AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: Amrita Acharyya & Bikash Das (2020) Actions of cofinite groups on cofinite graphs, AKCE International Journal of Graphs and Combinatorics, 17:1, 198-205, DOI: 10.1016/j.akcej.2019.02.006

To link to this article: https://doi.org/10.1016/j.akcej.2019.02.006

Published online: 09 Jun 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 1, 198–205
https://doi.org/10.1016/j.akcej.2019.02.006

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AKCE International Journal of Graphs and Combinatorics

Actions of cofinite groups on cofinite graphs

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Received 24 December 2017; received in revised form 13 February 2019; accepted 22 February 2019

Abstract

We defined group actions on cofinite graphs to characterize a unique way of uniformly topologizing an abstract group with profinite topology, induced by the cofinite graphs, so that the aforesaid action becomes uniformly equicontinuous.

Keywords: Profinite graph; Cofinite graph; Profinite group; Cofinite group; Equicontinuous group action

1. Introduction

1.1. Topological graphs

Definition 1.1 (*Topological Graphs*)**.** A topological graph [1] is a topological space Γ that is partitioned into two closed subsets $V(\Gamma)$ and $E(\Gamma)$ together with two continuous functions $s, t\colon E(\Gamma) \to V(\Gamma)$ and a continuous function $^{-}\colon E(\Gamma) \to E(\Gamma)$ satisfying the following properties: for every $e \in E(\Gamma)$,

(1) $\overline{e} \neq e$ and $\overline{\overline{e}} = e$;
(2) $t(\overline{e}) = s(e)$ and $s(\overline{e}) = t(e)$.

The elements of $V(\Gamma)$ are called *vertices*. An element $e \in E(\Gamma)$ is called a (*directed*) *edge* with *source* $s(e)$ and *target* $t(e)$; the edge $\overline{e}$ is called the *reverse* or *inverse* of e.

A *map of graphs* $f : \Gamma \to \Delta$ is a function that maps vertices to vertices, edges to edges, and preserves sources, targets, and inverses of edges. Analogously, we will call a map of graphs a *graph isomorphism* if and only if it is a bijection.

An *orientation* of a topological graph Γ is a closed subset $E^{+}(\Gamma)$ consisting of exactly one edge in each pair $\{e, \overline{e}\}$. In this situation, setting $E^{-}(\Gamma) = \{e \in E(\Gamma) \mid \overline{e} \in E^{+}(\Gamma)\}$ we see that $E(\Gamma)$ is a disjoint union of the two closed (hence also open) subsets $E^{+}(\Gamma)$, $E^{-}(\Gamma)$.

Note 1.2. *Let Γ be a topological graph. The following are equivalent:*

Peer review under responsibility of Kalasalingam University.

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https://doi.org/10.1016/j.akcej.2019.02.006

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(1) Γ admits an orientation;
(2) there exists a continuous map of graphs from Γ to the discrete graph with a single vertex and a single edge and its inverse;
(3) there exists a continuous map of graphs $f: \Gamma \to \Delta$ for some discrete graph Δ.

Conceivably there are topological graphs that do not admit closed orientations. However such graphs will not concern us. Therefore, unless otherwise stated, by a topological graph we will henceforth mean a topological graph that admits an orientation.

We will be interested in equivalence relations on graphs that are compatible with the graph structure:

Definition 1.3 (*Compatible Equivalence Relation*)**.** An equivalence relation R on a graph Γ is *compatible* if the following properties hold:

(1) $R = R_V \cup R_E$ where R_V, R_E are equivalence relations on $V(\Gamma)$, $E(\Gamma)$, precisely the restriction of R;
(2) if $(e_1, e_2) \in R$, then $(s(e_1), s(e_2)) \in R$, $(t(e_1), t(e_2)) \in R$, and $(\overline{e}_1, \overline{e}_2) \in R$;
(3) for all $e \in E(\Gamma)$, $(e, \overline{e}) \notin R$;

Note 1.4. *If K is a compatible equivalence relation on Γ, then there is a unique way to make Γ/K into a graph such that the canonical map $\Gamma \to \Gamma/K$ is a map of graphs. It is defined by setting $s(K[e]) = K[s(e)]$, $t(K[e]) = K[t(e)]$, and $\overline{K[e]} = K[\overline{e}]$.*

Conversely, if Δ is a graph and $f: \Gamma \to \Delta$ is a surjective map of graphs, then $K = f^{-1}f = \{(a, b) \in \Gamma \times \Gamma \mid f(a) = f(b)\}$ is a compatible equivalence relation on Γ and f induces an isomorphism of graphs such that $\Gamma/K \cong \Delta$.

Note 1.5. *If R_1 and R_2 are compatible equivalences on Γ, then so is $R_1 \cap R_2$.*

Theorem 1.6. *Let R be any cofinite equivalence relation on a topological graph Γ. Then there exists a compatible cofinite equivalence [2] relation S on Γ such that $S \subseteq R$.*

Proof. Extend the source and target maps $s, t: E(\Gamma) \to V(\Gamma)$ to all of Γ so that they are both the identity map on $V(\Gamma)$. Then $s, t: \Gamma \to \Gamma$ are continuous maps satisfying the following properties:

- $s^2 = s$, $t^2 = t$, $st = t$, and $ts = s$;
- $s(x) = x \iff t(x) = x \iff x \in V(\Gamma)$.

Similarly, extend the edge inversion map $^-: E(\Gamma) \to E(\Gamma)$ to all of Γ by also letting it be the identity map on $V(\Gamma)$. Then $^-: \Gamma \to \Gamma$ is a continuous map satisfying the following conditions for all $x \in \Gamma$:

- $\overline{\overline{x}} = x$;
- $\overline{x} = x \iff x \in V(\Gamma)$;
- $s(\overline{x}) = t(x)$ and $t(\overline{x}) = s(x)$.

Now define $S_1 = \{(x, y) \in \Gamma \times \Gamma \mid (s(x), s(y)) \in R\} = (s \times s)^{-1}[R]$, $S_2 = \{(x, y) \in \Gamma \times \Gamma \mid (t(x), t(y)) \in R\} = (t \times t)^{-1}[R]$, and $S_3 = \{(x, y) \in \Gamma \times \Gamma \mid (\overline{x}, \overline{y}) \in R) = (^- \times ^-)^{-1}[R]$. Then, by the Correspondence Theorem [2], S_1, S_2, S_3 are cofinite equivalence relations on Γ. Let $S_4 = R \cap S_1 \cap S_2 \cap S_3$ and observe that

(i) S_4 is a cofinite equivalence relation on Γ;
(ii) if $(e_1, e_2) \in S_4$, then $(s(e_1), s(e_2)) \in S_4$, $(t(e_1), t(e_2)) \in S_4$, and $(\overline{e}_1, \overline{e}_2) \in S_4$.

Finally, choose a closed orientation $E^+(\Gamma)$ of Γ and form the restrictions $S_V = S_4 \cap [V(\Gamma) \times V(\Gamma)]$, $S_{E^+} = S_4 \cap [E^+(\Gamma) \times E^+(\Gamma)]$, and $S_{E^-} = S_4 \cap [E^-(\Gamma) \times E^-(\Gamma)]$. Then it is easy to check that $S = S_V \cup S_{E^+} \cup S_{E^-}$ is a compatible cofinite equivalence relation on Γ and $S \subseteq R$, as required. □

The previous proof actually shows a little more, which is worth noting. Given a closed orientation $E^+(\Gamma)$ for Γ, we say that a compatible equivalence relation R on Γ is *orientation preserving* if whenever $(e, e') \in R$ and $e \in E^+(\Gamma)$, then also $e' \in E^+(\Gamma)$. Since the equivalence relation S that we constructed in the proof of Theorem 1.6 is also orientation preserving, we proved the following stronger result.

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Corollary 1.7. *Let Γ be a topological graph with a specified closed orientation $E^+(\Gamma)$. Then for any cofinite equivalence relation R on Γ, there exists a compatible orientation preserving cofinite equivalence relation S on Γ such that $S \subseteq R$.*

Corollary 1.8. *If Γ is a compact Hausdorff totally disconnected topological graph, then its compatible cofinite equivalence relations form a fundamental system of entourages for the unique uniform structure that induces the topology of Γ [3].*

1.2. Cofinite graphs

Definition 1.9 (*Cofinite Graph*)**.** A *cofinite graph* [2] is an abstract graph Γ endowed with a Hausdorff uniformity such that the compatible cofinite entourages [2] of Γ form a fundamental system of entourages (i.e. every entourage of Γ contains a compatible cofinite entourage).

A group G is said to act uniformly equicontinuously over a cofinite graph Γ if and only if for each entourage W over Γ there exists an entourage [2], V over Γ such that for all g in G, $(g \times g)[V] \subseteq W$, where $(g \times g)[V] = \{(g \cdot x, g \cdot y) : (x, y) \in V\}$ and $g \cdot x$ is the image of $x \in \Gamma$ under the group action of $g \in G$. In this case the group action induces a (Hausdorff) uniformity over G if and only if the aforesaid action is faithful.

Suppose that G is a group acting faithfully and uniformly equicontinuously on a cofinite graph Γ, then the action $G \times \Gamma \to \Gamma$ is uniformly continuous. Also in that case $\widehat{G}$, the [4] profinite completion of G, acts on $\widehat{\Gamma}$, the [2] profinite of completion of Γ, uniformly equicontinuously. Following is an example of uniform equicontinuous group action.

Example 1.10. Let Γ be an abstract graph with $V(\Gamma) = \{x : x \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers. Let, $E^+(\Gamma) = \{e_x : x \in \mathbb{Z}\}, s(e_x) = x, t(e_x) = x + 1$. Let, $E^-(\Gamma)$ be the set of all edges reversing the edges of $E^+(\Gamma)$, that is $E^-(\Gamma) = \{\overline{e_x} : x \in \mathbb{Z}\}$ and $s(\overline{e_x}) = t(e_x), t(\overline{e_x}) = s(e_x)$. Let p be any prime. Then for any positive integer n, consider Γ_n as the cycle of length p^n. One can say that $V(\Gamma_n) = \{[0]_n, [1]_n, [2]_n ... [p^n - 1]_n\}$, where $[x]_n$ is the congruence class of x modulo p^n and $E^+(\Gamma_n) = \{e_{[x]_n} : x \in V(\Gamma_n)\}, s(e_{[x]_n}) = [x]_n, t(e_{[x]_n}) = [x + 1]_n$. Let $E^-(\Gamma_n)$ be the set of edges reversing the edges in $E^+(\Gamma_n)$, that is $E^-(\Gamma_n) = \{\overline{e_{[x]_n}} : x \in V(\Gamma_n)\}$ and $s(\overline{e_{[x]_n}}) = t(e_{[x]_n}), t(\overline{e_{[x]_n}}) = s(e_{[x]_n})$. Now, consider the map of graphs $q_n : \Gamma \to \Gamma_n$ as $q_n[x] = [x]_n$ and $q_n(e_x) = e_{[x]_n}$. Let, $R_n = Ker\ q_n = \{(\gamma, \delta) \in \Gamma \times \Gamma : q_n(\gamma) = q_n(\delta)\}$. Then R_n is a compatible equivalence relation over Γ [2]and since there is a one-one, onto map of graphs from Γ/R_n to Γ_n, $|\Gamma/R_n| < \infty$. And $I = \{R_n : n \in N\}$ is a fundamental system of entourages over Γ. The corresponding topology induced by I is also Hausdorff, since for any two distinct $\gamma, \delta \in \Gamma$, there exists sufficiently large natural number n so that $R_n[x] \bigcap R_n[y] = \phi$.Thus Γ turns to be a cofinite graph. Consider the additive group of integers $(\mathbb{Z}, +)$ and a natural group action $\mathbb{Z} \times \Gamma \mapsto \Gamma$ by translation of vertices and edges as follows: For any $g \in \mathbb{Z}, x \in V(\Gamma), g.x = g + x$ and for any $e_x \in E^+(\Gamma), g.e_x = e_{g+x}$, for any $\overline{e_x} \in E^-(\Gamma), g.e_x = \overline{e_{g+x}}$. For any entourage U over Γ, as I is a fundamental system of entourage over Γ, there exists $n \in N$ so that $R_n \subseteq U$ and for all $g \in \mathbb{Z}, (g \times g)[R_n] \subseteq R_n$. For if $x, y \in R_n$, without loss of generality let us assume that $x, y \in V(\Gamma)$. So, $[x]_n = [y]_n$ which implies $[g + x]_n = [g + y]_n$ and that implies $(g.x, g.y) \in R_n$. Thus the above action is uniformly equicontinuous.

2. Groups acting on cofinite graphs

Let G be a group and Γ be a cofinite graph. We say that the group G acts over Γ if and only if

(1) For all x in Γ, for all g in G, $g.x$ is in Γ
(2) For all x in Γ, for all g_1, g_2 in G, $g_1.(g_2.x) = (g_1 g_2).x$
(3) For all x in Γ, $1.x = x$, where $1 \in G$ is the identity element of G.
(4) For all v in $V(\Gamma)$, for all g in G, $g.v$ is in $V(\Gamma)$ and for all e in $E(\Gamma)$, for all g in G, $g.e$ is in $E(\Gamma)$.
(5) For all e in $E(\Gamma)$, for all g in G, $g.s(e) = s(g.e), g.t(e) = t(ge)$,
$g.(\overline{e}) = \overline{g.e}$. We say, $s(e), t(e), \overline{e}$ are the source of e, target of e and inversion of e respectively, such that $s(\overline{e}) = t(e), t(\overline{e}) = s(e)$ and $\overline{\overline{e}} = e$.
(6) There exists a $G-$invariant orientation $E^+(\Gamma)$ of Γ.

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Note that the aforesaid group action restricted to {g} can be treated as a well defined map of graphs, $\Gamma \to \Gamma$ taking $x \mapsto g.x$.

Definition 2.1 (*Uniform Equicontinuous Group Action*)**.** A group G is said to act uniformly equicontinuously over a cofinite graph Γ, if and only if for each entourage W over Γ there exists an entourage V over Γ such that for all g in G, $(g \times g)[V]$ is a subset of W.

Example 2.2. Let Γ be an abstract graph with $V(\Gamma) = \{x : x \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers. Let, $E^+(\Gamma) = \{e_x : x \in \mathbb{Z}\}, s(e_x) = x, t(e_x) = x + 1$. Let, $E^-(\Gamma)$ be the set of all edges reversing the edges of $E^+(\Gamma)$, that is $E^-(\Gamma) = \{\overline{e_x} : x \in \mathbb{Z}\}$ and $s(\overline{e_x}) = t(e_x), t(\overline{e_x}) = s(e_x)$. Let $\mathcal{N}$ be a separating filter base [2] of finite index normal subgroups of $(\mathbb{Z}, +)$, the additive group of integers. Then for any subgroup $n\mathbb{Z} \in \mathcal{N}$, consider Γ_n as the cycle of length n. One can say that $V(\Gamma_n) = \{[0]_n, [1]_n, [2]_n ... [n-1]_n\}$, where $[x]_n$ is the congruence class of x modulo n and $E^+(\Gamma_n) = \{e_{[x]_n} : x \in V(\Gamma_n)\}, s(e_{[x]_n}) = [x]_n, t(e_{[x]_n}) = [x+1]_n$. Let $E^-(\Gamma_n)$ be the set of edges reversing the edges in $E^+(\Gamma_n)$, that is $E^-(\Gamma_n) = \{\overline{e_{[x]_n}} : x \in V(\Gamma_n)\}$ and $s(\overline{e_{[x]_n}}) = t(e_{[x]_n}), t(\overline{e_{[x]_n}}) = s(e_{[x]_n})$. Now, consider the map of graphs $q_n : \Gamma \to \Gamma_n$ as $q_n[x] = [x]_n$ and $q_n(e_x) = e_{[x]_n}$. Let, $R_n = Ker\ q_n = \{(\gamma, \delta) \in \Gamma \times \Gamma : q_n(\gamma) = q_n(\delta)\}$. Then R_n is a compatible equivalence relation over Γ [2] and since there is a one-one, onto map of graphs from Γ/R_n to Γ_n, $|\Gamma/R_n| < \infty$. And $I = \{R_n : n\mathbb{Z} \in \mathcal{N}\}$ is a fundamental system of entourages over Γ. The corresponding topology induced by I is also Hausdorff, since for any two distinct $\gamma, \delta \in \Gamma$, there exists sufficiently large natural number n so that $R_n[x] \bigcap R_n[y] = \phi$.Thus Γ turns to be a cofinite graph. Consider the additive group of integers $(\mathbb{Z}, +)$ and a natural group action $\mathbb{Z} \times \Gamma \mapsto \Gamma$ by translation of vertices and edges as follows: For any $g \in \mathbb{Z}, x \in V(\Gamma), g.x = g + x$ and for any $e_x \in E^+(\Gamma), g.e_x = e_{g+x}$, for any $\overline{e_x} \in E^-(\Gamma), g.e_x = \overline{e_{g+x}}$. For any entourage U over Γ, as I is a fundamental system of entourage over Γ, there exists $n\mathbb{Z} \in \mathcal{N}$ so that $R_n \subseteq U$ and for all $g \in \mathbb{Z}, (g \times g)[R_n] \subseteq R_n$. For if $x, y \in R_n$, without loss of generality let us assume that $x, y \in V(\Gamma)$. So, $[x]_n = [y]_n$ which implies $[g+x]_n = [g+y]_n$ and that implies $(g.x, g.y) \in R_n$. Thus the above action is uniformly equicontinuous.

Lemma 2.3. *If a group G acts uniformly equicontinuously over a cofinite graph Γ, then there exists a fundamental system of entourages consisting of G-invariant compatible cofinite entourages over Γ, i.e. for any entourage U over Γ there exists a compatible cofinite entourage R over Γ such that for all $g \in G, (g \times g)[R] \subseteq R \subseteq U$.*

Proof. Let U be any cofinite entourage [2] over Γ. Then as G acts uniformly equicontinuously over Γ, there exists a compatible cofinite entourage S over Γ such that for all $g \in G, (g \times g)[S] \subseteq U$. Choose a G-invariant orientation $E^+(\Gamma)$ of Γ. Without loss of generality, we can assume that our compatible equivalence relation S on Γ is *orientation preserving* i.e. whenever $(e, e') \in R$ and $e \in E^+(\Gamma)$, then also $e' \in E^+(\Gamma)$. Now $S \subseteq \cup_{g \in G}(g \times g)[S] \subseteq U$. Now if $S_0 = \cup_{g \in G}(g \times g)[S]$ and $T = \langle S_0 \rangle$, where $\langle S_0 \rangle$ is the smallest unique equivalence relation on Γ containing S_0, namely, the intersection of all equivalence relations that contains S_0. Note that $S \subseteq T \subseteq U$. Since for all $h \in G, (h \times h)[S_0] = S_0$ and $S_0^{-1} = S_0$ it follows that T is in the transitive closure of S_0. Let $(x, y) \in T$. Then there exists a finite sequence $x_0, x_1, \ldots, x_n$ such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, \ldots, n-1$ and $x = x_0, y = x_n$. Hence $(gx_i, gx_{i+1}) \in S_0$, for all $i = 0, 1, 2, \ldots, n-1$, for all $g \in G$. Thus $(gx_0, gx_n) = (gx, gy) \in T$, for all $g \in G$. Hence for all $g \in G, (g \times g)[T] \subseteq T$ and our claim that T is a G-invariant cofinite entourage, follows. It remains to check that T is compatible. Let $(x, y) \in T$. If $(x, y) \in S_0$, then there is $(t, s) \in S = S_V \cup S_E$ and $g \in G$ such that $(gt, gs) = (x, y)$. Without loss of generality let $(t, s) \in S_V$. Then $(t, s) \in V(\Gamma) \times V(\Gamma)$ which implies that $(x, y) \in T_V$. Now let $(x, y) \in T \setminus S_0$. Then there exists a finite sequence $x_0, x_1, \ldots, x_n$ such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, \ldots, n-1$ and $x = x_0, y = x_n$. Hence by the previous argument if $(x_0, x_1) \in T_V$ then $(x_i, x_{i+1}) \in T_V$, for all $i = 1, 2, \ldots, n-1$. Thus $(x, y) \in T_V$. If $(x_0, x_1) \in T_E$ then $(x_i, x_{i+1}) \in T_E$, for all $i = 1, 2, \ldots, n-1$, which implies $(x, y) \in T_E$. Let $(e_1, e_2) \in T_E$. If $(e_1, e_2) \in S_0$, then there is $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (e_1, e_2)$. Then $(s(p), s(q)) \in S$. So $(s(e_1), s(e_2))$ which equals $(gs(p), gs(q))$ is in $(g \times g)[S] \subseteq S_0$ so that $(s(e_1), s(e_2)) \in T$. Now let $(e_1, e_2) \in T \setminus S_0$. Then there exists a finite sequence $x_0, x_1, \ldots, x_n$ such that $(x_i, x_{i+1}) \in S_0, \forall i = 0, 1, 2, \ldots, n-1$ and $e_1 = x_0, e_2 = x_n$. Hence by the previous argument $(s(x_i), s(x_{i+1})) \in T, \forall i = 0, 1, 2, \ldots, n-1$ and thus $(s(e_1), s(e_2)) \in T$. Similarly, $(t(e_1), t(e_2)) \in T$ and $(\overline{e_1}, \overline{e_2}) \in T$. Finally, to show that for any $e \in E^+(\Gamma), (e, \overline{e}) \notin T$, if possible let $(e, \overline{e}) \in T$. If $(e, \overline{e}) \in S_0$, then there is $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (e, \overline{e})$. Then $\overline{e} = \overline{gp} = g\overline{p} = gq$ which implies that $\overline{p} = q$, so $(p, \overline{p}) \in S$, a contradiction. Now let $(e, \overline{e}) \in T \setminus S_0$. Then there exists a finite sequence $x_0, x_1, \ldots, x_n$ such that $(x_i, x_{i+1}) \in S_0$, for all $i = 0, 1, 2, \ldots, n-1$ and $e = x_0, \overline{e} = x_n$. Now let there be $(p, q) \in S$ and $g \in G$ such that $(gp, gq) = (x_0, x_1)$. Without loss of generality we may assume $(p, q) \in E^+(\Gamma) \times E^+(\Gamma)$. Then $(gp, gq) = (x_0, x_1) \in E^+(\Gamma) \times E^+(\Gamma)$. Hence $(x_i, x_{i+1}) \in E^+(\Gamma) \times E^+(\Gamma)$, for all $i = 1, 2, \ldots, n-1$ which implies that $(e, \overline{e}) \in E^+(\Gamma) \times E^+(\Gamma)$, a contradiction. Our claim follows. □

Note that in reference to Example 2.2, I is in fact a fundamental system of G-invariant compatible cofinite entourages over Γ.

Note 2.4. *Let G be a group and Γ be a cofinite graph. Let S be an equivalence relation over G then $S[g] = \{h \in G : (g, h) \in S\}$ is the equivalence class of $g \in G$. Similarly, if S is an equivalence relation on Γ then $S[\gamma] = \{\rho \in \Gamma : (\gamma, \rho) \in S\}$ is the equivalence class of $\gamma \in G$. Let G act on Γ. Let R be a cofinite entourage. We define $N_R = \{(g_1, g_2) \in G \times G : g_1 R[\gamma] = g_2 R[\gamma], \forall \gamma \in \Gamma\}$, and $N_R[1] = \{g \in G : (1, g) \in N_R\}$, [4]. In the following lemmas we will show that N_R is a congruence of G and $N_R[1]$ is a normal subgroup of G with finite index and we denote it by $N_R[1] \lhd_f G$.*

Lemma 2.5. *$N_R[1]$ is a finite index normal subgroup of G and $G/N_R[1]$ is isomorphic with G/N_R. More generally, if N is a congruence on G, then $N[1]$ is a normal subgroup of G and $G/N[1] \cong G/N$.*

Proof. Let us first see that $N_R[1] \lhd_f G$ for all G-invariant compatible cofinite entourage R over Γ. Let $g, h \in N_R[1]$. This implies $(1, g) \in N_R$ and hence $(g, 1), (1, h) \in N_R$. Thus $(g, h) \in N_R$. This implies $(g.x, h.x)$ is in R, for all $x \in \Gamma$ and so $(x, g^{-1}h.x) \in R$, for all $x \in \Gamma$. Hence, $(1, g^{-1}h)$ is in N_R and thus $g^{-1}h \in N_R[1]$. So, $N_R[1] \leq G$. For all $g \in G$, for all $x \in \Gamma, g.x \in \Gamma$. Hence for all $k \in N_R[1], (x, k.x) \in R$, hence $(k.x, x)$ is in R. Thus $(kg.x, g.x) \in R$ and $(g^{-1}kg.x, g^{-1}g.x) = (g^{-1}kg.x, x) \in R$. Hence $(g^{-1}kg, 1) \in N_R$. So, $g^{-1}kg \in N_R[1]$ and thus $N_R[1] \lhd G$. Now let us define η from $G/N_R[1]$ to G/N_R via $\eta(gN_R[1]) = N_R[g]$. Then, $gN_R[1]$ is equal to $hN_R[1]$ if and only if $h^{-1}g \in N_R[1]$ if and only if $(1, h^{-1}g) \in N_R$ if and only if $(x, h^{-1}g.x) \in R$ if and only if $(h.x, g.x) \in R$ if and only if $(h, g) \in N_R$ if and only if $N_R[h] = N_R[g]$, for all x in Γ. Thus η is a well defined injection and hence $|G/N_R[1]| \leq |G/N_R| < \infty$. Hence $N_R[1] \lhd_f G$. It follows that G/N_R is a group and let us define $\zeta : G/N_R[1] \to G/N_R$ via $\zeta(gN_R[1]) = N_R[g]$. Then for g_1, g_2 in $G, g_1N_R[1] = g_2N_R[1]$ if and only if $g_2^{-1}g_1 \in N_R[1]$ if and only if $(1, g_2^{-1}g_1) \in N_R$ if and only if $(x, g_2^{-1}g_1.x) \in R$ if and only if $(g_2.x, g_1.x) \in R$ if and only if $(g_2, g_1) \in N_R$ if and only if $N_R[g_2]$ equals $N_R[g_1]$. Hence ζ is a well defined injection. Also for all $N_R[g]$ in G/N_R, there exists $gN_R[1] \in G/N_R[1]$ such that $\zeta(gN_R[1]) = N_R[g]$. Thus ζ is surjective as well. Also for $g_1N_R[1], g_2N_R[1] \in G/N_R[1]$, we have $\zeta(g_1N_R[1]g_2N_R[1]) = \zeta(g_1g_2N_R[1])$ and that equals $N_R[g_1g_2]$ which equals $N_R[g_1]N_R[g_2] = \zeta(g_1N_R[1])\zeta(g_2N_R[1])$. Hence ζ is a group homomorphism and thus a group isomorphism. Also, both $G/N_R[1], G/N_R$, are finite discrete topological groups, so ζ is an isomorphism of cofinite groups as well. □

Lemma 2.6. *Let a group G act on a cofinite graph Γ uniformly equicontinuously. Then G acts on Γ/R and G/N_R acts on Γ/R as well, where R is a G-invariant compatible cofinite entourage over Γ and Γ/R is the quotient graph of Γ with respect to R. If I is a fundamental system of G-invariant compatible cofinite entourages over Γ, then $\{N_R \mid R \in I\}$ forms a fundamental system of cofinite congruences [5] for some uniformity over G.*

Proof. Let R be a G-invariant compatible cofinite entourage over Γ. Let us define a group action $G \times \Gamma/R \to \Gamma/R$ via $g.R[x] = R[g.x]$, for all $g \in G$, for all $x \in \Gamma$. Now let $R[x] = R[y]$ so $(x, y) \in R$ which implies that $(g.x, g.y) \in R$. Then $R[g.x] = R[g.y]$. Hence the induced group action is well defined.

Let us now consider the group action $G/N_R \times \Gamma/R \to \Gamma/R$, defined via $N_R[g].R[x] = R[g.x]$, for all $x \in \Gamma$, for all $g \in G$. Now let $(N_R[g], R[x]) = (N_R[h], R[y])$ which implies that $(g, h) \in N_R, (x, y)$ is in R. Then $(g.x, h.x) \in R$, as $h^{-1} \in G, (h^{-1}g.x, h^{-1}h.x) \in R$. So $(h^{-1}g.x, y) \in R$. Thus $(g.x, h.y) \in R$ which implies that $R[g.x]$ equals $R[h.y]$. Hence the induced group action is well defined. Let us now show that N_R is an equivalence relation over G, for all G-invariant compatible cofinite entourage R over Γ.

(1) for all $g \in G$, for all $x \in \Gamma, (g.x, g.x) \in R$. Hence $(g, g) \in N_R$, for all $g \in G$ which implies that $D(G) \subseteq N_R$.

(2) Now $(g, h) \in N_R \Leftrightarrow (g.x, h.x) \in R$, for all $x \in \Gamma$
$\Leftrightarrow (h.x, g.x) \in R$, for all $x \in \Gamma$.
$\Leftrightarrow (h, g) \in N_R$. Thus $N_R^{-1} = N_R$.
(3) Let $(g, h), (h, k) \in N_R$. This implies $(g.x, h.x), (h.x, k.x)$ is in $R, \forall x \in \Gamma$. Hence $(g.x, k.x) \in R$, for all $x \in \Gamma$. So $(g, k) \in N_R$ which implies that $(N_R)^2 \subseteq N_R$.

Also we now check that N_R is a congruence over G. For, let us take $(g_1, g_2), (g_3, g_4) \in N_R$. Then for all $x \in \Gamma, (g_1.x, g_2.x), (g_3.x, g_4.x) \in R$; for all $x \in \Gamma$, $g_3.x \in \Gamma$ and so $(g_1g_3.x, g_2g_3.x) \in R$ and $(g_2g_3.x, g_2g_4.x)$ is in R, since R is G-invariant. Thus $(g_1g_3.x, g_2g_4.x) \in R$, for all $x \in \Gamma$ so that $(g_1g_3, g_2g_4) \in N_R$. Thus our claim follows. Let us now show that G/N_R is finite. Furthermore, define $g\colon \Gamma/R \to \Gamma/R$ as g maps $(R[x])$ into $R[g.x]$. Now, $R[x] = R[y] \iff (x, y) \in R$ if and only if $(g.x, g.y) \in R \iff R[g.x] = R[g.y]$. Hence the map g is a well defined injection. Now for all $R[x] \in \Gamma/R$ there exists $g^{-1}R[x] \in \Gamma/R$ such that $g(g^{-1}R[x])$ equals $R[x]$. Hence $g \in Sym(\Gamma/R)$, where $Sym(\Gamma/R)$ is the collection of all graph isomorphisms from $\Gamma/R \to \Gamma/R$, [2]. Now let us define a map $\theta\colon G/N_R \to Sym(\Gamma/R)$ via $\theta(N_R[g]) = g$. Now $N_R[g_1]$ equals $N_R[g_2]$ if and only if $(g_1, g_2) \in N_R$ if and only if $(g_1.x, g_2.x) \in R$ for all $x \in \Gamma$. Hence $(g_1.x, g_2.x) \in R$ if and only if $R[g_1.x] = R[g_2.x]$ if and only if $g_1(R[x]) = g_2(R[x])$ if and only if $g_1 = g_2$ in $Sym(\Gamma/R)$. Hence θ is a well defined injection. Thus $|G/N_R| \leq |Sym(\Gamma/R)| < \infty$ as $|\Gamma/R| < \infty$. So, next we would like to show that $\{N_R \mid R \in I\}$ forms a fundamental system of cofinite congruences over G.

(1) $D(G) \subseteq N_R$, for all $R \in I$, as N_R is reflexive.
(2) Now for some $R, S \in I, (g_1, g_2) \in N_R \bigcap N_S$ if and only if $(g_1.x, g_2.x) \in R \bigcap S$, for all $x \in \Gamma \Leftrightarrow (g_1, g_2) \in N_{R\bigcap S}$. Thus $N_R \bigcap N_S = N_{R\bigcap S}$.
(3) For all N_R, $N_R^2 = N_R$, as N_R is transitive.
(4) For all N_R, $N_R^{-1} = N_R$, as N_R is symmetric.

Hence our claim follows. □

Note 2.7. *Let us refer back to Example 2.2 and define a group action $\mathbb{Z} \times \Gamma_n \mapsto \Gamma_n$ as following $g.[x]_n = [g+x]_n$, for any $x \in V(\Gamma_n)$, $g.e_{[x]_n} = e_{[g+x]_n}$, $g.\overline{e_{[x]_n}} = \overline{e_{g+x}}$, for any $\overline{e_x} \in E^-(\Gamma_n)$. Thus for any n, where $n\mathbb{Z} \in \mathcal{N}$, $\mathbb{Z}/N_{R_n}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.*

Definition 2.8. We say a group G acts on a cofinite graph Γ faithfully, if for all g in $G \setminus \{1\}$ there exists x in Γ such that gx is not equal to x in Γ.

Lemma 2.9. *The induced uniform topology over G as in Lemma 2.6 is Hausdorff if and only if G acts faithfully over Γ.*

Proof. Let us first assume that G acts faithfully over Γ. Now let $g \neq h$ in G. Then $h^{-1}g \neq 1$. So there exists $x \in \Gamma$ such that $h^{-1}g.x \neq x$ implying that $g.x \neq h.x$. Then there exists a G-invariant compatible cofinite entourage R over Γ such that $(g.x, h.x) \notin R$, as Γ is Hausdorff. Hence $(g, h) \notin N_R$. Thus G is Hausdorff.

Conversely, let us assume that G is Hausdorff and let $g \neq 1$ in G. Then there exists some G-invariant compatible cofinite entourage R over Γ such that $(1, g) \notin N_R$. Hence there exists $x \in \Gamma$ such that $(x, g.x) \notin R$. Hence $R[x] \neq R[g.x]$ so that $x \neq g.x$. Our claim follows. □

Lemma 2.10. *Suppose that G is a group acting uniformly equicontinuously on a cofinite graph Γ and give G the induced uniformity as in Lemma 2.6. Then the action $G \times \Gamma \to \Gamma$ is uniformly continuous.*

Proof. Let R be a G-invariant cofinite entourage over Γ. If I is a fundamental system of G-invariant compatible cofinite entourages over Γ. Then $\{N_R \times R : R \in I\}$ ia a fundamental system of entourage for a uniform structure over $G \times \Gamma$, [2]. Now let $((g, x), (h, y)) \in N_R \times R$, i.e. $(g, h) \in N_R$, $(x, y) \in R$. Now x in Γ and $(gx, hx) \in R$ this implies $(h^{-1}gx, x) \in R$. We have $(h^{-1}gx, y) \in R$ and hence $(gx, hy) \in R$. Thus our claim. □

Let us define a directed order '$\leq$' on I, a fundamental system of G-invariant entourages on a cofinite graph Γ as in Lemma 2.6. We say, $R \leq S$ in I, then $S \subseteq R$. Let $(g_1, g_2) \in N_S$. Then $(g_1x, g_2x) \in S$, for all $x \in \Gamma$ and

Please cite this article as: A. Acharyya and B. Das, Actions of cofinite groups on cofinite graphs, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.02.006.

hence $(g_1x, g_2x) \in R$, for all $x \in \Gamma$ which implies $(g_1, g_2) \in N_R$. Thus $N_S \subseteq N_R$. For all $R \leq S$, in I, let us define $\psi_{RS}\colon G/N_S \to G/N_R$ via $\psi_{RS}(N_S[g]) = N_R[g]$. Then ψ_{RS} is a well defined uniformly continuous group isomorphism, as each of G/N_R, G/N_S is finite discrete groups. If $R = S$, then $\psi_{RR} = id_{G/N_R}$. And if $R \leq S \leq T$, then $\psi_{RS}\psi_{ST} = \psi_{RT}$. Then $\{G/N_R \mid R \in I, \psi_{RS}, R \leq S \in I\}$, forms an inverse system of finite discrete groups. Let $\widehat{\Gamma} = \varprojlim_{R\in I} \Gamma/R$ and $\widehat{G} = \varprojlim_{R\in I} G/N_R$, where $\psi_R\colon \widehat{G} \to G/N_R$ is the corresponding canonical projection map, [2]. Now if I_1, I_2 are two fundamental systems of G-invariant cofinite entourages over Γ, clearly I_1, I_2 will form fundamental systems of cofinite congruences, for two induced uniformities, over G. Now let N_{R_1} be a cofinite congruence over G for some $R_1 \in I_1$. Then there exists a R_2, cofinite entourage over Γ, such that $R_2 \in I_2$ and $R_2 \subseteq R_1$. Hence $N_{R_2} \subseteq N_{R_1}$. Now let N_{S_2} be a cofinite congruence over G for some $S_2 \in I_2$. Then there exists S_1, cofinite entourage over Γ, such that $S_1 \in I_1$ and $S_1 \subseteq S_2$. Hence $N_{S_1} \subseteq N_{S_2}$. Thus any cofinite congruence corresponding to the directed set I_1 is a cofinite congruence corresponding to the directed set I_2 and vice versa. Thus the two induced uniform structures over G are equivalent and so the completion of G with respect to the induced uniformity, from the cofinite graph Γ, is unique up to both algebraic and topological isomorphism.

Theorem 2.11. *If G acts on Γ, as in Lemma 2.6, faithfully then $\widehat{G}$ acts on $\widehat{\Gamma}$ uniformly equicontinuously.*

Proof. Let a group G act on Γ uniformly equicontinuously. We fix a G-invariant orientation $E^+(\Gamma)$ of Γ. By Lemma 2.10 the action is uniformly continuous as well. Let $\chi\colon G \times \Gamma \to \Gamma$ be this group action. Now since Γ is topologically embedded in $\widehat{\Gamma}$ by the inclusion map, say, i, the map $i \circ \chi\colon G \times \Gamma \to \widehat{\Gamma}$ is a uniformly continuous. Then there exists a unique uniformly continuous map $\widehat{\chi}\colon \widehat{G} \times \widehat{\Gamma} \to \widehat{\Gamma}$ that extends χ. We claim that $\widehat{\chi}$ is the required group action. We can take $\widehat{\Gamma} = \varprojlim \Gamma/R$ and $\widehat{G} = \varprojlim G/N_R$, where R runs throughout all G-invariant compatible cofinite entourages of Γ that are orientation preserving. Then $\widehat{G} \times \widehat{\Gamma} = \varprojlim(G/N_R \times \Gamma/R) \cong \varprojlim(G/N_R \times \Gamma/R)$ [6] and we define a group action of G over Γ coordinatewise as follows $(N_R[g_R])_R.(R[x_R])_R = (R[g_R.x_R])_R$. If possible let, $((N_R[g_R])_R, (R[x_R])_R) = ((N_R[h_R])_R, (R[y_R])_R)$. So, $N_R[g_R]$ equals $N_R[h_R]$ and $R[x_R] = R[y_R], \forall R \in I, (g_R, h_R) \in N_R$ and $(x_R, y_R) \in R$. This implies that $(g_R.x_R, h_R.x_R) \in R$ which further ensures that $(h_R^{-1}g_R.x_R, x_R) \in R$. Then $(h_R^{-1}g_R.x_R, y_R) \in R$ and $(g_R.x_R, h_R.y_R) \in R$. Hence $(R[g_R.x_R])_R = (R[h_R.y_R])_R$. So, the action is well defined. Let $g = (N_R[g_R])_R$ and $h = (N_R[h_R])_R$ in $\widehat{G}$, $x = (R[x_R])_R \in \widehat{\Gamma}$. Now $h.(g.x) = h.(R[g_R.x_R])_R = (R[h_Rg_R.x_R])_R$ which then equals $(N_R[h_Rg_R])_R.x = (hg).x$. Hence the action is associative. Now $(N_R[1])_R.(R[x_R])_R = (R[1x_R])_R = (R[x_R])_R$. Furthermore for all vertex $v = (R[v_R])_R \in V(\widehat{\Gamma})$ and for all $g = (N_R[g_R])_R \in \widehat{G}$ one can say that $g.v = (R[g_R.v_R])_R \in V(\widehat{\Gamma})$ as each $g_R.v_R \in V(\Gamma)$. Similarly, for all $e = (R[e_R])_R$ in $E(\widehat{\Gamma})$ and for all $g = (N_R[g_R])_R$ in $\widehat{G}$, $g.e = (R[g_Re_R])_R$ in $E(\widehat{\Gamma})$. For all $e = (R[e_R])_R$ in $E(\widehat{\Gamma})$, for all $g = (N_R[g_R])_R$ in $\widehat{G}$, we have $s(g.e) = s((R[g_Re_R])_R)$ and so $(R[g_Rs(e_R)])_R$ equals $(g.(R[s(e_R)])_R)$ and that equals $g.s(e)$. Hence the properties $t(g.e) = g.t(e)$ and $\overline{g.e} = g.\overline{e}$ follow similarly. Finally, let $E^+(\widehat{\Gamma})$ consist of all the edges $(R[e_R])_R$, where $e_R \in E^+(\Gamma)$. Since each R is orientation preserving, it follows that $E^+(\widehat{\Gamma})$ is an orientation of $\widehat{\Gamma}$. Since $E^+(\Gamma)$ is G-invariant, we see that $E^+(\widehat{\Gamma})$ is $\widehat{\Gamma}$-invariant. Hence this is a well defined group action. Also for all $g \in G$, and $x \in \Gamma$, $(N_R[g])_R.(R[x])_R$ equals $(R[g.x])_R$ which equals $g.x$ in Γ, (please see [6], for any further clarification on how to embed G in $\widehat{G}$ and Γ in $\widehat{\Gamma}$. We use the notations $(N_R[g])_R$ and $(R[x])_R$, $R[gx]_R$ to refer to the Rth coordinates of g and x, gx in $\widehat{G}$ and $\widehat{\Gamma}$, respectively). Thus the restriction of this group action agrees with the group action χ. Now $\{R \mid R \in I\}$ is a fundamental system of cofinite entourages over Γ, and $\{N_R \mid R \in I\}$ is a fundamental system of cofinite congruences over G. Hence $\{\overline{R} \mid R \in I\}$ is a fundamental system of cofinite entourages over $\widehat{\Gamma}$ and $\{\overline{N_R} \mid R \in I\}$ is a fundamental system of cofinite congruences over $\widehat{G}$ respectively, where $\overline{R}$ is the topological closure of R in $\Gamma \times \Gamma$. Let us now see that the aforesaid group action is uniformly continuous. For let us consider the group action $G/N_R \times \Gamma/R \to \Gamma/R$ defined via $N_R[g]R[x] = R[g.x]$, which is uniformly continuous as both $G/N_R \times \Gamma/R$ and Γ/R are finite discrete uniform topological spaces. Hence the group action, $\widehat{G} \times \widehat{\Gamma} \to \widehat{\Gamma}$ is uniformly continuous. Thus the aforesaid group action is our choice of $\widehat{\chi}$, by the uniqueness of $\widehat{\chi}$, [2]. So the restriction of the aforesaid action $\{\widehat{g}\} \times \widehat{\Gamma} \to \widehat{\Gamma}$ is a uniformly continuous map of graphs, for all $\widehat{g} \in \widehat{G}$. We check that for all $(x, y) \in R$ and for all $\widehat{g} \in \widehat{G}$ the ordered pair $(\widehat{g}.x, \widehat{g}.y) \in \overline{R}$. For, let $\widehat{g} = (N_R[g_R])_R \in \widehat{G}$ and for $x, y \in \Gamma, ((R[x])_R, (R[y])_R) \in R$. Now $\overline{R}[(R[g_R.x])_R] = \overline{R}[g_R.x] = \overline{R}[g_R.y] = \overline{R}[(R[g_R.y])_R]$. So, $((N_R[g_R])_R \cdot (R[x])_R, (N_R[g_R])_R \cdot (R[y])_R) \in \overline{R}$. This implies $(\widehat{g} \times \widehat{g})[R]$ is a subset of $\overline{R}$. Thus for all $\widehat{g} \in \widehat{G}$ we observe that $(\widehat{g} \times \widehat{g})[\overline{R}]$ is a subset of $\overline{\widehat{g} \times \widehat{g}[R]}$ which is a subset of $\overline{\overline{R}} = \overline{R}$. Hence $\overline{R}$ is $\widehat{G}$ invariant. $\square$

Please cite this article as: A. Acharyya and B. Das, Actions of cofinite groups on cofinite graphs, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.02.006.

Thus $\Phi_1 = \{N_{\overline{R}} \mid R \in I\}$ and $\Phi_2 = \{\overline{N_R} \mid R \in I\}$ form fundamental systems of cofinite congruences over $\widehat{G}$. Let $\tau_{\Phi_1}, \tau_{\Phi_2}$ be the topologies induced by Φ_1, Φ_2 respectively.

Theorem 2.12. *The uniformities on $\widehat{G}$ obtained by Φ_1 and Φ_2 are equivalent.*

Proof. Let us first show that $N_{\overline{R}} \cap G \times G = N_R$. For, let $(g, h) \in N_R$. Then for all $x \in \Gamma, (g.x, h.x) \in R \subseteq \overline{R}$. Now let $(R[x_R])_R \in \widehat{\Gamma}$. Then $\overline{R}[g(R[x_R])_R] = \overline{R}[g.x_R] = \overline{R}[h.x_R] = \overline{R}[h(R[x_R])_R]$ which implies that $(g, h) \in N_{\overline{R}} \cap G \times G$. Thus, $N_R \subseteq N_{\overline{R}} \cap G \times G$. Again, if (g, h) belongs to $N_{\overline{R}} \cap G \times G$, then for all $x \in \Gamma \subseteq \widehat{\Gamma}$, and so $(g.x, h.x) \in \overline{R} \cap \Gamma \times \Gamma = R$ and this implies $(g, h) \in N_R$. Our claim follows. Then as uniform subgroups $(G, \tau_{\Phi_1}) \cong (G, \tau_{\Phi_2})$, both algebraically and topologically, their corresponding completions $(\widehat{G}, \tau_{\Phi_1}) \cong (\widehat{G}, \tau_{\Phi_2})$, both algebraically and topologically. Since for all $S \in I$, $\psi_S \colon G \to G/N_S$ is a uniform continuous group homomorphism and G/N_S is discrete, there exists a unique uniform continuous extension of ψ_S, namely, $\widehat{\psi_S} \colon \widehat{G} \to G/N_S$. Let us define $\lambda_S \colon \widehat{G} \to G/N_S$ via $\lambda_S(g) = N_S[g_S]$, where $g = (N_R[g_R])_R$, [6]. Now let $g = (N_R[g_R])_R, h = (N_R[h_R])_R \in \widehat{G}$ be such that $g = h$ which implies that $N_S[g_S] = N_S[h_S]$ and hence λ_S is well defined. Now let $(g, h) \in N_{\overline{S}}$. First of all $N_{\overline{S}}[g_S] = N_{\overline{S}}[g] = N_{\overline{S}}[h] = N_{\overline{S}}[h_S]$. So, $(g_S, h_S) \in N_{\overline{S}} \bigcap G \times G = N_S$. Hence $N_S[g_S] = N_S[h_S]$ which implies that $\lambda_S(g) = \lambda_S(h)$, so $(\lambda_S(g), \lambda_S(h)) \in D(G/N_R)$. Thus $N_{\overline{S}}$ is a subset of $(\lambda_S \times \lambda_S)^{-1} D(G/N_R)$. Hence λ_S is uniformly continuous. Now for all $g, h \in \widehat{G}, \lambda_S(gh) = N_S[g_S h_S] = N_S[g_S] N_S[h_S] = \lambda_S(g)\lambda_S(h)$ and for all $g \in G, \lambda_S(g) = \lambda_S((N_R[g])_R) = N_S[g] = \psi_S(g)$. Thus λ_S is a well defined uniformly continuous group homomorphism that extends ψ_S. Then by the uniqueness of the extension, $\widehat{\psi_S} = \lambda_S$. Now $N_{\overline{S}}$ is a closed subspace of $\widehat{G}$, then $\overline{N_{\overline{S}} \cap G \times G} = \overline{N_S}$ which implies that $\overline{N_S}$ is a subset of $\overline{N_{\overline{S}}}$ which equals $N_{\overline{S}}$. Let us define θ from $\widehat{G}/N_{\overline{S}}$ to G/N_S as θ takes $N_{\overline{S}}[g]$ into $N_S[g_S]$, where $g = (N_R[g_R])_R$. Now $N_{\overline{S}}[g] = N_{\overline{S}}[h]$ in $\widehat{G}/N_{\overline{S}}$ will imply (g_S, h_S) is in $N_{\overline{S}}$ and this implies for all x in Γ the ordered pair $(g_S x, h_S x)$ is in $\overline{S} \bigcap \Gamma \times \Gamma$ which is eventually equal to S. Thus $(g_S, h_S) \in N_S$. Then $\theta(N_{\overline{S}}[g]) = N_S[g_S]$ which is equal to $N_S[h_S]$ and that equals $\theta(N_{\overline{S}}[h])$. Hence θ is well defined. On the other hand let $N_{\overline{S}}[g]$, $N_{\overline{S}}[h]$ be such that $\theta(N_{\overline{S}}[g])$ equals $\theta(N_{\overline{S}}[h])$. Thus $N_S[g_S] = N_S[h_S]$ implies that $(g_S, h_S) \in N_S \subseteq N_{\overline{S}}$. Hence $N_{\overline{S}}[g] = N_{\overline{S}}[g_S] = N_{\overline{S}}[h_S] = N_{\overline{S}}[h]$. So, θ is injective as well. Also for all $N_S[g] \in G/N_S$ there exists $N_{\overline{S}}[g] \in \widehat{G}/N_{\overline{S}}$ such that $\theta(N_{\overline{S}}[g]) = N_S[g]$. So θ is surjective. Finally, $\theta(N_{\overline{S}}[g] N_{\overline{S}}[h])$ equals $\theta(N_{\overline{S}}[gh])$ and that equals $N_S[g_S h_S]$ which is $N_S[g_S] N_S[h_S]$ and finally that equals $\theta(N_{\overline{S}}[g])\theta(N_{\overline{S}}[h])$. So θ is a well defined group isomorphism, both algebraically and topologically. Hence $\widehat{G}/N_{\overline{S}} \cong G/N_S \cong \widehat{G}/\overline{N_S}$ which implies that $\left|\widehat{G}/N_{\overline{S}}[1]\right|$ is equal to $\left|\widehat{G}/\overline{N_S}[1]\right|$. But since $\overline{N_S} \subseteq N_{\overline{S}}$ one obtains $\overline{N_S}[1] \leq N_{\overline{S}}[1] \leq \widehat{G}$ and thus $\left|\widehat{G}/N_{\overline{S}}[1]\right| \left|N_{\overline{S}}[1] : \overline{N_S}[1]\right|$ equals $\left|\widehat{G}/\overline{N_S}[1]\right|$. Hence $\left|N_{\overline{S}}[1] : \overline{N_S}[1]\right| = 1$ which implies that $N_{\overline{S}}[1] = \overline{N_S}[1]$ and thus $N_{\overline{S}} = \overline{N_S}$ as each of them is congruences. Thus our claim. □

Note 2.13. *Thus referring back to Example 2.2, the action $\mathbb{Z} \times \Gamma \mapsto \Gamma$ has a unique uniform equicontinuous extension from $\widehat{\mathbb{Z}} \times \widehat{\Gamma} \mapsto \widehat{\Gamma}$, where $\widehat{\Gamma} = \varprojlim \Gamma/R_n, \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N_{R_n}$ are the respective profinite completions of Γ and $\mathbb{Z}$.*

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Please cite this article as: A. Acharyya and B. Das, Actions of cofinite groups on cofinite graphs, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.02.006.