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Vertex irregular reflexive labeling of prisms and wheels

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Abstract

For a graph G we define k-labeling ρ such that the edges of G are labeled with integers $\{1, 2, \dots, k_e\}$ and the vertices of G are labeled with even integers $\{0, 2, \dots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$. The labeling ρ is called a *vertex irregular reflexive k-labeling* if distinct vertices have distinct weights, where the vertex weight is defined as the sum of the label of that vertex and the labels of all edges incident this vertex. The smallest k for which such labeling exists is called the *reflexive vertex strength* of G.

In this paper, we give exact values of reflexive vertex strength for prisms, wheels, fan graphs and baskets.

Keywords: Vertex irregular reflexive labeling; Reflexive vertex strength; Prism; Wheel; Fan graph

1. Introduction

All graphs considered in this article are connected, finite and undirected. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ or just V and E when the graph G is clear. The graphs are also simple though they have their origin in multigraphs. In [1] the problem was posed "In a loopless multigraph, determine the fewest parallel edges required to ensure that all vertices have distinct degree". In terms of simple graphs the problem becomes a graph labeling problem in which the number of parallel edges is represented as a positive integer on an edge and irregularity requires that the sum of all edge labels at vertices be pairwise distinct. The problem may be now expressed "assign positive values to the edges of a simply connected graph of order at least 3, in such a way that the graph becomes irregular. What is the minimum largest label over all such irregular assignments"? This minimum largest label is known as *irregularity strength*. Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [2–7].

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Once the problem was considered as an edge labeling it is a simple step to pose it as a problem in total labeling in which both vertices and edges are labeled. This was first introduced by Bača et al. [8] where the authors defined vertex weight as the sum of all incident edge labels along with the label of the vertex. Now the problem is the same as in paragraph 1 except that the positive values are ascribed to both vertices and edges and we can remove the restriction of the graph being of order at least 3. The question remains one of finding the minimum largest label over all assignments. Such labeling is known as *vertex irregular total k-labeling* and *total vertex irregularity strength* of graph is the minimum k for which the graph has a vertex irregular total k-labeling. The bounds for the total vertex irregularity strength given in [8] were then improved in [9,10] and recently by Majerski and Przybylo in [11].

In [12] the authors combined the total labeling problem with the original multigraph problem by identifying the vertex labels as representing loops. They referred to this labeling as an *irregular reflexive labeling*. This helped pose the problem in terms of real world networks but also had an effect on the vertex labels. Firstly, the vertex labels were required to be non-negative even integers (since each loop adds 2 to the vertex degree) and secondly, the vertex label 0 was permitted as representing a loopless vertex.

For a graph G we define k-labeling ρ such that the edges of G are labeled with integers $\{1, 2, \dots, k_e\}$ and the vertices of G are labeled with even integers $\{0, 2, \dots, 2k_v\}$, where $k = \max\{k_e, 2k_v\}$.

Specifically, under a total labeling ρ the weight of a vertex u, denoted by $wt_\rho(u)$, is defined as

$$wt_\rho(u) = \rho(u) + \sum_{uv \in E(G)} \rho(uv)$$

while the weight of an edge uv, denoted by $wt_\rho(uv)$, is defined as

$$wt_\rho(uv) = \rho(u) + \rho(v) + \rho(uv).$$

A labeling ρ is said to be a *vertex irregular reflexive k-labeling* (resp. *edge irregular reflexive k-labeling*) if for $u, v \in V(G)$ is $wt_\rho(u) \neq wt_\rho(v)$ (resp. for $e, f \in E(G)$ is $wt_\rho(e) \neq wt_\rho(f)$). The smallest k for which such labelings exist is called the *reflexive vertex strength* (resp. *reflexive edge strength*).

In this paper we provide exact values for the reflexive vertex strength for prisms, wheels, fans and baskets.

2. Vertex irregular reflexive labeling of prisms

Before we give the exact value of reflexive vertex strength for prisms we first prove one auxiliary lemma.

Lemma 1. *The largest vertex weight of a graph G of order p and the minimum degree δ under any vertex irregular reflexive labeling is at least*

1. $p + \delta - 1$ *if* $p \equiv 0 \pmod 4$, $p \equiv 1 \pmod 4$ *and* $\delta \equiv 0 \pmod 2$, *or* $p \equiv 3 \pmod 4$ *and* $\delta \equiv 1 \pmod 2$,
2. $p + \delta$ *otherwise.*

Proof. Let f be a vertex irregular reflexive labeling of a graph G of order p and the minimum degree δ. Let us denote the vertices of G by the symbols $v_1, v_2, \dots, v_p$ such that $wt_f(v_i) < wt_f(v_{i+1})$ for $i = 1, 2, \dots, p-1$.

Then the vertex weight of a vertex v_i is

$$wt_f(v_i) = f(v_i) + \sum_{uv_i \in E(G)} f(uv_i) \geq 0 + \sum_{uv_i \in E(G)} 1 \geq \delta.$$

As the vertex weights are distinct we get

$$wt_f(v_p) \geq wt_f(v_1) + p - 1 \geq p + \delta - 1.$$

Let us consider that $wt_f(v_p) = p + \delta - 1$ which means that

$$\{wt_f(v_i) : i = 1, 2, \dots, p\} = \{\delta, \delta + 1, \dots, p + \delta - 1\}.$$

Thus the sum of all vertex weights is

$$\sum_{i=1}^{p} wt_f(v_i) = \sum_{i=1}^{p} (\delta + i - 1) = \tfrac{p(p+2\delta-1)}{2}.$$

Evidently, this sum must be an even integer as

$$\sum_{i=1}^{p} wt_f(v_i) = \sum_{i=1}^{p} f(v_i) + 2 \sum_{e \in E(G)} f(e),$$

and every vertex label is even. Thus

$$p(p + 2\delta - 1) \equiv 0 \pmod 4,$$

but it is not possible if $p \equiv 1 \pmod 4$ and $\delta \equiv 1 \pmod 2$, $p \equiv 2 \pmod 4$, or $p \equiv 3 \pmod 4$ and $\delta \equiv 0 \pmod 2$. □

For regular graphs we immediately deduce the following corollary.

Corollary 1. *Let G be an r-regular graph of order p. Then*

$$\mathrm{rvs}(G) \geq \begin{cases} \left\lceil \dfrac{p+r-1}{r+1} \right\rceil & \text{if } p \equiv 0, 1 \pmod 4, \\ \left\lceil \dfrac{p+r}{r+1} \right\rceil & \text{if } p \equiv 2, 3 \pmod 4. \end{cases}$$

The prism D_n, $n \geq 3$, is a trivalent graph which can be defined as the Cartesian product $P_2 \Box C_n$ of a path on two vertices with a cycle on n vertices. We denote the vertex set and the edge set of D_n such that $V(D_n) = \{x_i, y_i : i = 1, 2, \dots, n\}$ and $E(D_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i : i = 1, 2, \dots, n\}$, where indices are taken modulo n.

Theorem 1. *For $n \geq 3$,*

$$\mathrm{rvs}(D_n) = \left\lceil \tfrac{n}{2} \right\rceil + 1.$$

Proof. As the prism D_n is a 3-regular graph of order $2n$, by Corollary 1 we obtain that $\mathrm{rvs}(D_n) \geq \left\lceil \frac{n}{2} \right\rceil + 1$.

We define the total labeling f of D_n in the following way

$$\begin{aligned}
f(x_i) = f(y_i) &= 0 && i = 1, 2, \dots, \left\lceil \tfrac{n}{2} \right\rceil + 1, \\
f(x_i) = f(y_i) &= \left\lceil \tfrac{n}{2} \right\rceil && i = \left\lceil \tfrac{n}{2} \right\rceil + 2, \left\lceil \tfrac{n}{2} \right\rceil + 3, \dots, n, \\
& && \quad \text{and } n \equiv 0, 3 \pmod 4, \\
f(x_i) = f(y_i) &= \left\lceil \tfrac{n}{2} \right\rceil + 1 && i = \left\lceil \tfrac{n}{2} \right\rceil + 2, \left\lceil \tfrac{n}{2} \right\rceil + 3, \dots, n, \\
& && \quad \text{and } n \equiv 1, 2 \pmod 4, \\
f(x_i x_{i+1}) &= 1 && i = 1, 2, \dots, n-1, \\
f(x_1 x_n) &= 1, \\
f(y_i y_{i+1}) &= \left\lceil \tfrac{n}{2} \right\rceil + 1 && i = 1, 2, \dots, n-1, \\
f(y_1 y_n) &= \left\lceil \tfrac{n}{2} \right\rceil + 1, \\
f(x_i y_i) &= i && i = 1, 2, \dots, \left\lceil \tfrac{n}{2} \right\rceil + 1, \\
f(x_i y_i) &= i - \left\lceil \tfrac{n}{2} \right\rceil && i = \left\lceil \tfrac{n}{2} \right\rceil + 2, \left\lceil \tfrac{n}{2} \right\rceil + 3, \dots, n, \\
& && \quad \text{and } n \equiv 0, 3 \pmod 4, \\
f(x_i y_i) &= i - 1 - \left\lceil \tfrac{n}{2} \right\rceil && i = \left\lceil \tfrac{n}{2} \right\rceil + 2, \left\lceil \tfrac{n}{2} \right\rceil + 3, \dots, n, \\
& && \quad \text{and } n \equiv 1, 2 \pmod 4.
\end{aligned}$$

Evidently f is $\left(\left\lceil \frac{n}{2} \right\rceil + 1\right)$-labeling and the vertices are labeled with even numbers.

For the vertex weights of the vertices x_i, $i = 1, 2, \dots, \left\lceil \frac{n}{2} \right\rceil + 1$ in D_n under the labeling f we have

$$wt_f(x_i) = 0 + 1 + 1 + i = i + 2.$$

If $i = \left\lceil \frac{n}{2} \right\rceil + 2, \left\lceil \frac{n}{2} \right\rceil + 3, \dots, n$ and $n \equiv 0, 3 \pmod 4$ then

$$wt_f(x_i) = \left\lceil \tfrac{n}{2} \right\rceil + 1 + 1 + \left(i - \left\lceil \tfrac{n}{2} \right\rceil\right) = i + 2$$

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and for $i = \lceil \frac{n}{2} \rceil + 2, \lceil \frac{n}{2} \rceil + 3, \dots, n$ and $n \equiv 1, 2 \pmod 4$)

$$wt_f(x_i) = \left(\lceil \tfrac{n}{2} \rceil + 1\right) + 1 + 1 + \left(i - 1 - \lceil \tfrac{n}{2} \rceil\right) = i + 2.$$

Thus $\{wt_f(x_i) : i = 1, 2, \dots, n\} = \{3, 4, \dots, n + 2\}$.

For the vertex weights of the vertices y_i, $i = 1, 2, \dots, n$ in D_n under the labeling f we have the following

$$\begin{aligned}
wt_f(y_i) &= 0 + \left(\lceil \tfrac{n}{2} \rceil + 1\right) + \left(\lceil \tfrac{n}{2} \rceil + 1\right) + i = i + 2 + 2\lceil \tfrac{n}{2} \rceil \\
&\qquad \text{for } i = 1, 2, \dots, \lceil \tfrac{n}{2} \rceil + 1, \\
wt_f(y_i) &= \lceil \tfrac{n}{2} \rceil + \left(\lceil \tfrac{n}{2} \rceil + 1\right) + \left(\lceil \tfrac{n}{2} \rceil + 1\right) + \left(i - \lceil \tfrac{n}{2} \rceil\right) = i + 2 + 2\lceil \tfrac{n}{2} \rceil \\
&\qquad \text{for } i = \lceil \tfrac{n}{2} \rceil + 2, \lceil \tfrac{n}{2} \rceil + 3, \dots, n, \text{ and } n \equiv 0, 3 \pmod 4, \\
wt_f(y_i) &= \left(\lceil \tfrac{n}{2} \rceil + 1\right) + \left(\lceil \tfrac{n}{2} \rceil + 1\right) + \left(\lceil \tfrac{n}{2} \rceil + 1\right) + \left(i - 1 - \lceil \tfrac{n}{2} \rceil\right) \\
&= i + 2 + 2\lceil \tfrac{n}{2} \rceil \\
&\qquad \text{for } i = \lceil \tfrac{n}{2} \rceil + 2, \lceil \tfrac{n}{2} \rceil + 3, \dots, n, \text{ and } n \equiv 1, 2 \pmod 4.
\end{aligned}$$

Which means

$$\begin{aligned}
\{wt_f(y_i) : i = 1, 2, \dots, n\} &= \{2\lceil \tfrac{n}{2} \rceil + 3, 2\lceil \tfrac{n}{2} \rceil + 4, \dots, n + 2\lceil \tfrac{n}{2} \rceil + 2\} \\
&= \begin{cases} \{n + 3, n + 4, \dots, 2n + 2\} & \text{for } n \text{ even}, \\ \{n + 4, n + 5, \dots, 2n + 3\} & \text{for } n \text{ odd}. \end{cases}
\end{aligned}$$

Thus the vertex weights are all distinct, that is f is a vertex irregular reflexive $\left(\lceil \frac{n}{2} \rceil + 1\right)$-labeling of a prism D_n. □

3. Vertex irregular reflexive labeling of wheels

The wheel W_n, $n \geq 3$, is a graph obtained by joining all vertices of C_n to a further vertex called the center. We denote the vertex set and the edge set of W_n such that $V(W_n) = \{x, x_i : i = 1, 2, \dots, n\}$ and $E(W_n) = \{x_i x_{i+1}, x x_i : i = 1, 2, \dots, n\}$, where indices are taken modulo n. The wheel is of order $n + 1$ and size $2n$. We prove the following result for wheels.

Theorem 2. *For $n \geq 3$,*

$$\operatorname{rvs}(W_n) = \begin{cases} \lceil \frac{n+2}{4} \rceil & \text{if } n \not\equiv 2 \pmod 8, \\ \lceil \frac{n+2}{4} \rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

Proof. Let $n \geq 3$. As $\delta(W_n) = 3$ then the smallest vertex weight is at least 3. The wheel W_n contains n vertices of degree 3 thus the largest weight over all vertices of degree 3 is at least $n + 2$. Every vertex weight of a vertex of degree 3 is the sum of four labels from which at least one is even thus we have

$$\operatorname{rvs}(W_n) \geq \lceil \tfrac{n+2}{4} \rceil.$$

However, if $n = 8t + 2$, $t \geq 1$, we get that the fraction

$$\lceil \tfrac{n+2}{4} \rceil = \lceil \tfrac{8t+4}{4} \rceil = 2t + 1$$

is odd. The number $n + 2 = 8t + 4$ can be realizable as the sum of four labels not greater than $2t + 1$ only in the following way

$$n + 2 = 8t + 4 = (2t + 1) + (2t + 1) + (2t + 1) + (2t + 1),$$

but this is a contradiction as the vertex label must be even. Thus, for $n \equiv 2 \pmod 8$ we obtain

$$\operatorname{rvs}(W_n) \geq \lceil \tfrac{n+2}{4} \rceil + 1.$$

Let

$$R = \begin{cases} \lceil \frac{n+2}{4} \rceil & \text{if } n \not\equiv 2 \pmod 8, \\ \lceil \frac{n+2}{4} \rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

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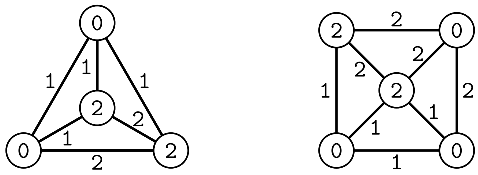

Fig. 1. The vertex irregular reflexive 2-labelings of wheels W_3 and W_4.

Let us denote by K the largest even number not greater than R. Thus

$$K = \begin{cases} R & \text{if } n \equiv 2, 3, 4, 5, 6 \pmod 8, \\ R - 1 & \text{if } n \equiv 0, 1, 7 \pmod 8. \end{cases}$$

For $n = 3{,}4$ we get that $\mathrm{rvs}(W_n) \geq 2$. The corresponding vertex irregular reflexive 2-labelings for W_3 and W_4 are illustrated on Fig. 1.

For $n \geq 5$ we define the total R-labeling f of W_n such that

$$\begin{aligned}
f(x) &= K, && \\
f(x_i) &= 0 && i = 1, 2, \ldots, 2R + K - 2,\ i \leq n - 1, \\
f(x_i) &= K && i = 2R + K - 1, 2R + K, \ldots, n, \\
f(x_i x) &= \left\lceil \tfrac{i}{3} \right\rceil && i = 1, 2, \ldots, 2R + K - 2,\ i \leq n - 1, \\
f(x_i x) &= i + 3 - 2R - K && i = 2R + K - 1, 2R + K, \ldots, n - 1, \\
f(x_n x) &= R, && \\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-1}{3} \right\rceil + 1 && i = 1, 2, \ldots, 2R + K - 2,\ i \leq n - 1, \\
f(x_i x_{i+1}) &= R && i = 2R + K - 1, 2R + K, \ldots, n - 1, \\
f(x_1 x_n) &= 1. &&
\end{aligned}$$

For the weight of vertices of degree 3 under the labeling f we obtain

$$\begin{aligned}
wt_f(x_1) &= 0 + 1 + 1 + 1 = 3, \\
wt_f(x_i) &= 0 + \left\lceil \tfrac{i}{3} \right\rceil + \left(\left\lceil \tfrac{i-2}{3} \right\rceil + 1 \right) + \left(\left\lceil \tfrac{i-1}{3} \right\rceil + 1 \right) = i + 2 \\
&\qquad \text{for } i = 2, 3, \ldots, 2R + K - 2,\ i \leq n - 1, \\
wt_f(x_{2R+K-1}) &= K + 2 + \left(\left\lceil \tfrac{2R+K-3}{3} \right\rceil + 1 \right) + R = 2R + K + 2, \\
wt_f(x_i) &= K + (i + 3 - 2R - K) + R + R = i + 3 \\
&\qquad \text{for } i = 2R + K, 2R + K + 1, \ldots, n - 1, \\
wt_f(x_n) &= K + 1 + R + R = 2R + K + 1.
\end{aligned}$$

It is easy to see that the weights of vertices x_i, $i = 1, 2, \ldots, n$, $n \geq 5$ and $n \neq 10$ are distinct numbers from the set $\{3, 4, \ldots, n + 2\}$. For $n = 10$ we have $\{wt_f(x_i) : i = 1, 2, \ldots, 10\} = \{3, 4, \ldots, 11, 13\}$.

The weight of the vertex x is

$$\begin{aligned}
wt_f(x) &= f(x) + \sum_{i=1}^{n} f(x_i x) = K + \sum_{i=1}^{n} f(x_i x) = K + R + \sum_{i=1}^{n-1} f(x_i x) \\
&> K + R + n - 1.
\end{aligned}$$

Evidently, for $n \geq 5$, the vertex weights are distinct. $\square$

A *fan graph* F_n is obtained from wheel W_n if one rim edge, say $x_1 x_n$ is deleted. A *basket* B_n is obtained by removing a spoke, say $x x_n$, from wheel W_n. Before we will give the exact value of reflexive vertex strength of fans and baskets we give the following observation.

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Observation 1. *Let f be a vertex irregular reflexive k-labeling of a graph G. If there exists an edge uv in G such that*

$$wt_f(u) - f(uv) \notin \{wt_f(x) : x \in V(G) - \{v\}\},$$
$$wt_f(v) - f(uv) \notin \{wt_f(x) : x \in V(G) - \{u\}\}$$

then f is a vertex irregular reflexive k-labeling of a graph $G - \{uv\}$.

Proof. The proof is trivial. □

Immediately from this observation we get the following corollary.

Corollary 2. *Let* $\mathrm{rvs}(G) = k$ *and let f be the corresponding vertex irregular reflexive k-labeling of a graph G. If the vertex weights of vertices u, v are the smallest over all vertex weights under the labeling f and $uv \in E(G)$ then*

$$\mathrm{rvs}(G - \{uv\}) \le \mathrm{rvs}(G).$$

Proof. Let f be a vertex irregular reflexive k-labeling of a graph G. Let u, v be two adjacent vertices and let the vertex weights of these vertices be the smallest over all vertices in G. Without loss of generality assume $wt_f(u) < wt_f(v)$. This means that for every $w \in V(G) - \{u, v\}$,

$$wt_f(u) < wt_f(v) < wt_f(w). \tag{1}$$

Let g be the restriction of the labeling f on $G - \{uv\}$. Evidently

$$\begin{aligned} wt_g(u) &= wt_f(u) - f(uv),\\ wt_g(v) &= wt_f(v) - f(uv),\\ wt_g(w) &= wt_f(w) && \text{for every } w \in V(G) - \{u, v\}. \end{aligned}$$

Combining with (1) we obtain

$$wt_g(u) = wt_f(u) - f(uv) < wt_f(v) - f(uv) = wt_g(v) < wt_f(w) = wt_g(w).$$

Thus, immediately using Observation 1 we have $\mathrm{rvs}(G - \{uv\}) \le \mathrm{rvs}(G)$. □

For the fan graph F_n we prove

Theorem 3. *For $n \ge 3$,*

$$\mathrm{rvs}(F_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil & \text{if } n \not\equiv 3 \pmod 8,\\ \lceil \frac{n+1}{4} \rceil + 1 & \text{if } n \equiv 3 \pmod 8. \end{cases}$$

Proof. The fan graph F_n contains two vertices of degree 2, thus the smallest vertex weight is at least 2. The fan graph F_n contains $n-2$ vertices of degree 3, thus the largest weight of a vertex of degree 3 is at least n.

If all vertex weights of vertices of degree 3 are at most n, then one of the vertices of degree 2 has to have weight at least $n+1$ and thus $\mathrm{rvs}(F_n) \ge \lceil \frac{n+1}{3} \rceil$. If a vertex of degree 3 has weight greater than n then $\mathrm{rvs}(F_n) \ge \lceil \frac{n+1}{4} \rceil$. As we are trying to minimize the parameter k for which there exists vertex irregular reflexive k-labeling of F_n we obtain

$$\mathrm{rvs}(F_n) \ge \lceil \tfrac{n+1}{4} \rceil,$$

which can be obtained when both vertices of degree 2 in F_n will have weights less than n.

According to the proof of Theorem 2 and Corollary 2, for $n \ge 5$, we get

$$\mathrm{rvs}(F_n) \le \mathrm{rvs}(W_n) = \begin{cases} \lceil \frac{n+2}{4} \rceil & \text{if } n \not\equiv 2 \pmod 8,\\ \lceil \frac{n+2}{4} \rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

Moreover, we can derive a vertex irregular reflexive $\mathrm{rvs}(W_n)$-labeling of F_n from a vertex irregular reflexive $\mathrm{rvs}(W_n)$-labeling of W_n.

Combining the previous facts we have that for $n \not\equiv 2, 3, 7 \pmod 8$

$$\mathrm{rvs}(F_n) = \lceil \tfrac{n+1}{4} \rceil$$

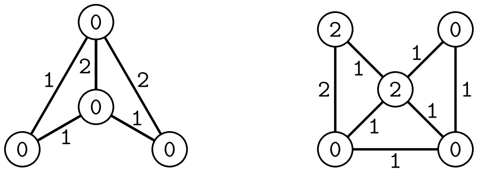

Fig. 2. The vertex irregular reflexive 2-labelings of fans F_3 and F_4.

and for $n \equiv 2, 3, 7 \pmod 8$

$$\lceil \tfrac{n+1}{4} \rceil \le \mathrm{rvs}(F_n) \le \lceil \tfrac{n+1}{4} \rceil + 1.$$

For $n = 3,4$ we get that $\mathrm{rvs}(F_n) \ge 2$. The corresponding vertex irregular reflexive 2-labelings for F_3 and F_4 are illustrated on Fig. 2.

Let $n = 8t + 3$, $t \ge 1$, then $\lceil \frac{n+1}{4} \rceil = 2t + 1$. As this value is odd we cannot get the number $n + 1 = 8t + 4$ as the sum of four labels less or equal to $2t + 1$ from which at least one is even. Thus $\mathrm{rvs}(F_n) = \lceil \frac{n+1}{4} \rceil + 1$ but in this case

$$\mathrm{rvs}(W_n) = \lceil \tfrac{n+2}{4} \rceil = \lceil \tfrac{n+1}{4} \rceil + 1$$

and we are done.

We denote the vertex set and the edge set of F_n such that $V(F_n) = \{x, x_i : i = 1, 2, \ldots, n\}$ and $E(F_n) = \{x_i x_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x x_i : i = 1, 2, \ldots, n\}$.

If $n = 8t + 2$, $t \ge 1$ then $\lceil \frac{n+1}{4} \rceil = 2t + 1$. We define a $(2t + 1)$-labeling of F_n such that

$$\begin{aligned}
f(x) &= 2t, & & \\
f(x_i) &= 0 & & i = 1, 2, \ldots, 6t, \\
f(x_i) &= 2t & & i = 6t + 1, 6t + 2, \ldots, 8t + 2, \\
f(x_1 x) &= 1, & & \\
f(x_i x) &= \left\lceil \tfrac{i-1}{3} \right\rceil & & i = 2, 3, \ldots, 6t, \\
f(x_i x) &= i - 6t & & i = 6t + 1, 6t + 2, \ldots, 8t + 1, \\
f(x_{8t+2} x) &= 2t + 1, & & \\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 & & i = 1, 2, \ldots, 6t, \\
f(x_i x_{i+1}) &= 2t + 1 & & i = 6t + 1, 6t + 2, \ldots, 8t + 1.
\end{aligned}$$

It is easy to verify that the set of all vertex weights is $\{2, 3, \ldots, 8t + 3, 8t^2 + 8t + 3\}$.

If $n = 8t + 7$, $t \ge 0$ then $\lceil \frac{n+1}{4} \rceil = 2t + 2$. We define $(2t + 2)$-labeling of F_n in the following way

$$\begin{aligned}
f(x) &= 2t + 2, & & \\
f(x_i) &= 0 & & i = 1, 2, \ldots, 6t + 4, \\
f(x_i) &= 2t + 2 & & i = 6t + 5, 6t + 6, \ldots, 8t + 7, \\
f(x_1 x) &= 1, & & \\
f(x_i x) &= \left\lceil \tfrac{i-1}{3} \right\rceil & & i = 2, 3, \ldots, 6t + 4, \\
f(x_i x) &= i - 6t - 4 & & i = 6t + 5, 6t + 6, \ldots, 8t + 6, \\
f(x_{8t+7} x) &= 2t + 2, & & \\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 & & i = 1, 2, \ldots, 6t + 4, \\
f(x_i x_{i+1}) &= 2t + 2 & & i = 6t + 5, 6t + 6, \ldots, 8t + 6.
\end{aligned}$$

Evidently the vertex weights are distinct. □

Theorem 4. *For $n \ge 3$,*

$$\mathrm{rvs}(B_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil & \text{if } n \not\equiv 3 \pmod 8, \\ \lceil \frac{n+1}{4} \rceil + 1 & \text{if } n \equiv 3 \pmod 8. \end{cases}$$

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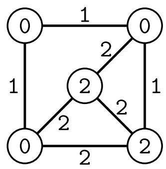

Fig. 3. The vertex irregular reflexive 2-labeling of basket B_4.

Proof. The basket B_n contains one vertex of degree 2, therefore the smallest vertex weight is at least 2 and it contains $n-1$ vertices of degree 3, hence the largest weight of a vertex of degree 3 is at least $n+1$. Thus

$$\mathrm{rvs}(B_n) \geq \lceil \tfrac{n+1}{4} \rceil.$$

Analogously as in the proof of the previous theorem, using Theorem 2 and Observation 1, for $n \geq 5$, we have

$$\mathrm{rvs}(B_n) \leq \mathrm{rvs}(W_n) = \begin{cases} \lceil \frac{n+2}{4} \rceil & \text{if } n \not\equiv 2 \pmod 8, \\ \lceil \frac{n+2}{4} \rceil + 1 & \text{if } n \equiv 2 \pmod 8. \end{cases}$$

Moreover, we can derive a vertex irregular reflexive $\mathrm{rvs}(W_n)$-labeling of B_n from the vertex irregular reflexive $\mathrm{rvs}(W_n)$-labeling of W_n defined in the proof of Theorem 2 by deleting the spoke x_1x in W_n.

Combining the previous facts we get that for $n \not\equiv 2, 3, 7 \pmod 8$

$$\mathrm{rvs}(B_n) = \lceil \tfrac{n+1}{4} \rceil$$

and $n \equiv 2, 3, 7 \pmod 8$

$$\lceil \tfrac{n+1}{4} \rceil \leq \mathrm{rvs}(B_n) \leq \lceil \tfrac{n+1}{4} \rceil + 1.$$

For $n = 3{,}4$ we get that $\mathrm{rvs}(B_n) \geq 2$. The basket B_3 is isomorphic to the fan F_3. The vertex irregular reflexive 2-labelings for B_4 is illustrated on Fig. 3.

Let $n = 8t+3$, $t \geq 1$, then $\lceil \frac{n+1}{4} \rceil = 2t+1$. As this is odd we cannot get the number $n+1 = 8t+4$ as the sum of four labels less or equal to $2t+1$ from which at least one is even. Thus $\mathrm{rvs}(B_n) = \lceil \frac{n+1}{4} \rceil + 1$ but in this case

$$\mathrm{rvs}(W_n) = \lceil \tfrac{n+2}{4} \rceil + 1 = \lceil \tfrac{n+1}{4} \rceil + 1$$

and we are done.

Let us denote the vertex set and the edge set of the basket B_n such that $V(B_n) = \{x, x_i : i = 1, 2, \ldots, n\}$ and $E(B_n) = \{xx_i : i = 2, 3, \ldots, n\} \cup \{x_ix_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{x_nx_1\}$.

If $n = 8t+2$, $t \geq 1$ then $\lceil \frac{n+1}{4} \rceil = 2t+1$. We define $(2t+1)$-labeling of B_n such that

$$\begin{aligned}
f(x) &= 2t, && \\
f(x_i) &= 0 && i = 1, 2, \ldots, 6t+1, \\
f(x_i) &= 2t && i = 6t+2, 6t+3, \ldots, 8t+2, \\
f(x_ix) &= \left\lceil \tfrac{i-1}{3} \right\rceil && i = 2, 3, \ldots, 6t+1, \\
f(x_ix) &= i - 6t && i = 6t+2, 6t+3, \ldots, 8t+1, \\
f(x_{8t+2}x) &= 2t+1, && \\
f(x_ix_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 && i = 1, 2, \ldots, 6t, \\
f(x_ix_{i+1}) &= 2t+1 && i = 6t+1, 6t+2, \ldots, 8t+1, \\
f(x_1x_{8t+2}) &= 1. &&
\end{aligned}$$

If $n = 8t + 7$, $t \geq 0$ then $\lceil \frac{n+1}{4} \rceil = 2t + 2$. We define $(2t + 2)$-labeling of B_n in the following way

$$
\begin{aligned}
f(x) &= 2t + 2, && \\
f(x_i) &= 0 && i = 1, 2, \ldots, 6t + 5, \\
f(x_i) &= 2t + 2 && i = 6t + 6, 6t + 7, \ldots, 8t + 7, \\
f(x_i x) &= \left\lceil \tfrac{i-1}{3} \right\rceil && i = 2, 3, \ldots, 6t + 5, \\
f(x_i x) &= i - 6t - 4 && i = 6t + 6, 6t + 7, \ldots, 8t + 6, \\
f(x_{8t+7} x) &= 2t + 2, && \\
f(x_i x_{i+1}) &= \left\lceil \tfrac{i-2}{3} \right\rceil + 1 && i = 1, 2, \ldots, 6t + 5, \\
f(x_i x_{i+1}) &= 2t + 2 && i = 6t + 6, 6t + 7, \ldots, 8t + 6, \\
f(x_1 x_{8t+7}) &= 1. &&
\end{aligned}
$$

It is not difficult to show that in both cases the described labelings have desired properties. □

4. Conclusion

In this paper we determined exact values of the reflexive vertex strength for prisms D_n, wheels W_n, fan graphs F_n and for baskets B_n, $n \geq 3$.

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References

[1] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular networks, Cong. Numer. 64 (1988) 187–192.
[2] M. Aigner, E. Triesch, Irregular assignments of trees and forests, SIAM J. Discrete Math. 3 (1990) 439–449.
[3] D. Amar, O. Togni, Irregularity strength of trees, Discrete Math. 190 (1998) 15–38.
[4] M. Anholcer, C. Palmer, Irregular labelings of circulant graphs, Discrete Math. 312 (23) (2012) 3461–3466.
[5] T. Bohman, D. Kravitz, On the irregularity strength of trees, J. Graph Theory 45 (2004) 241–254.
[6] A. Frieze, R.J. Gould, M. Karonski, F. Pfender, On graph irregularity strength, J. Graph Theory 41 (2002) 120–137.
[7] P. Majerski, J. Przybyło, On the irregularity strength of dense graphs, SIAM J. Discrete Math. 28 (1) (2014) 197–205.
[8] M. Bača, S. Jendrol', M. Miller, J. Ryan, Total irregular labelings, Discrete Math. 307 (2007) 1378–1388.
[9] J. Przybyło, Linear bound on the irregularity strength and the total vertex irregularity strength of graphs, SIAM J. Discrete Math. 23 (2009) 511–516.
[10] M. Anholcer, M. Kalkowski, J. Przybyło, A new upper bound for the total vertex irregularity strength of graphs, Discrete Math. 309 (2009) 6316–6317.
[11] P. Majerski, J. Przybyło, Total vertex irregularity strength of dense graphs, J. Graph Theory 76 (1) (2014) 34–41.
[12] J. Ryan, B. Munasinghe, D. Tanna, Reflexive irregular labelings, preprint.