{c1}^k, N_{c1}^k, M_{c1}^k, ε_2^k, and δ_2^k, which satisfy (52) and (53), and minimize the objective function

$$\text{Trace} \begin{pmatrix} Z_{c1}^k S_{c1}^{k-1} + Z_{c1}^{k-1} S_{c1}^k + \Pi_{c1}^k \Pi_{c2}^{k-1} \\ + \Pi_{c1}^{k-1} \Pi_{c2}^k + Q_{c1}^k U_{c1}^{k-1} \\ + Q_{c1}^{k-1} U_{c1}^k + N_{c1}^k M_{c1}^{k-1} \\ + N_{c1}^{k-1} M_{c1}^k + \varepsilon_2^k \delta_2^{k-1} + \varepsilon_2^{k-1} \delta_2^k \end{pmatrix}.$$

Calculate the minimized value Tr_c^k and take $\Delta Tr_c^k = |Tr_c^{k-1} - Tr_c^k|$.

Step 3: If $\Delta Tr_c < \in_c$, terminate the process. Otherwise, proceed to the next step.

Step 4: If $k_c > k_c^*$, terminate the process. Otherwise, proceed to step 2.

5. Simulation results

We consider a system with the following fast time-varying delay and uncertain parameters [22]:

$$A = \begin{pmatrix} -1.427 & 0.076 \\ -1.419 & -0.944 \end{pmatrix},$$

$$A_d = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0.3 \end{pmatrix}, C = (0 \quad 1),$$

$$W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, D_1 = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}, E_d = (0.09 \quad 0.12),$$

$$E_1 = (0.52 \quad 0.73), E_3 = 0.02, \text{ and } E_4 = 0.01,$$

where $F(t)$ satisfies $|F(t)| < 0.6$, and the system noise is $w(t) = 0.8 \cos 90t$.

Because the fault occurs on the input channel, we take $G = B$ and $E_2 = E_3$; thus, $G^* = -I$. We consider a fast time-varying delay case, where $d(t) = 0.4 - 0.2 \cos 7t$ and the rate of change $\tau = 1.4 > 1$. By solving the conditions in Theorem 3.2, we find that the minimized H_∞ performance indicator is 0.2006, the minimum objective function is 15.1304, and the relevant parameters of the fault observer are as follows.

$$P_1 = \begin{pmatrix} 0.2432 & 0.0115 \\ 0.0115 & 1.5507 \end{pmatrix}, P_2 = 0.0035,$$

$$Q_1 = \begin{pmatrix} 0.0109 & 0.0064 \\ 0.0064 & 0.1132 \end{pmatrix}, Z_1 = \begin{pmatrix} 0.0874 & 0.0201 \\ 0.0201 & 0.0979 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0.2349 & 0.1194 \\ 0.1194 & 0.1873 \end{pmatrix}, \Pi_2 = \begin{pmatrix} 0.2438 & 0.0284 \\ 0.0284 & 1.0512 \end{pmatrix},$$

$$K_1 = -5.4655, K_2 = -89.8355, \varepsilon_1 = 5.6152,$$

$$L = \begin{pmatrix} -4.2970 \\ 8.8049 \end{pmatrix}, \text{ and } H = \begin{pmatrix} -0.0390 \\ 0.0767 \end{pmatrix}.$$

The relaxation matrices are

$$N_2 = \begin{pmatrix} 0.1456 & 0.0336 \\ 0.0336 & 0.1632 \end{pmatrix}$$

$$\text{and } N_1 = \begin{pmatrix} -0.1456 & -0.0336 \\ -0.0336 & -0.1632 \end{pmatrix}$$

The introduced symmetric positive definite matrices are

$$U_3 = \begin{pmatrix} 4.1399 & -0.0306 \\ -0.0306 & 0.6498 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 4.1399 & -0.0306 \\ -0.0306 & 0.6498 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.5807 & -0.5587 \\ -0.5587 & 21.0206 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 1.7832 & 0.0474 \\ 0.0474 & 0.0493 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} 0.2476 & -1.1831 \\ -1.1831 & 15.9021 \end{pmatrix},$$

$$J_1 = \begin{pmatrix} 6.3231 & 0.4704 \\ 0.4704 & 0.0985 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} 6.3573 & -4.0546 \\ -4.0546 & 7.9726 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0.1533 & -0.0093 \\ -0.0093 & 1.0602 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0.1533 & -0.0093 \\ -0.0093 & 1.0602 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 6.5824 & 0.0574 \\ 0.0574 & 0.9520 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 4.1499 & -0.1119 \\ -0.1119 & 0.9627 \end{pmatrix}, \text{ and } \delta_1 = 0.1797$$

By solving the conditions in Theorem 4.2, we find that the minimized H_∞ performance indicator is 0.4716, and the minimum objective function is 9.0507. We can obtain R_{c1} and R_{c2} by the singular value decomposition of $I - Y_1 Y_2$. Accordingly, the relevant parameters of the fault-tolerant controller are as follows:

$$Y_1 = \begin{pmatrix} 0.5344 & 0.5744 \\ 0.5744 & 1.7509 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 1.8542 & -0.2878 \\ -0.2878 & 0.3444 \end{pmatrix},$$

$$R_{c1} = \begin{pmatrix} -0.6202 & 0.0987 \\ -0.7844 & -0.9951 \end{pmatrix},$$

$$R_{c2} = \begin{pmatrix} 0.1187 & 0 \\ 0 & 0.6180 \end{pmatrix},$$

$$Q_{c1} = \begin{pmatrix} 0.1440 & 0.1178 \\ 0.1178 & 0.8677 \end{pmatrix},$$

$$Z_{c1} = \begin{pmatrix} 0.5399 & 0.0280 \\ 0.0280 & 0.9009 \end{pmatrix},$$

$$\Pi_{c1} = \begin{pmatrix} 0.2313 & 0.1910 \\ 0.1910 & 1.0483 \end{pmatrix},$$

$$A_c = \begin{pmatrix} -1.9110 & -2.1913 \\ 29.2412 & -5.9374 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 0.3847 \\ 11.0875 \end{pmatrix}, C_c = \begin{pmatrix} 41.4716 & -8.6916 \end{pmatrix},$$

$$D_c = -5.5614,$$

and $\varepsilon_2 = 1.4580$.

The relaxation matrices are

$$N_{c1} = \begin{pmatrix} 0.1222 & 0.1158 \\ 0.1158 & 0.6008 \end{pmatrix}$$
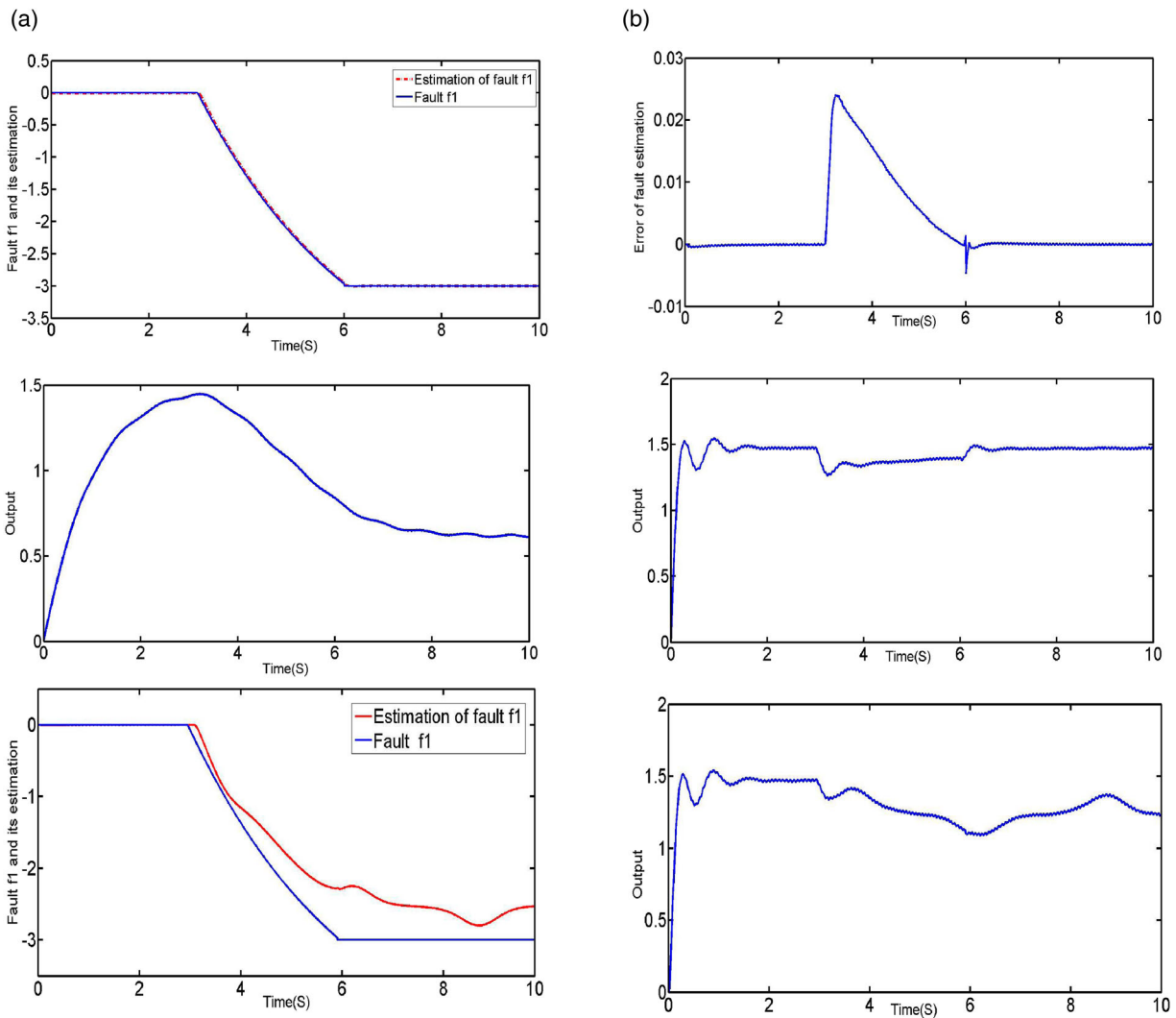
Figure 1. Simulation results of fault $f_1(t)$: (a) Estimated and actual values of fault $f_1(t)$. (b) Fault estimation error $f_1(t) - \hat{f}_1(t)$. (c) Output curve under a normal control of fault $f_1(t)$. (d) Output curve under a fault-tolerance control of fault $f_1(t)$. (e) Estimation of fault $f_1(t)$ using CAFE. (f) Output curve under a fault-tolerance control obtained using CAFE.

and $N_{c2} = \begin{pmatrix} 0.2097 & 0.0781 \\ 0.0866 & 0.6128 \end{pmatrix}$

The introduced symmetric positive definite matrices are

$$S_{c1} = \begin{pmatrix} 1.8659 & -0.0579 \\ -0.0579 & 1.1179 \end{pmatrix},$$

$$\Pi_{c2} = \begin{pmatrix} 5.1199 & -0.9328 \\ -0.9328 & 1.1291 \end{pmatrix},$$

$$U_{c1} = \begin{pmatrix} 7.8538 & -1.0663 \\ -1.0663 & 1.3035 \end{pmatrix},$$

$$M_{c1} = \begin{pmatrix} 10.0679 & -1.9411 \\ -1.9411 & 2.0478 \end{pmatrix}, \text{ and } \delta_2 = 0.6899.$$

In the simulation, the input to the system is $r(t) = 5$, and we assume that the actuator has the following fault

$f_1(t)$:

$$f_1(t) = \begin{cases} 0 & t \in [0,3] \\ 5\exp(-0.3(t-3)) - 1 & t \in [3,6] \\ -3 & t \in [6,10] \end{cases}.$$

Figure 1 shows the simulation results of fault $f_1(t)$ for $\Gamma = 30$. Figure 1(a) shows the simulation of the fault observer. Figure 1(b) shows the fault estimation error. Figure 1(c,d) show the output curves under normal and fault-tolerance controls of $f_1(t)$, respectively. Figure 1(e,f) show the corresponding simulation results of CAFE.

Remark 5.1: The higher the value of the adaptive learning factor Γ, the better. However, if it is too high, it is difficult to implement in practice. Thus, we select it in the same order of magnitude as given elsewhere [23].

In Figure 1(a), the blue curve represents the actual value of the fault, and the red one represents the
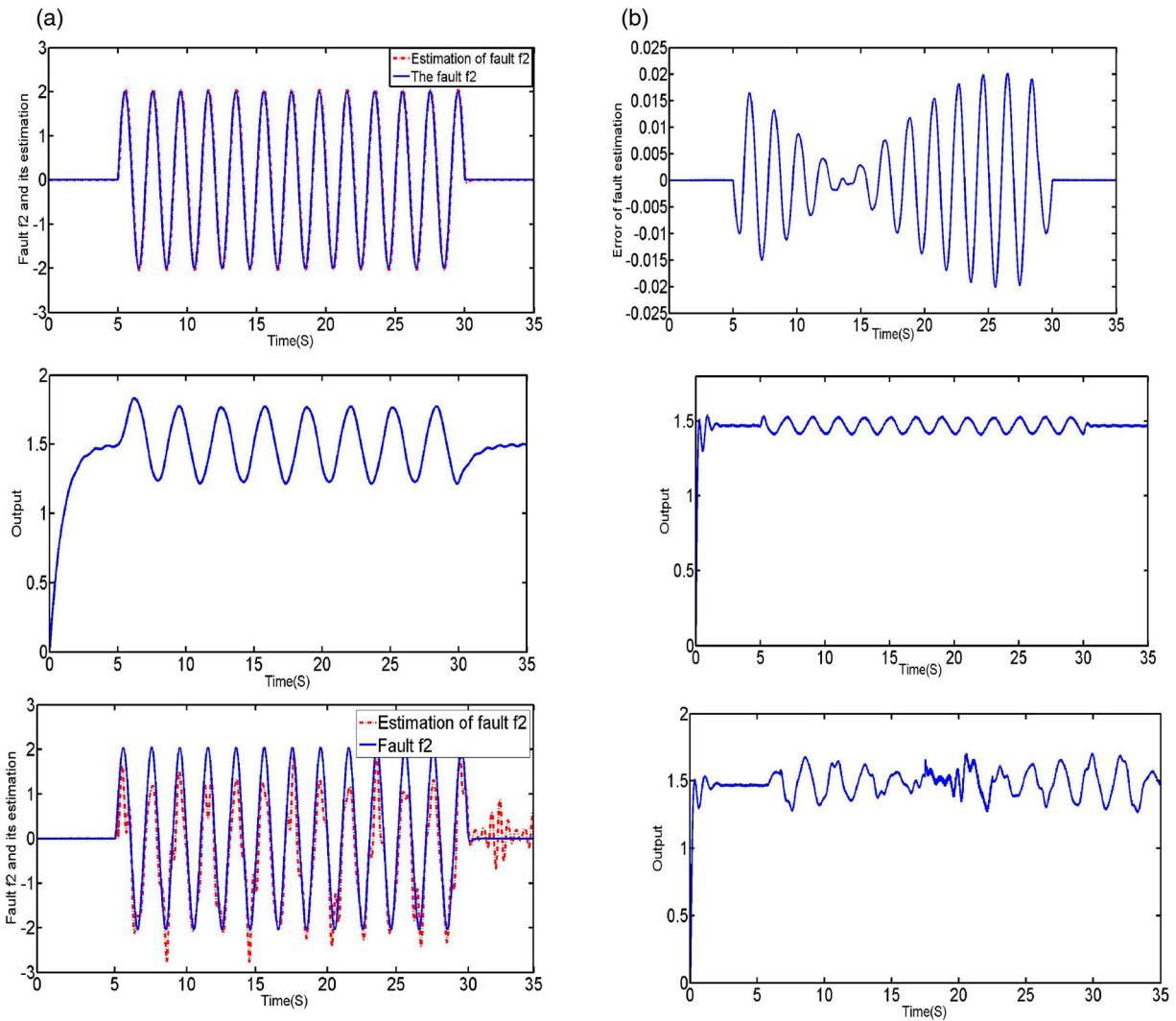
Figure 2. Simulation results of fault $f_2(t)$: (a) Estimated and actual values of fault $f_2(t)$. (b) Error in estimating the fault $f_1(t) - \hat{f}_2(t)$. (c) Output curve under a normal control of fault $f_2(t)$. (d) Output curve under a fault-tolerance control of fault $f_2(t)$. (e) Estimation of fault $f_2(t)$ using CAFE. (f) Output curve under a fault-tolerance control obtained using CAFE.

estimated. Figure 1(a,b) show that the fault observer designed in this study can estimate the fault $f_1(t)$ with low error. The fault $f_1(t)$ occurs during 3–10 s. Figure 1(c) shows that the system will be in an abnormal state during this time under normal control. Figure 1(d) shows that the fault-tolerance controller designed in this study effectively accommodates the fault $f_1(t)$.

Figure 1(e,f) show that the proposed algorithm outperforms the CAFE.

We assume that the actuator exhibits the following periodic time-varying fault $f_2(t)$:

$$f_2(t) = \begin{cases} 0 & t \in [0, 5] \\ 2 - \sin(2t - 10) & t \in [5, 30] \end{cases}.$$

Figure 2 shows the simulation results of fault $f_2(t)$ for $\Gamma = 30$. Figure 2(a) shows the simulation of the fault observer. Figure 2(b) shows the error in estimating the fault. Figure 2(c,d) show the output curves

under normal and fault-tolerance controls of the fault $f_2(t)$, respectively. Figure 2(e,f) show the corresponding simulation result of CAFE.

Figure 2(a,b) show that the fault observer designed in this study can accurately estimate the periodic time-varying fault $f_2(t)$. Furthermore, by comparing Figure 2(c,d), we find that the fault-tolerance controller designed in this study effectively accommodates the periodic time-varying fault $f_2(t)$.

Compared with the CAFE algorithm, the improved FAFE algorithm has better performance in fault estimation and compensation for the periodic time-varying fault.

We assume that the actuator has the following fault $f_3(t)$:

$$f_3(t) = \begin{cases} 0 & t \in [0, 10] \\ 5(1 - \exp(-0.5(t - 10))) & t \in [10, 20] \\ 5 - 8(1 - \exp(-0.5(t - 20))) & t \in [20, 40] \end{cases}.$$
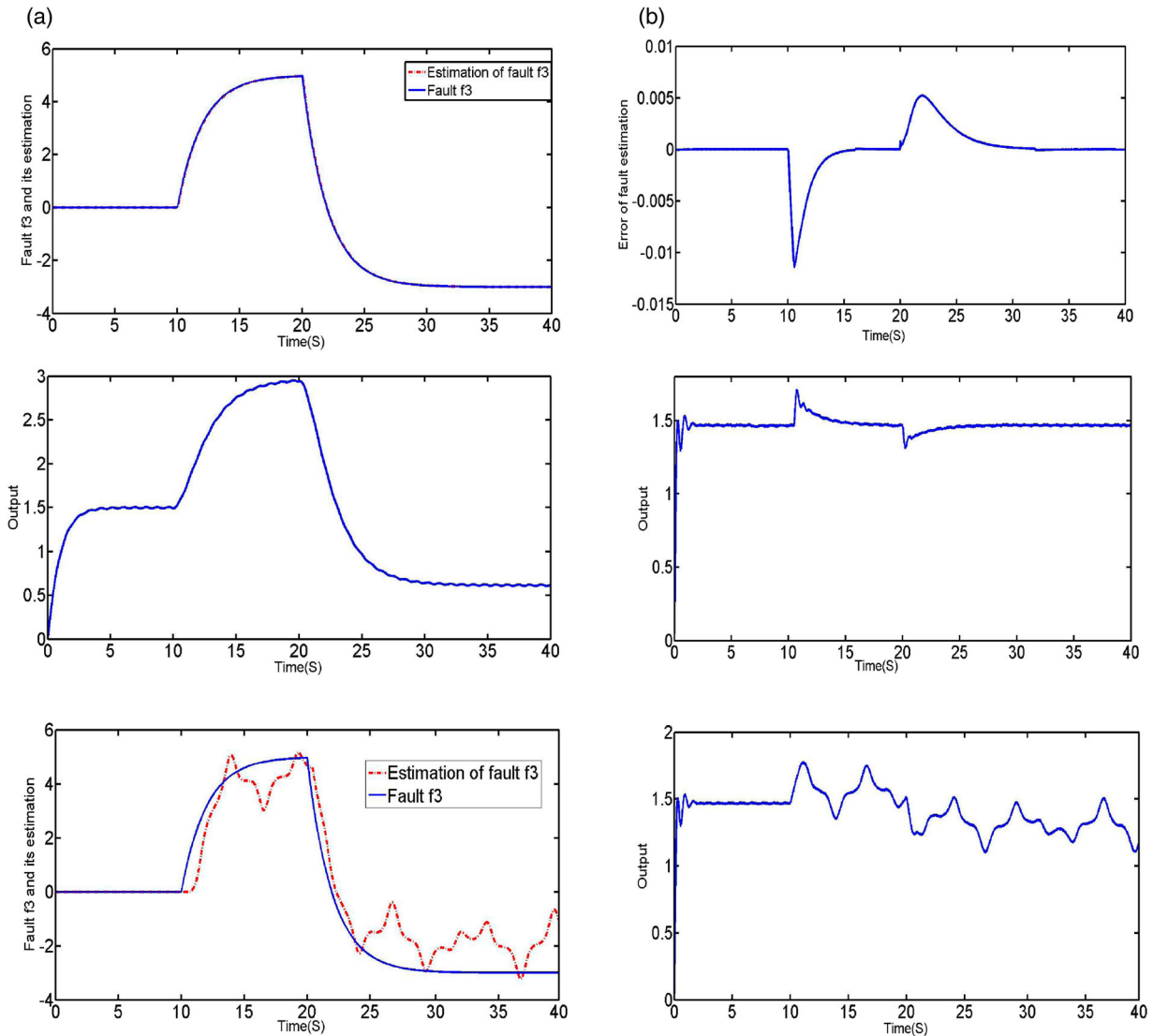

Figure 3. Simulation results of fault $f_3(t)$: (a) Estimated and actual values of fault $f_3(t)$. (b) Error in estimating the fault $f_3(t) - \hat{f}_3(t)$. (c) Output curve under a normal control of fault $f_3(t)$. (d) Output curve under a fault-tolerance control of fault $f_3(t)$. (e) Estimation of fault $f_3(t)$ using CAFE. (f) Output curve under a fault-tolerance control obtained using CAFE.
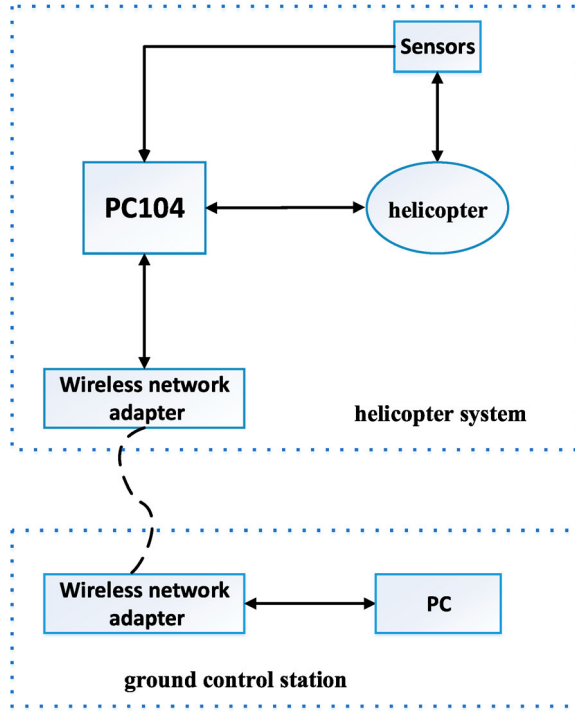
Figure 4. Structure of a helicopter platform.

Figure 3 shows the simulation results of fault $f_3(t)$ for $\Gamma = 30$. Figure 3(a) shows the fault observer simulation. Figure 3(b) shows the error in estimating the fault. Figure 3(c,d) show the output curves under normal and fault-tolerance controls of fault $f_3(t)$, respectively. Figure 3(e,f) show the corresponding simulation result of CAFE.

Figure 3(a,b) show that the fault observer designed in this study can estimate the fault $f_3(t)$ with low error. When the fault $f_3(t)$ occurs during 10–40 s, the system responds abnormally under normal control. From Figure 3(d), we find that the proposed fault-tolerance controller effectively accommodates the fault $f_3(t)$.

Figure 3(e,f) show that the proposed algorithm has better performance in fault estimation and compensation for the fault $f_3(t)$.

6. Real example

In this section, we show a real example of an unmanned helicopter experimental platform.

Figure 4 shows the structure of the experimental platform. A control board (PC104) is used to receive the control and fault signals collected by the sensors and send them to the helicopter. A wireless network is used to realize the communication between the ground control station and the helicopter system.

The ground control station can receive the status signals of the helicopter through the PC104 and can send the control signals to the helicopter system. Moreover, the ground control station can send the fault signals to simulate the actuator fault.

The linearization equation for the helicopter at the equilibrium point is

$$
\begin{cases}
\dot{x}(t) = (A + D_1 F(t) E_1) x(t) \\
\quad + (A_d + D_1 F(t) E_d) x(t - d(t)) \\
+ (B + D_1 F(t) E_2) u(t) + (G + D_1 F(t) E_3) f(t) \\
\quad + (W + D_1 F(t) E_4) w(t) \\
y(t) = C x(t)
\end{cases}
$$

Here $x(t) = (x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)\ x_5(t)\ x_6(t))^T$, where x_1, x_2, and x_3 denote the lifting, convergence, and path angles, respectively, and x_4, x_5, and x_6 denote their angular velocities, respectively. The time-varying parameters are $F(t) = 0.05 \sin t$, $d(t) = 0.2 + 0.1 \sin 9t$, and $w(t) = 0.1 \sin 90t$. $u(t) = (u_1(t)\ u_2(t))^T$, where u_1 and u_2 denote the voltages of the front and rear thrusters, respectively, and $y(t) = (x_1(t)\ x_2(t)\ x_3(t))^T$.

The detailed parameters of the system are as follows:

$$
A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2.131 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0.1 & 0.07 \\
0.62 & -0.62 \\
0 & 0
\end{pmatrix},
$$

$$
G = \begin{pmatrix}
0.32 \\
-0.41 \\
0.22 \\
-0.06 \\
0.09 \\
0.1
\end{pmatrix}, \quad
D_1 = \begin{pmatrix}
0 \\
0 \\
0.2 \\
0.3 \\
0 \\
0.1
\end{pmatrix},
$$

$$
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},
$$

$$
E_1 = \begin{pmatrix} 0.2 & 0.04 & 0.12 & -0.11 & 0.01 & 0.32 \end{pmatrix},
$$

$$
E_d = \begin{pmatrix} 0 & 0 & 0 & 0.21 & 0 & -0.12 \end{pmatrix},
$$

$$
E_2 = \begin{pmatrix} 0.03 & 0.2 \end{pmatrix}, E_3 = 0.56, E_4 = 0.29,
$$

$$
A_d = \begin{pmatrix}
0 & 0 & 0 & 0.1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.21 \\
0 & 0.01 & 0 & 0 & 0 & 0 \\
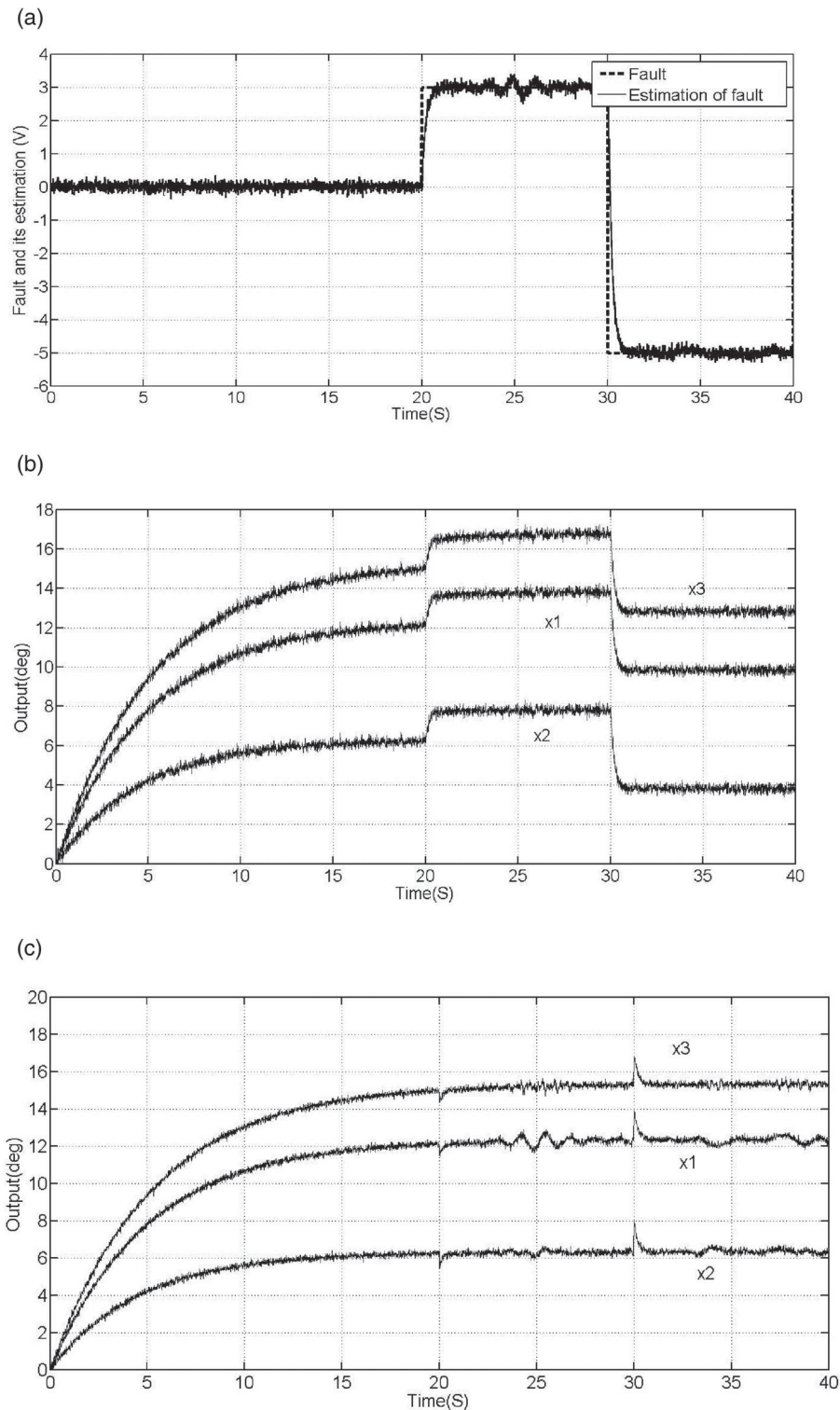0 & 0 & 0 & -0.13 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

(a)

(b)

(c)

Figure 5. Experimental result of fault estimation of a helicopter platform: (a) Estimated and actual values of fault. (b) Output curve under a normal control. (c) Output curve under a fault-tolerance control.

and $W = \begin{pmatrix} 0 \\ 0 \\ 0.3 \\ 0.13 \\ 0.2 \\ 0.1 \end{pmatrix}$

Using the above method, we can solve the gain matrices of the observer as follows:

$$L = \begin{pmatrix} 9.1 & 0.001 & 20.12 \\ 1.22 & 120.1 & 0.33 \\ 7.7 & 90.1 & 0.21 \\ 12.5 & 0.33 & 9.12 \\ -0.11 & 0.89 & 133.98 \\ 0.21 & -10 & 1.3 \end{pmatrix} \text{ and}$$

$$H = \begin{pmatrix} 0.41 & 0.91 & 0.19 \\ 0 & 0.052 & 0 \\ 1.71 & 0 & 0.01 \\ 0 & 0.01 & 0.03 \\ 0.03 & 0 & 0 \\ 0.2 & 0.11 & 0 \end{pmatrix}.$$

The parameter matrices of the controller are as follows:

$$A_c = \begin{pmatrix} 3.1 & 0 & 0 & 2.3 & 0 & 0 \\ 0.01 & 0 & 7.2 & 0.35 & 1.33 & 0 \\ 0 & 0.17 & 0 & 0 & 0.91 & 0.1 \\ 0 & 0 & 3.6 & 0 & 0.33 & 0 \\ 0.9 & 0 & 0 & 0.01 & 9.61 & 0 \\ 0 & 2.72 & 0 & 1 & 0 & 0.55 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 0 & 0 \\ 0 & 0.01 \\ 0.02 & 0 \\ 2.2 & 0.57 \\ 3.11 & 0.01 \\ 0 & -0.01 \end{pmatrix},$$

$$C_c = \begin{pmatrix} 0 & 0 & 0.19 & 0 & 0.76 & 1.1 \\ 0.01 & 0.33 & 5.1 & 0 & 0.44 & 0 \end{pmatrix},$$

and $D_c = \begin{pmatrix} 0 & 0.1 & 10.01 \\ 6.11 & 0.51 & 1.07 \end{pmatrix}$

We send the following fault signal to the helicopter:

$$f(t) = \begin{cases} 0 & 0 \leq t < 20 \\ 3 & 20 \leq t < 30 \\ -5 & 30 \leq t \leq 40 \end{cases}$$

Figure 5(a) shows the fault estimation. Figure 5(b,c) show the output curves under normal and fault-tolerance controls, respectively.

The fault occurs in 20–40 s. From Figure 5(a), we can confirm that the observer designed in this study accurately estimates the fault. By comparing Figure 5(b,c), we find that the helicopter does not work properly in the presence of the fault under normal control and that the proposed fault-tolerance controller maintains the helicopter in a normal state.

7. Conclusions

This paper reports on the fault estimation and fault-tolerant control method for a system with fast time-varying delay and time-varying parameters. The FAFE algorithm is improved, and relaxation matrices are introduced to design a fault observer. The role of the improved FAFE algorithm is to reduce the constraints, and the introduction of the relaxation matrices helps estimate faults for a system with a fast time-varying delay. Based on online fault estimates, an output feedback controller is designed to accommodate the faults. The relaxation matrices help realize a fault-tolerant control for a system with a fast time-varying delay. The solutions to the observer and controller are realized using an iterative algorithm derived in the study.

Disclosure statement

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References

[1] Klimkowski K, Dybkowski M. A fault tolerant control structure for an induction motor drive system. Automatika. 2016;57(3):638–647.

[2] González-Contreras B M, Osorio-Gordillo G L, López-Estrada F R, et al. Using the second-order information for reconfigurability analysis and design in the fault tolerant framework. Automatika. 2018;59(1):51–62.

[3] Xiong J, Chang X H, Yi X. Design of robust nonfragile fault detection filter for uncertain dynamic systems with quantization. Appl Math Comput. 2018;338:774–788.

[4] Liu L, Wang Z, Zhang H. Data-based adaptive fault estimation and fault-tolerant control for MIMO model-free systems using generalized fuzzy hyperbolic model. IEEE Trans Fuzzy Syst. 2018;26(6):3191–3205.

[5] Lu C, Yuan H, Ma J. Fault detection, diagnosis, and performance assessment scheme for multiple redundancy aileron actuator. Mech Syst Signal Process. 2018;113:199–221.

[6] Liu X, Gao X, Han J. Observer-based fault detection for high-order nonlinear multi-agent systems. J Franklin Inst. 2016;353(1):72–94.

[7] Zhou Z, Tan Y, Shi P. Fault detection of a sandwich system with dead-zone based on robust observer. Syst Control Lett. 2016;96:132–140.

[8] Majdzik P, Akielaszek-Witczak A, Seybold L, et al. A fault-tolerant approach to the control of a battery assembly system. Control Eng Pract. 2016;55:139–148.

[9] Dimassi H, Loria A, Belghith S. Continuously-implemented sliding-mode adaptive unknown-input observers under noisy measurements. Syst Control Lett. 2012;61(12):1194–1202.

[10] Alwi H, Edwards C. Robust fault reconstruction for linear parameter varying systems using sliding mode observers. Int J Robust Nonlinear Control. 2014;24(14):1947–1968.

[11] Boussaid B, Aubrun C, Jiang J, et al. FTC approach with actuator saturation avoidance based on reference management[J]. Int J Robust Nonlin. 2014;24(17):2724–2740.

[12] Du M, Mhaskar P. Isolation and handling of sensor faults in nonlinear systems. Automatica (Oxf). 2014;50(4):1066–1074.

[13] Gao Z. Fault estimation and fault-tolerant control for discrete-time dynamic systems. IEEE Trans Ind Electron. 2015;62(6):3874–3884.

[14] Tabatabaeipour S M, Bak T. Robust observer-based fault estimation and accommodation of discrete-time piecewise linear systems. J Franklin Inst. 2014;351(1):277–295.

[15] Davila J. Exact tracking using backstepping control design and high-order sliding modes. IEEE Trans Autom Control. 2013;58(8):2077–2081.

[16] Li L, Yang Y, Ding S X, et al. On fault-tolerant control configurations for a class of nonlinear systems. J Franklin Inst. 2015;352(4):1397–1416.

[17] Han J, Zhang H, Wang Y, et al. Robust fault estimation and accommodation for a class of T–S fuzzy systems with local nonlinear models. Circuits Syst Signal Process. 2016;35(10):3506–3530.

[18] Deutscher J. Fault detection for linear distributed-parameter systems using finite-dimensional functional observers. Int J Control. 2016;89(3):550–563.

[19] Cherni H, Boulaabi I, Sellami A, et al. Robust actuator and sensor faults reconstruction based on new sliding mode observer for a class of output time-delay systems. IETE J Res. 2016;62(6):812–821.

[20] Lin C, Wang QG, Lee TH. Stabilization of uncertain fuzzy time-delay systems via variable structure control approach. IEEE Trans Fuzzy Syst. 2005;13(6): 787–798.

[21] Khalil HK, Grizzle JW. Nonlinear systems. Upper Saddle River (NJ): Prentice hall; 2002.

[22] Qian H, Peng Y, Yang G. Reduced-order observer-based fault estimation and fault-tolerant control for a class of discrete Lipschitz nonlinear systems. Optim Control Appl Methods. 2016;37(6):1236–1262.

[23] You F, Li H, Wang F, et al. Robust fast adaptive fault estimation for systems with time-varying interval delay. J Franklin Inst. 2015;352(12):5486–5513.