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Optimal Dividends Paid in a Foreign Currency for a Lévy Insurance Risk Model

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This article considers an optimal dividend distribution problem for an insurance company where the dividends are paid in a foreign currency. In the absence of dividend payments, our risk process follows a spectrally negative Lévy process. We assume that the exchange rate is described by a an exponentially Lévy process, possibly containing the same risk sources like the surplus of the insurance company under consideration. The control mechanism chooses the amount of dividend payments. The objective is to maximize the expected dividend payments received until the time of ruin and a penalty payment at the time of ruin, which is an increasing function of the size of the shortfall at ruin. A complete solution is presented to the corresponding stochastic control problem. Via the corresponding Hamilton–Jacobi–Bellman equation we find the necessary and sufficient conditions for optimality of a single dividend barrier strategy. A number of numerical examples illustrate the theoretical analysis.

1. INTRODUCTION

In the public eye, dividend payments are holding the title to be one of the most important signs of financial health and future stability of shares-issuing companies. Thus, a forecast of opulent future dividends, compared to a benchmark such as 10 year government bonds, will most likely attract new investors, clients, and business partners. Therefore, it is natural to consider future dividend payments as a risk measure quantifying company's future profitability and debt sustainability. Because the pathbreaking work of Bruno de Finetti in 1957, substantial research has been carried out on finding the optimal dividend strategy in the framework of the classical risk model or diffusion approximation as a surplus process for an insurance company. The survey by Albrecher and Thonhauser (2009) sums up the most important results for these types of surplus. Avram, Palmowski, and Pistorius (2007) generalized de Finetti's problem to spectrally negative Lévy processes as surplus. Loeffen (2008, 2009) extended their results and added transaction costs. Loeffen and Renaud (2009) modified the optimization problem by adding an affine penalty function at ruin.

Despite severe differences in modeling the surplus and additional constraints, the above works have one feature in common: the discounting factor or rather the preference rate. The preference rate is usually assumed to remain constant and positive on time, signalizing the setup "money today is more preferable to money tomorrow." However, in the times of negative interest rates, like nowadays, a perpetual positive interest rate will lead to deterioration of results. For instance, Akyildirim et al. (2013), Eisenberg (2015), and Jiang and Pistorius (2012) incorporated stochastic interest rate into the dividend optimization framework.

Another aspect that has not been studied until now in the framework of dividend maximization is foreign interest rates. Big insurance companies have clients and shareholders all on the world. For instance, top global reinsurance companies, including such giants like Munich Re and Swiss Re, established themselves in the Middle East more than a decade ago and have been recently expanding to Asia and Latin America, while "local" reinsurance companies are still in their infancy. In most cases, the dividends are declared in the domestic currency of companies or in U.S. dollars and are paid to the shareholders in the local currency using the actual exchange rate.

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Currency fluctuations are a natural consequence of the floating exchange rate system (i.e., a currency's value is allowed to fluctuate in response to foreign exchange market events), which is used in most economies. Indeed, just a few countries worldwide are currently using the fixed rate approach, where the domestic currency is pegged to a stronger currency or a basket of them. Many factors impact a foreign exchange rate; for instance, relative supply and demand of the two currencies, a forecast for inflation, etc. Thus, any noticeable changes in the underlying economy affect the exchange rates and the economic activities of almost all domestic market participants. For shareholders of an insurance company, such events might become crucial because the affected company can decide to shorten dividend payments due to an unfavorable market situation. Thus, the changes with the exchange rate and the shortening of dividends can have the same risk component. Note that the impact type described above is rather of a continuous nature, reflecting infinitesimal economic changes on the daily basis; see, for instance, Mouna and Anis (2016). In the surplus of an insurance company this continuous dependency can be modeled via adding a Brownian motion. A classical risk model (compound Poisson process) perturbed by a diffusion component was first suggested by Gerber (1970) in order to model some uncertainties. Therefore, this type of model has been studied quite intensively Because its introduction in 1970.

In recent years, a number of incidents known as "flash crashes" have been shaking the global financial market. Christensen, Oomen, and Renò (2018) stated that the number of flashes will be even increasing. The sudden market crashes will again affect both the exchange rates and insurance companies. These changes are of a jump nature, occurring at discrete times but on a regular basis. By modeling the surplus process of an insurance company, the factor describing the dependence of the surplus on the financial crushes can be modeled, for instance, by continuous-time Markov chains. Another example of jump dependence provide countries with regular occurring catastrophic events; for instance, earthquakes in Japan and Mexico and floods in the UK, China, and Egypt. After a catastrophic event, the affected country experienced a downturn in the domestic currency. Damage in both the public and the private sector is (partly) taken on by (re)insurance companies. A common home insurance usually does not include the conage of losses due to a catastrophic event, in particular in areas most vulnerable to natural hazards. However, insurance companies can, in addition to a standard contract, offer a partial conage in case of a catastrophic event by introducing a cap for the losses. The same effect may be achieved by purchasing reinsurance. It is intuitively clear that catastrophic and non-catastrophic claims must be modeled separately. Thus, once a natural disaster occurs, the number of claims of an insurance company and the exchange rate have a joint jump. Li et al. (2009) give empirical proofs that a large group of U.S. insurers are exposed to foreign exchange movements against the seven largest U.S. trade partners in insurance services; for example from the UK, Japan, and Switzerland. Thus, many insurance companies are exposed to foreign exchange rate risks. These risks might put a strain on the surplus of an insurance company and therefore also impact the dividend payment strategies.

In the present article we describe the surplus process of an insurance company by a Lévy process containing a diffusion part and a jump part. We assume that the insurance company under consideration targets to maximize the expected discounted amount of dividends paid in a foreign currency. The exchange rate is assumed to follow a Lévy process featuring a dependence on the surplus process. Because Lévy processes can be decomposed into a diffusion part and a jump part, we distinguish two cases for dependencies: dependence of the continuous and jump parts. The article is organized as follows. In Section 2, we introduce the basic notation and describe the model we deal with. Subsection 2.1 is dedicated to the related one-sided and two-sided problems. In Section 3, we present the verification theorem, necessary and sufficient conditions for the barrier strategy to be optimal. Section 4 presents two detailed examples. For the sake of clarity of presentation, the proofs are provided in Section 5.

2. THE MODEL

Recall the classical Cramér-Lundberg model

$$R_t - R_0 = ct - S_t, \quad S_t = \sum_{k=1}^{N_t} C_k, \tag{1}$$

which is used in collective risk theory to describe the surplus $R = \{R_t, t \in \mathbb{R}_+\}$ of an insurance company. Here, C_k are independent and identically distributed. positive random variables representing the claims and S_t denotes the aggregate claims up to time t. The claim number $N = \{N_t, t \in \mathbb{R}_+\}$ is modeled via a homogeneous Poisson process with intensity λ and is independent of the claims. Finally, c represents the premium rate fulfilling $c > \lambda m > 0$ and $m = \mathbb{E}[C_1] < \infty$, in order to allow the process to remain nonnegative with a positive probability.

In 1970, Gerber introduced some uncertainty into the Cramér-Lundberg model by adding a Brownian motion. The "perturbed model" is then $R_t - R_0 := \sigma B_t + ct - S_t$, where B_t denotes a standard Brownian motion, describing small random fluctuations of the surplus.

A further very important generalization is to replace the aggregate claim amount S by a general subordinator (a non-decreasing Lévy process, with Lévy measure $\nu_R(\mathrm{d}x), x \in \mathbb{R}_+,$ which may have infinite mass). Under this model, the "fluctuations" can arise either continuously, due to the Brownian motion, or due to the infinite jump-activity.

Assuming S to be a pure jump martingale with independent and identically distributed increments and negative jumps with Lévy measure $\nu_R(\mathrm{d}x),$ one arrives at a general integrable spectrally negative Lévy process $R = \{R_t,\ t \in \mathbb{R}_+\}$; that is, a stochastic process with stationary independent increments and no positive jumps and càdlàg paths with R_t integrable for any $t \geq 0$ and $\mathbb{E}[R_1] > 0$ in order for the surplus to be profitable; see Kyprianou (2006) for details. The corresponding Lévy-Khintchin triple is (c, σ, ν_R) and $R_0 = x$; that is, the generator of R is given by

$$\mathfrak{A}_1 f(x) = cf'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \big(f(x+h) - f(x) - f'(x)h\mathbb{1}_{[|h|\leq 1]}\big)\nu_R(\mathrm{d}h)$$

for a suitable function f from the domain of the generator.

We further assume that exchange rate process denoted by $Y = \{Y_t,\ t \in \mathbb{R}_+\}$ is a spectrally negative Lévy process with a corresponding triple (p, δ, ν_Y) and $Y_0 = l$; that is, the generator of Y has the form

$$\mathfrak{A}_2 f(l) = pf'(l) + \frac{\delta^2}{2} f''(l) + \int_{\mathbb{R}} \big(f(l+h) - f(l) - f'(l)h\mathbb{1}_{[|h|\leq 1]}\big)\nu_Y(\mathrm{d}h)$$

for a suitable function f from the domain of the generator $\mathfrak{A}_2$.

Both processes are defined on some common probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{\{t\geq 0\}}, \mathbb{P}),$ where $\mathcal{F} = \{\mathcal{F}_t\}_{\{t\geq 0\}}$ is the natural filtration satisfying the usual conditions of right-continuity and completeness generated by bivariate Lévy process $X = \{X_t := (R_t, Y_t),\ t \in \mathbb{R}_+\}$. To avoid degeneracies, we exclude the case that R or Y has monotone paths. We denote by $\nu(\mathrm{d}z, \mathrm{d}y)$ the jump measure of the process X. Note that

$$\nu(\mathrm{d}z, (-\infty, \infty)) = \nu_R(\mathrm{d}z) \quad \text{and} \quad \nu([0,\infty), \mathrm{d}y) = \nu_Y(\mathrm{d}y).$$

We denote by $\{\mathbb{P}_{\underline{x}} = \mathbb{P}_{(x,l)}, \underline{x} = (x,l) \in \mathbb{R}^2\}$ the family of probability measures that correspond to the translations of X by a vector; that is, $\mathbb{P}[X_0 = \underline{x}] = 1$. Later, when it will be clear, we skip underlining of x to note the only dependence on x. In this case, by $\mathbb{E}_{\underline{x}}$ and $\mathbb{E}_x$ we denote the corresponding expectations. Finally, we will use the notation $\mathbb{P}_0 = \mathbb{P}$ and $\mathbb{E}_0 = \mathbb{E}$ as well.

To ensure that R_t and Y_t have finite means for fixed $t \geq 0,$ the Lévy measure ν is assumed to satisfy the integrability condition

$$\int_{[\mathbb{R}\ (-1,1)]^2} ||\underline{x}||\nu(\mathrm{d}\underline{x}) < \infty.$$

As stated in the Introduction, the processes R and Y are assumed to be dependent. Because the continuous part and the jump part of a Lévy process are independent, it is enough to consider the dependence structure of the continuous and discontinuous parts separately. The generator of the process X in case of both types of dependency is given by

$$\begin{aligned}\mathfrak{A}f(x,l) =& cf_x(x,l) + \frac{\sigma^2}{2} f_{xx}(x,l) + pf_l(x,l) + \frac{\delta^2}{2} f_{ll}(x,l) + \rho\sigma\delta f_{xl}(x,l)\\ &+ \int_{\mathbb{R}^2} f(x+h_2, l+h_1) - f(x,l) - f_x(x,l)h_1\mathbb{1}_{[|h_1|\leq 1]} - f_l(x,l)h_2\mathbb{1}_{[|h_2|\leq 1]}\nu(\mathrm{d}h_1, \mathrm{d}h_2).\end{aligned}$$

If R and Y depend just on the jump part, $\rho\sigma\delta f_{xl}(l,x)$ disappears. By dependency just on the continuous part, the integral above transforms to

$$\begin{aligned}&\int_{\mathbb{R}} f(x, l+h) - f(x,l) - f_l(x,l)h\mathbb{1}_{[|h|\leq 1]}\nu_Y(\mathrm{d}h)\\ &+ \int_{\mathbb{R}} f(x+h, l) - f(x,l) - f_x(x,l)h\mathbb{1}_{[|h|\leq 1]}\nu_R(\mathrm{d}h).\end{aligned}$$

For more details, see Bäuerle, Blatter, and Müller (2008).

We assume that the considered insurance company pays dividends and the ex-dividend process is given by

$$R_t^\pi = R_t - L_t^\pi,$$

where π denotes a strategy chosen from the set Π of all admissible dividend controls, resulting in dividend process L_t^π denoting the accumulated dividends under π paid up to time t. An admissible dividend strategy π generates the dividend process $L^\pi = \{L_t^\pi, t \in \mathbb{R}_+\}$, which is càdlàg, adapted to the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$, and at any time preceding the ruin, the dividend payment is smaller than the size of the available reserves ($L_t^\pi - L_{t-}^\pi < R_{t-}^\pi$); that is, the ruin cannot be caused by a dividend payment.

The object of interest is the expected discounted amount of dividends paid in a domestic currency and declared in a foreign currency

$$\mathcal{D}(\pi) := \int_0^{T^\pi} e^{-Y_t} \, \mathrm{d}L_t^\pi$$

and the expected discounted penalty payment (so-called Gerber-Shiu function)

$$\mathcal{W}(\pi) := e^{-Y_{T^\pi}} w(R_{T^\pi}^\pi).$$

Here, $T^\pi := \inf\{t \geq 0 : R_t^\pi < 0\}$ is the ruin time and w is a penalty function acting on the negative half-line. Later, unless it is necessary, we will write T instead of T^π to simplify the notation.

Note that in the definition of $\mathcal{D}$ we mean that the integrand is taken at the time t in order for the integral to be well-defined. We will use this notation without further mention throughout the article.

The process Y_t apart from the interpretation as an exchange rate also describes discounting. In particular, if $Y_t = qt$ the q could be interpreted as a given discount or rather a preference rate, describing the monetary preferences of the considered insurance company. Our objective is to maximize

$$V_\pi(x,l) := \mathbb{E}_{(x,l)}[\mathcal{D}(\pi)] + \mathbb{E}_{(x,l)}[\mathcal{W}(\pi)]$$

on all admissible strategies; that is, to find the so-called value function

$$V(x,l) := \sup_{\pi\in\Pi} V_\pi(l,x), \tag{2}$$

and the optimal strategy $\pi_* \in \Pi$, if it exists, such that

$$V(x,l) = V_{\pi_*}(x,l) \qquad \text{for all } x \geq 0, \ l \in \mathbb{R}.$$

2.1. Preliminaries

In this section, we summarize the basic definitions and properties of Lévy processes and some other concepts we will use in our modeling.

We conjecture that the optimal dividend payment strategy will be of a barrier type. This means that the dividends are paid as the excess of the surplus above a certain constant level, say $a > 0$. If the surplus is above the level a, the excess will be immediately distributed as a lump sum dividend payment and the surplus amounts to a. By starting below a, the insurance company will not pay any dividends until the surplus attains a; the considerations stop if the surplus attains 0 before attaining a. Therefore, we will need the following first passage times:

$$\tau_a^+ := \inf\{t \geq 0 : R_t \geq a\} \quad \text{and} \quad \tau_0^- := \inf\{t \geq 0 : R_t < 0\}.$$

We will now formally define auxiliary functions Δ for which the following exit identity holds true

$$\mathbb{E}_{(x,l)}\left[e^{-Y_{\tau_a^+}} \mathbb{1}_{[\tau_a^+ < \tau_0^-]}\right] = \frac{\Delta(x)}{\Delta(a)} e^{-l}, \tag{3}$$

where $x \in (0,a)$.

In the following, we recall some results from the fluctuation theory for spectrally negative Lévy processes. For more details, see Kyprianou (2006, 2013) and references therein.

Let

$$\mathbb{E}_{(x,l)}\left[e^{\langle\theta,X_t\rangle}\right] = e^{t\psi(\theta)+\theta_1 x+\theta_2 l}$$

for $\theta=(\theta_1,\theta_2)\in D\subseteq\mathbb{R}_+\times\mathbb{R}$ and some set D for which the above expectation is well defined and $\langle .,.\rangle$ is a scalar product. For any $\theta\in D$ we denote by $\mathbb{P}^\theta$ an exponential tilting of measure $\mathbb{P}$ with the Radon-Nikodym derivative given by

$$\left.\frac{\mathrm{d}\mathbb{P}^\theta}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\langle\theta,X_t\rangle-\psi(\theta)t\right\}.$$

Under the measure $\mathbb{P}^\theta$ the process X is still a bivariate Lévy process with the Laplace exponent $\phi_\theta(s)$ with $s\in\mathbb{R}^2$ given by:

$$\phi_\theta(s) = \log\left(\mathbb{E}^\theta[e^{\langle s,X_1\rangle}]\right) = \psi(s+\theta)-\psi(\theta), \tag{4}$$

where $\mathbb{E}^\theta$ denotes the expectation with respect to $\mathbb{P}^\theta$. From now on we assume that there exists an $\alpha\geq 0$ such that

$$\psi(\alpha,-1)=0. \tag{5}$$

We denote by

$$\psi_R(\beta) := \phi_{(\alpha,-1)}(\beta,0) = \log\left(\mathbb{E}^{(\alpha,-1)}[e^{\beta R_1}]\right)$$

the Laplace exponent of R under $\mathbb{P}^{(\alpha,-1)}$. Note that under $\mathbb{P}^{(\alpha,-1)}$ the process R has the following Lévy-Khintchin triple:

$$(\tilde{c},\sigma,\mu_R), \tag{6}$$

where

$$\tilde{c} := c+\alpha\sigma^2-\rho\sigma\delta+\int_{\mathbb{R}^2}\{e^{\alpha h_1-h_2}-1\}h_1\mathbb{1}_{[|h_1|\leq 1]}\nu(\mathrm{d}h_1,\mathrm{d}h_2),$$
$$\mu_R(A) := \int_{A\times\mathbb{R}} e^{\alpha h_1-h_2}\nu(\mathrm{d}h_1,\mathrm{d}h_2) \text{ for all Borel sets } A.$$

Further, there exists a function $W^\alpha:[0,\infty)\to[0,\infty)$, called the scale function, (see, e.g., Bertoin 1997), continuous and increasing with the Laplace transform

$$\int_0^\infty e^{-\beta y}W^\alpha(y)\,\mathrm{d}y = \psi_R(\beta)^{-1}. \tag{7}$$

The domain of W^α is extended to the entire real axis by setting $W^\alpha(z)=0$ for $z<0$.

2.1.1. Assumption 1

Throughout the article we assume that the following (regularity) condition is satisfied:

$$W^\alpha\in\mathcal{C}^2(0,\infty). \tag{8}$$

To obtain it we can assume that either

$$\mu_R(-\infty,-x),\quad x\geq 0 \text{ has a completely monotone density} \tag{9}$$

(see Loeffen 2008; Chan, Savov, and Kyprianou 2011, p. 695)[1]

[1] A function f with the domain $(0,\infty)$ is said to be completely monotone, if the derivatives $f^{(n)}(x)$ exist for all $n=0,1,2,3,\ldots$, and $(-1)^n f^{(n)}(x)\geq 0$ for all $x>0$.

$$\sigma^2 > 0$$

(see Chan, Savov, and Kyprianou 2011, theorem 1) or that R_t is given in (1) with

$$\mu_R(-\infty, -x) \in \mathcal{C}^1(0,\infty)$$

(see Kyprianou 2006, problem 8.4 (ii)).

The function W^α plays a key role in the solution of the two-sided exit problem as shown by the following classical identity:

$$\mathbb{P}_x^{(\alpha,-1)}\left[\tau_0^- > \tau_a^+\right] = \frac{W^\alpha(x)}{W^\alpha(a)} \tag{10}$$

that holds for $x \in [0,a]$; see Kyprianou (2006).The function Δ defined in (3) is related to the above scale function W^α in the following way.

Lemma 1. *It holds that*

$$\Delta(z) = e^{\alpha z} W^\alpha(z).$$

Proof. Note that by (10) we have

$$\mathbb{E}_{(x,l)}\left[e^{-Y_{\tau_a^+}} \mathbb{1}_{\left[\tau_a^+ < \tau_0^-\right]}\right] = e^{\alpha(x-a)} e^{-l} \cdot \mathbb{P}^{(\alpha,-1)}\left[\tau_a^+ < \tau_0^-\right] = e^{-l} \cdot \frac{e^{\alpha x} W^\alpha(x)}{e^{\alpha a} W^\alpha(a)},$$

which completes the proof. □

In order for the optimization problem to be well defined, we require the following condition.

2.1.2. Assumption 2

$$\psi(0,-1) < 0. \tag{11}$$

We first show that without this assumption the value function could be infinite. Indeed, let $\psi(0,-1) > 0$ and we assume without loss of generality. $w=0$ and $\mathbb{E}[R_1] < \infty$. Letting $b := \mathbb{E}[R_1]/2$ and defining π^b to be the strategy with the dividend payout $L_t^{\pi^b} = bt$ yields, using Tonelli's theorem,

$$\begin{aligned} V(x,l) \geq V_{\pi^b}(x,l) &= b\mathbb{E}_{(x,l)}\left[\int_0^{T^{\pi^b}} e^{-Y_t} \, \mathrm{d}t\right] = b\int_0^\infty \mathbb{E}_{(x,l)}\left[e^{-Y_t}\mathbb{1}_{\left[T^{\pi^b} > t\right]}\right] \mathrm{d}t \\ &= be^{-l}\int_0^\infty e^{\psi(0,-1)t} \cdot \mathbb{P}_x^{(0,-1)}\left[T^{\pi^b} > t\right] \mathrm{d}t. \end{aligned}$$

Because the ex-dividend process fulfills $\mathbb{E}[R_1^{\pi^b}] > 0$ (see the prior definition of R), it holds that $\mathbb{P}[T^{\pi^b} = \infty] > 0$ and accordingly $\mathbb{P}_x^{(0,-1)}[T^{\pi^b} = \infty] > 0$. Thus, we immediately get $V(x,l) = \infty$.

On the other hand, under Assumption (11) our value function is well-defined. Indeed, this assumption yields that $\alpha > 0$ for α solving (5). A dividend process L_t^π is nondecreasing so that using integration by parts one gets

$$\int_0^{T^\pi} e^{-Y_t} \, \mathrm{d}L_t^\pi = L_{T^\pi-}^\pi e^{-Y_{T^\pi-}} - \int_0^{T^\pi} L_t^\pi \, \mathrm{d}e^{-Y_t} - L_0 e^{-l} - [e^{-Y}, L]_t.$$

Note that because L is nondecreasing and Y is spectrally negative, the square bracket $[e^{-Y}, L]_t$ is nonnegative (see corollary II.6.2 in Protter [2005] and theorem I.4.52 in Jacod and Shiryaev [2003]). Applying Ito's formula on e^{-Y_t} and building the expectations yields

$$\mathbb{E}_{(x,l)}\left[\int_0^{T^\pi} e^{-Y_t}\ \mathrm{d}L_t^\pi\right] = \mathbb{E}_{(x,l)}\left[L_{T^\pi-}^\pi e^{-Y_{T^\pi-}}\right] - \mathbb{E}_{(x,l)}\left[\int_0^{T^\pi} \psi(0,-1)e^{-Y_t}L_t^\pi\ \mathrm{d}t\right] - L_0 e^{-l}.$$

By definition of L_t^π it holds that $R_t \geq L_t^\pi$ for all $t < T^\pi$. Because there is an $\alpha > 0$ with $\psi(\alpha,-1) = 0$ and because e^x is a convex function, it holds that

$$\mathbb{E}_{(x,l)}\left[R_{T^\pi-}e^{-Y_{T^\pi-}}\right] \leq \frac{1}{\alpha}\mathbb{E}_{(x,l)}\left[e^{\alpha R_{T^\pi-}-Y_{T^\pi-}}\right] = \frac{1}{\alpha}e^{\alpha x - l} < \infty. \tag{12}$$

Furthermore, due to the continuity of ψ, there is an $a \in (0,\alpha)$ such that $\psi(a,-1) < 0$. Then,

$$\begin{aligned}
\mathbb{E}_{(x,l)}\left[\int_0^{T^\pi} e^{-Y_t}L_t^\pi\ \mathrm{d}t\right] &\leq \left[\int_0^{T^\pi} e^{-Y_t}R_t\ \mathrm{d}t\right] \leq \int_0^\infty \mathbb{E}_{(x,l)}\left[e^{-Y_t}R_t\mathbb{1}_{[T^\pi > t]}\right]\ \mathrm{d}t \\
&\leq \frac{1}{a}\int_0^\infty \mathbb{E}_{(x,l)}\left[e^{aR_t - Y_t}\mathbb{1}_{[T^\pi > t]}\right]\ \mathrm{d}t = \frac{e^{ax-l}}{a}\int_0^\infty e^{\psi(a,-1)t}\mathbb{E}_x^{(a,-1)}\left[\mathbb{1}_{[T^\pi > t]}\right]\ \mathrm{d}t \\
&\leq \frac{e^{ax-l}}{a}\int_0^\infty e^{\psi(a,-1)t}\ \mathrm{d}t < \infty.
\end{aligned} \tag{13}$$

Using $\psi(0,-1) < 0$ and building the supremum on π yields $V(l,x) < \infty$.

2.1.3. Penalty Functions

Throughout the article we will also consider the penalty functions belonging to the family of functions $\mathcal{R}$, which is defined in the following way. $\mathcal{R}$ is the set of càdlàg functions $w : (-\infty, 0] \to \mathbb{R}$ that are left-continuous at 0, admit a finite first left-derivative $w'_-(0)$ at 0 and satisfy the integrability condition

$$\sup_{y>1}\int_{[y,\infty)} \sup_{u\in[y-1,y]} |w(u-z)|e^{\alpha z}\nu_R(\mathrm{d}z) < \infty.$$

Let

$$G_w(x) := e^l \cdot \mathbb{E}_{(x,l)}\left[e^{-Y_{\tau_0^-}}w(R_{\tau_0^-})\right] = \mathbb{E}_x^{(\alpha,-1)}\left[w(R_{\tau_0^-})\right]. \tag{14}$$

For $w \in \mathcal{R}$, from proposition 4.9 in Avram, Palmowski, and Pistorius (2016), we have the following lemma.

Lemma 2. *Let $w \in \mathcal{R}$. For any $x \in \mathbb{R}$ it holds that*

$$\begin{aligned}
G_w(x) &= F_w(x) - W^\alpha(x)\kappa_w, \quad \text{with} \\
\kappa_w &:= \left[\frac{\sigma^2}{2}w'(0-) + \frac{1}{\mathbb{E}^{(\alpha,-1)}[R_1]}w(0) - \mathcal{L}w_\nu\right],
\end{aligned}$$

where $\mathcal{L}w_\nu = \int_0^\infty\int_x^\infty [w(x-z) - w(0)]e^{\alpha z}\nu_R(\ \mathrm{d}z)\ \mathrm{d}x$ and the function $F_w : \mathbb{R} \to \mathbb{R}$ is given by $F_w(x) = w(x)$ for $x < 0$, and by

$$\begin{aligned}
F_w(x) &= w(0) + w'_-(0)x - \int_0^x W^\alpha(x-y)J_w(y)\ \mathrm{d}y, x \in \mathbb{R}_+, \text{with} \\
J_w(x) &= w'_-(0)c + \int_x^\infty \{w(x-z) - w(0) + w'_-(0)(z-x)\}e^{\alpha z}\nu_R(\mathrm{d}z).
\end{aligned}$$

3. MAIN RESULTS

In this section we will present the main result of the article, namely, we show that the optimal strategy among all admissible strategies Π, defined previously, is of a constant barrier type. For that purpose, we consider the corresponding Hamilton-Jacobi-Bellman (HJB) equation, which has been derived using heuristic arguments (see, e.g., Schmidli 2008).

$$\begin{aligned}\max\Big\{ & cV_x(x,l)+\frac{\sigma^2}{2}V_{xx}(x,l)+pV_l(x,l)+\frac{\delta^2}{2}V_{ll}(x,l)+\rho\delta\sigma V_{xl}(x,l)\\ &+\int_{\mathbb{R}^2}V(x+h_2,l+h_1)-V(x,l)-V_x(x,l)h_2\mathbb{1}_{[|h_2|\le 1]}-V_l(x,l)h_1\mathbb{1}_{[|h_1|\le 1]}\nu(\mathrm{d}h_1,\mathrm{d}h_2),\\ & e^{-l}-V_x(x,l)\Big\}=0,\end{aligned} \tag{15}$$

subject to the boundary condition

$$\begin{cases} V(x,l)=e^{-l}w(x), & \text{for all } x<0,\\ V(0,l)=e^{-l}w(0), & \text{in the case } \sigma^2>0 \text{ or } \int_{-1}^0 ye^{\alpha y}\nu_R(\mathrm{d}y)=\infty. \end{cases} \tag{16}$$

The second part of the HJB equation (15), multiplied by e^l yields $1-e^lV_x(x,l)$, which can result in a constant barrier strategy for the surplus if $e^lV_x(x,l)$ does not depend on l. Because we conjecture that the optimal strategy is of a constant barrier type, the value function should have the form $e^{-l}F(x)$ and the HJB equation becomes

$$\begin{aligned}\max\Big\{ & cF'(x)+\frac{\sigma^2}{2}F''(x)-pF(x)+\frac{\delta^2}{2}F(x)-\rho\delta\sigma F'(x)\\ &+\int_{\mathbb{R}^2}e^{-h_1}F(x+h_2)-F(x)-F'(x)h_2\mathbb{1}_{[|h_2|\le 1]}+F(x)h_1\mathbb{1}_{[|h_1|\le 1]}\nu(\mathrm{d}h_1,\mathrm{d}h_2),\\ & 1-F'(x)\Big\}=0.\end{aligned} \tag{17}$$

subject to the boundary condition

$$\begin{cases} F(x)=w(x), & \text{for all } x<0,\\ F(0)=w(0), & \text{in the case } \sigma^2>0 \text{ or } \int_{-1}^0 ye^{\alpha y}\nu_R(\mathrm{d}y)=\infty. \end{cases} \tag{18}$$

If $w(x)=0$ for all $x<0$, which corresponds to the Gerber-Shiu function $\mathcal{W}(\pi)=0$, then the boundary condition (18) is equivalent to the requirement that F equals zero on the negative half-line.

In order to prove the optimality of a barrier strategy, we consider the HJB equation (17) with boundary conditions (18) first.

Theorem 1 (Verification Theorem).

Let π be an admissible dividend strategy such that V_π is twice-continuously differentiable and ultimately dominated by some affine function. If (15) and (16) hold true for V_π, then $V_\pi(x,l)=V(x,l)$ for all $x\ge 0, l\in\mathbb{R}$.

Proof. For the sake of clarity of presentation, the proof is postponed to Section 5. □

Now, we will focus on the set of barrier strategies paying out any excess above a given level as dividends. Let $a>0$ and π_a denote a barrier and the corresponding barrier strategy. In the following, we will investigate the properties of the return functions corresponding to barrier strategies in order to apply Theorem 1. For simplicity, we will denote the return function corresponding to the strategy π_a by $V_a(x,l)=e^{-l}F_a(x)$; that is,

$$F_a(x):=e^l\mathbb{E}_{(x,l)}\left[\int_0^{T^{\pi_a}}e^{-Y_t}\;\mathrm{d}L_t^{\pi_a}+\mathcal{W}(\pi_a)\right].$$

The above representation is possible because the underlying barrier does not depend on Y.

Theorem 2
It holds that

$$F_a(x) = \begin{cases} \dfrac{\Delta(x)}{\Delta'(a)}\big(1 - G_w'(a)\big) + G_w(x), & x \le a, \\ x - a + F_a(a), & x > a \end{cases} \tag{19}$$

with G_w defined in (14). The function F_a is continuously differentiable with respect to x on $[0,\infty)$.

Proof. See Section 5. □

Let

$$H_\alpha'(y) := \frac{1 - G_w'(y)}{\Delta'(y)}$$

and define a candidate for the optimal dividend barrier by

$$a^* := \sup\{a \ge 0 : H_\alpha'(a) \ge H_\alpha'(x) \text{ for all } x \ge 0\}, \tag{20}$$

where $H_\alpha'(0) = \lim_{x\downarrow 0} H_\alpha'(x)$.

Now, using the above two theorems we can give necessary and sufficient conditions for the barrier strategy to be optimal.

Theorem 3
The value function $V_{a^*}(x,l) = e^{-l}F_{a^*}(x)$ under the barrier strategy π_{a^*} is in the domain of the full generator $\mathfrak{A}$. The barrier strategy π_{a^*} is optimal and $V_{a^*}(x,l) = V(x,l)$ for all $x \ge 0$ and $l \in \mathbb{R}$ if and only if

$$\mathfrak{A}\big[e^{-l}F_{a^*}(x)\big] \le 0 \quad \text{for all } x > a^*. \tag{21}$$

Proof. See Section 5. □

Remark 1

1. The optimal level defined in (20) is uniquely defined. However, in general it might happen that there is another barrier producing the same value function. We believe that in the case when condition (21) is not satisfied, the optimal strategy is a band strategy involving several continuation bands $[b_i, a_i)$, with upper reflecting boundaries b_i, separated by lump-sum dividend payment bands $[a_i, b_{i+1})$ of jumping to the next reflecting barrier below a_i by paying all of the excess as a lump-sum payment. The proof will most probably follow Avram, Palmowski, and Pistorius (2016) and will be very complex and long. Therefore, we decided to skip this analysis here and investigate the possibility of a band strategy in the future in a separate article. Even an example yielding a band strategy as the optimal strategy would require a lot of background knowledge; see, for instance, the example with a constant preference rate given in Azcue and Muler (2005).
2. Here, we would like to emphasize that in our case the barrier might even completely disappear, meaning that the value function equals x on $[0,\infty)$. This is, for example, the case if we choose $R_t = x + ct + \sigma W_t$ and $Y_t = l + pt + \delta B_t$, where W and B are two standard Brownian motions with correlation $\rho = 0.3$, $c = 1.3$, $p = 0.6$, $\sigma = 5$, and $\delta = 1$. The value function is given by $V(l,x) = e^{-l}x$.
   The interpretation is the following. The financial market component in the surplus process is too risky. Thus, the ruin probability is so high that in order to maximize the discounted dividends it is better to pay out the entire capital as dividends than to wait and get ruined without any payment.

Theorem 4
Suppose that

$$H_\alpha'(a) \ge H_\alpha'(b) \quad \text{for all } a^* \le a \le b. \tag{22}$$

Then the barrier strategy with the barrier a^* is the optimal strategy; that is, $V(x,l) = e^{-l}F_{a^*}(x)$ for all $x \ge 0$.

Proof. See Section 5. □

Corollary 1.
Assume that $w(x) = 0$ (there is no penalty function) and that (9) holds true. Then π_{a^*} is the optimal strategy.

Proof. See Section 5. □

Remark 2
If $w(x) = 0$ for $x \leq 0$—that is, there is no penalty function—then under the assumption that f^α is monotone decreasing, we have

$$V(x,l) = V_{a^*}(x,l) = e^{-l}F_{a^*}(x) = e^{-l} \cdot \frac{\Delta(x)}{\Delta'(a^*)}, \tag{23}$$

where a^* maximizes $H'_\alpha(x) = 1/\Delta'(x)$ and hence solves

$$\Delta''(a^*) = 0,$$

which is equivalent to the requirement that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} V(a^*, l) = 0. \tag{24}$$

In other words, knowing the barrier strategy is optimal, identifying the value function could be based on solving the HJB equation (17) (without any boundary conditions) and finding a^* via (24) and using the boundary condition $\frac{\mathrm{d}}{\mathrm{d}x} V(a^*, l) = e^{-l}$ or, equivalently, $F'_{a^*}(a^*) = 1$.

4. EXAMPLES

In this section we pick up the idea of continuous dependence and flash crashes on the global market impacting both the exchange rate and the surplus of an insurance company. In the first example below we deal with the continuous dependency case; Example 2 considers the flash crashes. By assuming that the jumps in the considered Lévy processes are exponentially distributed we are able to rewrite the HJB equation in terms of an ordinary differential equation of order 3. In this case, we can show that the problem of finding the optimal barrier and the value function transforms in solving the underlying differential equation with corresponding boundary conditions.

Example 1.
Let us first consider the following example. The classical model of risk theory describes the surplus of an insurance entity up to infinity. We let N_t be the jump number Poisson process with intensity λ, c the premium rate, C_i independent and identically distributed claim sizes Exp(γ)-distributed. We let the surplus be given by the perturbed classical risk model and the exponential expression of the exchange rate by a Brownian motion with drift.

$$R_t = x + ct - \sum_{i=1}^{N_t} C_i + \sigma B_t \quad \text{and} \quad Y_t = l + pt + \delta W_t,$$

where B and W are Brownian motions with correlation coefficient ρ. Further, in order to guarantee the well-posedness of our problem, we assume $p > \frac{\delta^2}{2}$; see Assumption (11).

In the following, we first derive the value function directly from the HJB equation and show in the second part the derivation of the value function and the optimal barrier via scale functions.

Derivation of the value function via HJB

The HJB equation (17) has the following form:

$$\max\left\{cF'(x)+\frac{\sigma^2}{2}F''(x)+\lambda\int_0^x F(x-y)\ \mathrm{d}G(y)-\left(\lambda+p-\frac{\delta^2}{2}\right)F(x)-\rho\delta\sigma F'(x),\right.$$
$$\left.1-F'(x)\right\}=0,$$

where $G(y)=1-e^{-\gamma y}$. Let $g(x):=\int_0^x F(y)e^{\gamma y}\ \mathrm{d}y$. Then

$$\lambda\gamma g(x)+\left(\frac{\delta^2+\sigma^2\gamma^2}{2}-p-\lambda-\gamma c+\rho\gamma\delta\sigma\right)g'(x)+(c-\delta\rho\sigma-\gamma\sigma^2)g''(x)$$
$$+\frac{\sigma^2}{2}g''\prime(x)=0.$$

For the sake of clarity

$$a_2:=\frac{2}{\sigma^2}\left(c-\delta\rho\sigma-\gamma\sigma^2\right),$$
$$a_1:=\frac{2}{\sigma^2}\left(p+\frac{\delta^2}{2}-\lambda+\frac{\sigma^2\gamma^2}{2}-\gamma c+\rho\gamma\delta\sigma\right),$$
$$a_0:=\frac{2\lambda\gamma}{\sigma^2}.$$

Define

$$P(s):=s^3+a_2s^2+a_1s+a_0.$$

If s_i, $1\le i\le n$ are different zeros of $P(s)$ and λ_i $1\le i\le n$ the corresponding multiplicities with $n\le 3$, then due to Kamke (1983, p. 105) or Walter (1998), all solutions to the above differential equation are given by

$$e^{s_1x}P_{\lambda_1-1}(x)+\ldots+e^{s_nx}P_{\lambda_n-1}(x),$$

where P_h is a polynomial of the degree $\le h$. Concerning the zeros of $P(s)$, we can distinguish between two cases: $P(s)$ has three real zeros, $P(s)$ has one real and two complex zeros (complex conjugates). In the second case, the general solution is $e^{s_1x}C_1+e^{s_2x}\sin(x)C_2+e^{s_2x}\cos(x)C_3$.

Considering again the equation

$$cF'(x)+\frac{\sigma^2}{2}F''(x)+\lambda\int_0^x F(x-y)\ \mathrm{d}G(y)-\left(\lambda+p-\frac{\delta^2}{2}\right)F(x)-\rho\delta\sigma F'(x)=0$$

yields $F''(x)>0$ if $F'(x)=0$ and $F''\prime(x)>0$ if $F''(x)=0$ and $F'(x)>0$. Therefore, for an a^* fulfilling $F''(a^*)=0$ and $F'(a^*)=1$ we have $F'(x)>1$ on $[0,a^*)$; that is, F fulfills the HJB equation. To identify the optimal level a^*, note that by Remark 2 the boundary conditions are given by the following equations: $g(0)=0, g'(0)=0, F'(a^*)=1$, and $F''(a^*)=0$ for some a^*.

For instance, for $c=1.3$, $p=0.6$, $\sigma=1$, $\delta=1$, $\rho=0.3$, $\lambda=2$, and $\gamma=2$. Then,

$$g(x)=C_1e^{s_1x}+C_2e^{s_2x}+C_3e^{s_3x},$$

where $s_1=1.697007, s_2=2.327991$, and $s_3=-2.024999$. The boundary conditions yield the unique solution, (see Walter 1998, p. 199)

$$g(x)=(-6.898735\cdot e^{s_1x}+5.898735\cdot e^{s_2x}+e^{s_3x})0.111605.$$

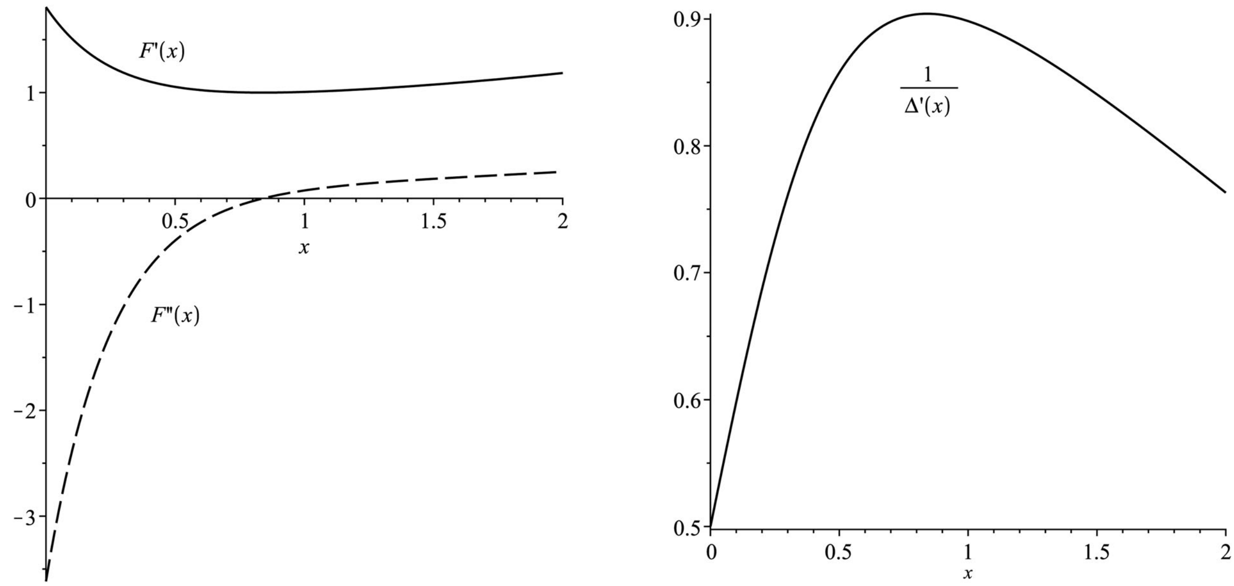

FIGURE 1. **(a)** The Derivatives $F'(x)$ and $F''(x)$ on the Interval $[0, 2]$ and **(b)** $\frac{1}{\Delta'(x)}$ with the Optimal Barrier $a^* = 0.840599$.

The solution to the HJB equation (17), $F(x)$, is then given by

$$F(x) = g'(x)e^{-\gamma x} = -1.306593e^{-0.302992x} + 1.532594e^{0.327991x} - 0.226002e^{-4.024999x}.$$

The boundary conditions yield the optimal dividend barrier $a^* = 0.840599$. Figure 1(a) illustrates the first and second derivatives of the value function $F(x)$, where we see that $F'(x) > 1$ and $F''(x) < 0$ for $x \in [0, a^*)$ and $F'(a^*) = 1, F''(a^*) = 0$.

Because the function $e^{-l}F(x)$ is twice-continuously differentiable with respect to x, it can be shown using the standard methods, (see, e.g., Schmidli 2008) that $e^{-l}F(x)$ is the value function.

Derivation via scale functions

Coming from the other side, using Theorem 2 we can derive the value function and the optimal barrier via scale functions. First of all, find $\alpha \geq 0$ that sets the Laplace exponent of the bivariate Lévy process (R, Y) to zero:

$$\psi(\alpha, -1) = (c - \rho\delta\sigma)\alpha + \frac{\sigma^2\alpha^2}{2} + \lambda\left(\frac{\gamma}{\gamma + \alpha} - 1\right) + \frac{\delta^2}{2} - p = 0.$$

Having identified, $\alpha = 0.32799143$ one can calculate the function $\psi_R(\beta)$ due to (4) (or by [6]):

$$\psi_R(\beta) = \psi(\beta + \alpha, -1) = (c + \alpha\sigma^2 - \rho\delta\sigma)\beta + \frac{\sigma^2\beta^2}{2} - \frac{\lambda\gamma}{\gamma + \alpha} \cdot \frac{\beta}{\gamma + \alpha + \beta}.$$

Now, using (7), we can get $W^\alpha(x)$. Noting that the zeros of ψ_R are given by $\tilde{s}_1 = 0, \tilde{s}_2 = -0.630984, \tilde{s}_3 = -4.352991$ and using the inverse Laplace transform, we get

$$W^\alpha(x) = -0.249971e^{-4.352991x} - 1.445168e^{-0.630984x} + 1.695139.$$

Therefore, we can conclude

$$\Delta(x) = e^{\alpha x}W^\alpha(x) = -0.249971e^{-4.024999x} - 1.445168e^{-0.302992x} + 1.695139e^{0.327991x}.$$

By (6) and the form of ψ_R given above, the density $f^\alpha(y) = (\gamma + \alpha)e^{-(\gamma+\alpha)y}$ of the generic jump size C of the surplus under $\mathbb{P}^{(\alpha,-1)}$ is completely monotone. Hence, from Corollary 1 the barrier strategy π_{a^*} is optimal. Due to (19), the value function and the optimal strategy are given by

$$V(x,l) = \begin{cases} e^{-l}\dfrac{\Delta(x)}{\Delta'(a^*)} & : x \le a^*, \\ V(a^*,l) + x - a^* & : x > a^*, \end{cases}$$

$$a^* = \sup\left\{a \ge 0 : \frac{1}{\Delta'(a)} \ge \frac{1}{\Delta'(x)} \text{ for all } x \ge 0\right\} = 0.840599.$$

Figure 1a illustrates that $\frac{1}{\Delta'(x)}$ has the global maximum at 0.840599. Because all assumptions of Corollary 1 are satisfied, the value function is given in (23) and it is consistent with the previous analysis.

Example 2

In this example, we consider a company acting in a country exposed to some natural disaster. We assume that the damages connected to this kind of natural disasters are modeled via the Weibull distribution with parameters ξ and 0.5; that is, the distribution function is

$$\begin{cases} 1 - e^{-\frac{\sqrt{x}}{\sqrt{\xi}}} & : x \ge 0 \\ 0 & : x < 0. \end{cases}$$

Imagine now that the insurance company under consideration (the first insurer) buys reinsurance, so that the self-insurance function—the function applied on the total claim due to the catastrophic event to be paid by the first insurer—is given by $\sqrt{x}$. This means in particular that the total catastrophic claims are exponentially distributed. We assume that the catastrophic events happen on a regular basis and model the number of events by a Poisson process.

In this example, we again assume that the surplus process of the considered insurance company R_t is given by a perturbed classical risk model and the exponential of the exchange rate Y_t by a continuous drift and a jump part, where the number of jumps is correlated with the number of jumps in the surplus. Let

$$R_t = x + ct + \sigma B_t - \sum_{i=1}^{N_t} C_i \quad \text{and} \quad Y_t = l + pt - \sum_{i=1}^{M_t} Z_i,$$

where B_t is a standard Brownian motion, C_i describe the jumps in the surplus, and Z_i describe jumps in the exchange rate, where the sequences $(C_i)_{i\ge1}$ and $(Z_i)_{i\ge1}$ are independent. Because we assume that the crashes are not severe, we let C_i have the distribution function $G(x) = 1 - e^{-\gamma x}$ and the distribution function of Z_i is $H(x) = 1 - e^{-\eta x}$; that is, the jumps are not heavy-tailed. Further, let $\bar{N}_t$ be a Poisson process with parameter $\bar{\lambda}$ independent of the Poisson process M_t with parameter θ. We let $N_t = \bar{N}_t + M_t$; that is, N_t is again a Poisson process with parameter $\lambda = \bar{\lambda} + \theta$.

In order for the problem to be well defined (Assumption [11]), we require

$$\psi(0,-1) = -p + \theta\frac{\eta}{\eta - 1} - \theta < 0.$$

Derivation of the value function via HJB

In this case, the HJB equation (17) (divided by e^{-1}) has the form

$$\max\left\{cF'(x) + \frac{\sigma^2}{2}F''(x) - pF(x) + \int_{\mathbb{R}^2} e^{-h_2}F(x+h_1) - F(x)\nu(\,\mathrm{d}h_1,\ \mathrm{d}h_2), 1 - F'(x)\right\} = 0.$$

The integral in the above equation can be written as follows:

$$\begin{aligned} \int_{\mathbb{R}^2} e^{-h_2}F(x+h_1) - F(x)\nu(\,\mathrm{d}h_1,\ \mathrm{d}h_2) &= \theta\int_0^\infty\int_0^x e^z F(x-y)\ \mathrm{d}G(y)\ \mathrm{d}H(z) \\ &\quad + \bar{\lambda}\int_0^x F(x-y)\ \mathrm{d}G(y) - \lambda F(x) \\ &= \frac{\theta\gamma\eta e^{-\gamma x}}{\eta - 1}\int_0^x F(y)e^{\gamma y}\ \mathrm{d}y \\ &\quad + \bar{\lambda}\gamma e^{-\gamma x}\int_0^x F(y)e^{\gamma y}\ \mathrm{d}y - \lambda F(x). \end{aligned}$$

As in the previous example, consider the differential equation with

$$g(x) = \int_0^x F(y)e^{\gamma y}\ \mathrm{dy} :$$

$$\left(\frac{\theta\gamma\eta}{\eta-1}+\bar{\lambda}\gamma\right)g(x)+\left(\frac{\sigma^2\gamma^2}{2}-p-\lambda-\gamma c\right)g'(x)+\left(c-\gamma\sigma^2\right)g''(x)+\frac{\sigma^2}{2}g'''(x)=0.$$

Let now $c=1.6$, $p=0.6$, $\sigma=1$, $\gamma=2$, $\theta=2$, $\bar{\lambda}:=0.5$, $\eta=5$.

The solution is given by

$$g(x) = (-4.598667e^{s_1 x} + 3.598667e^{s_2 x} + e^{s_3 x})0.09447$$

with $s_1 = 1.329911, s_2 = 2.557360, s_3 = -3.087271$. The solution to the HJB equation is then given by

$$F(x) = \left(-4.598667s_1 e^{(s_1-2)x} + 3.598667s_2 e^{(s_2-2)x} + s_3 e^{(s_3-2)x}\right)0.09447.$$

The boundary conditions yield $a^* = 0.684809$. Figure 2a illustrates the derivatives F' and F'' on the interval $[0,2]$: $F'(x)>1$ and $F''(x)<0$ on $[0, 0.684809)$.

Because the solution to the HJB equation, $e^{-1}F(x)$ is twice-continuously differentiable with respect to x, using Ito's formula (see Schmidli 2008) one can prove that $e^{-1}F(x)$ is indeed the value function.

Derivation via scale functions

Consider first the Laplace exponent of the bivariate Lévy process (R, Y). Find $\alpha \ge 0$ setting the Laplace exponent to zero:

$$\psi(\alpha,-1) = \frac{\sigma^2\alpha^2}{2} + c\alpha + \theta\left(\frac{\gamma}{\gamma+\alpha}\cdot\frac{\eta}{\eta-1}-1\right) + \bar{\lambda}\left(\frac{\gamma}{\gamma+\alpha}-1\right) - p = 0.$$

It holds that $\alpha = 0.557360$, leading to

$$\psi_R(\beta) = \psi(\beta+\alpha,-1) = \frac{\sigma^2\beta^2}{2} + \beta(c+\sigma^2\alpha) + \frac{\gamma}{\gamma+\alpha}\left(\frac{\theta\eta}{\eta-1}+\bar{\lambda}\right)\left(\frac{\gamma+\alpha}{\gamma+\alpha+\beta}-1\right).$$

Using the inverse Laplace transform, one gets due to (7) and Lemma 1

$$\Delta(x) = e^{\alpha x}W^\alpha(x) = e^{0.557360x}\left(-0.2476417475e^{-5.644632x} - 0.490573e^{-1.22745x} + 0.738215\right).$$

From the representation of $\psi_R(\beta)$ above, we find that the density of the jumps in the surplus under the measure $\mathbb{P}^{(\alpha,-1)}$ is given by $f^\alpha(x) = (\alpha+\gamma)e^{-(\alpha+\gamma)x}$. Because f^α is completely monotone, by Corollary 1 and Remark 2 the optimal strategy is of barrier type.

The value function and the optimal strategy are

$$V(x,l) = \begin{cases} e^{-l}\dfrac{\Delta(x)}{\Delta'(a^*)} & : x \le a^*, \\ V(a^*,l) + x - a^* & : x > a^*, \end{cases}$$

$$a^* = \sup\left\{a \ge 0 : \frac{1}{\Delta'(a)} \ge \frac{1}{\Delta'(x)}\ \text{for all}\ x \ge 0\right\} = 0.684809.$$

The function $\frac{1}{\Delta'(x)}$ is illustrated in Figure 2b. And the achieved results are in line with the results derived via solving the HJB equation directly. □

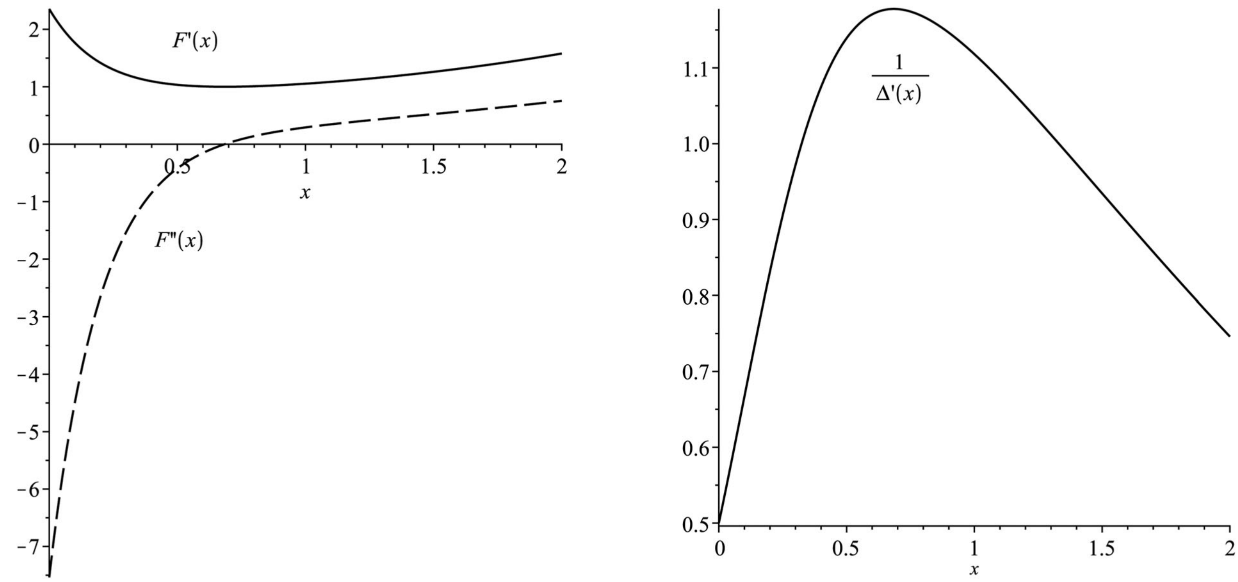

FIGURE 2. **(a)** The Derivatives $F'(x)$ and $F''(x)$ on the Interval $[0, 2]$ and **(b)** $\frac{1}{\Delta'(x)}$ with the Optimal Barrier $a^* = 0.684809$.

5. PROOFS

5.1. Proof of Verification Theorem 1

The proof is based on a representation of v as the pointwise minimum of a class of "controlled" supersolutions to the HJB equation. We start with the observation that the value function satisfies the following dynamic programming equation.

Lemma 3.
After extending V to the negative half-axis by $V(x) = w(x)$ for $x < 0$, we have, for any stopping time τ,

$$V(x, l) = \sup_{\pi \in \Pi} \mathbb{E}_x \left[V(R^{\pi}_{\tau \wedge T}, Y_{\tau \wedge T}) + \int_0^{\tau \wedge T} e^{-Y_t} \, dL^{\pi}_t \right].$$

Proof. This follows by a straightforward adaptation of classical arguments,(see e.g., Azcue and Muler 2005, pp. 276–277).We will prove that v is a supersolution to the HJB equation (15). □

Lemma 4.
The process

$$V^{\pi}_t := V(R^{\pi}_{t \wedge T}, Y_{t \wedge T}) + \int_0^{t \wedge T} e^{-Y_s} \, dL^{\pi}_s \tag{25}$$

is a uniformly integrable (UI) supermartingale.

Proof. Fix arbitrary $\pi \in \Pi, x \geq 0$ and $s, t \geq 0$ with $s < t$. The process V^{π}_t is $\mathcal{F}_t$-measurable, and is UI. Indeed, by Lemma 3 we have

$$\mathbb{E}_{(x,l)} \left[V^{\pi}_t \right] \leq \sup_{\pi \in \Pi} \mathbb{E}_{(x,l)} \left[V(R^{\pi}_{t \wedge T}, Y_{t \wedge T}) + \int_0^{t \wedge T} e^{-Y_s} dL^{\pi}_s \right] = V(x, l).$$

Now, using the linearity of Lévy processes in the initial value and inequalities (12) and (13), we get

$$V(x, l) \leq (Ax + B) e^{-l}, \tag{26}$$

for some constants $A, B > 0$.

Let W_t^π be the following value process:

$$W_s^\pi := ess\ \sup_{\tilde{\pi}\in\Pi_s} J_s^{\tilde{\pi}}, \qquad J_s^{\tilde{\pi}} := \mathbb{E}\left[\int_0^{T^{\tilde{\pi}}} e^{-Y_u}\ dL_u^{\tilde{\pi}} + e^{-Y_{T^{\tilde{\pi}}}} w(R_{T^{\tilde{\pi}}}^{\tilde{\pi}})|\mathcal{F}_s\right],$$
$$\Pi_s := \left\{\tilde{\pi} = (\pi, \bar{\pi}) = \{L_u^{\pi,\bar{\pi}}, u \geq 0\} : \bar{\pi} \in \Pi\right\}, \qquad L_u^{\pi,\bar{\pi}} := \left\{ L_u^\pi, \quad u \in \left[0, s\right[, L_s^\pi + L_{u-s}^{\bar{\pi}}(R_s^\pi), u \geq s, \right. \tag{27}$$

where $L^{\bar{\pi}}(x)$ denotes the process of cumulative dividends of the strategy $\bar{\pi}$ corresponding to the initial capital x.

The fact that V^π is a supermartingale is a direct consequence of the following $\mathbb{P}$-a.s. relations:

(a) $V_s^\pi = W_s^\pi$, (b) $W_s^\pi \geq \mathbb{E}[W_t^\pi|\mathcal{F}_s]$, where W^π is the process defined in (27).

Point (b) follows by classical arguments, because the family $\{J_t^{\tilde{\pi}}, \tilde{\pi} \in \Pi_t\}$ of random variables is upwards directed; see Neveu (1975) and Avram, Palmowski, and Pistorius (2016, Lemma 3.1(ii)) for details.

To prove (a), note that because of the Markov property of R^π and Y_t it also follows that conditional on R_s^π, $\{R_u^{\tilde{\pi}} - R_s^{\tilde{\pi}}, u \geq s\}$ is independent of $\mathcal{F}_s$. As a consequence, the following identity holds on the set $\{s < T^{\tilde{\pi}}\}$:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T^{\tilde{\pi}}} e^{-Y_u}\ dL_u^{\tilde{\pi}} + e^{-Y_{T^{\tilde{\pi}}}} w(R_{T^{\tilde{\pi}}}^{\tilde{\pi}})\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}_{(R_s^\pi, Y_s)}\left[\int_0^{T^{\bar{\pi}}} e^{-Y_u}\ dL_u^{\bar{\pi}} + e^{-Y_{T^{\bar{\pi}}}} w(R_{T^{\bar{\pi}}}^{\bar{\pi}})\right] + \int_0^s e^{-Y_u}\ dL_s^\pi \\
&= V_{\bar{\pi}}(R_s^\pi, Y_s) + \int_0^s e^{-Y_u}\ dL_u^\pi,
\end{aligned}$$

and then we have the following representation:

$$J_s^{\tilde{\pi}} = V_{\bar{\pi}}(R_{s\wedge T}^\pi, Y_{s\wedge T}) + \int_0^{s\wedge T} e^{-Y_u}\ dL_u^\pi,$$

which completes the proof on taking the essential supremum on the relevant family of strategies. □

To prove that the value function V is a solution to the HJB equation (15), we will denote by $\mathcal{G}$ the family of functions g for which

$$M^{g,T_I} := \{g(R_{t\wedge T_I}, Y_{t\wedge T_I}), t \geq 0\}, \quad T_I := \inf\{t \geq 0 : R_t \notin I\}, \tag{28}$$

is a supermartingale for any closed interval $I \subset [0, \infty)$ and such that

$$\frac{g(x,l) - g(y,l)}{x-y} \geq e^{-l} \quad \text{for all } x > y \geq 0, \qquad g(x,l) \geq e^{-l} w(x) \quad \text{for } x < 0 \tag{29}$$

and g is ultimately dominated by some linear function.

Lemma 5.
We have $V \in \mathcal{G}$.

Proof. Taking a strategy of not paying any dividends, by Lemma 5.2 we find that the process (28) with $g = v$ is a supermartingale. We will show now that

$$V(x,l) - V(y,l) \geq e^{-l}(x-y) \quad \text{for all } x > y \geq 0 \text{ and } l \in \mathbb{R}.$$

Denote by $\pi^\epsilon(y)$ an ϵ-optimal strategy for the case $R_0^\pi = y$. Then we take the strategy of paying $x - y$ immediately and subsequently following the strategy $\pi^\epsilon(y)$ (note that such a strategy is admissible), so that the following holds:

$$V(x,l) \geq (x-y)e^{-l} + V_{\pi^\epsilon}(y,l) \geq (x-y)e^{-l} + V(y,l) - \epsilon.$$

Because this inequality holds for any $\epsilon > 0,$ the stated lower bound follows. Linear domination of v in x by some affine function in x follows from (26). □

We now give the dual representations of the value function on a closed interval I. Assume that $\mathcal{H}_I$ is a family of functions k for which

$$\tilde{M}_t^{k,\pi} := e^{-Y_{t\wedge\tau_I^\pi}} k(R^\pi_{t\wedge\tau_I^\pi}, Y_{t\wedge\tau_I^\pi}) + \int_0^{t\wedge\tau_I^\pi} e^{-Y_s} dL_s^\pi$$

is an UI supermartingale for $\tau_I^\pi := \inf\{t \geq 0 : R_t^\pi \notin I\}$ and

$$k(x,l) \geq V(x,l) \qquad \text{for } x \notin I.$$

Then

$$V(x,l) = \min_{k\in\mathcal{H}_I} k(x,l) \qquad \text{for } x \in I. \tag{30}$$

Indeed, let $\pi \in \Pi, k \in \mathcal{H}_I$ and $x \in I$. Then the optional stopping theorem applied to the UI Dynkin martingale yields

$$\begin{aligned} k(x,l) \geq& \lim_{t\to\infty} \mathbb{E}_{(x,l)}\left[e^{-Y_{\tau_I^\pi\wedge t}} k(R^\pi_{t\wedge\tau_I^\pi}, Y_{t\wedge\tau_I^\pi}) + \int_0^{t\wedge\tau_I^\pi} e^{-Y_s}\ dL^\pi(s)\right] \\ \geq& \mathbb{E}_{(x,l)}\left[e^{-Y_{\tau_I^\pi}} V(R^\pi_{\tau_I^\pi}, Y_{\tau_I^\pi}) + \int_0^{\tau_I^\pi} e^{-Y_s}\ dL^\pi(s)\right], \end{aligned}$$

where the convention $\exp\{-\infty\} = 0$ is used.

Taking the supremum on all $\pi \in \Pi$ shows that $k(x,l) \geq V(x,l)$. Because $k \in \mathcal{H}_I$ was arbitrary, it follows that

$$\inf_{k\in\mathcal{H}_I} k(x,l) \geq V(x,l).$$

This inequality is in fact an equality because V is a member of $\mathcal{H}_I$ by Lemma 4 The value function V admits a more important representation from which the Verification Theorem 1 follows.

Proposition 1.
We have

$$V(x,l) = \min_{g\in\mathcal{G}} g(x,l).$$

Proof. Because $v \in \mathcal{G}$ in view of Lemma 5, by (30) it suffices to prove that $\mathcal{G} \subset \mathcal{H}_{[0,\infty)}$. The proof of this fact is similar to the proof of the shifting lemma (Avram, Palmowski, and Pistorius 2016, Lemma 5.5). For completeness, we give the main steps. Fix arbitrary $g \in \mathcal{G}, \pi \in \Pi,$ and $s,t \geq 0$ with $s < t$. Note that $\tilde{M}^{g,\pi}$ is adapted and UI by the linear growth condition and arguments in the proof of Lemma 5.2 and by Avram, Palmowski, and Pistorius (2016, section 8). Furthermore, the following (in)equalities hold true:

$$\mathbb{E}\left[\tilde{M}_t^{g,\pi}|\mathcal{F}_{s\wedge T}\right] \overset{(a)}{=} \lim_{n\to\infty} \mathbb{E}\left[\tilde{M}_t^{g,\pi_n}|\mathcal{F}_{s\wedge T}\right] \overset{(b)}{\leq} \lim_{n\to\infty} \tilde{M}^{g,\pi_n}_{s\wedge T} \overset{(c)}{=} \tilde{M}^{g,\pi}_{s\wedge T} \overset{(d)}{=} \tilde{M}^{g,\pi}_s,$$

where the sequence $(\pi_n)_{n\in\mathbb{N}}$ of strategies is defined by $\pi_n = \{L_t^{\pi_n}, t \geq 0\}$ with $L_0^{\pi_n} = L_0^\pi$ and

$$L_u^{\pi_n} := \begin{cases} \sup\{L_v^\pi : v < u, v \in \mathbb{T}_n\}, & 0 < u < T, \\ L^{\pi_n}_{T-}, & u \geq T, \end{cases}$$

$$\mathbb{T}_n := \left(\left\{t_k := s + (t-s)\frac{k}{2^n}, k \in \mathbb{Z}\right\} \cup \{0\}\right) \cap \mathbb{R}_+,$$

where the above T is calculated for the strategy π. Because s and t are arbitrary, it follows that $\tilde{M}^{g,\pi}$ is a supermartingale, which will complete the proof.

Points (a), (c), and (d) follow from the monotone and dominated convergence theorems. To prove (b), let $T_i := T \wedge t_i$, denote $\tilde{M}^{g,\pi_n} = M,\ L^{\pi_n} = L$ and observe that

$$\begin{aligned} M_t - M_s &= \sum_{i=1}^{2^n} Q_i + \sum_{i=1}^{2^n} Z_i, \quad \text{with} \\ Q_i &:= g\big(R_{T_i-}, Y_{T_i}\big) - g\big(R_{T_{i-1}}, Y_{T_{i-1}}\big), \\ Z_i &:= \big(g(R_{T_i}, Y_{T_i}) - g(R_{T_i-}, Y_{T_i}) + \Delta L_{T_i}\big)\mathbb{1}_{[\Delta L_{T_i} > 0]}. \end{aligned}$$

The strong Markov property of R and Y and the definition of R^π imply

$$\mathbb{E}\big[g(R_{T_{i-}}, Y_{T_i}) - g(R_{T_{i-1}}, Y_{T_{i-1}}) | \mathcal{F}_{T_{i-1}}\big] = \mathbb{E}_{(R_{T_{i-1}}, Y_{T_{i-1}})}\big[g(R_{\tau_i}, Y_{\tau_i}) - g(R_0, Y_0)\big], \tag{31}$$

with $\tau_i := T_i \circ \theta_{T_{i-1}}$, where θ denotes the shift operator. The right-hand side of (31) is non-positive because $g \in \mathcal{G}$. Furthermore, it follows from (29) that all Z_i are non-positive. The tower property of conditional expectation then yields

$$\mathbb{E}[M_t - M_s \mid \mathcal{F}_s] \le 0.$$

This establishes inequality (b) and the proof is complete.

Finally, we are ready to prove the verification theorem.

Proof of Verification Theorem 1.
Because V_π is twice-continuously differentiable and dominated by an affine function, the function $h(x,l) := V_\pi(x,l)$ is in the domain of the extended generator of $X = (R, Y)$. This means that the process

$$V_\pi(R_{t\wedge T_I}, Y_{t\wedge T_I}) e^{-\int_0^{t\wedge T_I} \frac{\mathfrak{A}h(X_s)}{h(X_s)} \mathrm{d}s}$$

is a martingale for any closed interval $I \in [0, \infty)$. By (15) it follows that $\frac{\mathfrak{A}h(X_s)}{h(X_s)} \le 0$ and hence $V_\pi \in \mathcal{G}$, which completes the proof. □

5.2. On the Return Function for a Barrier Strategy

Proof of Theorem 2
Note that for the barrier strategy, until the first hitting of the barrier a, the regulated process R^{π_a} behaves like the process R. By the strong Markov property of R_t and by (3) for $x \in [0, a]$, we have

$$V_a(x,l) = \frac{\Delta(x)}{\Delta(a)} V_a(a,l) + \mathbb{E}_{(x,l)}\Big[e^{-Y_{\tau_0^-}} w(R_{\tau_0^-}) \mathbb{1}_{[\tau_0^- < \tau_a^+]}\Big].$$

Moreon, again using the strong Markov property and (3) we can derive

$$\mathbb{E}_{(x,l)}\Big[e^{-Y_{\tau_0^-}} w(R_{\tau_0^-}) \mathbb{1}_{[\tau_0^- < \tau_a^+]}\Big] = \left(G_w(x) - G_w(a)\frac{\Delta(x)}{\Delta(a)}\right) e^{-l}.$$

Hence,

$$V_a(x,l) = \frac{\Delta(x)}{\Delta(a)}\big(V_a(a,l) - e^{-l} G_w(a)\big) + e^{-l} G_w(x).$$

Note that $L_t^{\pi_a} = (\sup_{s \le t} R_s - a) \vee 0$. Thus, using the classical arguments for the Lévy dividend problem, (see, e.g., Avram, Palmowski, and Pistorius 2016, eq. (5.12)), it follows that □

$$\frac{\mathrm{d}}{\mathrm{d}x}V_a(a,l) = e^{-l},$$

from which the assertion of Theorem 2 immediately follows.

5.3. Proofs of Necessary and Sufficient Conditions for Optimality of a Barrier Strategy

Proof of Theorem 3.
To prove sufficiency, we need to show that V_{a^*} satisfies the conditions of the Verification Theorem 1. From Theorem 2 it follows that V_{a^*} is ultimately linear and by Assumption (8) is twice-continuously differentiable. Moreon, by the choice of the optimal barrier a^* we know that $V'_{a^*}(x) \geq 1$. Finally, by definition of Δ and G_w in (3) and (14), respectively, and the strong Markov property of the risk process R, it follows that

$$e^{-Y_{t\wedge T}}\Delta(R_{t\wedge T\wedge\tau^+_{a^*}}), \qquad e^{-Y_{t\wedge T}}G_w(R_{t\wedge T})$$

are martingales. Hence,

$$e^{-Y_{t\wedge T}}F_{a^*}(R_{t\wedge T\wedge\tau^+_{a^*}})$$

is a martingale. This means that F_{a^*} is in the domain of the full generator of R stopped on exiting $[0,a^*]$ and that $\mathfrak{A}(F_{a^*}(x)e^{-l}) = 0$ for $x \leq a^*$ and $l \in \mathbb{R}$. The remaining part of the HJB equation follows from assumption (21).

To prove necessity we assume that condition (21) is not satisfied. By the continuity of the function $x \mapsto \mathfrak{A}(F_{a^*}(x)e^{-l})$ there exists an open and bounded interval $\mathrm{J} \subset (a^*,\infty)$ such that $\mathfrak{A}(F_{a^*}(x)e^{-l}) > 0$ for all $x \in \mathrm{J}$. Let $\tilde{\pi}$ be the strategy of paying nothing if the reserve process $R^{\tilde{\pi}}$ takes a value in J, and following the strategy π_{a^*} otherwise. If we extend V_{a^*} to the negative half-axis by $F_{a^*}(x) = w(x)$ for $x < 0$, we have

$$V_{\tilde{\pi}}(x,l) = \begin{cases} \mathbb{E}_{(x,l)}\left[e^{-Y_{T_\mathrm{J}}}F_{a^*}(R_{T_\mathrm{J}})\right], & x \in \mathrm{J}, \\ e^{-l}F_{a^*}(x), & x \notin \mathrm{J}, \end{cases}$$

where T_J is defined by (28).

By the optional stopping theorem applied to the process $e^{-Y_t}F_{a^*}(R_t)$, for all $x \in \mathrm{J}$, we obtain

$$V_{\tilde{\pi}}(x,l) = \mathbb{E}_{(x,l)}\left[e^{-Y_{T_\mathrm{J}}}F_{a^*}(R_{T_\mathrm{J}})\right] = e^{-l}F_{a^*}(x) + \mathbb{E}_{(x,l)}\left[\int_0^{T_\mathrm{J}} \mathfrak{A}(F_{a^*}(R_s)e^{-Y_s})\ \mathrm{d}s\right] > e^{-l}F_{a^*}(x).$$

This leads to a contradiction and consequently proves the optimality of the strategy π_{a^*}. □

Proof of Theorem 4.
In the first step, we will show that

$$\lim_{y\uparrow x}\ \mathfrak{A}\left[(F_{a^*}-F_x)(y)e^{-l}\right] \leq 0 \qquad \text{for all } x > a^*, l \in \mathbb{R}. \tag{32}$$

Let $x > a^*$. By the dominated convergence theorem we obtain

$$\begin{aligned}\lim_{y\uparrow x}\ \mathfrak{A}\left[(F_{a^*}-F_x)(y)e^{-l}\right] =& e^{-l}\left\{c - \rho\delta\sigma - \int_{\mathbb{R}^2} h_2 \mathbb{1}_{[|h_2|\leq 1]}\nu(\mathrm{d}h_1,\mathrm{d}h_2)\right\}(F'_{a^*} - F'_x)(x) \\ &- e^{-l}\left\{p - \frac{\delta^2}{2} + \int_{\mathbb{R}^2} 1 - h_1 \mathbb{1}_{[|h_1|\leq 1]}\nu(\mathrm{d}h_1,\mathrm{d}h_2)\right\}(F_{a^*} - F_x)(x)) \\ &+ e^{-l}\int_{\mathbb{R}^2} e^{-h_1}[(F_{a^*} - F_x)(x+h_2)]\nu(\mathrm{d}h_1,\mathrm{d}h_2). \\ &+ e^{-l}\frac{\sigma^2}{2}\left(F''_{a^*}(x) - \lim_{y\uparrow x}F''_x(y)\right).\end{aligned}$$

By (19) we have for $x > a^*$:

i. $(F'_{a^*} - F'_x)(x) = 0$.
ii. $(F'_{a^*} - F'_x)(b) = \Delta(b)(H'_\alpha(a^*) - H'_\alpha(x)) \geq 0$ for $b \in [0, a^*]$ by the definition of a^*.
iii. $(F'_{a^*} - F'_x)(u) = \Delta(u)(H'_\alpha(u) - H'_\alpha(x)) \geq 0$ for $u \in [a^*, x]$ by Assumption (22).
iv. $(F_{a^*} - F_x)(a^*) \geq 0$, thus by iii, $(F_{a^*} - F_x)(x) \geq 0$.
v. $(F_{a^*} - F_x)(x + z) \leq (F_{a^*} - F_x)(x)$ for all $z \leq 0$ by ii and iii.
vi. Assumption (11) yields $-p + \frac{\delta^2}{2} + \int_{\mathbb{R}^2} -1 + h_1 \mathbb{1}_{[|h_1| \leq 1]} \nu(\mathrm{d}h_1, \mathrm{d}h_2) < -\int_{\mathbb{R}^2} e^{-h_1} \nu(\mathrm{d}h_1, \mathrm{d}h_2)$.
vii. If $\sigma > 0$, then by our Assumption (22) we have $\lim_{y \uparrow x} F''_x(y) \geq 0 = F''_{a^*}(x)$.

Thus, we have shown (32).

Now assume that (21) does not hold. Then there exists an $x > a^*$ such that

$$\mathfrak{A}(F_{a^*}(x)e^{-l}) > 0.$$

By the continuity of $\mathfrak{A}(F_{a^*}e^{-l})$ we deduce that $\lim_{y \uparrow x} \mathfrak{A}(F_x(y)e^{-l}) > 0$, which contradicts (32).

Proof of Corollary 1.
It is well known that the scale function of a spectrally negative Lévy process that does not go to minus infinity is equal (up to a multiplicative constant appearing in the local time) to the renewal function of the descending ladder height process. Following Loeffen (2008) and Assumption (9), we conclude that $W^\alpha(x)$ is completely monotone (see the footnote on p. 5 for definition), and, because it is non-negative, it is also a Bernstein function. Thus (see, e.g., Jacob 2001, ch. 3.9),

$$W^\alpha(x) = a + bx + \int_0^\infty (1 - e^{-xt})\xi(\mathrm{d}t), \quad x > 0,$$

where $a, b \geq 0$ and ξ is a measure on $(0, \infty)$ satisfying the integrability condition:

$$\int_0^\infty (t \wedge 1)\xi(\mathrm{d}t) < \infty.$$

From Lemma 1 it follows that

$$\Delta(x) = \left(e^{\alpha x}(a + bx) + \int_0^\infty \left(e^{\alpha x} - e^{-x(t-\alpha)}\right)\xi(\mathrm{d}t) \right).$$

By repeatedly using the dominated convergence theorem, we can now deduce

$$\Delta'''(x) = \left[g'''(x) + \int_0^\infty \left(\alpha^3 e^{\alpha x} x + (t-\alpha)^3 e^{-x(t-\alpha)} \right) \xi(\mathrm{d}t) \right],$$

where $g(x) = e^{\alpha x}(a + bx)$. Hence, $\Delta'''(x) > 0$ for all $x > 0$ and so $\Delta'(x)$ is strictly convex on $(0, \infty)$. We can now apply Theorem 4 to deduce that the barrier strategy at a^* is optimal.

CONCLUSIONS

In this article, we sought to maximize the amount of expected dividends paid in a foreign currency for dependent Lévy risk processes as a surplus and exchange rate. We found some sufficient and necessary conditions for a constant barrier strategy to be optimal.

It would be interesting to consider a more general exchange rates Y_t; for examples, ones gonned by stochastic differential equations. In the examples, we demonstrated how the optimal strategy and the value function can be found through direct solution of the HJB equation (classical method) and via scale functions (the method presented in this article). Of course, solution via classical methods by guessing a twice-continuously differentiable function solving the HJB equation can be applied in just few cases by dealing with Lévy processes; for instance, if the jumps are assumed to be exponentially distributed. In the remaining cases one has to rely on the method presented in this article.

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