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Perfect 2-colorings of the cubic graphs of order less than or equal to 10

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Abstract

Perfect coloring is a generalization of the notion of completely regular codes, given by Delsarte. A perfect m-coloring of a graph G with m colors is a partition of the vertex set of G into m parts $A_1, \ldots, A_m$ such that, for all $i, j \in \{1, \ldots, m\}$, every vertex of A_i is adjacent to the same number of vertices, namely, a_{ij} vertices, of A_j. The matrix $A = (a_{ij})_{i,j\in\{1,\ldots,m\}}$, is called the parameter matrix. We study the perfect 2-colorings (also known as the equitable partitions into two parts) of the cubic graphs of order less than or equal to 10. In particular, we classify all the realizable parameter matrices of perfect 2-colorings for the cubic graphs of order less than or equal to 10.

Keywords: Perfect coloring; Equitable partition; Cubic graph; Parameter matrix

1. Introduction

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see [1]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6, 3)$, $J(7, 3)$, $J(8, 3)$, $J(8, 4)$, and $J(v, 3)$ (v odd) (see [2–4]).

Fon-Der-Flass enumerated the parameter matrices of n-dimensional cube for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix (see [5–7]). In [8–11] perfect 2-colorings and perfect 3-colorings of some graphs were described. So for cubic graphs of order less than or equal to 10, the problem of existence of perfect 2-colorings was open. In this

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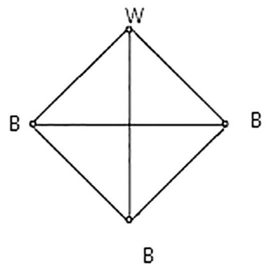

Fig. 1. Connected cubic graph of order 4.

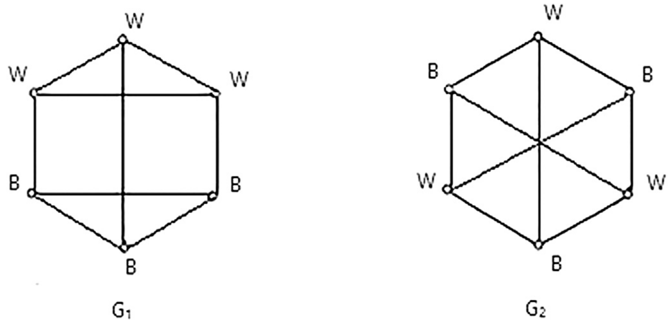

Fig. 2. Connected cubic graphs of order 6.

article, we enumerate the parameter matrices of all perfect 2-colorings of the cubic graphs of order less than or equal to 10.

2. Preliminaries

Definition 2.1. For each graph G and each integer m, a mapping $T : V(G) \to \{1, \ldots, m\}$ is called a perfect m-coloring with matrix $A = (a_{ij})_{i,j\in\{1,\ldots,m\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the parameter matrix of a perfect coloring. When $m = 2$, we denote the two colors by W and B representing white and black respectively.

A cubic graph is a 3-regular graph. Cubic graphs of order less than or equal to 10 are given in Figs. 1–4. Wherever possible the vertices of the graph are labeled with B or W giving a perfect 2-coloring.

Now, we first give some results concerning necessary conditions for the existence of perfect 2-colorings of a k-regular graph with a given parameter matrix $A = (a_{ij})_{i,j=1,2}$.

The simplest condition for the existence of a perfect 2-colorings of a k-regular graph with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is:

$$a_{11} + a_{12} = a_{21} + a_{22} = k.$$

If G is connected, then a_{12} and a_{21} are both non-zero. By the given conditions, we can see that a parameter matrix of a perfect 2-coloring of cubic graphs must be one of the following matrices: $A_1 = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $A_5 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, $A_6 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

The next proposition gives a formula for calculating the number of white vertices in a perfect 2-coloring.

Proposition 2.2 (*[2]*)**.** *If W is the set of white vertices in a perfect 2-coloring of a graph G with matrix $A = (a_{ij})_{i,j=1,2}$, then*

$$|W| = |V(G)|\frac{a_{21}}{a_{12} + a_{21}}.$$

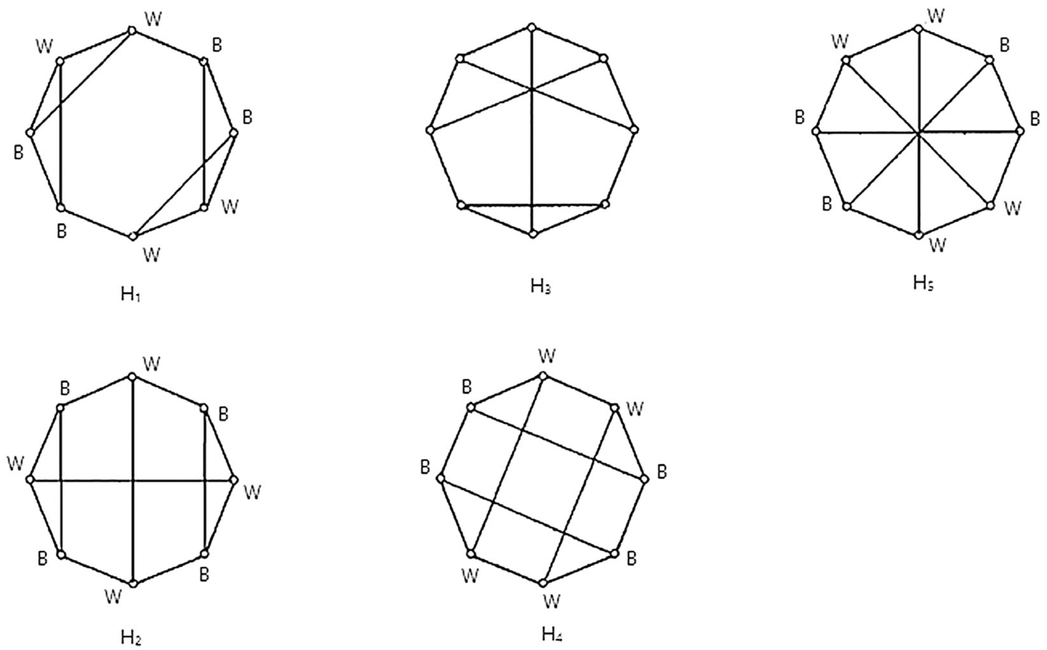

Fig. 3. Connected cubic graphs of order 8.

The number λ is called an eigenvalue of a graph G, if λ is an eigenvalue of the adjacency matrix of this graph. The number θ is called an eigenvalue of a perfect coloring T into m colors with the matrix A, if θ is an eigenvalue of A. The next theorem demonstrates the connection between the introduced notions.

Theorem 2.3 (*[12]*). *If T is a perfect coloring of a graph G in m colors, then any eigenvalue of T is an eigenvalue of G.*

Corollary 2.4. *Every perfect 2-coloring of a k-regular graph with parameter matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two eigenvalues: one is k, and the other is $a - c$ such that we obviously have $a - c \neq k$. So, from Theorem 2.3, we conclude that $a - c$ is an eigenvalue of a k-regular connected graph which is not equal to k.*

3. Main results

In this section, we enumerate the parameter matrices of all perfect 2-colorings of the cubic connected graphs of order less than or equal to 10. Any cubic graph of order 4 is isomorphic to K_4 and for the perfect 2-coloring of this graph given in Fig. 1, the parameter matrix is A_5. Also, it follows from Proposition 2.2 that the matrices A_2 and A_6 cannot be parameter matrices. In the next theorems, we introduce the parameter matrices for cubic graphs of order 6, 8, and 10.

Theorem 3.1. *The graph G_1 has no perfect 2-coloring with parameter matrix A_4.*

Proof. Suppose there exists a perfect 2-coloring T of G_1 with matrix A_4. Then each vertex with color 1 has one adjacent vertex with color 2. We now have two possibilities:
(1) $T(a_2) = T(a_6) = 1$ and $T(a_4) = 2$.
(2) $T(a_4) = T(a_5) = 1$ and $T(a_2) = 2$.

In both cases we get $T(a_3) = T(a_5) = 2$, which is a contradiction with the second row of A_4. Hence G_1 has no perfect 2-coloring with matrix A_4.

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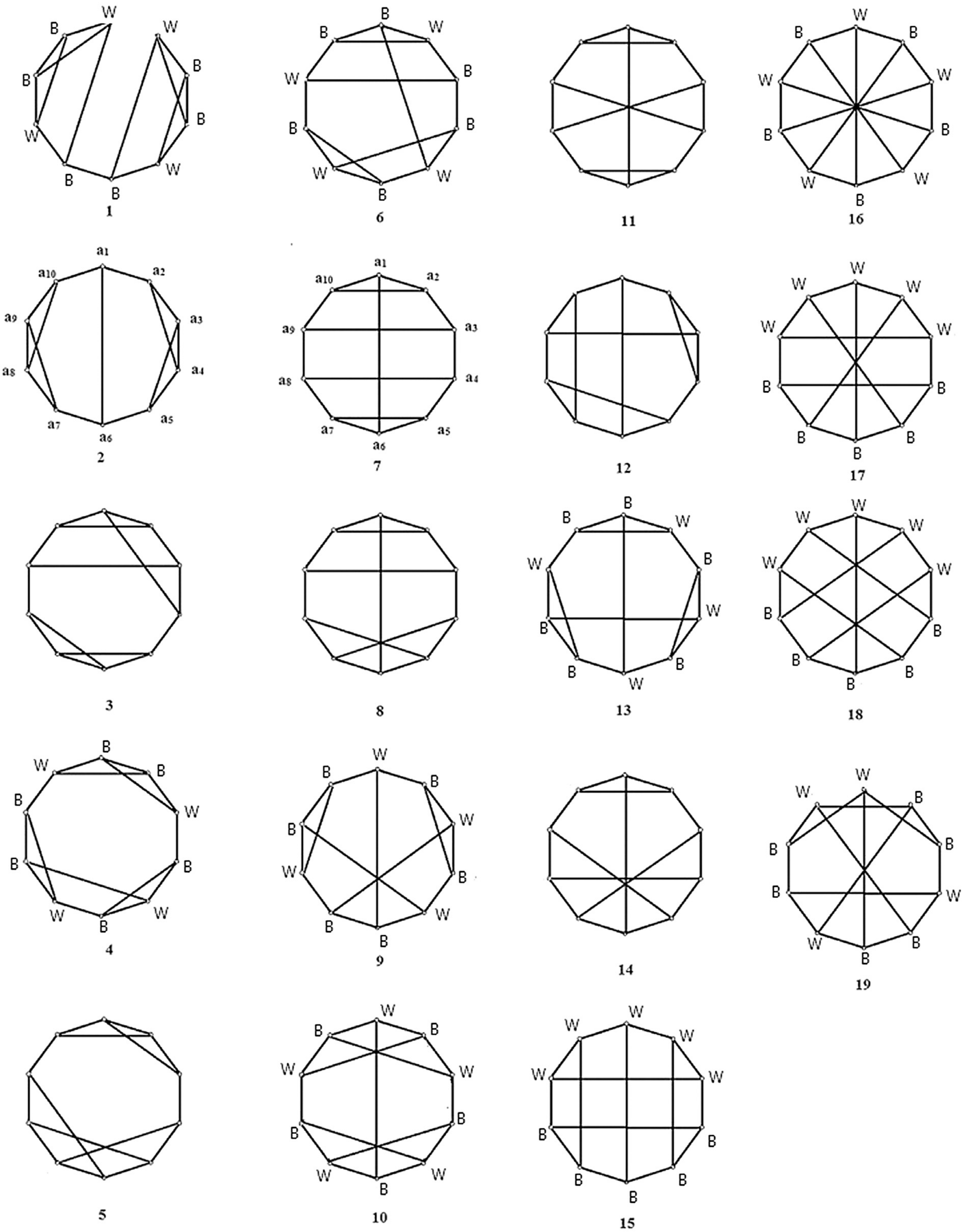

Fig. 4. Connected cubic graphs of order 10.

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Theorem 3.2. *The graph G_2 has no perfect 2-coloring with parameter matrix A_6.*

Proof. The proof is similar to the proof of Theorem 3.1.

Theorem 3.3. *The graph G_1 has perfect 2-coloring with parameter matrices $\{A_3, A_6\}$ and graph G_2 has perfect 2-coloring with parameter A_1.*

Proof. For the perfect coloring of G_1 and G_2 given in Fig. 2, the corresponding parameter matrices are A_3 and A_1 respectively. Also the mapping defined by $T(a_1) = T(a_2) = T(a_6) = 1$ and $T(a_3) = T(a_4) = T(a_5) = 2$, gives a perfect 2-coloring of G_1 with matrix A_3.

Theorem 3.4. *The graph H_1 has no perfect 2-coloring with parameter matrix A_3.*

Proof. Suppose there exists a perfect 2-coloring T of H_1 with matrix A_3. Then each vertex with color 1 has two adjacent vertices with color 1. We now have three possibilities:
(1) $T(a_2) = T(a_3) = 1$ and $T(a_8) = 2$.
(2) $T(a_8) = T(a_2) = 1$ and $T(a_3) = 2$.
(3) $T(a_8) = T(a_3) = 1$ and $T(a_2) = 2$.

In case 1 we get $T(a_4) = 2$, which is a contradiction with the second row of matrix A_3. In other both cases we get a contradiction with the second row of A_4. Hence H_1 has no perfect 2-coloring with matrix A_3.

Theorem 3.5. *The graph H_2 has no perfect 2-coloring with parameter matrices A_3 and A_6.*

Proof. The proof is similar to the proof of Theorem 3.4.

Theorem 3.6. *The graph H_5 has no perfect 2-coloring with parameter matrix A_5.*

Proof. The proof is similar to the proof of Theorem 3.4.

Theorem 3.7. *The Graphs H_1, H_2, H_4 and H_5 of order 8 have perfect 2-colorings.*

Proof. For the perfect coloring of $\{H_1, H_2\}$ and $\{H_4, H_5\}$ given in Fig. 3, the corresponding parameter matrices are A_4 and A_3 respectively. Also the mapping defined by:

$$T(a_2) = T(a_7) = 1, T(a_1) = T(a_3) = T(a_4) = T(a_5) = T(a_6) = T(a_8) = 2,$$

gives a perfect 2-coloring of H_1 with matrix A_5, the mappings defined by:

$$T_1(a_1) = T_1(a_3) = T_1(a_5) = T_1(a_7) = 1, T_1(a_2) = T_1(a_4) = T_1(a_6) = T_1(a_8) = 2,$$
$$T_2(a_1) = T_2(a_3) = T_2(a_6) = T_2(a_8) = 1, T_2(a_2) = T_2(a_4) = T_2(a_5) = T_2(a_7) = 2,$$
$$T_3(a_1) = T_3(a_6) = 1, T_3(a_2) = T_3(a_3) = T_3(a_4) = T_3(a_5) = T_3(a_7) = T_3(a_8) = 2,$$

gives a perfect 2-colorings of H_4 with matrices A_1, A_4 and A_5, respectively and the mappings defined by:

$$T(a_1) = T(a_3) = T(a_5) = T(a_7) = 1, T(a_2) = T(a_4) = T(a_6) = T(a_8) = 2,$$

gives a perfect 2-coloring of H_5 with matrix A_4.

Theorem 3.8. *The parameter matrices of cubic graphs of order 10 are listed in Table 1.*

Proof. As it has been shown in the previous section, only matrices A_1 , A_2, ..., A_6 can be parameter matrices. With consideration the eigenvalues of cubic graphs, and using Proposition 2.2 and Theorem 2.3, it can be seen that the connected cubic graphs with 10 vertices can have perfect 2-coloring with matrices A_1, A_2, A_3 and A_4 which

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Table 1
The parameter matrices of cubic graphs of order 10.

Graphs	Matrix A_1	Matrix A_2	Matrix A_3
1	×	√	×
2	×	×	×
3	×	×	×
4	×	√	×
5	×	√	×
6	×	√	×
7	×	×	×
8	×	×	×
9	×	√	×
10	√	√	×
11	×	×	×
12	×	×	×
13	×	√	×
14	×	×	×
15	×	×	×
16	√	×	×
17	×	×	×
18	×	×	×
19	×	√	√

Table 2
The possibility of existence a perfect 2-coloring to cubic graphs of order 10.

Graphs	Matrix A_1	Matrix A_2	Matrix A_3	Matrix A_4
1	×	√	√	√
2	×	√	√	√
3	×	×	√	×
4	×	√	√	×
6	×	√	√	√
7	×	×	×	√
9	×	√	√	√
10	√	√	√	√
13	×	√	×	√
14	×	×	×	√
15	×	×	×	√
16	√	×	×	×
17	×	×	×	√
18	×	√	√	√
19	×	√	×	√

is represented by Table 2. For the perfect coloring of graphs {1, 2, 4, 6, 7, 9, 13, 15, 17, 18, 19} and graphs {10, 16} given in Fig. 4, the corresponding parameter matrices are A_2 and A_1 respectively. Also the mapping defined by:

$$T(a_1) = T(a_3) = T(a_4) = T(a_6) = T(a_8) = T(a_9) = 1,$$
$$T(a_2) = T(a_5) = T(a_7) = T(a_{10}) = 2.$$

gives a perfect 2-coloring of graph 10 with matrix A_2 and the mapping defined by:

$$T(a_2) = T(a_3) = T(a_4) = T(a_7) = T(a_8) = 1,$$
$$T(a_1) = T(a_5) = T(a_6) = T(a_9) = T(a_{10}) = 2.$$

gives a perfect 2-coloring of graph 19 with matrix A_3.

There are no perfect 2-colorings with the matrices A_3 for graph 1. Suppose there exists a perfect 2-coloring T of 1 with matrix A_3. Then each vertex with color 1 has two adjacent vertices with color 1. We now have three possibilities: **(1)** $T(a_3) = T(a_5) = 1$ and $T(a_2) = 2$.

(2) $T(a_2) = T(a_5) = 1$ and $T(a_3) = 2$.
(3) $T(a_2) = T(a_3) = 1$ and $T(a_5) = 2$.

In all three cases, the vertex by color 2 has two adjacent vertex with color 1, which is a contradiction with the second row of A_3. Hence graph 1 has no perfect 2-coloring with matrix A_3.

About other graphs in Fig. 4, similarly, we can get the same result as in Table 1.

References

[1] C. Godsil, Compact graphs and equitable partitions, in: Linear Algebra and Its Application, Vol. 255, 1997, pp. 259–266.
[2] S.V. Avgustinovich, I. Yu. Mogilnykh, Perfect 2-colorings of Johnson graphs J(6, 3) and J(7, 3), Lecture Notes in Comput. Sci. 5228 (2008) 11–19.
[3] S.V. Avgustinovich, I.Yu. Mogilnykh, Perfect colorings of the Johnson graphs J(8, 3) and J(8, 4) with two colors, J. Appl. Ind. Math. 5 (2011) 19–30.
[4] A.L. Gavrilyuk, S.V. Goryainov, On perfect 2-colorings of Johnson graphs J(v,3), J. Combin. Des. 21 (2013) 232–252.
[5] D.G. Fon-Der-Flaass, A bound on correlation immunity, Siberian Electron. Math. Rep. J. 4 (2007) 133–135.
[6] D.G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube, Sib. Math. J. 4 (2007) 923–930.
[7] D.G. Fon-der Flaass, Perfect 2-colorings of a 12-dimensional Cube that achieve a bound of correlation immunity, Sib. Math. J. 4 (2007) 292–295.
[8] M. Alaeiyan, A. Mehrabani, Perfect 3-colorings of the cubic graphs of order 10, Electron. J. Graph Theory Appl. 5 (2) (2017) 194–206.
[9] M. Alaiyan, A. Mehrabani, Perfect 3-colorings of the platonic graph, Iranian J. Sci. Technol., Trans. A Sci. (2017) 1-9.
[10] M. Alaeiyan, H. Karami, Perfect 2-colorings of the generalized Petersen graph, Proc. Math. Sci. 126 (2016) 1–6.
[11] M. Alaiyan, A. Mehrabani, Perfect 3-colorings of cubic graphs of order 8, Armenian J. Math. 10 (2) (2018) 1–11.
[12] F.C. Bussemaker, S. Cobeljic, D.M. Cvetkovic, J.J. Seidel, Computer investigation of cubic graphs, Tech. Hogeschool Eindhoven Ned. Onderafedeling Wisk. (1976).

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