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On the edge irregularity strength of grid graphs

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Abstract

For a simple graph G, a vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called a vertex k-labeling. For any edge xy in G, its weight $w_\phi(xy) = \phi(x) + \phi(y)$. If all the edge weights are distinct, then ϕ is called an edge irregular k-labeling of G. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$.

In this paper, we determine an exact value of edge irregularity strength for triangular grid graph L_n^m, zigzag graph Z_n^m and Cartesian product $P_n \square P_m \square P_2$.

Keywords: Edge irregularity strength; Triangular grid graph; Zigzag graph; Cartesian product

1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set and edge set denoted by $V(G)$ and $E(G)$ respectively. Motivated by irregular assignments, irregularity strength of graphs was introduced by Chartrand et al. [1] and papers [2–18], Ahmad, Al-Mushayt and Bača [19] defined the notion of an edge irregular k-labeling of a graph G to be labeling of the vertices of G, $\phi : V(G) \to \{1, 2, \ldots, k\}$ such that, the edge weights $w_\phi(vu) = \phi(v) + \phi(u)$ are distinct for every edges. The minimum k for which the graph $G = (V, E)$ has an edge irregular k-labeling is called an edge irregularity strength of G, denoted by $es(G)$.

The following theorem established the lower bound of edge irregularity strength of a graph G:

Theorem 1 (*[19]*). *Let $G = (V, E)$ be a simple graph with maximum degree $\Delta = \Delta(G)$. Then*

$$es(G) \geq \max\left\{\left\lceil \frac{|E(G)| + 1}{2} \right\rceil, \Delta(G)\right\}.$$

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In [19], the authors estimated the bounds of the edge irregularity strength es and then determined its exact values for several families of graphs namely, paths, stars, double stars and Cartesian product of two paths. Mushayt [20] determined the edge irregularity strength of Cartesian product of star, cycle with path P_2 and strong product of path P_n with P_2. Tarawneh et al. [21–23] investigated the exact value of edge irregularity strength of corona product of graph with paths and cycles; and corona product of cycle with isolated vertices, respectively. Ahmad [24] determined the exact value of the edge irregularity strength of corona graph $C_n \odot K_1$ (or sun graph S_n). Very recently, Ahmad et al. [25] obtained the exact value of the edge irregularity strength of Toeplitz graphs.

In this paper, we determine the exact value of edge irregularity strength for some families of grid graphs.

2. Triangle grid graph

For $n \geq 2,\ m \geq 1$, let L_n^m be triangle grid graph with the vertex set $V(L_n^m) = \{x_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m+1\}$ and the edge set $E(L_n^m) = \{x_{i,j}x_{i+1,j} : 1 \leq i \leq n-1,\ 1 \leq j \leq m+1\} \cup \{x_{i,j}x_{i,j+1} : 1 \leq i \leq n,\ 1 \leq j \leq m\} \cup \{x_{i+1,j}x_{i,j+1} : 1 \leq i \leq n-1,\ 1 \leq j \leq m \text{ and j is odd}\} \cup \{x_{i,j}x_{i+1,j+1} : 1 \leq i \leq n-1,\ 1 \leq j \leq m \text{ andj is even}\}$.

In the following theorem, we determine the exact value of the edge irregularity strength of triangle grid graph $L_n = L_n^1, n \geq 2$.

Theorem 2. *For any integer $n \geq 2$, $es(L_n) = 2n$.*

Proof. Let L_n be a graph with the vertex set $V(L_n)$ and the edge set $E(L_n)$. Clearly $\Delta(L_n) = 4$. From Theorem 1, we have $es(L_n) \geq \max\{2n-1, 4\} = 2n-1$. Since the edges $x_{i,j}$, $x_{i+1,j}$ and $x_{i,j+1}$ are parts of complete graph K_3, the smallest edge weight has to be at least 3. Thus, the edge weights under the labeling ϕ_1 successively attain values $\{3, 4, \ldots, 2n\}$. This implies that $es(L_n) \geq 2n$. To prove the inequality $es(L_n) \leq 2n$, we define a suitable vertex labeling $\phi_1 : V(L_n) \to \{1, 2 \ldots, 2n\}$

$$\phi_1(x_{i,j}) = \frac{1}{2}((-1)^j - 1) + 2i,\ 1 \leq i \leq n \text{ and } 1 \leq j \leq 2.$$

Since $w_{\phi_1}(x_{i,j}x_{i+1,j}) = 4i + (-1)^j + 1,\ 1 \leq i \leq n-1,\ 1 \leq j \leq 2$, $w_{\phi_1}(x_{i,j+1}x_{i+1,j}) = 4i+1,\ 1 \leq i \leq n-1,\ j = 1$ and $w_{\phi_1}(x_{i,j}x_{i,j+1}) = 4i - 1,\ 1 \leq i \leq n,\ j = 1$. so the edge weights are different for all pairs of distinct edges. Thus, the vertex labeling ϕ_1 is an optimal edge irregular $2n$-labeling. This completes the proof. □

In Theorem 2, we determined the exact value of edge irregularity strength es of triangle grid graph $L_n = L_n^1$, $n \geq 2$. So far we are not able to find es of L_n^m for $m \geq 2$. So we conclude the following open problem.

Problem 1. Determine the exact value of $es(L_n^m)$ for $m \geq 2$.

3. Zigzag graph

For $n, m \geq 2$, the zigzag graph Z_n^m is given in Fig. 1. Its vertex set and edge set are given by m rows and n columns.

The symbols $V(Z_n^m)$ and $E(Z_n^m)$ denote the vertex set and the edge set of zigzag graph Z_n^m. Moreover $V(Z_n^m) = \{x_{i,j} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ with $|V(Z_n^m)| = mn$ and $E(Z_n^m) = \{x_{i,j}x_{i+1,j} : 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m\} \cup \{x_{i,j}x_{i-1,j+1} : 2 \leq i \leq n \text{ and } 1 \leq j \leq m-1\}$ with $|E(Z_n^m)| = 2mn - 2m - n + 1$.

In [4], the authors determined the exact values of the total edge irregularity strength tes of zigzag graphs Z_n^m. Motivated by this paper, we determine the exact values of the edge irregularity strength es of zigzag graphs Z_n^m in the next theorem.

Theorem 3. *For $m, n \geq 2$, $es(Z_n^m) = \left\lceil \frac{2mn-2m-n+2}{2} \right\rceil$.*

Proof. Let Z_n^m be a zigzag graph with the vertex set $V(Z_n^m)$ and edge set $E(Z_n^m)$. Note that $|V(Z_n^m)| = nm$, $|E(Z_n^m)| = 2mn - 2m - n + 1$.

We deal with zigzag graph Z_n^m for all $n, m \geq 2$. In [19], it was proved that $es(G) \geq max\left\{\left\lceil \frac{|E(G)|+1}{2} \right\rceil, \triangle(G)\right\}$. Since $\triangle(Z_n^m) = 4$, it follows from Theorem 1 that $es(G) \geq \left\lceil \frac{|E(G)|+1}{2} \right\rceil$. To show that $\left\lceil \frac{2mn-2m-n+2}{2} \right\rceil$ is an upper bound for the $es(Z_n^m)$, we describe an edge irregular $\left\lceil \frac{2mn-2m-n+2}{2} \right\rceil$-labeling for Z_n^m.

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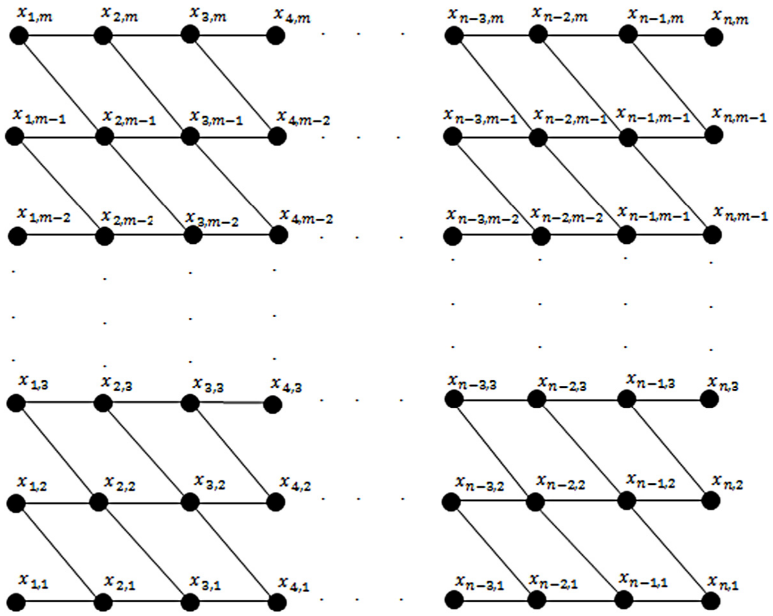

Fig. 1. Zigzag graph.

Let $\phi_1 : V(Z_n^m) \to \{1, 2, \ldots, \lceil \frac{2mn-2m-n+2}{2} \rceil\}$ be the vertex labeling defined by

$$\phi_1(x_{i,j}) = \lceil \frac{i}{2} \rceil + (j-1)(n-1), \text{ for } 1 \le i \le n, 1 \le j \le m.$$

The weight of the edges are as follows:

$$w_{\phi_1}(x_{i,j}x_{i+1,j}) = i + 1 + 2(j-1)(n-1), for 1 \le i \le n-1, \ 1 \le j \le m,$$
$$w_{\phi_1}(x_{i,j}x_{i-1,j+1}) = i + 1 + 2(j-1)(n-1), for 2 \le i \le n, \ 1 \le j \le m-1.$$

Since all the edge weights are distinct, the vertex labeling ϕ_1 is an optimal edge irregular $\left\lceil \frac{|E(G)|+1}{2} \right\rceil$-labeling. This completes the proof. □

4. $m \times n \times l$ grid graph

In the next theorem, we determine the exact value of the edge irregularity strength of $P_n \Box P_m \Box P_2$.

Theorem 4. *Let $G = P_n \Box P_m \Box P_2$ where $m, n \ge 2$. Then*

$$es(G) = \left\lceil \frac{5mn - 2m - 2n + 1}{2} \right\rceil.$$

Proof. Let $V(G) = \{(x_i, y_j, z_r) : 1 \le i \le n, 1 \le j \le m, 1 \le r \le 2\}$ and the edge set $E(G) = \{(x_i, y_j, z_r)(x_{i+1}, y_j, z_r) : 1 \le i \le n-1, 1 \le j \le m, 1 \le r \le l\} \cup \{(x_i, y_j, z_r)(x_i, y_{j+1}, z_r) : 1 \le i \le n, 1 \le j \le m-1, 1 \le r \le l\} \cup \{(x_i, y_j, z_1)(x_{i+1}, y_j, z_2) : 1 \le i \le n, 1 \le j \le m\}$. We divide the edge set $P_n \Box P_m \Box P_2$ into mutually disjoint subset A_i^r, B_j^r and C, where $A_i^r = \{(x_i, y_j, z_r)(x_i, y_{j+1}, z_r) : 1 \le i \le n, 1 \le j \le m-1, 1 \le r \le 2\}$, $B_j^r = \{(x_i, y_j, z_r)(x_{i+1}, y_j, z_r) : 1 \le i \le n-1, 1 \le j \le m, 1 \le r \le 2\}$ and $C = \{(x_i, y_j, z_1)(x_i, y_{j+1}, z_2) : 1 \le i \le n, 1 \le j \le m\}$.

Clearly that, $|A_i^r| = (m-1)$, $|B_j^r| = (n-1)$ and $|C| = mn$. Hence $|E(P_n \Box P_m \Box P_2)| = 5mn - 2m - 2n$.

From Theorem 1, it follows that $es(P_n \Box P_m \Box P_2) \ge \lceil \frac{5mn-2m-2n+1}{2} \rceil$. Assume that $k = \lceil \frac{5mn-2m-2n+1}{2} \rceil$. Since the graph $P_n \Box P_m \Box P_2$, $P_n \Box P_2 \Box P_m$, $P_m \Box P_n \Box P_2$, $P_m \Box P_2 \Box P_n$, $P_2 \Box P_n \Box P_m$ and $P_2 \Box P_m \Box P_n$ are isomorphic, without loss

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of generality we assume that $n \geq m$. Now, for $n \geq m \geq 2$, $1 \leq j \leq m$ and $1 \leq i \leq n$, we define the function ϕ_3 as follows:

$$\phi_3(x_i y_j z_r) = \begin{cases} \frac{i-1}{2}(5m-2) + \lfloor \frac{r}{2} \rfloor (m-1) + \lceil \frac{j+r-1}{2} \rceil, & \text{if } i \text{ is odd} \\ \frac{i-2}{2}(5m-2) + 4m + r - \lfloor \frac{j-r+1}{2} \rfloor - 2\lfloor \frac{j+r-2}{2} \rfloor - 3, & \text{if } i \text{ is even} \end{cases}$$

The edge weights are as follows:
for $1 \leq i \leq n,\ 1 \leq j \leq m-1,\ 1 \leq r \leq 2$,

$$w_{\phi_3}(A_i^r) = \begin{cases} (i-1)(5m-2) + 2\lfloor \frac{r}{2} \rfloor (m-1) + j + r, & \text{if } i \text{ is odd} \\ (i-1)(5m-2) + 8m - 3j - r - 1, & \text{if } i \text{ is even} \end{cases}$$

for $1 \leq i \leq n-1,\ 1 \leq j \leq m,\ 1 \leq r \leq 2$,

$$w_{\phi_3}(B_j^r) = (i-1)(5m-2) + (r+3)m + \lceil \frac{j+r-1}{2} \rceil - \lfloor \frac{j-r+1}{2} \rfloor - 2\lfloor \frac{j+r-2}{2} \rfloor - 2$$

for $1 \leq i \leq n,\ 1 \leq j \leq m$,

$$w_{\phi_3}(C) = \begin{cases} (i-1)(5m-2) + m + j, & \text{if } i \text{ is odd} \\ (i-1)(5m-2) + 8m - 3j, & \text{if } i \text{ is even} \end{cases}$$

We can see that all vertex labels are at most k and the weights of the edge from the subset A_i^r, B_j^r and C are pairwise different. This completes the proof. □

In Theorem 4, we determined the exact value of the edge irregularity strength es of $P_n \square P_m \square P_l$ for $m, n \geq 2$ and $l = 2$. We have tried to find $P_n \square P_m \square P_2$ for $l \geq 3$ but so far without success. So we pose the following open problem.

Problem 2. Determine the exact value of $es(P_n \square P_m \square P_l)$ for $m, n \geq 2$ and $l \geq 3$.

5. Conclusion

In this paper, we discussed the edge irregularity strength es, as a modification of the well-known irregularity strength, total edge irregularity strength and total vertex irregularity strength (see [19–25]). We obtained the precise values for edge irregularity strength of triangular grid graph L_n^m, zigzag graph Z_n^m and Cartesian product $P_n \square P_m \square P_2$. It seems to be a very challenging problem to find the exact value for the edge irregularity strength of families of graph.

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