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Some properties of Square element graphs over semigroups

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Abstract

The Square element graph over a semigroup S is a simple undirected graph $\mathbb{S}q(S)$ whose vertex set consists precisely of all the non-zero elements of S, and two vertices a, b are adjacent if and only if either ab or ba belongs to the set $\{t^2 : t \in S\} \setminus \{1\}$, where 1 is the identity of the semigroup (if it exists). In this paper, we study the various properties of $\mathbb{S}q(S)$. In particular, we concentrate on square element graphs over three important classes of semigroups. First, we consider the semigroup Ω_n formed by the ideals of $\mathbb{Z}_n$. Afterwards, we consider the symmetric groups S_n and the dihedral groups D_n. For each type of semigroups mentioned, we look into the structural and other graph-theoretic properties of the corresponding square element graphs.

Keywords: Square element graph; Semigroup; Semigroup of ideals; Symmetric group; Dihedral group

1. Introduction

Graphs defined over algebraic structures reveal interesting interplay between graph-theoretic and algebraic properties. For example, zero-divisor graphs [1] have shown that the set of zero-divisors of a ring has many underlying properties which are significant from a graph-theoretic perspective.

Like the set of zero-divisors, we can consider another interesting set in an algebraic structure R, viz., the set of squares of R (i.e., the set $T = \{x^2 \mid x \in R\}$). It is interesting to observe that exactly like the set of zero-divisors, the set of squares of a commutative ring is not closed under addition (in general) but is closed under multiplication. Using the set of squares, Sen Gupta and Sen defined the square element graph over a finite commutative ring [2], where the set of all non-zero elements of a finite commutative ring R is taken as the vertex set, and two vertices are adjacent if and only if their sum is a square of some non-zero element of R. Later, Sen Gupta and Sen generalized the square element graphs over arbitrary rings [3]. Now once the set of squares of a ring is determined, the square element graph essentially uses only one operation of a ring. Hence, like the zero-divisor graphs, the square element graphs can also be defined over a semigroup. We define the square element graph over a semigroup in the following way:

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Definition 1.1. Let S be a semigroup. We consider a simple undirected graph $G = (V, E)$, where $V = S - \{0\}$, and for any two elements $a, b \in V$, $ab \in E$ if and only if $\{ab, ba\} \cap \{t^2 \mid t \in S, t^2 \neq 1\} \neq \emptyset$. Here 1 and 0 are respectively the identity and the zero-element of S (if they exist). This simple undirected graph is called the *Square element graph* over the semigroup S, and is denoted by $\mathbb{S}q(S)$.

Remark 1.2. From the definition of $\mathbb{S}q(S)$, it is easy to see that if S has a zero-element 0, then the zero-divisor graph $\Gamma(S)$ over S (studied in [4,5]) is a subgraph of the graph $\mathbb{S}q(S)$ (since $0 = 0^2$). Consequently, there is a path between any two zero-divisor vertices in $\mathbb{S}q(S)$, since $\Gamma(S)$ is always connected.

Example 1.3. Let $S = \{(i, a, \lambda) \mid i, \lambda \in \{1, 2\} \text{ and } a \in \{0, 1\}\} \cup \{0\}$. We consider the matrix $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ whose entries are from the set $\{0, 1\}$. Let $p_{\lambda j}$ denote the (λ, j)th entry of M. We define an operation '$\cdot$' on S as follows:

$$(i, a, \lambda).(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $(S, .)$ becomes a completely 0−semisimple semigroup. Here, each non-zero element of $(S, .)$ is a zero-divisor. Now the zero-divisor graph $\Gamma(S)$ is connected. Since $\Gamma(S)$ is a subgraph of $\mathbb{S}q(S)$ with the same vertex set as that of $\mathbb{S}q(S)$, it follows that $\mathbb{S}q(S)$ is also connected. Now $\mathbb{S}q(S)$ and $\Gamma(S)$ are shown below (see Figs. 1 and 2):

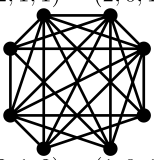

Fig. 1. $\Gamma(S)$.

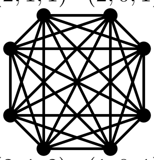

Fig. 2. $\mathbb{S}q(S)$.

We observe that $\Gamma(S)$ is not complete, as $(1, 0, 1)$ is not adjacent to $(2, 1, 2)$ in $\Gamma(S)$. Now we note that $(i, a, \lambda)^2 = (i, ap_{\lambda j}a, \lambda)$. So if $a = 0$, then $(i, 0, \lambda)^2 \neq 0$ if and only if $(\lambda, i) = (1, 2)$ or $(2, 1)$. The same is true if $a = 1$. Hence the square elements in S are 0, (1, 0, 2), (2, 0, 1), (1, 1, 2), (2, 1, 1). So $(1, 1, 1) \leftrightarrow (2, 1, 2)$ in $\mathbb{S}q(S)$. From Fig. 3, $\mathbb{S}q(S)$ is seen to be isomorphic to the complete graph K_8.

We now give some results regarding the connectedness of $\mathbb{S}q(S)$.

Theorem 1.4. *If S is a union of groups of odd order, then $\mathbb{S}q(S)$ is connected with $diam(\mathbb{S}q(S)) \leq 2$. In particular, if G is a group of odd order, then $\mathbb{S}q(G)$ is connected with $diam(\mathbb{S}q(S)) \leq 2$.*

Proof. Let $S = G_1 \cup G_2 \cup \cdots \cup G_n$, where the G_i's are groups of odd order. Let $x \in S$. Then $x \in G_i$ for some $i \in \{1, 2, \ldots, n\}$. Let $|G_i| = r$, where r is odd. Then $x^r = e_1$ where e_1 is the identity of G_i. So $x = (x^{\frac{r+1}{2}})^2$. This shows that every element of S is a square element. Let a, b be any two vertices of $\mathbb{S}q(S)$. If $a = b^{-1}$, then a, b must belong to the same group and hence we have a path $a \leftrightarrow e \leftrightarrow b$, where e is the identity of the group to which a, b belong. If $a \neq b^{-1}$, then ab is a non-identity square element belonging to S (as every element of S has been shown to be a square). Thus $a \leftrightarrow b$. Hence $\mathbb{S}q(S)$ is connected with diameter at most 2. In particular, it obviously follows that for a group G of odd order, $\mathbb{S}q(G)$ is connected with diameter at most 2. $\square$

The converse of the last part does not hold in general, but it holds true if the group is commutative, as shown in the next result.

Theorem 1.5. *Let G be a finite commutative group. Then the following are equivalent:*
(i) $\mathbb{S}q(G)$ is connected.
(ii) All elements of G are squares.
(iii) $|G|$ is odd.

Proof. $(i) \implies (ii)$: Let $\mathbb{S}q(G)$ be connected. Suppose S_1 is the set of all square elements of G. If $\mathbb{S}q(G)$ is a single vertex graph, then all elements of G are squares. So we now assume that G has at least two elements. If possible, let G contain non-squares. Now as $\mathbb{S}q(G)$ is connected, there must be some square element t^2, and some non-square m such that $t^2 \leftrightarrow m$ in $\mathbb{S}q(G)$. Then, we have that $t^2 m = s^2$ for some $s \in G$, which implies that $m = (st^{-1})^2$. This contradicts that m is a non-square. So all elements of G must be squares.

$(ii) \implies (iii)$: Suppose all elements of G are squares. If possible, let $|G| = n$, where n is an even integer. If $n = 2$, then $G \cong (\mathbb{Z}_2, +)$. This is not possible, since $(\mathbb{Z}_2, +)$ contains a non-square $\bar{1}$. So we assume that $n > 2$. Clearly, $n = 2^k m$, where $k \geq 1$ and m is odd with $km \neq 1$. Then G has an element a of order 2. Suppose $a = a_1^2$ for some a_1. Then $a_1^4 = e$. So $o(a_1)$ is 1, 2, or 4. Now $a, a_1 \neq e$, and $a_1^2 = a \neq e$. Hence, $o(a_1) = 4$. Again, let $a_1 = a_2^2$ for some a_2. Now since $a_2^8 = e$, we have that $o(a_2)$ is 1, 2, 4 or 8. It is easy to see that $o(a_2) = 8$. We continue this process and ultimately, we get an element a_k such that $o(a_k) = 2^{k+1}$. This is a contradiction, since order of any element in G has to be a divisor of $2^k m$. So a^t is non-square for some $1 \leq t \leq k-1$, which is again a contradiction as all elements of G are squares. Thus, $|G|$ is odd.

$(iii) \implies (i)$: This follows from Theorem 1.4. $\square$

In this paper, we concentrated on square element graphs defined over three special classes of semigroups. In [2], the ring $(\mathbb{Z}_n, +, \cdot)$ was considered. It seemed worthwhile to consider the semigroup Ω_n formed by the ideals of $\mathbb{Z}_n$. In Section 2, we studied the properties of $\mathbb{S}q(\Omega_n)$. Then we considered square element graphs over S_n. Finally, we looked at the dihedral groups D_n (which are noncommutative groups of even order) and looked into the various graph-theoretic properties of $\mathbb{S}q(D_n)$.

In this paper, $a \leftrightarrow b$ denotes that the vertices a, b are adjacent. Again, the symbols $diam(G)$, $gr(G)\chi(G)$, $\omega(G)$, $\alpha(G)$, $\gamma(G)$ respectively denote the diameter, the girth, the chromatic number, the clique number, the independence number and the domination number of the graph G. For other graph-theoretic terminologies, one may refer to [6]. For the algebraic terminologies, one may have a look at [7,8].

2. The graph $\mathbb{S}q(\Omega_n)$

In this section, we study the square element graphs over a special class of semigroups, viz. the semigroup formed by the ideals of a ring. Specifically, we here consider rings of the form $\mathbb{Z}_n$.

For a ring R with identity, let Ω_R denote the set of all left ideals of R. For any two ideals I, J of R, multiplication '$\cdot$' is defined by $I \cdot J = \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \mid a_i \in I, b_i \in J, i = 1, 2, \ldots, n, n \in \mathbb{N}\}$. Then $(\Omega_R, .)$ forms a semigroup. We are interested to study the square element graph over the semigroup Ω_R. In particular, we consider the ideals of $\mathbb{Z}_n$. For convenience, we denote the corresponding semigroup by Ω_n instead of $\Omega_{\mathbb{Z}_n}$. It is known that the distinct ideals of $\mathbb{Z}_n$ are precisely the ideals generated by the distinct divisors of n. Therefore, the number of ideals in $\mathbb{Z}_n$ equals the number of divisors of n. Hence, if $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $r_1, r_2, \ldots, r_k$ are nonnegative integers, then there are $(r_1+1)(r_2+1)\cdots(r_k+1)$ ideals in $\mathbb{Z}_n$. So $\Omega_n = \{\langle\bar{0}\rangle, \langle\bar{1}\rangle, \langle\overline{p_1}\rangle, \langle\overline{p_1 p_2}\rangle, \ldots, \langle\overline{p_1 p_2^{r_k}}\rangle, \ldots, \langle\overline{p_1^{r_1} \cdots p_k^{r_k - 1}}\rangle\}$ is the set of all ideals in $\mathbb{Z}_n$. Note that in Ω_n, no element is invertible except the identity element $\langle\bar{1}\rangle$.

For example, $\Omega_6 = \{\langle\bar{0}\rangle, \langle\bar{1}\rangle, \langle\bar{2}\rangle, \langle\bar{3}\rangle\}$, and $\mathbb{S}q(\Omega_6)$ is the following complete graph:

In general we obtain the following.

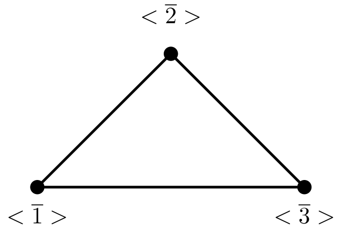

Fig. 3. $\mathbb{S}q(\Omega_6)$.

Theorem 2.1. *The graph $\mathbb{S}q(\Omega_n)$ is a complete graph if and only if $n = p_1p_2p_3\cdots p_r$ for some distinct primes $p_1,\ldots,p_r$.*

Proof. Let $n = p_1p_2p_3\ldots p_r$ for some distinct primes $p_1,\ldots,p_r$. If $r = 1$, then $\mathbb{S}q(\Omega_n)$ is a single-vertex graph and hence is complete. Next, let $r > 1$. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\times\mathbb{Z}_{p_{r-1}}\times\mathbb{Z}_{p_r}$ and hence there are $(1+1)(1+1)\cdots(1+1) = 2^r$ distinct ideals of $\mathbb{Z}_n$. If I is an ideal of $\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\times\mathbb{Z}_{p_{r-1}}\times\mathbb{Z}_{p_r}$, then $I = I_1\times I_2\times\cdots\times I_{r-1}\times I_r$ where I_k is some ideal of $\mathbb{Z}_{p_k}$ for $k = 1,2,\ldots,r$. Clearly, I_k is either $\{\bar{0}\}$ or $\mathbb{Z}_{p_k}$. Now $\mathbb{Z}_{p_k}^2 = \mathbb{Z}_{p_k}$ and $\{\bar{0}\}^2 = \{\bar{0}\}$. Since this implies that $I_k^2 = I_k$ for all $k = 1,2,\ldots,r$, we have that $I^2 = I$. Thus any ideal of $\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\times\mathbb{Z}_{p_{r-1}}\times\mathbb{Z}_{p_r}$ is a square element. Let I, J be two non-zero distinct ideals of $\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\times\mathbb{Z}_{p_{r-1}}\times\mathbb{Z}_{p_r}$, then $I\cdot J = (I\cdot J)^2 \neq \langle\bar{1}\rangle$. Hence $I \leftrightarrow J$. So the square element graph over the set of all ideals of $\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\cdots\times\mathbb{Z}_{p_{r-1}}\times\mathbb{Z}_{p_r}$ is isomorphic to K_{2^r-1}. Thus $\mathbb{S}q(\Omega_n)$ is a complete graph.

Conversely, let $\mathbb{S}q(\Omega_n)$ be a complete graph. If possible, let $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ with at least one $r_i > 1$. If $k = 1$ (and hence $r_1 > 1$), we have that the vertex $\langle\bar{1}\rangle$ is not adjacent to the vertex $\langle\overline{p}_1\rangle$, and hence $\mathbb{S}q(\Omega_n)$ is not complete in that case. Let $k > 1$. Now $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}}\times\mathbb{Z}_{p_2^{r_2}}\times\cdots\times\mathbb{Z}_{p_{r-1}^{r_{k-1}}}\times\mathbb{Z}_{p_r^{r_k}}$ and hence the square element graph over the set of all ideals of $\mathbb{Z}_{p_1^{r_1}}\times\mathbb{Z}_{p_2^{r_2}}\times\cdots\times\mathbb{Z}_{p_{r-1}^{r_{k-1}}}\times\mathbb{Z}_{p_r^{r_k}}$ is a complete graph. Without loss of generality, let $r_1 > 1$, then $\langle\overline{p_1}\rangle$ is not a square element in $\mathbb{Z}_{p_1^{r_1}}$. This implies that $\langle\overline{p_1}\rangle\times\{\bar{0}\}\times\cdots\times\{\bar{0}\}\times\{\bar{0}\}$ is not adjacent to $\langle\bar{1}\rangle\times\langle\bar{1}\rangle\times\cdots\times\langle\bar{1}\rangle$, which is a contradiction as the square element graph over the set of all ideals of $\mathbb{Z}_{p_1^{r_1}}\times\mathbb{Z}_{p_2^{r_2}}\times\cdots\times\mathbb{Z}_{p_{r-1}^{r_{k-1}}}\times Z_{p_r^{r_k}}$ is a complete graph. Thus each $r_i \leq 1$. Hence n must be of the form $p_1p_2p_3\cdots p_r$ where $p_1,\ldots,p_r$ are distinct primes. $\square$

Next, we consider the connectedness of $\mathbb{S}q(\Omega_n)$. We observe that $\mathbb{S}q(\Omega_n)$ is not always connected. For example, the graph $\mathbb{S}q(\Omega_9)$ is not connected, as shown below (see Fig. 4):

Fig. 4. $\mathbb{S}q(\Omega_9)$.

The following theorem gives the complete set of values of n for which $\mathbb{S}q(\Omega_n)$ is connected.

Theorem 2.2. *The graph $\mathbb{S}q(\Omega_n)$ is connected if and only if $n \neq p^2$ for any prime p.*

Proof. We consider the different values of n and look at the structure of $\mathbb{S}q(n)$ accordingly.

Case 1: Let $n = p_1p_2p_3\cdots p_r$ for some distinct primes $p_1,p_2,\ldots,p_r$. Then by Theorem 2.1, $\mathbb{S}q(\Omega_n)$ is connected.

Case 2: Let $n = p^k$ for some prime p and $k > 2$. Then $\Omega_n = \{\langle\bar{1}\rangle, \langle\overline{p}\rangle, \langle\overline{p^2}\rangle, \langle\overline{p^3}\rangle,\ldots,\langle\overline{p^{k-1}}\rangle, \langle\bar{0}\rangle\}$. Now in $\mathbb{S}q(\Omega_n)$, $\langle\overline{p^i}\rangle \leftrightarrow \langle\overline{p^{k-1}}\rangle$ for $i = 1,2,3,\ldots,k-2$; and $\langle\bar{1}\rangle \leftrightarrow \langle\overline{p^2}\rangle$. This shows that we have a path between any two vertices in $\mathbb{S}q(\Omega_n)$. Thus the graph $\mathbb{S}q(\Omega_n)$ is connected.

Case 3: Let $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ where $p_1,p_2,\ldots,p_k$ are distinct primes, $k \geq 2$, and $r_1,r_2,\ldots,r_k \in \mathbb{N}$ with at least one $r_i > 1$. Then $\Omega_n = \{\langle\bar{0}\rangle, \langle\bar{1}\rangle, \langle\overline{p_1}\rangle, \langle\overline{p_1p_2}\rangle,\ldots,\langle\overline{p_1p_2^{r_k}}\rangle,\ldots,\langle\overline{p_1^{r_1}\cdots p_k^{r_k-1}}\rangle\}$ is the set of all ideals of $\mathbb{Z}_n$. Without loss of generality, let $r_1 > 1$. In this case $\langle\overline{p_1^2}\rangle$ is a square element, which is adjacent to $\langle\bar{1}\rangle$. Consider a vertex of the form $\langle\overline{p_1^{s_1}\cdots p_k^{s_k}}\rangle$ with $s_1 \geq 1$. Then we have a path $\langle\overline{p_1^{s_1}\cdots p_k^{s_k}}\rangle \leftrightarrow \langle\overline{p_1^{r_1-1}p_2^{r_2}\cdots p_k^{r_k}}\rangle \leftrightarrow \langle\overline{p_1^2}\rangle$. Next, consider an element of the form $\langle\overline{p_2^{s_2}\cdots p_i^{s_i}\cdots p_k^{s_k}}\rangle$ with $s_i \geq 1$. Then we have a path $\langle\overline{p_2^{s_2}\cdots p_i^{s_i}\cdots p_k^{s_k}}\rangle \leftrightarrow$

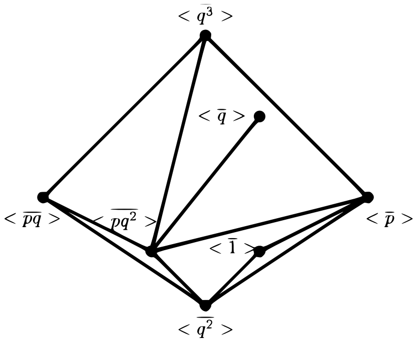

Fig. 5. $\mathbb{S}q(\Omega_{pq^3})$.

$\langle\overline{p_1^{r_1}p_2^{r_2}p_3^{r_3}\cdots p_i^{r_i-1}\cdots p_k^{r_k}}\rangle \leftrightarrow \langle\overline{p_1^{r_1-1}p_2^{r_2}\cdots p_k^{r_k}}\rangle \leftrightarrow \langle\overline{p_1^2}\rangle$. Thus every vertex is in the component to which the vertex $\langle\overline{p_1^2}\rangle$ belongs. Hence $\mathbb{S}q(\Omega_n)$ is a connected graph.

Case 4: Let $n = p^2$, where p is any prime. Then $\Omega_n = \{\langle\bar{0}\rangle, \langle\bar{1}\rangle, \langle\bar{p}\rangle\}$. Now $\langle\bar{1}\rangle$ is not adjacent to $\langle\bar{p}\rangle$ and thus it is easy to see that $\mathbb{S}q(\Omega_n)$ is not connected.

Having considered all possible cases, we find that $\mathbb{S}q(\Omega_n)$ is connected if and only if $n \neq p^2$ for some prime p. $\square$

Corollary 2.3. *When $\mathbb{S}q(\Omega_n)$ is connected, $diam(\mathbb{S}q(\Omega_n)) \leq 6$.*

Proof. From the proof of Theorem 2.2, we see that for $n = p_1p_2\cdots p_r$, all the vertices of $\mathbb{S}q(\Omega_n)$ are adjacent to each other. Also, for $n = p^k$ (where $k > 2$), there is a path of length at most 3 between any two vertices. Finally, for the remaining values of n (except for the form p^2 for some prime p), we have a path of length at most 6 between any two vertices (through the vertex $\langle\overline{p_j^2}\rangle$ if $r_j > 2$). So $diam(\mathbb{S}q(\Omega_n)) \leq 6$ when $\mathbb{S}q(\Omega_n)$ is connected. $\square$

In the next result, we consider the planarity of $\mathbb{S}q(\Omega_n)$.

Proposition 2.4. *$\mathbb{S}q(\Omega_n)$ is planar if and only if n is in one of the following forms:*

$$n = \begin{cases} pq^r & , 0 < r \leq 3 \\ p^2q^2 & \\ p^s & , s \leq 8. \end{cases}$$

where p, q are distinct primes.

Proof. First of all, we show that $\mathbb{S}q(\Omega_n)$ is indeed planar for these values of n as mentioned. Let p, q be distinct primes. The graphs $\mathbb{S}q(\Omega_{pq^2})$ and $\mathbb{S}q(\Omega_{pq^3})$ are shown in Figs. 6 and 5, respectively.

From the figures, it is clear that both the graphs $\mathbb{S}q(\Omega_{pq^3})$ and $\mathbb{S}q(\Omega_{pq^2})$ are planar. Similarly, it can be easily shown that the graph $\mathbb{S}q(\Omega_{pq})$ is planar as the number of the vertices of $\mathbb{S}q(\Omega_{pq})$ is 3.

Next, we consider the graphs $\mathbb{S}q(\Omega_{p^5})$, $\mathbb{S}q(\Omega_{p^6})$, $\mathbb{S}q(\Omega_{p^7})$, $\mathbb{S}q(\Omega_{p^8})$, and $\mathbb{S}q(\Omega_{p^2q^2})$ where p, q are distinct prime integers:

From Figs. 7, 8, 9, and 10, it is clear that the graphs $\mathbb{S}q(\Omega_{p^5})$, $\mathbb{S}q(\Omega_{p^6})$, $\mathbb{S}q(\Omega_{p^7})$ and $\mathbb{S}q(\Omega_{p^8})$ are all planar. Again, it is easy to see that the graph $\mathbb{S}q(\Omega_{p^r})$ for $r = 1, 2, 3, 4$ is planar as the number of the vertices of the graph $\mathbb{S}q(\Omega_{p^r})$ is ≤ 4 for $r = 1, 2, 3, 4$. The graph $\mathbb{S}q(\Omega_{p^2q^2})$ is also planar, as shown in Fig. 11.

Now we show that for the remaining values of n, $\mathbb{S}q(\Omega_n)$ is not planar.

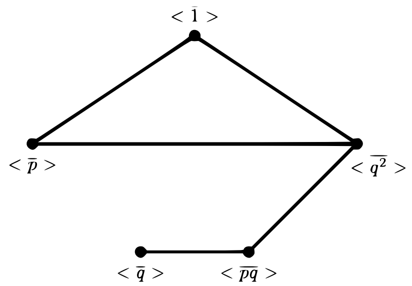

Fig. 6. $\mathbb{S}q(\Omega_{pq^2})$.

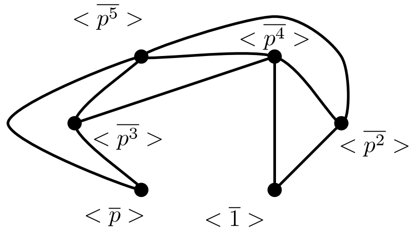

Fig. 7. $\mathbb{S}q(\Omega_{p^6})$.

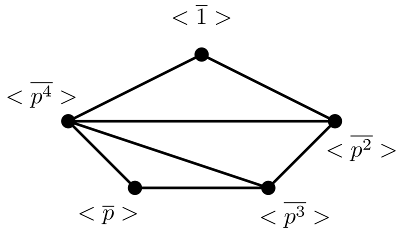

Fig. 8. $\mathbb{S}q(\Omega_{p^5})$.

Case I: If $n = p_1^{s_1}$ where $s_1 > 8$, then we have a subgraph induced by the subset $\{\langle\overline{1}\rangle, \langle\overline{p_1^2}\rangle, \langle\overline{p_1^4}\rangle, \langle\overline{p_1^6}\rangle, \langle\overline{p_1^8}\rangle\}$ which is isomorphic to K_5. Thus $\mathbb{S}q(\Omega_n)$ is not planar.

Case II: If $n = p_1 p_2^{s_2}$ where $s_2 > 3$, then we have a subgraph induced by the subset $\{\langle\overline{1}\rangle, \langle\overline{p_1}\rangle, \langle\overline{p_2^2}\rangle, \langle\overline{p_2^4}\rangle, \langle\overline{p_1 p_2^2}\rangle\}$ which is isomorphic to K_5 and hence $\mathbb{S}q(\Omega_n)$ is not planar.

Case III: If $n = p_1^2 p_2^3$, then the subgraph induced by the vertices $\{\langle\overline{p_1^2}\rangle, \langle\overline{p_1^2 p_2}\rangle, \langle\overline{p_1^2 p_2^2}\rangle, \langle\overline{p_2^3}\rangle, \langle\overline{p_2^2}\rangle, \langle\overline{p_1 p_2^3}\rangle\}$ has a subgraph which is isomorphic to $K_{3,3}$. So $\mathbb{S}q(\Omega_n)$ is not planar.

Case IV: If $n = p_1^3 p_2^3$, then the subgraph induced by the vertices $\{\langle\overline{p_1^2 p_2^2}\rangle, \langle\overline{p_1^3 p_2}\rangle, \langle\overline{p_1^3 p_2^2}\rangle, \langle\overline{p_1 p_2^2}\rangle, \langle\overline{p_1 p_2^3}\rangle, \langle\overline{p_1^2 p_2^3}\rangle\}$ has a subgraph which is isomorphic to $K_{3,3}$. So $\mathbb{S}q(\Omega_n)$ is not planar.

Case V: If $n = p_1^{s_1} p_2^{s_2}$ where $s_1 > 1$ and $s_2 > 3$, then there is a subgraph induced by the subset $\{\langle\overline{1}\rangle, \langle\overline{p_1^2}\rangle, \langle\overline{p_2^2}\rangle, \langle\overline{p_2^4}\rangle, \langle\overline{p_1^2 p_2^2}\rangle\}$ which is isomorphic to K_5. Thus $\mathbb{S}q(\Omega_n)$ is not planar.

Case VI: If $n = p_1 p_2 p_3 p_4^{s_4} \cdots p_r^{s_r}$ where $p_1, p_2, p_3, \ldots, p_r$ are distinct primes, $s_i \geq 0$ and $r \geq 3$, then we have a subgraph induced by the subset $\{\langle\overline{1}\rangle, \langle\overline{p_1}\rangle, \langle\overline{p_2}\rangle, \langle\overline{p_3}\rangle, \langle\overline{p_1 p_2}\rangle\}$ which is isomorphic to K_5. Hence $\mathbb{S}q(\Omega_n)$ is not planar.

Case VII: If $n = p_1 p_2 p_3^{s_3} \cdots p_r^{s_r}$ where $p_1, p_2, p_3, \ldots, p_r$ are distinct primes, $r \geq 3$ and $s_3 > 1$, then we have a subgraph induced by the subset $\{\langle\overline{1}\rangle, \langle\overline{p_1}\rangle, \langle\overline{p_2}\rangle, \langle\overline{p_1 p_2}\rangle, \langle\overline{p_3^2}\rangle\}$ which is isomorphic to K_5. Hence $\mathbb{S}q(\Omega_n)$ is not planar in this case as well.

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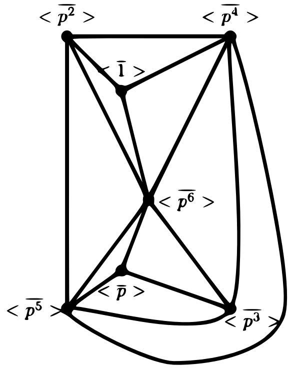

Fig. 9. $\mathbb{S}_q(\Omega_{p^7})$.

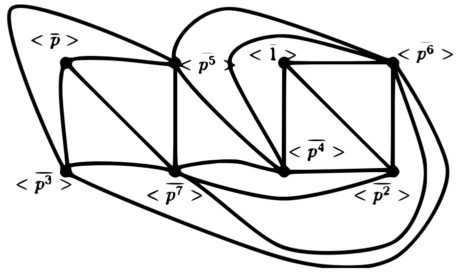

Fig. 10. $\mathbb{S}_q(\Omega_{p^8})$.

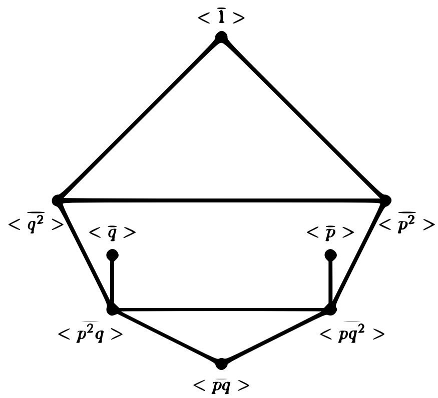

Fig. 11. $\mathbb{S}_q(\Omega_{p^2q^2})$.

Case VIII: If $n = p_1 p_2^{s_2} \cdots p_r^{s_r}$ where $p_1, p_2, p_3, \ldots, p_r$ are distinct primes, $r \geq 3$ and $s_2, s_3 > 1$, then we have a subgraph induced by the subset $\{\langle\overline{1}\rangle, \langle\overline{p_1}\rangle, \langle\overline{p_2^2}\rangle, \langle\overline{p_3^2}\rangle, \langle\overline{p_2^2 p_3^2}\rangle\}$ which is isomorphic to K_5. Thus $\mathbb{S}q(\Omega_n)$ is not planar.

Case IX: Finally, if $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$ where $p_1, p_2, p_3, \ldots, p_r$ are distinct primes, $r \geq 3$ and $s_1, s_2, s_3 > 1$, then we have a subgraph induced by the subset $\{\langle\overline{1}\rangle, \langle\overline{p_1^2}\rangle, \langle\overline{p_2^2}\rangle, \langle\overline{p_3^2}\rangle, \langle\overline{p_1^2 p_2^2}\rangle\}$ which is isomorphic to K_5. Thus $\mathbb{S}q(\Omega_n)$ is not planar.

So having considered all possible cases, we infer that $\mathbb{S}q(\Omega_n)$ is planar if and only if n is of the form pq^r $(0 < r \leq 3)$, p^s (for $s \leq 8$) or p^2q^2, where p, q are distinct primes. $\square$

We next consider the existence of cycles in $\mathbb{S}q(\Omega_n)$.

Please cite this article as: B. Biswas, R. Sen Gupta, M.K. Sen et al., Some properties of Square element graphs over semigroups, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.02.001.

Theorem 2.5. *$\mathbb{S}q(\Omega_n)$ is acyclic if and only if $n = p^k$ for some prime p and some $k \in \{1, 2, 3, 4\}$. For all other values of n, $gr(\mathbb{S}q(\Omega_n)) = 3$.*

Proof. It is easy to see that for a prime p, $\mathbb{S}q(\Omega_p) \cong K_1$, $\mathbb{S}q(\Omega_{p^2}) \cong 2K_1$, $\mathbb{S}q(\Omega_{p^3}) \cong P_3$ and $\mathbb{S}q(\Omega_{p^4}) \cong P_4$. So $\mathbb{S}q(\Omega_n)$ is acyclic if $n = p^k$ for some prime p and $k \in \{1, 2, 3, 4\}$. We next show that for all other values of n, $\mathbb{S}q(\Omega_n)$ contains a 3-cycle, and hence, is not acyclic.

If $n = p_1p_2$, then $\langle\overline{1}\rangle \leftrightarrow \langle\overline{p_1}\rangle \leftrightarrow \langle\overline{p_2}\rangle \leftrightarrow \langle\overline{1}\rangle$ is a 3-cycle.

If $n = p_1^2p_2$, then $\langle\overline{1}\rangle \leftrightarrow \langle\overline{p_1^2}\rangle \leftrightarrow \langle\overline{p_2}\rangle \leftrightarrow \langle\overline{1}\rangle$ is a 3-cycle.

If $n = p^rq$ with $r > 2$, then $\langle\overline{p^{r-1}}\rangle \leftrightarrow \langle\overline{pq}\rangle \leftrightarrow \langle\overline{p^{r-1}q}\rangle \leftrightarrow \langle\overline{p^{r-1}}\rangle$ is a 3-cycle.

If $n = p^rq^s$ with $r, s \geq 2$, then $\langle\overline{pq}\rangle \leftrightarrow \langle\overline{p^rq^{s-1}}\rangle \leftrightarrow \langle\overline{p^{r-1}q^s}\rangle \leftrightarrow \langle\overline{pq}\rangle$ is a 3-cycle.

If $n = p^2qr$, then $\langle\overline{p^2r}\rangle \leftrightarrow \langle\overline{pqr}\rangle \leftrightarrow \langle\overline{p^2q}\rangle \leftrightarrow \langle\overline{p^2r}\rangle$ is a 3-cycle.

If $n = p_1^{s_1}p_2^{s_2}\cdots p_r^{s_r}$ with $r > 2$ and $s_i > 1$ for some i, then $\langle\overline{p_i^{s_i-1}p_r^{s_r}}\rangle \leftrightarrow \langle\overline{p_1^{s_1}p_2^{s_2}\cdots p_i\cdots p_r^{s_r}}\rangle \leftrightarrow \langle\overline{p_1^{s_1}p_2^{s_2}\cdots p_i^{s_i}\cdots p_{r-1}^{s_{r-1}}}\rangle \leftrightarrow \langle\overline{p_i^{s_i-1}p_r^{s_r}}\rangle$ is a 3-cycle.

If $n = p_1p_2\cdots p_r$ with $r \geq 3$, then $\langle\overline{p_1p_3p_4\cdots p_r}\rangle \leftrightarrow \langle\overline{p_2p_3p_4\cdots p_r}\rangle \leftrightarrow \langle\overline{p_1p_2p_4\cdots p_r}\rangle \leftrightarrow \langle\overline{p_1p_3\cdots p_r}\rangle$ is a 3-cycle.

If $n = p^s$ with $s > 4$, then $\langle\overline{p^{s-1}}\rangle \leftrightarrow \langle\overline{p^{s-2}}\rangle \leftrightarrow \langle\overline{p^{s-3}}\rangle \leftrightarrow \langle\overline{p^{s-1}}\rangle$ is a 3-cycle.

Thus $gr(\mathbb{S}q(\Omega_n)) = 3$ for all the above cases. This completes the proof. □

3. Some results on $\mathbb{S}q(S_n)$

In this section, we discuss the graph $\mathbb{S}q(S_n)$, where S_n is the symmetric group on a finite set of n symbols. First, we give an interesting result, which is helpful in determining the adjacencies in $\mathbb{S}q(S_n)$. The result was proved by M. Snowden and J.M. Howie [9].

Theorem 3.1 (*Theorem 1, [9]*)**.** *An element α of S_n is a square if and only if for each even number k the decomposition of α into disjoint cycles involves an even number of cycles of length k.*

Remark 3.2. Using the above theorem we can show that the set of all squares in S_n is given by precisely the permutations belonging to the subgroup A_n. For example, a square in S_4 is either a 3-cycle, or a product of 2-cycles or the identity permutation ρ_0, i.e., the set of all squares of S_4 consists precisely of the elements belonging to the alternative group A_4.

Example 3.3. Let us consider the graph $\mathbb{S}q(S_3)$. The non-commutative group S_3 contains precisely three squares $\{e, (123), (132)\}$. From Fig. 12 it is seen that $\mathbb{S}q(S_3) \cong K_3 + \overline{K_1 + K_2}$.

We now give the general structure of $\mathbb{S}q(S_n)$.

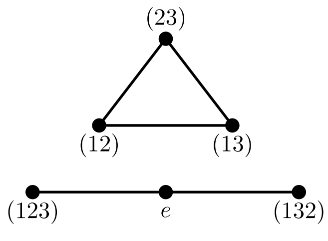

Fig. 12. $\mathbb{S}q(S_3)$.

Theorem 3.4. *For $n \geq 3$, $\mathbb{S}q(S_n)$ is a disjoint union of $\overline{mK_1 + \frac{n!-2m}{4}K_2}$ and $\overline{(p+1)K_1 + \frac{n!-2p-2}{4}K_2}$, where p is the number of those permutations in S_n which are the products of an even number of disjoint 2-cycles, and m is the number of those permutations in S_n which are the products of an odd number of disjoint 2-cycles.*

Proof. In the group S_n, the square elements are precisely the even permutations (by Remark 3.2). Then any two distinct elements $\rho, \alpha \in S_n$ are adjacent if and only if either $\rho\alpha \in A_n \setminus \{\rho_0\}$ or $\alpha\rho \in A_n \setminus \{\rho_0\}$ where ρ_0 is the identity permutation. If $\rho \in A_n$ and $\alpha \in S_n \setminus A_n$, then ρ and α are not adjacent to each other since neither of $\rho\alpha$ and $\alpha\rho$ belongs to $A_n \setminus \{\rho\}$. Therefore no element of A_n is adjacent to any element of $S_n \setminus A_n$. This implies that $\mathbb{S}q(S_n)$ is disconnected. Again, any element $\rho \in A_n$ is adjacent to the identity element ρ_0 as $\rho_0\rho = \rho$. Let $\rho_1, \rho_2 \in A_n$. Then $\rho_1\rho_2 \in A_n$. Let $\alpha \in A_n$ be a permutation of order greater than 2. Then $\alpha \neq \alpha^{-1}$. Thus α is adjacent to any other element of A_n except its inverse α^{-1}. Now let β be a permutation which is a product of an even number of disjoint 2-cycles. Then $\beta = \beta^{-1}$. Hence β is adjacent to any other element of A_n. Let p be the number of those permutations which are the products of an even number of disjoint 2-cycles. So the subgraph induced by A_n is isomorphic to $(p+1)K_1 + \frac{n!-2p-2}{4}K_2$. Again, consider two elements $\alpha, \beta(\neq \alpha^{-1}) \in S_n \setminus A_n$. Clearly, $\alpha\beta(\neq \rho_0)$ is an even permutation. So α and β are adjacent. Now let $\alpha, \alpha^{-1} \in S_n \setminus A_n$. Then there exists an element $x(\neq \alpha^{-1}) \in S_n \setminus A_n$ different from α, α^{-1} such that we have a path $\alpha \leftrightarrow x \leftrightarrow \alpha^{-1}$. Thus the subgraph induced by $S_n \setminus A_n$ is a connected subgraph. Let $\alpha \in S_n \setminus A_n$ be a permutation of order greater than 2. Then $\alpha \neq \alpha^{-1}$. So α is adjacent to any other element of $S_n \setminus A_n$ except α^{-1}. Again, let β be a permutation which is product of odd number of disjoint 2-cycles. Then $\beta = \beta^{-1}$ and hence β is adjacent to any other element of $S_n \setminus A_n$. So if m is the number of permutations which are product of odd number of disjoint 2-cycles, we have that the subgraph induced by $S_n \setminus A_n$ is isomorphic to $mK_1 + \frac{n!-2m}{4}K_2$. $\square$

Next, we find out the values of n for which $\mathbb{S}q(S_n)$ is planar.

Proposition 3.5. *$\mathbb{S}q(S_n)$ is planar if and only if $n \in \{2, 3\}$.*

Proof. Let $p = |\{(ab)(cd) \in S_n \mid a, b, c, d \text{ are distinct}\}|$. From the proof of Theorem 3.4, it follows that $\mathbb{S}q(S_n)$ has a subgraph isomorphic to K_{p+1} induced by the vertices of the form $(ab)(cd)$ and the identity permutation. If $n \geq 5$, then $p \geq 4$. Hence we have a subgraph in $\mathbb{S}q(S_n)$ which is isomorphic to K_5. Thus $\mathbb{S}q(S_n)$ is not planar for $n \geq 5$. Now for $n = 4$, consider the set $S = \{(12)(34), (13)(24), (14)(23), (123), \rho_0\}$. Then the subgraph induced by S is isomorphic to K_5. So $\mathbb{S}q(S_4)$ is not planar. Again it is easy to see that $\mathbb{S}q(S_n)$ is planar for $n = 2, 3$. Thus the graph $\mathbb{S}q(S_n)$ is planar if and only if $n = 2$ or 3. $\square$

We now consider the domination number of $\mathbb{S}q(S_n)$. It is interesting to note that the domination number of $\mathbb{S}q(S_n)$ is same for all $n > 1$, as we show next.

Proposition 3.6. *$\gamma(\mathbb{S}q(S_n)) = 2$ for $n \geq 2$.*

Proof. Since the graph $\mathbb{S}q(S_n)$ is a disjoint union of two components (by Theorem 3.4), we have that $\gamma(\mathbb{S}q(S_n)) \geq 2$. It is easy to see that $\mathbb{S}q(S_2) \cong 2K_1$, so $\{\rho, (1, 2)\}$ forms a minimal dominating set. For $n \geq 3$, consider a 2-cycle $\rho \in S_n \setminus A_n$. Let $\alpha(\neq \rho) \in S_n \setminus A_n$. Then $\alpha\rho$ is an even permutation as both α and ρ are odd permutations. Hence, $\alpha\rho(\neq \rho_0) \in A_n$ as $\alpha \neq \rho^{-1}(= \rho)$. So $\alpha \leftrightarrow \rho$. Since α is arbitrary, it follows that every element in $S_n \setminus A_n$ is adjacent to ρ. Now we consider the set $D = \{\rho, \rho_0\}$ where ρ_0 is the identity element. Any vertex from $S_n \setminus \{A_n \setminus D\}$ is adjacent to ρ and any vertex from $A_n \setminus D$ is adjacent to ρ_0. Hence D is a dominating set. Thus $\gamma(\mathbb{S}q(S_n)) \leq 2$. Since we have already shown that $\gamma(\mathbb{S}q(S_n)) \geq 2$, it follows that $\gamma(\mathbb{S}q(S_n)) = 2$. $\square$

4. The structure and some properties of $\mathbb{S}q(D_n)$

In this section, we study the square element graphs over the dihedral groups D_n. Before looking at the properties of $\mathbb{S}q(D_n)$, we start the section by giving a structural result which holds for $\mathbb{S}q(G)$ defined over any group G whenever the set of all squares of G forms a (normal) subgroup of G.

Lemma 4.1. *Let H be the set of all squares of a group G. If H is a (normal) subgroup of G, then the elements belonging to distinct cosets of H are not adjacent to each other in $\mathbb{S}q(G)$.*

Proof. Let $G = \{e, x_1, x_2, \ldots, x_n\}$. Since H forms a subgroup of G (which can be easily proved to be a normal subgroup), the product of two squares in G is also a square in G. Clearly, $yH = y^{-1}H$ for any $y \in G$. If possible, let there exist vertices p, q belonging to distinct cosets aH and bH (respectively), such that $p \leftrightarrow q$ in $\mathbb{S}q(G)$. Suppose $p = ax_r^2, q = bx_s^2$. So without loss of generality we have that $ax_r^2bx_s^2 = x_t^2$ for some $t \in \{1, 2, \ldots, n\}$. This implies that $ax_r^2b = x_t^2x_s^{-2} \in H$, which gives that $x_r^2b \in a^{-1}H = aH = Ha$. So $b \in x_r^{-2}Ha = Ha = aH$. However, this is a contradiction since $b \in bH$ and distinct cosets of H are disjoint. So p and q can be adjacent to each other only if $aH = bH$. Thus two vertices belonging to distinct cosets of H cannot be adjacent to each other in $\mathbb{S}q(G)$. $\square$

It is easily seen that the set of all squares of D_n forms a subgroup of D_n. So the above lemma is applicable for D_n. Using the above lemma, we can find the structure of $\mathbb{S}q(D_n)$ for an odd integer n.

Theorem 4.2. *If n is an odd integer, then* $\mathbb{S}q(D_n) \cong \overline{K_1 + \frac{n-1}{2}K_2} + K_n$.

Proof. It is known that we can write $D_n = \{e, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\}$, where $a^n = e = b^2$ and $ab = ba^{n-1}$. Clearly, the set of all squares of D_n is given by $H = \{e, a, a^2, \dots, a^{n-1}\}$, which is a cyclic subgroup of odd order. Also, $D_n = H \cup bH$. By Lemma 4.1, vertices from H and vertices from bH are in different components in $\mathbb{S}q(D_n)$. Now $e \leftrightarrow a^m$ for all $m = 1, 2, \dots, n-1$; and $a^i \leftrightarrow a^j$ if and only if $i + j \neq n$. Thus, the vertices belonging to H induce a subgraph G_n where $\overline{G_n}$ consists of $\frac{n-1}{2}$ disjoint copies of K_2 and one isolated vertex. Next, we consider the coset bH. Noting that every element in bH is of order 2 and that H is a normal subgroup, we see that the vertices from bH induce a complete graph in $\mathbb{S}q(D_n)$. Hence, $\mathbb{S}q(D_n) \cong \overline{K_1 + \frac{n-1}{2}K_2} + K_n$. □

Next, we give the structure of $\mathbb{S}q(D_n)$ when n is even.

Theorem 4.3. *If n is an even integer, then*

$$\mathbb{S}q(D_n) \cong \begin{cases} 2(\overline{K_1 + \dfrac{n-2}{4}K_2}) + 2K_{\frac{n}{2}} & \text{if } \dfrac{n}{2} \text{ is odd} \\[2ex] \overline{\dfrac{n}{4}K_2} + \overline{2K_1 + \dfrac{n-4}{4}K_2} + 2K_{\frac{n}{2}} & \text{if } \dfrac{n}{2} \text{ is even.} \end{cases}$$

Proof. Let $D_n = \langle a, b\rangle$ where $a^n = b^2 = e$ and $ab = ba^{n-1}$. So we can write $D_n = \{a, a^2, a^3, \dots, a^{n-1}, a^n(= e), b, ba, ba^2, \dots, ba^{n-1}\}$. Since n is even, the set of all squares of D_n is given by $H = \{a^2, a^4, \dots, a^{n-2}, e\}$. It can be shown that H is a normal subgroup of D_n. It is easy to see that there are 4 distinct cosets H, aH, bH, baH of H which partition the group D_n. Clearly, $aH = \{a, a^3, a^5, a^7, \dots, a^{n-1}\}$, $bH = \{ba^2, ba^4, ba^6, \dots, ba^{n-2}\}$ and $baH = \{ba, ba^3, ba^5, ba^7, \dots, ba^{n-1}\}$. Now the subgraphs induced by these 4 cosets H, aH, bH, baH are disjoint from each other by Lemma 4.1. In $\mathbb{S}q(D_n)$, any two elements of the form a^i and a^j are adjacent if and only if $i + j\ (\neq n)$ is even; and two distinct elements of the form ba^i, ba^j are adjacent if and only if $ba^i ba^j = ba^i ba^i a^{j-i} = (ba^i)^2 a^{j-i} = a^{j-i}$ is a non-identity square, i.e., if and only if $j - i$ is an even number. So in bH, we note that $(ba^i)^2 = e$ and any two distinct vertices ba^i, ba^j are adjacent to each other as $ba^i ba^j = a^{i-j} \in H \setminus \{e\}$ (as $i - j$ is even). Again in baH, $(ba^{2k+1})^2 = e$ and any two distinct vertices ba^{2k+1}, ba^{2m+1} are adjacent to each other as $ba^{2k+1}ba^{2m+1} = a^{2(k-m)} \in H \setminus \{e\}$. This implies that the subgraphs induced by bH and baH are both isomorphic to $K_{\frac{n}{2}}$.

First, let $\frac{n}{2}$ be an even integer. Then $H = \{a^2, a^4, \dots, a^{\frac{n}{2}}, \dots, a^{n-2}, e\}$ and $aH = \{a, a^3, \dots, a^{\frac{n}{2}+1}, \dots, a^{n-1}\}$. In H, e and $a^{\frac{n}{2}}$ are the only self-invertible squares. Thus e and $a^{\frac{n}{2}}$ are adjacent to all other vertices in the subgraph induced by H except themselves. For any other element v of H, v is adjacent to all other vertices of the subgraph induced by H except itself and its own inverse. Therefore the subgraph induced by H is isomorphic to $\overline{2K_1 + \frac{n-4}{4}K_2}$. In the subgraph induced by aH, any vertex is adjacent with all other vertices of that subgraph except itself and its own inverse. Thus the subgraph induced by aH is isomorphic to $\overline{\frac{n}{4}K_2}$.

Next, let $\frac{n}{2}$ be an odd integer. So $H = \{a^2, a^4, \dots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \dots, a^{n-2}, e\}$ and $aH = \{a, a^3, \dots, a^{\frac{n}{2}}, \dots, a^{n-1}\}$. In H, e is the only element which is self-invertible. So e is adjacent to any other vertex in H, and any $v(\neq e)$ is adjacent to all other vertices in that subgraph except its own inverse. Thus the subgraph induced by H is isomorphic to $\overline{K_1 + \frac{n-2}{4}K_2}$. In aH, $a^{\frac{n}{2}} = (a^{\frac{n}{2}})^{-1}$ and so $a^{\frac{n}{2}}$ is adjacent to all other vertices in the subgraph induced by aH. Any other vertex $a^{2k+1}(\neq a^{\frac{n}{2}})$ is not self-invertible and hence is adjacent to all other vertices in that subgraph except its own inverse a^{n-2k-1}. Hence the subgraph induced by aH is isomorphic to $\overline{K_1 + \frac{n-2}{4}K_2}$. So the subgraphs induced by H and aH are isomorphic to $\overline{2K_1 + \frac{n-4}{4}K_2}$ and $\overline{\frac{n}{4}K_2}$ (respectively) if $\frac{n}{2}$ is even, and are both isomorphic to $\overline{K_1 + \frac{n-2}{4}K_2}$ if $\frac{n}{2}$ is odd. Therefore, $\mathbb{S}q(D_n) \cong \overline{\frac{n}{4}K_2} + \overline{(2K_1 + \frac{n-4}{4}K_2)} + 2K_{n/2}$ if $\frac{n}{2}$ is even and $\mathbb{S}q(D_n) \cong 2(\overline{K_1 + \frac{n-2}{4}K_2}) + 2K_{n/2}$ if $\frac{n}{2}$ is odd. □

Now we consider the planarity of $\mathbb{S}q(D_n)$.

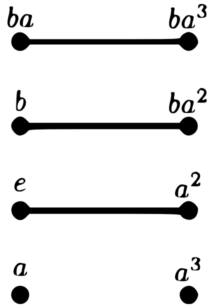

Fig. 13. $\mathbb{S}q(D_4)$.

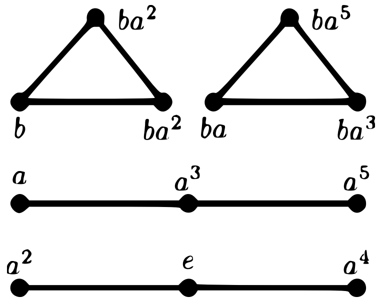

Fig. 14. $\mathbb{S}q(D_6)$.

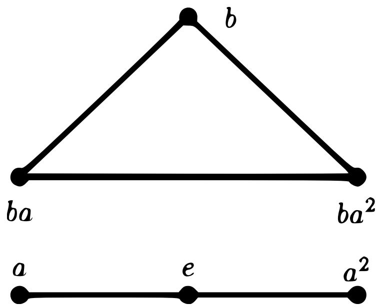

Fig. 15. $\mathbb{S}q(D_3)$.

Theorem 4.4. *$\mathbb{S}q(D_n)$ is planar if and only if $n \in \{1, 2, 3, 4, 6, 8\}$.*

Proof. If n is an odd integer and $n \geq 5$, then there is a subgraph of $\mathbb{S}q(D_n)$ which is isomorphic to K_5 (by Theorem 4.2). Hence in this case $\mathbb{S}q(D_n)$ is not planar. Now $\mathbb{S}q(D_1)$ has only two vertices and hence is planar. Again, $\mathbb{S}q(D_3)$ is also seen to be planar (cf. Fig. 15). Now let n be even. Then there is a subgraph of $\mathbb{S}q(D_n)$ which is isomorphic to $K_{\frac{n}{2}}$ (by Theorem 4.3). In this case, if $n \geq 10$, then there exists a subgraph of $\mathbb{S}q(D_n)$ which is isomorphic to K_5. Hence $\mathbb{S}q(D_n)$ is not planar for any even $n \geq 10$. Finally, we consider $\mathbb{S}q(D_n)$ for $n = 2, 4, 6, 8$. The graphs $\mathbb{S}q(D_4)$, $\mathbb{S}q(D_6)$, and $\mathbb{S}q(D_8)$, as shown in Figs. 13, 14, and 16, are planar. Having considered all the possible cases, we see that $\mathbb{S}q(D_n)$ is planar if and only if $n \in \{1, 2, 3, 4, 6, 8\}$. □

Moving on, we find the chromatic number of $\mathbb{S}q(D_n)$ for different values of n.

Theorem 4.5.

$$\chi(\mathbb{S}q(D_n)) = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

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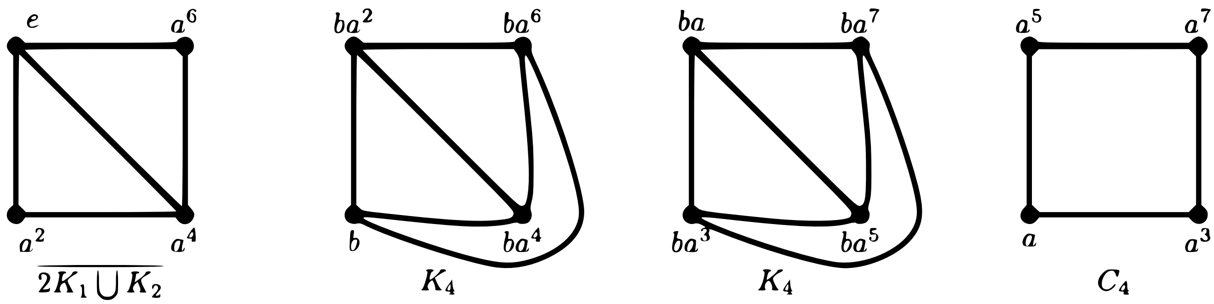

Fig. 16. $\mathbb{S}q(D_8)$.

Proof. Let $D_n = \langle a, b\rangle$ where $a^n = b^2 = e$ and $ab = ba^{n-1}$. First, let n be odd. Then it follows from the proof of Theorem 4.2 that in $\mathbb{S}q(D_n)$ there are exactly two components (induced by H and bH). In this case, the subgraph induced by bH is isomorphic to K_n. Now we associate n different colours $c_1, c_2, \ldots, c_n$ with the n distinct vertices of bH, and we also correspond those n colours $c_1, c_2, \ldots, c_n$ to the n distinct vertices of H. In this way we are able to colour every vertex of $\mathbb{S}q(D_n)$ such that no two adjacent vertices have the same colour. Therefore $\chi(\mathbb{S}q(D_n)) \le n$. Again, $\omega(\mathbb{S}q(D_n)) \ge |V(K_n)| = n$. So $\chi(\mathbb{S}q(D_n)) \ge \omega(\mathbb{S}q(D_n)) \ge n$. Thus $\chi(\mathbb{S}q(D_n)) = n$.

Next, let us assume that n is even. Then by the proof of Theorem 4.3, we have that $\mathbb{S}q(D_n)$ is a disjoint union of 4 subgraphs (induced by the cosets H, aH, bH and baH). In this case we need to associate $\frac{n}{2}$ different colours $c_1, c_2, \ldots, c_{\frac{n}{2}}$ to the $\frac{n}{2}$ vertices of bH as the subgraph induced by bH is isomorphic to $K_{\frac{n}{2}}$. We note that the subgraphs induced by H, aH, baH and bH are disjoint components having $\frac{n}{2}$ vertices each. So for each component, we can correspond those $\frac{n}{2}$ colours to the distinct vertices. Hence $\chi(\mathbb{S}q(D_n)) \le \frac{n}{2}$. Again, $\omega(\mathbb{S}q(D_n)) \ge |V(K_{\frac{n}{2}})| = \frac{n}{2}$. Hence $\chi(\mathbb{S}q(D_n)) \ge \omega(\mathbb{S}q(D_n)) \ge \frac{n}{2}$. So $\chi(\mathbb{S}q(D_n)) = \frac{n}{2}$. □

In the next result, we consider the domination number of $\mathbb{S}q(D_n)$.

Proposition 4.6.

$$\gamma(\mathbb{S}q(D_n)) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 5 & \text{if } n \text{ is even and } \dfrac{n}{2} \text{ is also even} \\ 4 & \text{if } n \text{ is even and } \dfrac{n}{2} \text{ is odd.} \end{cases}$$

Proof. Let $D_n = \langle a, b\rangle$ where $a^n = b^2 = e$ and $ab = ba^{n-1}$. If n is odd, then we consider the set $D = \{e, b\}$. From the proof of Theorem 4.2, we can easily see that D is a minimal dominating subset of the graph $\mathbb{S}q(D_n)$. Thus $\gamma(\mathbb{S}q(D_n)) = 2$. Again, if n is even and $\frac{n}{2}$ is odd, then from Theorem 4.3, $\mathbb{S}q(D_n)$ is disjoint union of 4 components. So $\gamma(\mathbb{S}q(D_n)) \ge 4$. From the proof of Theorem 4.3, it is easily seen that the set $A_1 = \{e, a, b, ba\}$ is a dominating subset. So $\gamma(\mathbb{S}q(D_n)) \le 4$. Therefore $\gamma(\mathbb{S}q(D_n)) = 4$. Next, let both n and $\frac{n}{2}$ be even. If H is the set of all squares of D_n, then from the proof of Theorem 4.3 we have that $\mathbb{S}q(D_n)$ is a disjoint union of 4 subgraphs induced by the 4 cosets H, aH, bH and baH. Let us consider the set $A_2 = \{e, b, ba, a, a^{n-1}\}$. Now e is adjacent to any element of $H \setminus \{e\}$, b is adjacent to any element of $bH \setminus \{b\}$, ba is adjacent to any element of $baH \setminus \{ba\}$. Also, any element of $aH \setminus \{a, a^{n-1}\}$ is adjacent to either a or a^{n-1}. Hence A_2 is a dominating set for $\mathbb{S}q(D_n)$. It can be easily checked that none of $\{A_2 \setminus \{a\}, A_2 \setminus \{a^{n-1}\}, A_2 \setminus \{e\}, A_2 \setminus \{b\}, A_2 \setminus \{ba\}\}$ is a dominating set (note that a is not adjacent to a^{n-1}). Therefore A_2 is a minimal dominating set of the graph $\mathbb{S}q(D_n)$. Thus $\gamma(\mathbb{S}q(D_n)) = 5$ in this case. This completes the proof. □

We conclude the paper by considering the independence number (i.e., the cardinality of a maximal set of independent vertices) $\alpha(\mathbb{S}q(D_n))$ of $\mathbb{S}q(D_n)$.

Proposition 4.7.

$$\alpha(\mathbb{S}q(D_n)) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 6 & \text{if } n \text{ is even.} \end{cases}$$

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Proof. Let $D_n = \langle a, b\rangle$ where $a^n = b^2 = e$ and $ab = ba^{n-1}$. If n is odd, then we consider the set $I = \{b, a, a^{n-1}\}$. From the proof of Theorem 4.2, we see that I is a maximal set of independent vertices of the graph $\mathbb{S}q(D_n)$. Thus $\alpha(\mathbb{S}q(D_n)) = 3$. Next, let n be even. As seen in Theorem 4.3, $\mathbb{S}q(D_n)$ is a disjoint union of 4 subgraphs induced by the vertices belonging to the 4 cosets H, aH, bH and baH, where H is the set of all squares of D_n. Now any two vertices in the component induced by bH are adjacent to each other. The same is true for the component induced by baH. Thus in any independent set of vertices of $\mathbb{S}q(D_n)$, there can be only one element each from these cosets. Considering the set H, any subset containing at least three elements is not independent, and the same is true for aH as well. Hence the cardinality of any independent set is at most $1 + 1 + 2 + 2 = 6$. In other words $\alpha(\mathbb{S}q(D_n)) \leq 6$. Now we consider the set $I_1 = \{a, a^{n-1}, a^2, a^{n-2}, b, ba\}$. It is easy to see that I_1 is an independent set of vertices and since $|I_1| = 6$, it follows that $\alpha(\mathbb{S}q(D_n)) \geq 6$. Hence $\alpha(\mathbb{S}q(D_n)) = 6$. $\square$

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