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Sunflowers and L-intersecting families

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Abstract

Let $f(k, r, s)$ stand for the least number so that if $\mathcal{F}$ is an arbitrary k-uniform, L-intersecting set system, where $|L| = s$, and $\mathcal{F}$ has more than $f(k, r, s)$ elements, then $\mathcal{F}$ contains a sunflower with r petals.

We give an upper bound for $f(k, 3, s)$.

Keywords: Δ-system; L-intersecting families; Extremal set theory

1. Introduction

Let $[n]$ stand for the set $\{1, 2, \ldots, n\}$. We denote the family of all subsets of $[n]$ by $2^{[n]}$.

Let X be a fixed subset of $[n]$. For an integer $0 \le k \le n$ we denote by $\binom{X}{k}$ the family of all k element subsets of X.

We call a family $\mathcal{F}$ of subsets of $[n]$ *k-uniform*, if $|F| = k$ for each $F \in \mathcal{F}$.

A family $\mathcal{F} = \{F_1, \ldots, F_m\}$ of subsets of $[n]$ is a *sunflower* (or *Δ-system*) with m petals if

$$F_i \cap F_j = \bigcap_{t=1}^{m} F_t$$

for each $1 \le i, j \le m$.

The intersection of the members of a sunflower form its *kernel*. Clearly a family of disjoint sets is a sunflower with empty kernel.

Erdős and Rado gave an upper bound for the size of a k-uniform family without a sunflower with r petals in [1].

Theorem 1.1 (*Sunflower Theorem*)**.** *If $\mathcal{F}$ is a k-uniform set system with more than*

$$k!(r-1)^k \Big(1 - \sum_{t=1}^{k-1} \frac{t}{(t+1)!(r-1)^t}\Big)$$

members, then $\mathcal{F}$ contains a sunflower with r petals.

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Kostochka improved this upper bound in [2].

Theorem 1.2. *Let $r > 2$ and $\alpha > 1$ be fixed integers. Let k be an arbitrary integer. Then there exists a constant $D(r, \alpha)$ such that if $\mathcal{F}$ is a k-uniform set system with more than*

$$D(r, \alpha)k!\Big(\frac{(\log\log\log k)^2}{\alpha \log\log k}\Big)^k$$

members, then $\mathcal{F}$ contains a sunflower with r petals.

Erdős and Rado gave in [1] a construction of a k-uniform set system with $(r-1)^k$ members such that $\mathcal{F}$ does not contain any sunflower with r petals. Later Abbott, Hanson and Sauer improved this construction in [3] and proved the following result.

Theorem 1.3. *There exists a $c > 0$ positive constant and a k-uniform set system $\mathcal{F}$ such that*

$$|\mathcal{F}| > 2 \cdot 10^{k/2 - c \log k}$$

and $\mathcal{F}$ does not contain any sunflower with 3 petals.

Erdős and Rado conjectured also the following statement in [1].

Conjecture 1. *For each r, there exists a constant C_r such that if $\mathcal{F}$ is a k-uniform set system with more than C_r^k members, then $\mathcal{F}$ contains a sunflower with r petals.*

Erdős has offered 1000 dollars for the proof or disproof of this conjecture for $r = 3$ (see [4]).
We prove here Conjecture 1 in the case of some special L-intersecting and ℓ-intersecting families.
A family $\mathcal{F}$ is *ℓ-intersecting*, if $|F \cap F'| \geq \ell$ whenever $F, F' \in \mathcal{F}$. Specially, $\mathcal{F}$ is an *intersecting* family, if $F \cap F' \neq \emptyset$ whenever $F, F' \in \mathcal{F}$.
Erdős, Ko and Rado proved the following well-known result in [5]:

Theorem 1.4. *Let n, k, t be integers with $0 < t < k < n$. Suppose $\mathcal{F}$ is a t-intersecting, k-uniform family of subsets of $[n]$. Then for $n > n_0(k, t)$,*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

Further, $|\mathcal{F}| = \binom{n-t}{k-t}$ if and only if for some $T \in \binom{[n]}{t}$ we have

$$\mathcal{F} = \{F \in \binom{[n]}{k} : \ T \subseteq F\}.$$

Let L be a set of nonnegative integers. A family $\mathcal{F}$ is *L-intersecting*, if $|E \cap F| \in L$ for every pair E, F of distinct members of $\mathcal{F}$. In this terminology a k-uniform $\mathcal{F}$ set system is a t-intersecting family iff it is an L-intersecting family, where $L = \{t, t+1, \ldots, k-1\}$.
The following result gives a remarkable upper bound for the size of a k-uniform L-intersecting family (see [6]).

Theorem 1.5 (*Ray-Chaudhuri–Wilson*)**.** *Let $0 < s \leq k \leq n$ be positive integers. Let L be a set of s nonnegative integers and $\mathcal{F}$ an L-intersecting, k-uniform family of subsets of $[n]$. Then*

$$|\mathcal{F}| \leq \binom{n}{s}.$$

Deza proved the following result in [7].

Theorem 1.6 (*Deza*)**.** *Let $\lambda > 0$ be a positive integer. Let $L := \{\lambda\}$. If $\mathcal{F}$ is an L-intersecting, k-uniform family of subsets of $[n]$, then either*

$$|\mathcal{F}| \leq k^2 - k + 1$$

or $\mathcal{F}$ is a sunflower, i.e. all the pairwise intersections are the same set with λ elements.

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Our main result is the following generalization of Theorem 1.6 for L-intersecting families.

Theorem 1.7. *Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $\mathcal{F}$ does not contain any sunflowers with three petals. Let $L = \{\ell_1 < \cdots < \ell_s\}$ be a set of s non-negative integers. Suppose that $\mathcal{F}$ is a k-uniform, L-intersecting family. Then*

$$|\mathcal{F}| \le (k^2 - k + 1)8^{(s-1)}2^{(1+\frac{\sqrt{5}}{5})k(s-1)}.$$

We present our proofs in Section 2. We give some concluding remarks in Section 3.

2. Proofs of our main results

We start our proof with an elementary fact.

Lemma 2.1. *Let $0 \le r \le n$ be integers. Then*

$$\binom{n}{r} \le \binom{n-1}{r+1}$$

if and only if

$$r^2 + (1 - 3n)r + n^2 - 2n \ge 0.$$

Corollary 2.2. *Let $0 \le r \le n$ be integers. If $0 \le r \le \frac{3n-1-\sqrt{5}(n+1)}{2}$, then*

$$\binom{n}{r} \le \binom{n-1}{r+1}$$

We use the following easy lemma in the proof of our main results.

Lemma 2.3. *Let $0 \le \ell \le k - 1$ be integers. Then*

$$\binom{2k-\ell}{\ell+1} \le 8 \cdot 2^{(1+\frac{\sqrt{5}}{5})k}.$$

Proof. First suppose that

$$\ell \le \lceil (1 - \frac{\sqrt{5}}{5})k - (1 + \frac{2\sqrt{5}}{5})\rceil.$$

Then

$$\binom{2k-\ell}{\ell+1} \le \binom{2k - \lceil (1-\frac{\sqrt{5}}{5})k - (1+\frac{2\sqrt{5}}{5})\rceil}{\lceil (2-\frac{\sqrt{5}}{5})k - (1+\frac{2\sqrt{5}}{5})\rceil} \le$$

$$\le 2^{2k - \lceil (1-\frac{\sqrt{5}}{5})k - (1+\frac{2\sqrt{5}}{5})\rceil} \le 8 \cdot 2^{(1+\frac{\sqrt{5}}{5})k}.$$

The first inequality follows easily from Corollary 2.2. Namely if

$$\ell \le \lceil (1 - \frac{\sqrt{5}}{5})k - (1 + \frac{2\sqrt{5}}{5})\rceil,$$

then

$$\ell + 1 \le \lceil \frac{3-\sqrt{5}}{2}(2k - \ell) - \frac{1+\sqrt{5}}{2}\rceil$$

and we can apply Corollary 2.2 with the choices $r := \ell + 1$ and $n := 2k - \ell$.

Secondly, suppose that

$$\ell > \lceil (1 - \frac{\sqrt{5}}{5})k - (1 + \frac{2\sqrt{5}}{5})\rceil.$$

Then

$$2k - \ell \le 2k - \lceil (1 - \frac{\sqrt{5}}{5})k - (1 + \frac{2\sqrt{5}}{5}) \rceil \le \lceil 2 + (1 + \frac{\sqrt{5}}{5})k \rceil,$$

hence

$$\binom{2k-\ell}{\ell+1} \le \binom{2k - \lceil (1-\frac{\sqrt{5}}{5})k - (1+\frac{2\sqrt{5}}{5})\rceil}{\ell+1} \le$$

$$\le 2^{2k - \lceil (1-\frac{\sqrt{5}}{5})k-(1+\frac{2\sqrt{5}}{5})\rceil} \le 8 \cdot 2^{(1+\frac{\sqrt{5}}{5})k}. \quad \square$$

The soul of the proof of our main result is the following lemma.

Lemma 2.4. *Let $\mathcal{F}$ be an ℓ-intersecting, k-uniform family of subsets of $[n]$ such that $\mathcal{F}$ does not contain any sunflowers with three petals. Suppose that there exist $F_1, F_2 \in \mathcal{F}$ distinct subsets such that $|F_1 \cap F_2| = \ell$. Let $M := F_1 \cup F_2$. Then*

$$|F \cap M| > \ell$$

for each $F \in \mathcal{F}$.

Proof. Clearly $F \cap F_1 \subseteq F \cap M$ for each $F \in \mathcal{F}$, hence

$$|F \cap M| \ge \ell$$

for each $F \in \mathcal{F}$.

We prove by an indirect argument. Suppose that there exists an $F \in \mathcal{F}$ such that $|F \cap M| = \ell$. Clearly $F \ne F_1$ and $F \ne F_2$. Let $G := F_1 \cap F_2$. Then $|G| = \ell$ by assumption. It follows from $F \cap F_1 \subseteq F \cap M$ and $\ell \le |F \cap F_1| \le |F \cap M| = \ell$ that $F \cap F_1 = F \cap M$. Similarly $F \cap F_2 = F \cap M$. Consequently $F \cap F_1 = F \cap F_2$. We get that $F \cap F_2 = F \cap G = F \cap F_1$. Since $\ell = |F \cap F_1| = |F \cap G| \le |G| = \ell$ and $F \cap G \subseteq G$, hence $G = F \cap G = F \cap F_2 = F \cap F_1$, so $\{F, F_1, F_2\}$ is a sunflower with three petals, a contradiction. $\square$

Proof of Theorem 1.7. We apply induction on $|L| = s$. If $s = 1$, then our result follows from Theorem 1.6.

Suppose that Theorem 1.7 is true for $s - 1$ and now we attack the case $|L| = s$.

If $|F \cap F'| \ne \ell_1$ holds for each distinct $F, F' \in \mathcal{F}$, then $\mathcal{F}$ is actually an $L' := \{\ell_2, \dots, \ell_s\}$-intersecting system and the much stronger upper bound

$$|\mathcal{F}| \le (k^2 - k + 2)8^{(s-2)}2^{(1+\frac{\sqrt{5}}{5})k(s-2)}.$$

follows from the induction.

Hence we can suppose that there exist $F_1, F_2 \in \mathcal{F}$ such that $|F_1 \cap F_2| = \ell_1$. Let $M := F_1 \cup F_2$. Clearly $\mathcal{F}$ is an ℓ_1-intersecting family. It follows from Lemma 2.4 that

$$|F \cap M| > \ell_1 \tag{1}$$

for each $F \in \mathcal{F}$. Clearly $|M| = 2k - \ell_1$.

Let T be a fixed subset of M such that $|T| = \ell_1 + 1$. Define the family

$$\mathcal{F}(T) := \{F \in \mathcal{F} : \ T \subseteq M \cap F\}.$$

Let $L' := \{\ell_2, \dots, \ell_s\}$. Clearly $|L'| = s - 1$. Then $\mathcal{F}(T)$ is an L'-intersecting, k-uniform family, because $\mathcal{F}$ is an L-intersecting family and $T \subseteq F$ for each $F \in \mathcal{F}(T)$. The following Proposition follows easily from (1).

Proposition 2.5.

$$\mathcal{F} = \bigcup_{T \subseteq M, |T| = \ell_1 + 1} \mathcal{F}(T). \quad \square$$

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Let T be a fixed, but arbitrary subset of M such that $|T| = \ell_1 + 1$. Consider the set system

$$\mathcal{G}(T) := \{F \setminus T : \ F \in \mathcal{F}(T)\}.$$

Clearly $|\mathcal{G}(T)| = |\mathcal{F}(T)|$. Let $\overline{L} := \{\ell_2 - \ell_1 - 1, \ldots, \ell_s - \ell_1 - 1\}$. Here $|\overline{L}| = s - 1$. Since $\mathcal{F}(T)$ is an L'-intersecting, k-uniform family, thus $\mathcal{G}(T)$ is an $\overline{L}$-intersecting, $(k - \ell_1 - 1)$-uniform family. It follows from the inductional hypothesis that

$$|\mathcal{F}(T)| = |\mathcal{G}(T)| \le (k^2 - k + 2)8^{(s-2)}2^{(1+\frac{\sqrt{5}}{5})k(s-2)}.$$

Finally Proposition 2.5 implies that

$$|\mathcal{F}| \le \sum_{T \subseteq M, |T| = \ell_1 + 1} |\mathcal{F}(T)| \le \binom{2k - \ell_1}{\ell_1 + 1}(k^2 - k + 2)8^{(s-2)}2^{(1+\frac{\sqrt{5}}{5})k(s-2)}.$$

But

$$\binom{2k - \ell}{\ell + 1} \le 8 \cdot 2^{(1+\frac{\sqrt{5}}{5})k}$$

by Lemma 2.3, hence

$$|\mathcal{F}| \le (k^2 - k + 2)8^{(s-2)}2^{(1+\frac{\sqrt{5}}{5})k(s-2)} \cdot 8 \cdot 2^{(1+\frac{\sqrt{5}}{5})k} =$$

$$= (k^2 - k + 2)8^{(s-1)}2^{(1+\frac{\sqrt{5}}{5})k(s-1)},$$

which was to be proved. □

3. Concluding remarks

Define $f(k, r, s)$ as the least number so that if $\mathcal{F}$ is an arbitrary k-uniform, L-intersecting family, where $|L| = s$, then $|\mathcal{F}| > f(k, r, s)$ implies that $\mathcal{F}$ contains a sunflower with r petals. In Theorem 1.7 we proved the following recursion for $f(k, r, s)$:

$$f(k, 3, s) \le \max_{0 \le \ell \le k-1} \binom{2k - \ell}{\ell + 1} f(k - 1, 3, s - 1).$$

Our upper bound in Theorem 1.7 was a clear consequence of this recursion. It would be very interesting to give a similar recursion for $f(k, r, s)$ for $r > 3$.

On the other hand, it is easy to prove the following Proposition from Theorem 1.3.

Proposition 3.1. *Let $1 \le s < k$ be integers. Then there exists a $c > 0$ positive constant such that*

$$f(k, 3, s) > 2 \cdot 10^{s/2 - c \log s}$$

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