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Rainbow connection number of generalized composition

Fendy Septyanto*, Kiki Ariyanti Sugeng

Department of Mathematics and Natural Sciences, Universitas Indonesia, Depok 16424, Indonesia

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Abstract

Let G be a connected graph with $V(G) = \{v_1, \dots, v_n\}$. The rainbow connection number $rc(G)$ is the smallest k for which there is a map $\gamma : E(G) \to \{1, \dots, k\}$ such that any two vertices can be connected by a path whose edge colors are all distinct. The generalized composition $G[H_1, \dots, H_n]$ is obtained by replacing each v_i with the graph H_i. We prove $rc(G[H_1, \dots, H_n]) = diam(G)$ if each H_i has at least $diam(G) \geq 4$ vertices, improving known upper bounds of Basavaraju et al. and Gologranc et al. (2014). We prove the same result when $diam(G) = 3$ but with some additional conditions. When $diam(G) = 2$, we show that the largest possible value of $rc(G[H_1, \dots, H_n])$ is related to whether every vertex of G is contained in a triangle or not.

Keywords: Composition; Lexicographic product; Rainbow connection

1. Introduction

In 2008 Chartrand et al. [1] introduced new concepts that use edge-coloring to strengthen the connectedness property of a graph. An edge-coloring on a graph G is a map $E(G) \to \{1, \dots, k\}$ (also called "k-coloring"). A *rainbow path* is a path whose edge colors are all distinct. A *rainbow coloring* is an edge-coloring in which any two vertices can be connected by a rainbow path. The *rainbow connection number* $rc(G)$ is the smallest k for which G has a rainbow k-coloring. A *strong rainbow coloring* is an edge-coloring in which any two vertices can be connected by a rainbow geodesic. The *strong rainbow connection number* $src(G)$ is the smallest k for which G has a strong rainbow k-coloring.

We have [1]

$$diam(G) \leq rc(G) \leq src(G) \leq |E(G)|. \tag{1}$$

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* Corresponding author.
E-mail addresses: fendy.septyanto41@sci.ui.ac.id (F. Septyanto), kiki@sci.ui.ac.id (K.A. Sugeng).

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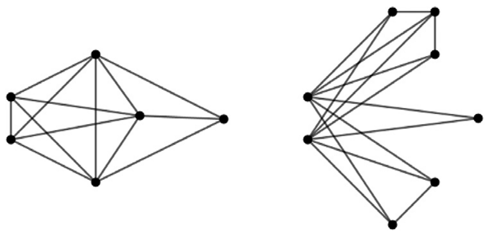

Fig. 1. More examples, $P_3[P_2, C_3, K_1]$ (left) and $K_{1,3}[2K_1, P_3, K_1, P_2]$ (right).

The reader is referred to [2] for a detailed survey. It is known that computing rc and src is NP-hard [3]. Many studies have focused on special classes of graphs or graph operations, such as Cartesian product, strong product, lexicographic product (see [4] and [5]), and graph join (see [6]).

We study generalized composition, which can be thought of as "blowing-up" vertices into individual graphs. Let $V(G) = \{v_1, \ldots, v_n\}$, and $H_1, \ldots, H_n$ be any graphs. The *generalized composition* $G[H_1, \ldots, H_n]$ is obtained by replacing each v_i with H_i and adding a new edge between every vertex of H_i and every vertex of H_j whenever $v_i v_j \in E(G)$. We call this operation as G-composition. See Fig. 1.

Examples. $P_2[H_1, H_2] = H_1 + H_2$ is the usual graph join, and a P_n-composition is known as a sequential join. The special case $G \circ H = G[H, H, \ldots, H]$ is known as composition or lexicographic product.

We always assume that G is non-trivial and connected. If $G[H_1, \ldots, H_n]$ is not a complete graph, then its diameter is $\max\{2, diam(G)\}$. So

$$rc(G[H_1, \ldots, H_n]) \geq diam(G). \tag{2}$$

If each H_i has at least $diam(G) \geq 4$ vertices, we show that (2) becomes an equality. When $H_1 = \cdots = H_n$, this improves the results of Basavaraju et al. [4] ($rc(G \circ H) \leq 2\,rad(G)$) and Gologranc et al. [5] ($rc(G \circ H) \leq 2\,diam(G) + 1$).

If $diam(G) \leq 3$, the bound (2) can be strict. However, we show that equality occurs when $diam(G) = 3$ and some conditions are met (either each H_i has at least one edge, or G has the property that there is a 3-walk between every pair $x, y \in V(G)$ possibly with $x = y$). When $diam(G) = 2$, we show that the largest possible value of $rc(G[H_1, \ldots, H_n])$ determines whether every vertex of G is contained in a triangle or not.

2. Results

2.1. A preliminary bound

Let $Q \subseteq V(G)$. Its *common neighborhood* $CN(Q)$ is the intersection of $N(v) = \{w : vw \in E(G)\}$ over all $v \in Q$. A set of vertices is *independent* if any two are non-adjacent, or *co-neighboring* if $N(v) = N(w) \neq \emptyset$ for all $v, w \in Q$.

Lemma 1. *Let $Q \subseteq V(G)$ be a co-neighboring set. Then*

(1) $src(G) \geq |Q|^{\frac{1}{|CN(Q)|}}$.
(2) *If moreover $CN(Q)$ is independent, then* $rc(G) \geq \min\left\{4, |Q|^{\frac{1}{|CN(Q)|}}\right\}$.

Proof. This is based on an idea in [1]. Let $CN(Q) = \{t_1, \ldots, t_b\}$. Given a k-coloring γ on G, define the *color code* of $v \in Q$ with respect to $CN(Q)$ as

$$code(v) = (\gamma(vt_1), \ldots, \gamma(vt_b)). \tag{3}$$

Note that there are at most k^b distinct codes.

Claim. *There is a rainbow geodesic between $v, w \in Q$ if and only if $code(v) \neq code(w)$.*

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In fact, any geodesic between v and w has the form $v - t - w$ with $t \in CN(Q)$, which is rainbow if and only if $\gamma(vt) \neq \gamma(wt)$. Now we prove the lower bounds.

(1) Let $k = \lceil \sqrt[b]{|Q|} \rceil - 1$. Suppose $src(G) \leq k$, so there is a strong rainbow k-coloring on G. Since $k^b < |Q|$, there are $v, w \in Q$ with the same code. By the claim, there are no rainbow geodesics between them, a contradiction.

(2) Let $k = \min\{3, \lceil \sqrt[b]{|Q|} \rceil - 1\}$. Suppose $rc(G) \leq k$, so there is a rainbow k-coloring on G. Since $k^b < |Q|$, there are $v, w \in Q$ with the same code. Let $L : v - x - \cdots - y - w$ be a rainbow geodesic between v and w. Then $x, y \in CN(Q)$, since Q is co-neighboring. By the claim, L is not a rainbow geodesic. So $x \neq y$. Since $CN(Q)$ is independent, $d_G(x, y) \geq 2$. The length of L is at least $2 + d_G(x, y) \geq 4$, contradicting $k \leq 3$. $\square$

Below is the reason we only study the rc of G-compositions, not the src.

Corollary 1. *The src of G-compositions cannot be bounded above in terms of G alone.*

Proof. Let $k, c \in \mathbb{N}$. Replace some $v \in V(G)$ with mK_1 such that $m > c^{k\deg(v)}$, and replace any other vertex with kK_1, to get a G-composition graph A. The set $Q = V(mK_1)$ is co-neighboring and $|CN(Q)| = k\deg(v)$, so by Lemma 1(1) we have $src(A) \geq |Q|^{\frac{1}{|CN(Q)|}} > c$. $\square$

2.2. Diameter at least four

Theorem 1. *Let $diam(G) \geq 4$ and $n = |V(G)|$. If each H_i has at least $diam(G)$ vertices, then $rc(G[H_1, \ldots, H_n]) = diam(G)$.*

We prove the upper bound separately for later use.

Lemma 2. *If each H_i has at least $\max\{4, diam(G)\}$ vertices, then*

$$rc(G[H_1, \ldots, H_n]) \leq \max\{4, diam(G)\}. \tag{4}$$

Proof. Let $A = G[H_1, \ldots, H_n]$ and $V(H_i) = \{(i, j) | 1 \leq j \leq n_i\}$. We will construct a rainbow u-coloring on A, where $u = \max\{4, diam(G)\}$. Define a map $\gamma : E(A) \to \{0, 1, \ldots, u-1\}$ arbitrarily on each $E(H_i)$, and put

$$\gamma\left((i, j)(i', j')\right) = j + j' \pmod{u} \tag{5}$$

for all i, i' adjacent in G. We show that γ is a rainbow coloring.

Let $x = (i, j)$, $y = (i', j')$ with i, i' non-adjacent in G. We will find a rainbow path between x and y.

Case 1: $d_G(i, i') = 0$ or 2.

Choose a common neighbor i_1 of i, i'. If $j \equiv j' \pmod{u}$, then the path $(i, j) \overset{2j}{-} (i_1, j) \overset{2j+1}{-} (i, j+1) \overset{2j+3}{-} (i_1, j+2) \overset{2j+2}{-} (i', j')$ is rainbow. Otherwise, the path $(i, j) \overset{2j}{-} (i_1, j) \overset{j+j'}{-} (i', j')$ is rainbow.

Case 2: $d_G(i, i') = 2m + 1 \geq 3$.

Choose a path $i - i_1 - \cdots - i_{2m} - i'$ in G. Define the numbers $j_1, j_2, \ldots, j_{2m} \in \{0, 1, \ldots, u-1\}$ as $j_{2k-1} = j + j' + m - k + 1 \pmod{u}$ and $j_{2k} = j - k \pmod{u}$, for each $k \in \{1, 2, \ldots, m\}$. Thus

$$j_1 = j + j' + m, \quad j_3 = j + j' + m - 1, \quad \ldots, \quad j_{2m-1} = j + j' + 1 \tag{6}$$

and

$$j_2 = j - 1, \quad j_4 = j - 2, \quad \ldots, \quad j_{2m} = j - m. \tag{7}$$

Consider the following path,

$$L : (i, j) - (i_1, j_1) - (i_2, j_2) - \cdots (i_{2m}, j_{2m}) - (i', j'). \tag{8}$$

The color sequence of this path is

$$2j + j' + m, 2j + j' + m - 1, 2j + j' + m - 2, \ldots, 2j + j' - m. \tag{9}$$

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The numbers above are $2m + 1$ consecutive integers in decreasing order. Since $2m + 1 \leq u$, these numbers are all different modulo u. So L is a rainbow path.

Case 3: $d_G(i, i') = 2m \geq 4$.
Use the path in Case 2 after deleting the penultimate vertex. $\square$

2.3. Diameter three

The conclusion of Theorem 1 can fail when $diam(G) = 3$ even if we control the number of vertices in each H_i. Below is a class of such examples.

Theorem 2. *If $diam(G) \leq 4$ and some vertex of G is not contained in any triangle, then for any fixed $k \geq 4$ we have*

$$\max_k rc(G[H_1, \ldots, H_n]) = 4 \tag{10}$$

where the maximum is taken over all possible graphs $H_1, \ldots, H_n$ such that each H_i has at least k vertices.

Proof. By Lemma 2, the maximum is at most 4. To build a tight example, choose some $v \in V(G)$ not in any triangle. Replace v with mK_1 where $m > 3^{k \deg(v)}$, and every other vertex with kK_1 to form a G-composition graph A. Then $Q = V(mK_1)$ is co-neighboring and $CN(Q)$ is independent (otherwise v would be in a triangle) so $rc(A) \geq \min\left\{|Q|^{\frac{1}{|CN(Q)|}}, 4\right\} > 3$ by Lemma 1. $\square$

If some additional conditions are met, we do have an exact result.

Theorem 3. *Let G be a connected graph with $diam(G) = 3$, and let $H_1, \ldots, H_n$ be arbitrary graphs with at least three vertices each. Suppose that one of the following holds,*

(1) *each H_i has at least one edge, or*
(2) *G contains a 3-walk between every pair $x, y \in V(G)$ (possibly with $x = y$).*

Then $rc(G[H_1, \ldots, H_n]) = 3$.

Proof. Let $V(G) = \{1, \ldots, n\}$ and $A = G[H_1, \ldots, H_n]$.
First, assume that (1) holds. We will construct a rainbow 3-coloring on A. We start with the following procedure.

(1) For each $i \in \{1, \ldots, n\}$, choose a spanning forest F_i for H_i.
(2) Choose a bipartition $V(F_i) = V_i \cup W_i$ so that $|V_i| \geq 1$ and $|W_i| \geq 2$. This is possible since we assumed $|V(H_i)| \geq 3$.
(3) Put all isolated vertices of H_i (if any) in W_i.
(4) Choose a non-isolated vertex of H_i (which exists, because we assumed $E(H_i) \neq \emptyset$), and put it in W_i. Denote that vertex by s_i.

Now we define a map $\gamma : E(A) \to \{0, 1, 2\}$ as follows.

(1) $\gamma(e) = 0$ for each $e \in E(H_i)$ or $e = xy$ with $x \in V_i \cup \{s_i\}$ and $y \in V_j$.
(2) $\gamma(xy) = 1$ if $x \in W_i \setminus \{s_i\}$ and $y \in V_j$.
(3) $\gamma(xy) = 2$ if $x \in W_i$ and $y \in W_j$.

We show that γ is a rainbow coloring. Let $x, y \in V(A)$ be non-adjacent. Then $x \in V(H_i)$ and $y \in V(H_j)$ for some non-adjacent i, j in G.

Case 1: $d_G(i, j) = 0$ or $d_G(i, j) = 2$.
Choose a walk $i - i_1 - j$ in G.

Subcase 1.1: $x \in V_i$ and $y \in V_j$.
Choose $w_1 \in W_i \cap N(x)$ and $w_2 \in W_{i_1} \setminus \{s_{i_1}\}$. Then $x \overset{0}{-} w_1 \overset{2}{-} w_2 \overset{1}{-} y$ is rainbow.

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Subcase 1.2: $x \in V_i$ and $y \in W_j$.

The path $x \overset{0}{-} s_{i_1} \overset{2}{-} y$ is rainbow.

Subcase 1.3: $x \in W_i$ is not isolated in H_i and $y \in W_j$.

Choose $v \in V_i \cap N(x)$ and $w \in W_{i_1} \setminus \{s_{i_1}\}$. Then $x \overset{0}{-} v \overset{1}{-} w \overset{2}{-} y$ is rainbow.

Subcase 1.4: $x \in W_i$ is isolated in H_i and $y \in W_j$ is isolated in H_j.

Choose adjacent $v \in V_{i_1}, w \in W_{i_1}$. The path $x \overset{1}{-} v \overset{0}{-} w \overset{2}{-} y$ is rainbow.

Case 2: $d_G(i, j) \geq 3$.

Choose a path $i - i_1 - i_2 - j$ in G.

Subcase 2.1: $x \in V_i$ and $y \in V_j$.

Choose $w \in W_{i_2} \setminus \{s_{i_2}\}$. Then the path $x \overset{0}{-} s_{i_1} \overset{2}{-} w \overset{1}{-} y$ is rainbow.

Subcase 2.2: $x \in V_i$ and $y \in W_j$.

Choose $v \in V_{i_1}$ and $w \in W_{i_2} \setminus \{s_{i_2}\}$. Then $x \overset{0}{-} v \overset{1}{-} w \overset{2}{-} y$ is rainbow.

Subcase 2.3: $x \in W_i \setminus \{s_i\}$ and $y \in W_j$.

Choose any $v \in V_{i_1}$. Then $x \overset{1}{-} v \overset{0}{-} s_{i_2} \overset{2}{-} y$ is rainbow.

Subcase 2.4: $x = s_i$ and $y \in W_j$.

Choose $v \in V_{i_1}$ and $w \in W_{i_2} \setminus \{s_{i_2}\}$. Then $x \overset{0}{-} v \overset{1}{-} w \overset{2}{-} y$ is rainbow.

This proves that if (1) holds, then $rc(G[H_1, \ldots, H_n]) = 3$. If (2) holds, we are done by the next lemma which we prove separately for later use. □

Lemma 3. *If G contains a 3-walk between any $x, y \in V(G)$ (possibly with $x = y$), and each H_i has at least 3 vertices, then $rc(G[H_1, \ldots, H_n]) \leq 3$.*

Proof. We will construct a rainbow 3-coloring on $A = G[H_1, \ldots, H_n]$. Let $V(H_i) = \{(i, j) | 1 \leq j \leq n_i\}$. Define a map $\gamma : E(A) \to \{0, 1, 2\}$ arbitrarily on each $E(H_i)$, and put

$$\gamma\left((i, j)(i', j')\right) = j + j' \pmod 3 \tag{11}$$

for all i, i' adjacent in G. We prove that γ is a rainbow coloring.

Let $x, y \in V(A)$ be non-adjacent, say $x = (i, j)$ and $y = (i', j')$ with i, i' non-adjacent in G. Choose any walk $i - i_1 - i_2 - i'$ in G. We use this walk to find a rainbow path between x and y.

If $j \equiv j' \pmod 3$, use $(i, j) \overset{2j+1}{-} (i_1, j+1) \overset{2j+3}{-} (i_2, j+2) \overset{2j+2}{-} (i', j')$. If $j \not\equiv j' \pmod 3$, use $(i, j) \overset{2j}{-} (i_1, j) \overset{j+j'}{-} (i_2, j') \overset{2j'}{-} (i', j')$ as a rainbow path. □

2.4. Diameter two

The rc of a graph can sometimes determine the structure of that graph. For instance, $rc(G) = 1$ if and only if G is a complete graph, while $rc(G) = |E(G)|$ if and only if G is a tree (see [1]). Below, we show that if $diam(G) = 2$ then the largest possible value of $rc(G[H_1, \ldots, H_n])$ determines whether every vertex of G is contained in a triangle or not.

Theorem 4. *If $diam(G) = 2$ and $k \geq 4$, then*

$$\max_k rc(G[H_1, \ldots, H_n]) = \begin{cases} 3, & \text{if every vertex of } G \text{ lies on a triangle} \\ 4, & \text{otherwise.} \end{cases} \tag{12}$$

where the maximum is taken over all possible graphs $H_1, \ldots, H_n$ such that each H_i has at least k vertices.

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Proof. Suppose that every vertex is contained in a triangle. This assumption together with $diam(G) = 2$ implies that G has a 3-walk between any $x, y \in V(G)$ (possibly $x = y$). So by Lemma 3 we have $rc(A) \leq 3$ for any G-composition graph A.

Now we build a tight example. Replace a vertex $v \in V(G)$ with mK_1 where $m > 2^{k \deg(v)}$, and every other vertex with kK_1, to get a G-composition graph A. Then $Q = V(mK_1)$ is co-neighboring and $|CN(Q)| = k \deg(v)$ so $src(A) \geq |Q|^{\frac{1}{|CN(Q)|}} > 2$ by Lemma 1(1), *i.e.* $src(A) \geq 3$. This implies $rc(A) \geq 3$, because any rainbow 2-coloring must also be a strong rainbow coloring.

If some vertex does not lie on any triangle, we are done by Theorem 2. $\square$

3. Concluding remarks and open problems

We have proved $rc(G[H_1, \ldots, H_n]) = diam(G)$ when each H_i has at least $diam(G) \geq 4$ vertices. We have also studied what happens when $diam(G) \leq 3$, but we did not consider the case that some H_i has less than $diam(G)$ vertices.

Theorem 3(2) is a partial converse to Theorem 2 because if G contains a 3-walk between any $x, y \in V(G)$ (possibly with $x = y$), then every vertex is contained in a triangle. These two statements are not equivalent, but it might be interesting to consider whether the weaker statement is enough. If it is, then the conclusion of Theorem 4 will also be true in the case $diam(G) = 3$.

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