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ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: Kiyoshi Hosono & Masatsugu Urabe (2020) Specified holes with pairwise disjoint interiors in planar point sets, AKCE International Journal of Graphs and Combinatorics, 17:1, 7-15, DOI: 10.1016/j.akcej.2018.08.003

To link to this article: https://doi.org/10.1016/j.akcej.2018.08.003

Published online: 01 Jul 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 1, 7–15
https://doi.org/10.1016/j.akcej.2018.08.003

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Specified holes with pairwise disjoint interiors in planar point sets

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Received 1 March 2018; accepted 14 August 2018

Abstract

A k-hole of a planar point set in general position is a convex k-gon whose vertices are elements of the set and whose interior contains no elements of the set. We discuss the minimum size of a point set that contains specified holes with disjoint interiors.

Keywords: The Erdős–Szekeres theorem; Empty convex polygons; Disjoint holes

1. Introduction

In 1935, Erdős and Szekeres [1] stated that for every integer $t \geq 3$ there is a smallest number $f(t)$ such that every set of at least $f(t)$ points in general position in the plane, contains a subset of points that are the vertices of a convex t-gon. The exact value of $f(t)$ is a long standing open problem. A construction due to Erdős and Szekeres [2] shows that $f(t) \geq 2^{t-2}+1$, which is also conjectured to be sharp. It is known that $f(4) = 5$, $f(5) = 9$ [3] and $f(6) = 17$ [4]. The best known upper bound is due to Tóth and Valtr [5], $f(t) \leq \binom{2t-5}{t-3} + 1$. For a more detailed description of the Erdős and Szekeres theorem and its many ramifications, see the surveys by Bárány and Károlyi [6] and Morris and Soltan [7].

Erdős [8] also asked the following combinatorial geometry problem in 1979: Find the smallest integer $n(k)$ such that any set of $n(k)$ points in general position in the plane, contains the vertices of a convex k-gon, whose interior contains no points of the set. Such a subset is called an *empty convex k-gon* or a *k-hole* of the set. Klein [1] found $n(4) = 5$, and $n(5) = 10$ was determined by Harborth [9]. Horton [10] constructed arbitrarily large point sets which do not contain any 7-holes, so $n(k)$ does not exist for $k \geq 7$. For the remaining case of $n(6)$, Overmars exhibited a set of 29 points, the largest known, with no empty convex hexagons [11]. About 10 years ago, the existence of $n(6)$ was proved by Gerken [12] and independently by Nicolás [13]. Later, Valtr [14] gave a similar version of Gerken's proof. And recently, Koshelev improved the upper bound to $n(6) \leq 463$ [15]. Therefore the current record of $n(6)$ is $30 \leq n(6) \leq 463$.

Peer review under responsibility of Kalasalingam University.
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https://doi.org/10.1016/j.akcej.2018.08.003

Please cite this article in press as: K. Hosono, M. Urabe, Specified holes with pairwise disjoint interiors in planar point sets, AKCE International Journal of Graphs and Combinatorics (2018), https://doi.org/10.1016/j.akcej.2018.08.003.

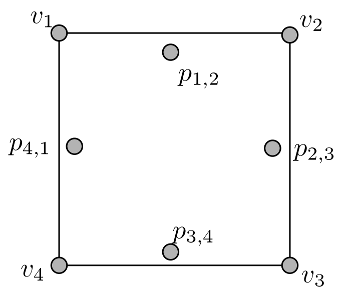

Fig. 1(a). $n(4, 4) \geq 9$.

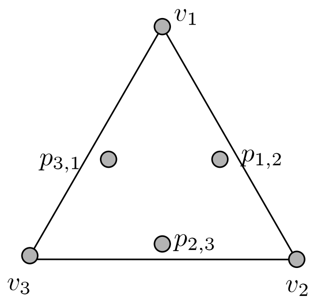

Fig. 1(b). $m(4, 4) \geq 7$.

A pair of holes is said to be *disjoint* if their convex hulls do not intersect. We denote $n(k, l)$ for $k \leq l$ by the smallest integer such that any set of $n(k, l)$ points in general position in the plane, contains both a k-hole and an l-hole that are disjoint. Clearly, $n(3, 3) = 6$ and Horton's result implies that $n(k, l)$ does not exist for all $l \geq 7$. For this function, we showed $n(3, 4) = 7$ [16] and $n(4, 4) = 9$ in [17], and also determined $n(3, 5) = 10$, $12 \leq n(4, 5) \leq 13$ and $17 \leq n(5, 5) \leq 20$ in [18,19]. Bhattacharya and Das [20] later tightened to $n(4, 5) = 12$ and also improved the upper bound of $n(5, 5)$ to 19 [21].

In [19], we considered several problems for disjoint holes. Let $n(k_1, k_2, \ldots, k_l)$ be the smallest integer such that any set of $n(k_1, \ldots, k_l)$ points contains a k_i-hole for each i, $1 \leq i \leq l$, where the holes are pairwise disjoint. We showed that $n(2, 3, 4) = 9$, $n(2, 3, 5) = 11$, $n(3, 4, 4) = 12$, $n(4, 4, 4) = 14$, $15 \leq n(4, 4, 5) \leq 17$ and more. In particular, any set of 15 points in general position in the plane is partitioned into a 1-hole, a 2-hole, a 3-hole, a 4-hole and a 5-hole which are pairwise disjoint, that is $n(1, 2, 3, 4, 5) = 15$.

In this paper, the related problem is considered as follows. A family of holes is *with disjoint interiors* if their interiors are pairwise disjoint. We define $m(k, l)$ for $k \leq l$ by the smallest integer such that any set of $m(k, l)$ points in general position in the plane contains both a k-hole and an l-hole with disjoint interiors. Clearly, $m(k, l) \leq n(k, l)$ holds for any k, l, and also $m(k, l)$ does not exist for all $l \geq 7$ by Horton's result.

For example, an 8-point set in Fig. 1(a) does not contain two disjoint 4-hole, implying that $n(4, 4) \geq 9$. However, it contains two holes with disjoint interiors, formed by $\{v_1, p_{1,2}, p_{3,4}, p_{4,1}\}$ and $\{v_2, p_{2,3}, p_{3,4}, p_{1,2}\}$. And Fig. 1(b) shows $m(4, 4) \geq 7$, that is, this 6-point set has no two 4-holes with disjoint interiors. We discuss two specified holes in Section 3 and three specified holes in Section 4.

2. Preliminaries

2.1. Notations and definitions

We first give notations and definitions used in the proofs. Throughout this work, we consider only planar point sets in general position. For such a point set P, we distinguish the vertices $V(P)$ on the convex hull boundary from the remaining *interior points* $I(P)$. Let $V(P) = \{v_1, v_2, \ldots, v_t\}$ in clockwise order. We remark that when indexing a

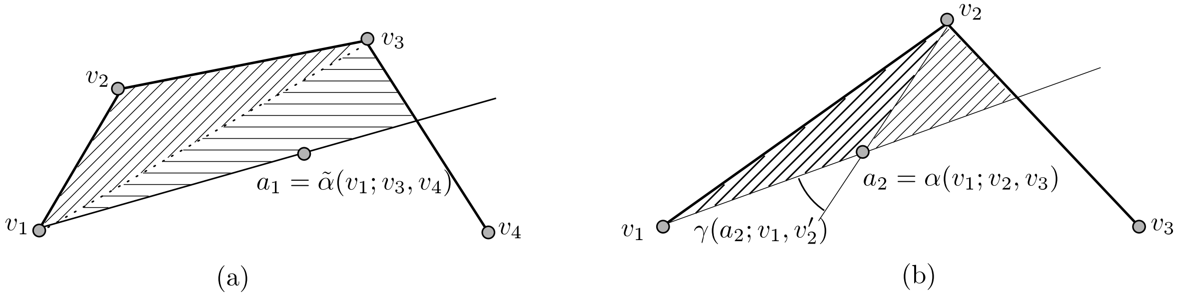

Fig. 2. Shaded areas are empty.

set of t points, we identify indices modulo t. If $I(P) = \emptyset$, the convex t-gon formed by $V(P)$ is called *empty*. A line segment $\overline{v_i v_{i+1}}$ is simply called an *edge* of $V(P)$. Let R be a closed region in the plane. A point of P in the interior of R is generally said to be an *interior point* of R, and R is *empty* if it contains no interior points.

We denote the closed *convex cone* by $\gamma(a; b, c)$ or $\gamma(a; c, b)$ such that a is the apex and b and c lie on the boundary. If $\gamma(a; b, c)$ is not empty, we define an *attack point* $\alpha(a; b, c)$ as the interior point of $\gamma(a; b, c)$ such that $\gamma(a; b, \alpha(a; b, c))$ is empty. A *quasi-attack point* $\tilde{\alpha}(a; b, c)$ in $\gamma(a; b, c)$ is the point c or $\alpha(a; b, c)$ if $\gamma(a; b, c)$ is empty or not, respectively. For $\delta = b$ or c of $\gamma(a; b, c)$, let δ' be a point collinear with a and δ so that a lies on the line segment $\overline{\delta\delta'}$. Then we can consider a convex cone of $\gamma(a; b, c')$ or $\gamma(a; b', c)$.

Let $l(a, b)$ be the line through a and b. Denote the closed half-plane bounded by the line $l(a, b)$ that contains c or does not contain c by $H(ab; c)$ or $H(ab; \overline{c})$, respectively. For any elements a, b, c of P, we let P_1 or P_2 be a subset of P on $H(ab; c)$ or $H(ab; \overline{c})$, respectively, where $P_1 \cap P_2 = \{a, b\}$. Then we say that the *cutting line* $l(a, b)$ divides P into P_1 and P_2.

An interior point $p_{i,i+1}$ of P is said to be a *friend* to the edge $\overline{v_i v_{i+1}}$ of $V(P)$ if $\gamma(v_i; v_{i+1}, p_{i,i+1}) \cup \gamma(v_{i+1}; v_i, p_{i,i+1})$ is empty, e.g. Fig. 1(a). We represent a k-hole H by $H = (v_1 \cdots v_k)_k$ if $V(H) = \{v_1, \ldots, v_k\}$ is in clockwise order.

2.2. Lemmas

We now present two lemmas used throughout the paper. Let P be a set of n points for $n \geq 4$, and $V(P) = \{v_1, \ldots, v_t\}$ in clockwise order.

Lemma 1. *If there exists an edge of $V(P)$ with no friend, then we have a cutting line which divides P into a 4-hole and the remaining $n - 2$ points.*

Proof. We consider any edge of $V(P)$, say $\overline{v_1 v_2}$. First, if $\triangle v_1 v_2 v_3$ is empty, then $\overline{v_1 v_2}$ has no friend. For the quasi-attack point $a_1 = \tilde{\alpha}(v_1; v_3, v_4)$, there exists a cutting line $l(v_1, a_1)$ which divides P into a 4-hole of $(v_1 v_2 v_3 a_1)_4$ and the remaining $n - 2$ points, see Fig. 2(a).

If $\triangle v_1 v_2 v_3$ is not empty, there is an attack point $a_2 = \alpha(v_1; v_2, v_3)$, see Fig. 2(b). We remark that if the convex cone $\gamma(a_2; v_1, v_2')$ is empty, then a_2 is the friend to the edge $\overline{v_1 v_2}$. Otherwise, for $a_3 = \alpha(a_2; v_1, v_2')$, there exists a cutting line $l(a_2, a_3)$ which divides P into a 4-hole of $(v_1 v_2 a_2 a_3)_4$ and the $n - 2$ points. □

We remark that for any i, a friend $p_{i,i+1}$ must lie in $\gamma(v_i; v_{i+1}, v_{i+2}) \cap \gamma(v_{i+1}; v_i, v_{i-1})$. Thus, any pair of consecutive edges does not have a common friend except for the case in which $|P| = 4$ and $|V(P)| = 3$. If $|V(P)| > |I(P)|$, then there exists an edge of $V(P)$ having no friend. Therefore we give the next lemma.

Lemma 2. *If $|V(P)| > |I(P)|$, then we have a cutting line which divides P into a 4-hole and the remaining $n - 2$ points.*

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3. Two holes with disjoint interiors

In this section, we discuss values of $m(k, l)$, that is we consider two holes with disjoint interiors. If $k = 3$, then the values are easily shown. For example, any set of four points has a 3-hole of $T = (abc)_3$ and the remaining point p. Since p can *see* some edge of T, say $\overline{ab}$, we obtain another 3-hole of $(abp)_3$ such that these two holes are with disjoint interiors. Thus,

Proposition 1. $m(3, 3) = 4$. □

Using $n(4) = 5$, any set of five points has a 4-hole. The remaining point can also see some edge of the 4-hole. Thus,

Proposition 2. $m(3, 4) = 5$. □

The next result is clearly shown by $n(5) = 10$.

Proposition 3. $m(3, 5) = 10$. □

The next result shows that a set of seven points has two 4-holes with disjoint interiors, and the value is tight.

Theorem 1. $m(4, 4) = 7$.

Proof. Fig. 1(b) shows the lower bound of $m(4, 4) \geq 7$. To prove the upper bound, let P be a set of seven points. If $|V(P)| \geq 4$, there exists a cutting line dividing P into a 4-hole and the remaining 5-point set S by Lemma 2. Then we can find another 4-hole of S using $n(4) = 5$.

The remaining case is for $|V(P)| = 3$. If there is an edge having no friend, we have a desired cutting line dividing into a 4-hole and the 5-point set containing another 4-hole by Lemma 1. Otherwise, there are three friends $p_{i,i+1}$ to each edge $\overline{v_i, v_{i+1}}$ of $V(P) = \{v_1, v_2, v_3\}$. Denote $T_i = \triangle p_{i-1,i} v_i p_{i,i+1}$ for $i = 1, 2, 3$. If the remaining point p lies in some T_i, say T_2, we obtain two 4-holes of $(pp_{3,1}v_1p_{1,2})_4$ and $(pp_{2,3}v_3p_{3,1})_4$ with disjoint interiors. If p lies in $\triangle p_{1,2}p_{2,3}p_{3,1}$, we also obtain $(pp_{3,1}v_1p_{1,2})_4$ and $(pp_{2,3}v_3p_{3,1})_4$. □

The next result is a set of ten points has a 4-hole and a 5-hole with disjoint interiors, and the value is tight. Since a 10-point set has a 5-hole, we consider configurations of the remaining five points to prove the upper bound.

Theorem 2. $m(4, 5) = 10$.

Proof. Any set of ten points has a 5-hole by $n(5) = 10$, so $m(4, 5) \geq 10$. To prove $m(4, 5) \leq 10$, let $F = (v_1v_2v_3v_4v_5)_5$ be a 5-hole of a given 10-point set and consider the closed convex cones $\gamma_i = \gamma(v_i; v'_{i-1}, v_{i+1})$ for $1 \leq i \leq 5$. Without loss of generality, we assume that γ_1 contains the largest number of interior points of all the γ_i's. Let $I(\gamma_i)$ be a set of interior points of γ_i for any i, and we have three cases according to the number of $I(\gamma_1)$.

Case 1: $|I(\gamma_1)| \geq 3$. Since there are at least five points on γ_1, we have F and a 4-hole on γ_1 by $n(4) = 5$.

Case 2: $|I(\gamma_1)| = 2$. If $\gamma(v_1; v'_5, v'_2)$ is not empty, we have F and a 4-hole on $H(v_1v_2; \overline{v_5})$. And more if γ_5 is not empty, there exist $(v_1v_3v_4v_5\alpha(v_5; v_1, v'_4))_5$ and a 4-hole on $\gamma(v_1; v_3, v'_5)$. Thus, we consider the case in which γ_5 is empty. By the same way, $\gamma_3 \setminus \gamma(v_4; v'_3, v'_5)$ is also empty, see Fig. 3. We have three subcases.

(a) $|I(\gamma_2)| = 0$: We obtain F and a 4-hole on $H(v_4v_5; \overline{v_1})$.

(b) $|I(\gamma_2)| = 1$: If $I(\gamma_1)$ lies in $\gamma(v_2; v'_3, v'_1)$, we have F and a 4-hole on $H(v_2v_3; \overline{v_1})$. Otherwise, we have a 6-hole of $(v_1wv_2v_3v_4v_5)_6$ for some point w of $I(\gamma_1)$. Then if $\gamma(v_2; v_3, v_4)$ is empty, we obtain $(v_1wv_2v_3v_4)_5$ and a 4-hole on $\gamma(v_4; v'_2, v_1)$. If not so, we obtain $(v_1wv_2v_4v_5)_5$ and a 4-hole on $\gamma(v_2; v'_1, v_4)$.

(c) $|I(\gamma_2)| = 2$: If $\gamma(v_2; v'_3, v'_1)$ is not empty, we obtain F and a 4-hole on $H(v_2v_3; \overline{v_1})$. Otherwise, we have a 5-hole of $(v_1wv_2v_4v_5)_5$ for some point w of $I(\gamma_1)$ and a 4-hole on $\gamma(v_2; v'_1, v_4)$.

Case 3: $|I(\gamma_i)| = 1$ for each i. Let w_i be precisely one interior point of γ_i.

(a) w_1 lies on $H(v_2v_3; v_1)$: Clearly, we have a 6-hole $(v_1w_1v_2v_3v_4v_5)_6$. If w_3 lies in $\gamma(v_2; v_3, v_4)$, we have $(v_1w_1v_2v_4v_5)_5$ and $(w_3v_4v_2v_3)_4$. Otherwise, we have $(v_1w_1v_2v_3v_4)_5$ and a 4-hole on $\gamma(v_4; v'_2, v_1)$.

(b) w_1 lies on $H(v_2v_3; \overline{v_1})$: If w_2 lies on $H(v_3v_4; v_2)$, we have a 6-hole $(v_1v_2w_2v_3v_4v_5)_6$ and we are done by the same way as in (a). Hence, w_2 lies on $H(v_3v_4; \overline{v_2})$. If w_2 is not contained in $\triangle v_2w_1v_3$, we have F and $(v_3v_2w_1w_2)_4$. Otherwise, we obtain F and $(v_3w_2w_1w_3)_4$. □

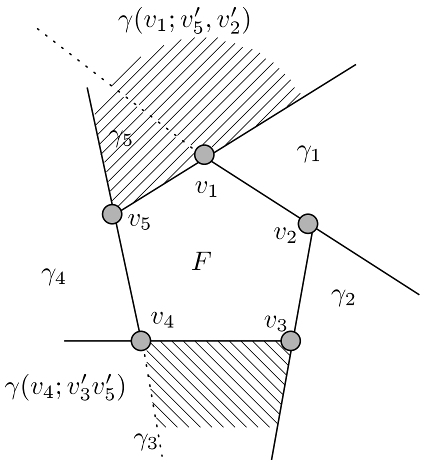

Fig. 3. Illustration of Case 2.

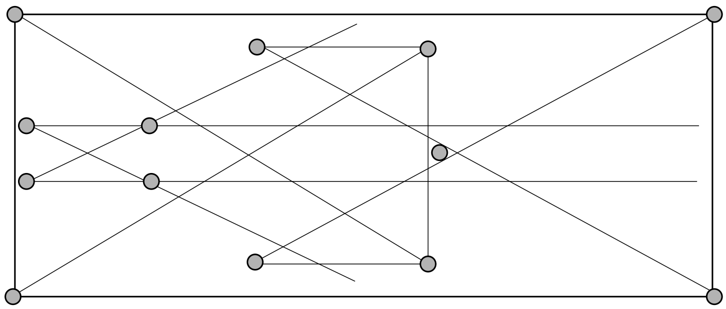

Fig. 4. $m(5, 5) \geq 14$.

We next consider the case of two 5-holes with disjoint interiors. The upper bound is showed by the simple way using $n(5) = 10$.

Theorem 3. $14 \leq m(5, 5) \leq 18$.

Proof. A 13-point set as shown in Fig. 4 gives $m(5, 5) \geq 14$. To prove the upper bound, we consider an 18-point set, and let v_1, v_2 and v_3 be three consecutive vertices of the set. Then there exists an interior point p such that each of $\gamma(v_2; v_1, p)$ and $\gamma(v_2; p, v_3)$ contains exactly ten points and it has a 5-hole by $n(5) = 10$. □

4. Three holes with disjoint interiors

In this section, we discuss the cases of three holes with disjoint interiors. Let $m(k_1, k_2, k_3)$ denote the smallest integer such that any set of $m(k_1, k_2, k_3)$ points contains a k_1-hole, a k_2-hole and a k_3-hole with disjoint interiors. We first consider some cases of $m(3, k, l)$ for $3 \leq k \leq l \leq 5$.

Proposition 4. $m(3, 3, 3) = 5$, $m(3, 3, 4) = 6$, $m(3, 3, 5) = 10$, $m(3, 4, 4) = 7$, $m(3, 4, 5) = 10$.

Proof. Let P be a set of $m(3, k, l)$ points. If P has a k-hole, an l-hole and the remaining points S, then some point p of S can *see* some edge of these holes. Therefore,
(i) $m(3, 3, 3) = 5$ holds by $m(3, 3) = 4$, (ii) $m(3, 3, 4) = 6$ holds by $m(3, 4) = 5$,
(iii) $m(3, 3, 5) = 10$ holds by $m(3, 5) = 10$, and (iv) $m(3, 4, 5) = 10$ holds by $m(4, 5) = 10$.

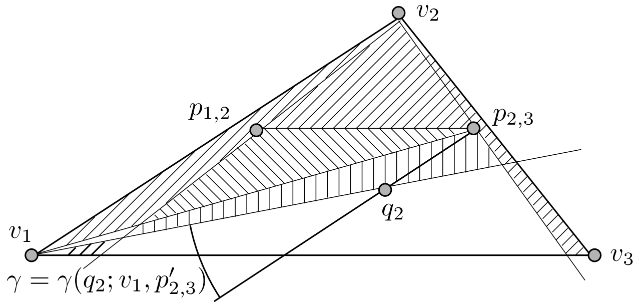

Fig. 5. We consider the convex cone $\gamma(q_2; v_1, p'_{2,3})$.

We show $m(3, 4, 4) = 7$. By $m(4, 4) = 7$, if the remaining point exists, the point sees some edge of 4-holes. Otherwise, two 4-holes have only the common vertex p, namely $(v_1v_2v_3p)_4$ and $(u_1u_2u_3p)_4$. Then we have a 3-hole of $(v_1pu_3)_3$ or $(u_1pv_3)_3$. Hence we can show the existence of desired holes. □

The next result shows a set of nine points has three 4-holes with disjoint interiors, and this value is tight.

Theorem 4. $m(4, 4, 4) = 9$.

Proof. The lower bound of $m(4, 4, 4)$ realizes an 8-point set as shown in Fig. 1(a), so $m(4, 4, 4) \geq 9$.

To prove the upper bound, let P be a set of nine points. We have the cases according to the number of vertices of P. If $|V(P)| \geq 5$, there exists a cutting line dividing P into a 4-hole and the remaining 7-point set S by Lemma 2. Then we have two 4-holes of S using $m(4, 4) = 7$.

Case 1: $|V(P)| = 4$. If an edge of $V(P)$ has no friend, we have a cutting line dividing into a 4-hole and the remaining seven points by Lemma 1, and we are done by $m(4, 4) = 7$.

Otherwise, every edge of P has its friend. Let $V(P) = \{v_1, v_2, v_3, v_4\}$ and $p_{i,i+1}$ is a friend to an edge $\overline{v_iv_{i+1}}$ for any i, $1 \leq i \leq 4$. We consider the position of the remaining point p of P.

Subcase 1A: p lies in some $T_i = \triangle p_{i-1,i}v_ip_{i,i+1}$ for $i = 1, 2, 3, 4$, say $\triangle p_{1,2}v_2p_{2,3}$. If p lies in $H(v_2p_{4,1}; v_1)$, then we have a 4-hole of $(pp_{4,1}v_1p_{1,2})_4$ and $H(pp_{4,1}; v_3)$ has seven points. Otherwise, p lies in $H(v_2p_{4,1}; v_3)$. Then we have $(pp_{2,3}v_3p_{3,4})_4$, $(pp_{3,4}v_4p_{4,1})_4$ and $(pp_{4,1}v_1p_{1,2})_4$.

Subcase 1B: p lies inside the quadrilateral $p_{1,2}p_{2,3}p_{3,4}p_{4,1}$. We obtain $(p_{4,1}v_1p_{1,2}p)_4$, $(p_{1,2}v_2p_{2,3}p)_4$ and $(p_{2,3}v_3p_{3,4}p)_4$ with disjoint interiors.

Case 2: $|V(P)| = 3$. We only consider the case in which every edge of $V(P)$ has its friend by Lemma 1. Let $V(P) = \{v_1, v_2, v_3\}$ and denote $T_i = \triangle p_{i-1,i}v_ip_{i,i+1}$ for $i = 1, 2, 3$. There are two subcases.

Subcase 2A: Some T_i, say T_2, is empty.

(i) $\triangle v_1p_{1,2}p_{2,3}$ is not empty: Since we have $q_1 = \alpha(p_{2,3}; p_{1,2}, v_1)$, there exists a cutting line $l(q_1, p_{2,3})$ dividing into a 4-hole $(p_{1,2}v_2p_{2,3}q_1)_4$ and the remaining seven points.

(ii) $\triangle v_1p_{1,2}p_{2,3}$ is empty: Since there is $q_2 = \alpha(v_1; p_{2,3}, p_{3,1})$, we consider the convex cone $\gamma = \gamma(q_2; v_1, p'_{2,3})$, see Fig. 5. If γ is not empty, for $q_3 = \alpha(q_2; v_1, p'_{2,3})$ we have a cutting line $l(q_3, q_2)$ dividing into two 4-holes of $(v_1p_{1,2}q_2q_3)_4$ and $(p_{1,2}v_2p_{2,3}q_2)_4$, and the remaining five points. There is a 4-hole of the 5-point set by $n(4) = 5$ and we are done.

If γ is empty, q_2 is a friend to the edge $\overline{v_1p_{2,3}}$ of $V(P')$ for $P' = P \backslash \{v_2, p_{1,2}\}$. We remark that $\triangle v_1p_{2,3}q_2$ cannot be contained in the convex hull of any 4-hole of P'. Thus we obtain $(p_{1,2}v_2p_{2,3}q_2)_4$ and two 4-holes of the 7-point set P'.

Subcase 2B: T_i contains only the point w_i of P for every $i = 1, 2, 3$.

We consider the position of w_2. If w_2 lies in $H(v_2w_1; v_1)$, we have a cutting line $l(w_1, w_2)$ dividing into $(v_1p_{1,2}w_2w_1)_4$ and the 7-point set. Also, by the symmetry, if w_2 lies in $H(v_2w_3; v_3)$, $l(w_2, w_3)$ is the cutting line. Otherwise, we have three 4-holes of $(p_{1,2}v_2w_2w_1)_4$, $(w_2v_2p_{2,3}w_3)_4$ and $(w_1w_2w_3p_{3,1})_4$ with disjoint interiors. □

We next consider the case of two 4-holes and one 5-hole with disjoint interiors, that is not exact value.

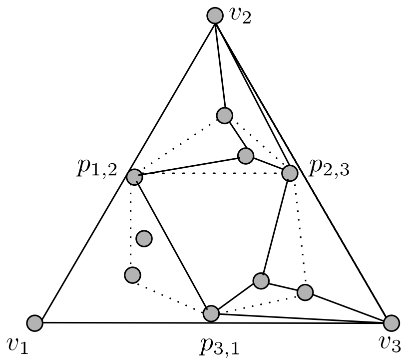

Fig. 6. The final configuration in case 2.

Theorem 5. $11 \leq m(4, 4, 5) \leq 12$.

Proof. The lower bound realizes a 10-point set P such that $|V(P)| = 5$ and each edge of $V(P)$ has its friend. To show $m(4, 4, 5) \leq 12$, let P be a 12-point set. If $|V(P)| \geq 7$, there exists a cutting line dividing P into a 4-hole and the remaining 10-point set S by Lemma 2. We have both a 4-hole and a 5-hole of S by $m(4, 5) = 10$.

For $3 \leq |V(P)| \leq 6$, we discuss under the condition in which every edge of $V(P)$ has its friend by Lemma 1. Recall that $V(P) = \{v_i\}_{i \geq 1}$ in clockwise order and $p_{i,i+1}$ is the friend to an edge $\overline{v_i v_{i+1}}$. We consider a triangle $T_i = \triangle p_{i-1,i} v_i p_{i,i+1}$ for any i.

Case 1: Some T_i, say T_2, is empty.

Subcase 1A: If $\triangle v_1 p_{1,2} p_{2,3}$ is not empty, we have a cutting line $l(q_1, p_{2,3})$ for $q_1 = \alpha(p_{2,3}; p_{1,2}, v_1)$ dividing into a 4-hole $(p_{1,2} v_2 p_{2,3} q_1)_4$ and the remaining ten points.

Subcase 1B: $\triangle v_1 p_{1,2} p_{2,3}$ is empty.

(i) $\triangle v_1 p_{2,3} v_3$ is not empty: Since there is $q_2 = \alpha(v_1; p_{2,3}, v_3)$, we consider $\gamma = \gamma(q_2; v_1, p'_{2,3})$. If γ is not empty, for $q_3 = \alpha(q_2; v_1, p'_{2,3})$ we have a cutting line $l(q_3, q_2)$ dividing into a 5-hole of $(v_1 p_{1,2} p_{2,3} q_2 q_3)_5$ and the remaining eight points. There are two 4-hole of the 8-point set by $m(4, 4) = 7$. If γ is empty, since q_2 is a friend to $\overline{v_1 p_{2,3}}$ of $V(P')$ for $P' = P \backslash \{v_2, p_{1,2}\}$, $\triangle v_1 p_{2,3} q_2$ cannot be contained in the convex hull of any 4-hole of P'. We obtain $(p_{1,2} v_2 p_{2,3} q_2)_4$ and both a 4-hole and a 5-hole of the 10-point set P'.

(ii) $\triangle v_1 p_{2,3} v_3$ is empty: Note that $|V(P)| \geq 4$. For $q_3 = \widetilde{\alpha}(v_1; v_3, v_4)$, we have a cutting line $l(v_1, q_3)$ dividing into $(v_1 p_{1,2} p_{2,3} v_3 q_3)_5$ and the remaining eight points.

Case 2: No T_i is empty for any i. Since $|P| \geq 3|V(P)|$, we consider the following two subcases.

Subcase 2A: $|V(P)| = 4$. Let w_i be only the point of P inside T_i for each i. If w_2 lies in $H(v_2 w_1; v_1)$, we have a 4-hole $(w_2 w_1 v_1 p_{1,2})_4$ and $l(w_1, w_2)$ is the cutting line. Otherwise, w_2 is in $H(v_2 w_1; v_3)$. Then we have $(w_1 p_{1,2} v_2 w_2)_4$, $(w_2 p_{2,3} w_3 p_{3,4})_4$ and $(w_1 w_2 p_{3,4} w_4 p_{4,1})_5$.

Subcase 2B: $|V(P)| = 3$. There are two cases according to the number of points of P inside T_i.

(i) Some T_i, say T_2 contains only the point w: If $\triangle p_{1,2} p_{2,3} v_1$ is empty, we have a cutting line $l(p_{2,3}, q_1)$ for $q_1 = \alpha(p_{2,3}; v_1, p_{3,1})$ dividing into $(p_{1,2} w p_{2,3} q_1 v_1)_5$ and the remaining eight points. If it is not empty, we consider $\gamma = \gamma(q_2; p_{2,3}, p'_{1,2})$ for $q_2 = \alpha(p_{2,3}; p_{1,2}, v_1)$. Then if γ is not empty, we have a cutting line $l(q_3, q_2)$ for $q_3 = \alpha(q_2; p_{2,3}, p'_{1,2})$ dividing into six points containing $(p_{1,2} w p_{2,3} q_3 q_2)_5$ and the remaining eight points. If γ is empty, then q_2 is a friend to $\overline{p_{1,2} p_{2,3}}$ of $V(P')$ for $P' = P \backslash \{v_2, w\}$. Hence we obtain $(p_{1,2} w p_{2,3} q_2)_4$, and a 4-hole and a 5-hole of the 10-point set P'.

(ii) Every triangle T_i contains exactly two points of P: If some T_i, say T_2 contains $\{w_1, w_2\}$ such that $Q = \{p_{1,2}, w_1, w_2, p_{2,3}\}$ is in convex position, for $q_4 = \widetilde{\alpha}(p_{1,2}; p_{2,3}, v_3)$ we have a cutting line $l(p_{1,2}, q_4)$ dividing into six points containing a 5-hole formed by $Q \cup \{q_4\}$ and the remaining eight points. Otherwise, we have a configuration as shown in Fig. 6 and we can obtain the desired holes. □

Please cite this article in press as: K. Hosono, M. Urabe, Specified holes with pairwise disjoint interiors in planar point sets, AKCE International Journal of Graphs and Combinatorics (2018), https://doi.org/10.1016/j.akcej.2018.08.003.

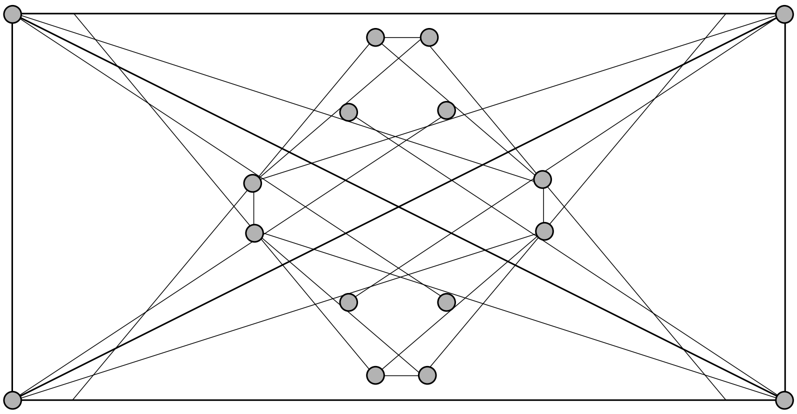

Fig. 7. $m(5, 5, 5) \geq 17$.

5. Conclusions

1. We showed several results for $m(k, l)$. In fact, the condition of integers k and l is for $3 \leq k \leq l \leq 6$, so the number of types for $m(k, l)$ are ten cases. However, $30 \leq n(6) \leq 463$ means that the function $m(k, 6)$ is not valid. Therefore, we checked out all the cases of $m(k, l)$ for $l \leq 5$.

For a set of three holes, we can easily show the following results by a simple method. Let v_1, v_2 be consecutive vertices on the convex hull of a given point set. We consider a point p such that the closed convex cone $\gamma(v_1; v_2, p)$ contains exactly ten points. Then we have a 5-hole on this convex cone by $n(5) = 10$. Therefore, $m(3, 5, 5) \leq 18$, $m(4, 5, 5) \leq m(4, 5) + 8 = 18$ and $m(5, 5, 5) \leq m(5, 5) + 8 \leq 26$. The lower bounds of $m(3, 5, 5)$ and $m(4, 5, 5)$ are shown by $14 \leq m(5, 5) \leq m(3, 5, 5) \leq m(4, 5, 5)$. And the lower bound of $m(5, 5, 5)$ realizes the configuration as shown in Fig. 7, which implies $n(5, 5) \geq 17$.

Proposition 5. $14 \leq m(3, 5, 5) \leq m(4, 5, 5) \leq 18$, $17 \leq m(5, 5, 5) \leq 26$. □

Hence, for a set of three holes, we estimated all the cases except for $m(k, l, 6)$.

2. The following theorem is announced in [22] without proof.

Theorem A. *Any point set P with $n = 2k + 3$ elements in general position contains the vertices of k empty convex quadrilaterals with disjoint interiors.*

Using this result, both $m(4, 4) = 7$ in Theorem 1 and $m(4, 4, 4) = 9$ in Theorem 4 are derived. However, because the proof has been not published for ten years, we prove only our results in this article to introduce a new problem. In addition, we can show that the lower bound of $m(4, 4, \ldots, 4)$ for k 4-holes is realized in the configuration of a $2k + 2$ point set P such that $|V(P)| = k + 1$ and each edge has its friend. Therefore,

Proposition 6. $m(\underbrace{4, 4, \ldots, 4}_{k}) \geq 2k + 3$ □

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