AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: Saad Ihsan Butt, Muhammad Numan, Sharafat Ali & Andrea Semaničová-Feňovčíková (2020) Face antimagic labelings of toroidal and Klein bottle grid graphs, AKCE International Journal of Graphs and Combinatorics, 17:1, 109-117, DOI: 10.1016/j.akcej.2018.09.005

To link to this article: https://doi.org/10.1016/j.akcej.2018.09.005

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Published online: 01 Jun 2020.

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AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 1, 109–117
https://doi.org/10.1016/j.akcej.2018.09.005

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AKCE International Journal of Graphs and Combinatorics

Face antimagic labelings of toroidal and Klein bottle grid graphs

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Received 5 June 2018; received in revised form 18 September 2018; accepted 18 September 2018

Abstract

In this paper we deal with the problem of labeling the vertices, edges and faces of a toroidal T_n^m and Klein bottle K_n^m grid graphs with mn 4-sided faces by the consecutive integers from 1 up to $|V(T_n^m)| + |E(T_n^m)| + |F(T_n^m)|$ and $|V(K_n^m)| + |E(K_n^m)| + |F(K_n^m)|$ in such a way that the label of a 4-sided face and the labels of the vertices and edges surrounding that face all together add up to a weight of that face. These face-weights then form an arithmetic progression with common difference d. The paper examines the existence of such labelings for several differences d.

Keywords: Toroidal grid; Klein bottle grid; Super d-antimagic labeling

1. Introduction and definitions

Let $G \in \mathbf{G}$ be a family of 4-regular graphs embedded on the surface of a torus or Klein bottle such that each of its face is 4-sided. Let $V(G)$, $E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of a graph $G \in \mathbf{G}$, where v, e and f denote the cardinality of vertex, edge and face set respectively.

A labeling of type $(1, 1, 1)$ is a bijection $f : V \cup E \cup F \to \{1, 2, \ldots, v + e + f\}$. The weight of a 4-sided face under a labeling of type $(1, 1, 1)$ is the sum of labels carried by that face and the edges and vertices surrounding it.

A labeling of type $(1, 1, 1)$ of graph $G \in \mathbf{G}$ is called *d-antimagic* if the set of weights of all 4-sided faces is $W = \{a, a + d, a + 2d, \ldots, a + (f - 1)d\}$ for some integers $a > 0$ and $d \geq 0$, where f is the number of the 4-sided faces.

The concept of the d-antimagic labeling of plane graphs was defined in [1]. The d-antimagic labeling of type $(1, 1, 1)$ for the generalized Petersen graph $P(n, 2)$, hexagonal planar maps and grids can be found in [2,3] and [4],

Peer review under responsibility of Kalasalingam University.

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https://doi.org/10.1016/j.akcej.2018.09.005
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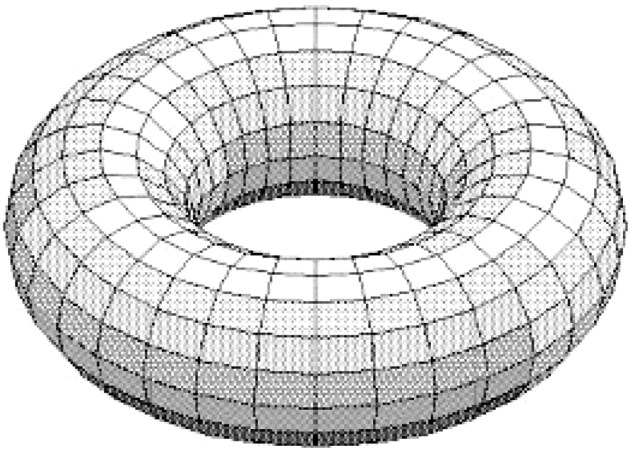

Fig. 1. Toroidal grid graph.

respectively. Lin et al. in [5] showed that prism D_n, $n \geq 3$, admits d-antimagic labeling of type $(1, 1, 1)$ for $d \in \{2, 4, 5, 6\}$. The d-antimagic labeling of type $(1, 1, 1)$ for D_n and for several $d \geq 7$ are described in [6].

In particular for $d = 0$, Lih in [7] calls such labeling *magic* and describes magic (0-antimagic) labeling of type $(1, 1, 0)$ for wheels, friendship graphs and prisms. Kathiresan and Gokulakrishnan [8] provided the 0-antimagic labeling of type $(1, 1, 1)$ for the families of planar graphs with 3-sided faces, 5-sided faces, 6-sided faces and one external infinite face.

A d-antimagic labeling of type $(1, 1, 1)$ is called *super* if the smallest possible labels appear on the vertices. The super d-antimagic labelings of type $(1, 1, 1)$ for antiprisms and for $d \in \{0, 1, 2, 3, 4, 5, 6\}$ are described in [9] and for disjoint union of prisms and for $d \in \{0, 1, 2, 3, 4, 5\}$ are given in [10]. The existence of a super d-antimagic labeling of type $(1, 1, 1)$ for the plane graphs containing a special Hamilton path is examined in [11] and a super d-antimagic labeling of type $(1, 1, 1)$ for disconnected plane graphs is given in [12]. For more details (see [13]). Plane graphs can also be embedded on other surfaces like torus, sphere, Klein bottle and projective plane (see [14]).

Motivated by the paper [15] we deal with the super d-antimagic labelings of type $(1, 1, 1)$ for the toroidal grid and we describe those labelings for several values of d.

Let L be a regular square lattice and P_n^m be an $m \times n$ quadrilateral section (with n squares on the top and bottom sides and m squares on the lateral sides,) cut from the regular square lattice L. First identify 2 lateral sides of P_n^m to form a cylinder, and finally identify the top and bottom sides of cylinder at their corresponding points; see Fig. 1.

Thus we get a toroidal grid graph T_n^m with mn 4-sided faces, mn vertices, and $2mn$ edges. More about *toroidal grid* can be found in ([16]).

In the case of Klein bottle grid first we identify lateral side of P_n^m to form a cylinder and then identify the top and bottom sides of the cylinder in opposite direction. By this identification of P_n^m, we get Klein bottle grid graph K_n^m with mn 4-sided faces, mn vertices, and $2mn$ edges. see Fig. 2.

2. Necessary conditions

In this section, we shall find bound for a feasible value of d for the super d-antimagic labeling of type $(1, 1, 1)$ for the toroidal grid T_n^m and Klein bottle grid K_n^m.

Let g be such a labeling. We consider weights of 4-sided faces of the toroidal grid separately for a vertex labeling, an edge labeling and a face labeling. For d-antimagic vertex labeling $\varphi_1 : V(T_n^m) \to \{1, 2, \ldots, |V(T_n^m)|\}$ the minimum possible weight of a 4-sided face is at least $1 + 2 + 3 + 4$ and the maximum weight of a 4-sided face is no more than

$$\sum_{i=1}^{4}(|V(T_n^m)| - i + 1) = 4nm - 6.$$

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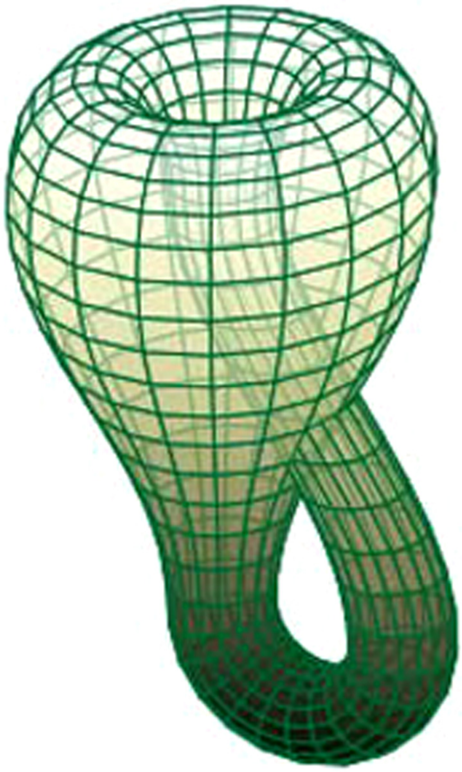

Fig. 2. Klein bottle grid graph.

Thus

$$a_4 + (f_4 - 1)d \leq 4nm - 6$$

and

$$d \leq 4 - \frac{12}{nm - 1}.$$

Lemma 1. *For every toroidal grid T_n^m, $n \geq 3$, $m \geq 3$, there is no d-antimagic vertex labeling with $d \geq 4$.*

Assume that T_n^m has a d-antimagic edge labeling φ_2 with $|E(T_n^m)|$ values from the set $\{|V(T_n^m)| + 1, |V(T_n^m)| + 2, \ldots, |V(T_n^m)| + |E(T_n^m)| + |F(T_n^m)|\}$. Then the minimum possible weight of 4-sided face is at least $\sum_{i=1}^{4}(|V(T_n^m)| + i) = 4mn + 10$ and the maximum weight of 4-sided face is no more than

$$\sum_{i=1}^{4} \left(|V(T_n^m)| + |E(T_n^m)| + |F(T_n^m)| + 1 - i\right) = 16nm - 6.$$

Hence

$$a_4 + (f_4 - 1)d \leq 16nm - 6.$$

It is easy to see that

$$d \leq 12 - \frac{4}{nm - 1}.$$

Lemma 2. *For every toroidal grid T_n^m, $n \geq 3$, $m \geq 3$, there is no d-antimagic edge labeling with $d \geq 12$.*

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According to Lemma 1, Lemma 2 and the fact that under a d-antimagic face labeling φ_3 with f_4 values from the set $\{|V(T_n^m)|+1, |V(T_n^m)|+2, \dots, |V(T_n^m)|+|E(T_n^m)|+|F(T_n^m)|\}$ the parameter d is no more than 3, we obtain the following theorem.

Theorem 1. *Let T_n^m, $n \geq 3$, $m \geq 3$, be a toroidal grid graph which admits d_1-antimagic vertex labeling φ_1, d_2-antimagic edge labeling φ_2 and d_3-antimagic face labeling φ_3. If the labelings φ_1, φ_2 and φ_3 combine to a super d-antimagic labeling of type* $(1, 1, 1)$ *then the parameter $d \leq 17$.*

Remark 1. Similarly, we can estimate the bound $d \leq 17$ for the Klein bottle grid graph.

3. *d*-antimagic labeling of toroidal grid graph

Let $V(T_n^m) = \{(x_s, y_t) : 0 \leq s \leq n-1,\ 0 \leq t \leq m-1\}$ be the vertex set of the graph T_n^m, $E(T_n^m) = \{(x_s, y_t)(x_{s+1}, y_t) : 0 \leq s \leq n-1,\ 0 \leq t \leq m-1\} \cup \{(x_s, y_t)(x_s, y_{t+1}) : 0 \leq s \leq n-1,\ 0 \leq t \leq m-1\}$ be the edge set and $F(T_n^m) = \{z_{s,t} : 0 \leq s \leq n-1,\ 0 \leq t \leq m-1\}$ be the face set with indices s and t taken modulo n and m respectively. The face $z_{s,t}$ is bounded by edges $(x_s, y_t)(x_{s+1}, y_t)$, $(x_{s+1}, y_t)(x_{s+1}, y_{t+1})$, $(x_s, y_{t+1})(x_{s+1}, y_{t+1})$, $(x_s, y_t)(x_s, y_{t+1})$. (See Fig. 3).

In this section we will use a similar idea which was used for an investigation of d-antimagic labeling of generalized prism in [17].

Lemma 3. *Let T_n^m $n, m \geq 3$ be toroidal grid and let $\alpha_1((x_s, y_t)) = \{tn+1+s,\ \ 0 \leq s \leq n-1,\ \ 0 \leq t \leq m-1\}$ and $\alpha_1((x_s, y_t)(x_{s+1}, y_t)) = \{mn+(m-t)n-s,\ \ 0 \leq s \leq n-1,\ \ 0 \leq t \leq m-1\}$. If $n \geq 3$ and $m \geq 3$, then the partial weights of $z_{s,t}$ under the labeling α_1 for every t, $0 \leq t \leq m-2$, constitute an arithmetic sequence of difference* 2 *and for $t = m-1$ the partial weights $z_{s,m-1}$ constitute the sequence* $n(5m-1)+4, n(5m-1)+6, \dots, n(5m+1)+2$.

Proof. Under the labeling α_1, for every t, $0 \leq t \leq m-1$, the partial weights of 4-sided faces $z_{s,t}$ are as follows:

$$\begin{aligned} wt_{\alpha_1}(z_{s,t}) &= \alpha_1((x_s, y_t)) + \alpha_1((x_{s+1}, y_t)) \\ &\quad + \alpha_1((x_s, y_{t+1})) + \alpha_1((x_{s+1}, y_{t+1})) \\ &\quad + \alpha_1((x_s, y_t)(x_{s+1}, y_t)) + \alpha_1((x_s, y_{t+1})(x_{s+1}, y_{t+1})) \end{aligned}$$

$$wt_{\alpha_1}(z_{s,t}) = \begin{cases} n(4m+1+2t)+6+2s, & \text{for } 0 \leq s \leq n-2,\ \ 0 \leq t \leq m-2 \\ n(4m+1+2t)+4, & \text{for } s = n-1,\ \ 0 \leq t \leq m-2 \\ n(5m-1)+6+2s, & \text{for } 0 \leq s \leq n-2,\ \ t = m-1 \\ n(5m-1)+4, & \text{for } s = n-1,\ \ t = m-1. \end{cases} \tag{1}$$

This shows that for every t, s, $0 \leq t \leq m-1$, $0 \leq s \leq n-1$, the partial weights of $z_{s,t}$ form the arithmetic sequence with difference 2 from $n(4m+1)+4$ up to $n(6m-1)+2$ and for every s, $0 \leq s \leq n-1$, the partial weights of $z_{s,m-1}$ form the arithmetic sequence with difference 2 from $n(5m-1)+4$ up to $n(5m+1)+2$. □

Lemma 4. *Let T_n^m $n, m \geq 3$ be toroidal grid and let for every t, $0 \leq t \leq m-1$*

$$\beta_1((x_s, y_t)(x_s, y_{t+1})) = \begin{cases} n(3m-t)-1-s, & \text{for } 0 \leq s \leq n-2 \\ n(3m-t), & \text{for } s = n-1 \end{cases}$$

$$\beta_1(z_{s,t}) = \begin{cases} n(3m+t)+3+s, & \text{for } 0 \leq s \leq n-3 \\ n(3m+t)+1, & \text{for } s = n-2 \\ n(3m+t)+2, & \text{for } s = n-1. \end{cases}$$

If $n \geq 3$ and $m \geq 3$, then under the labeling β_1 the partial weights of $z_{s,t}$, for every t, $0 \leq t \leq m-1$, form an arithmetic sequence of difference 1.

Proof. The partial weights of the 4-sided faces $z_{s,t}$ under the labeling β_1, for every t, $0 \leq t \leq m-1$, admit values

$$wt_{\beta_1}(z_{s,t}) = \beta_1((x_s, y_t)(x_s, y_{t+1})) + \beta_1(z_{s,t}) + \beta_1((x_{s+1}, y_t)(x_{s+1}, y_{t+1}))$$

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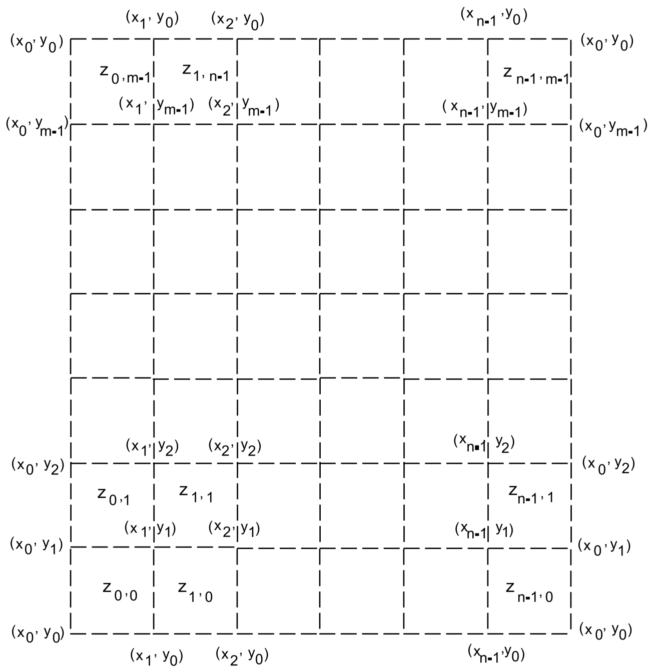

Fig. 3. Toroidal grid identification.

$$wt_{\beta_1}(z_{s,t}) = \begin{cases} n(9m-t)-s, & \text{for } 0 \le s \le n-2, \ \ 0 \le t \le m-1 \\ n(9m-t)+1, & \text{for } s = n-1, \ \ 0 \le t \le m-1. \end{cases} \tag{2}$$

This shows that the partial weights of $z_{s,t}$ form the arithmetic sequence with difference 1 from $8mn + 2$ up to $9mn + 1$. □

Lemma 5. *Let for every s,* $0 \le s \le n-1$

$$\beta_2((x_s, y_t)(x_s, y_{t+1})) = \begin{cases} n(2m+t+1)+1+s, & \text{for } 0 \le t \le m-2 \\ 2mn+1+s, & \text{for } t = m-1 \end{cases}$$

$$\beta_2(z_{s,t}) = n(4m-t)-s, \ \ \textit{for} \ \ 0 \le t \le m-1.$$

If $n \ge 3$ *and* $m \ge 3$*, then the partial weights of* $z_{s,t}$ *under the labeling* β_2*, for every t,* $0 \le t \le m-1$ *constitute an arithmetic sequence of difference* 1.

Proof. The partial weights of the 4-sided faces $z_{s,t}$ under the labeling β_2, for every t, $0 \le t \le m-1$, attain values

$$wt_{\beta_2}(z_{s,t}) = \beta_2((x_s, y_t)(x_s, y_{t+1})) + \beta_2(z_{s,t}) + \beta_2((x_{s+1}, y_t)(x_{s+1}, y_{t+1}))$$

$$wt_{\beta_2}(z_{s,t}) = \begin{cases} n(8m+2+t)+3+s, & \text{for } 0 \le s \le n-2, \ \ 0 \le t \le m-2 \\ n(8m+2+t)+2, & \text{for } s = n-1, \ \ 0 \le t \le m-2 \\ n(7m+1)+3+s, & \text{for } 0 \le s \le n-2, \ \ t = m-1 \\ n(7m+1)+2, & \text{for } s = n-1, \ \ t = m-1. \end{cases} \tag{3}$$

Thus, under the labeling β_2 the partial weights of 4-sided faces $z_{s,k}$, $0 \le s \le n-1$, $0 \le t \le m-2$, constitute the arithmetic sequence of difference 1 from $n(8m+2)+2$ up to $n(9m+1)+1$ and for $0 \le s \le n-1$ the partial weights $z_{s,m-1}$ attain consecutive values $n(7m+1)+2, n(7m+1)+3, \ldots, n(7m+2)+1$. □

Theorem 2. *For* $n \ge 3$ *and* $m \ge 3$*, the toroidal grid graph* T_n^m *has a super* 1*-antimagic labeling and a super* 3*-antimagic labeling of type* (1, 1, 1).

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Proof. *Case d=1.*

It follows from Lemmas 3 and 4 that under the labeling α_1 and β_1 the weights of all 4-sided faces are

$$wt(z_{s,t}) = wt_{\alpha_1}(z_{s,t}) + wt_{\beta_1}(z_{s,t}) = \begin{cases} n(13m+1+t)+6+s, & \text{for } 0 \le s \le n-2,\ 0 \le t \le m-2 \\ n(13m+1+t)+5, & \text{for } s = n-1,\ 0 \le t \le m-2 \\ 13mn+6+s, & \text{for } 0 \le s \le n-2,\ t = m-1 \\ 13mn+5, & \text{for } s = n-1,\ t = m-1. \end{cases}$$

This shows that the weights of the 4-sided faces form an arithmetic sequence with difference 1 starts from $13mn+5$ up to $14mn+4$.

Case d=3.

Taking into account Lemmas 3 and 5 we can see that under the labeling α_1 and β_2 the weights of 4-sided faces are

$$wt(z_{s,t}) = wt_{\alpha_1}(z_{s,t}) + wt_{\beta_2}(z_{s,t}) = \begin{cases} n(12m+3+3t)+9+3s, & \text{for } 0 \le s \le n-2,\ 0 \le t \le m-2 \\ n(12m+3+3t)+6, & \text{for } s = n-1,\ 0 \le t \le m-2 \\ 12mn+9+3s, & \text{for } 0 \le s \le n-2,\ t = m-1 \\ 12mn+6, & \text{for } s = n-1,\ t = m-1. \end{cases}$$

Thus the weights of all 4-sided faces constitute an arithmetic sequence of the difference 3, namely $12mn+6$ up to $15mn+3$. □

4. d-antimagic labeling of Klein bottle grid graph

Let $V(K_n^m) = \{(x_s, y_t) : 0 \le s \le n-1,\ 0 \le t \le m-1\}$ be the vertex set, $E(K_n^m) = \{(x_s, y_t)(x_{s+1}, y_t) : 0 \le s \le n-1,\ 0 \le t \le m-1\} \cup \{(x_s, y_t)(x_s, y_{t+1}) : 0 \le s \le n-1,\ 0 \le t \le m-2\} \cup \{(x_s, y_{m-1})(x_{n-s}, y_0) : 0 \le s \le n-1\} \cup \{(x_0, y_{m-1})(x_0, y_0)\}$ be the edge set and $F(K_n^m) = \{z_{s,t} : 0 \le s \le n-1,\ 0 \le t \le m-1\}$ be the face set. The face $z_{s,t}$ is bounded by edges $(x_s, y_t)(x_{s+1}, y_t)$, $(x_{s+1}, y_t)(x_{s+1}, y_{t+1})$, $(x_s, y_{t+1})(x_{s+1}, y_{t+1})$, $(x_s, y_t)(x_s, y_{t+1})$, for $0 \le s \le n-1$, $0 \le t \le m-2$ and $z_{s,m-1}$ is bounded by edges $(x_s, y_{m-1})(x_{s+1}, y_{m-1})$, $(x_s, y_{m-1})(x_{n-s}, y_0)$, $(x_{n-s}, y_0)(x_{n-(s+1)}, y_0)$, $(x_{s+1}, y_{m-1})(x_{n-(s+1)}, y_0)$ for $1 \le s \le n-1$ and $z_{0,m-1}$ is bounded by edges $(x_0, y_{m-1})(x_1 y_{m-1})$, $(x_1, y_{m-1})(x_{n-1} y_0)$, $(x_0, y_0)(x_{n-1} y_0)$, $(x_0, y_{m-1})(x_0 y_0)$, (see Fig. 4):

Lemma 6. *Let* $\lambda_1((x_s, y_t)) = \{tn+1+s,\ 0 \le s \le n-1,\ 0 \le t \le m-1\}$ *and* $\lambda_1((x_s, y_t)(x_{s+1}, y_t)) = \{n(2m-t)-s,\ 0 \le s \le n-1,\ 0 \le t \le m-1\}$. *If* $n \ge 3$ *and* $m \ge 3$, *then under the labeling* λ_1 *the partial weights of* $z_{s,t}$ *for every* t, s, $0 \le t \le m-2$, $0 \le s \le n-1$, *constitute an arithmetic sequence of difference* 2 *and the partial weights of* $z_{s,m-1}$ *for* $1 \le s \le n-2$ *are* $5mn+5$ *and for* $z_{0,m-1}$ *and* $z_{n-1,m-1}$ *are* $n(5m-1)+5$.

Proof. Under the labeling λ_1, for every t, $0 \le t \le m-2$, the partial weights of 4-sided faces $z_{s,t}$ are as follows:

$$\begin{aligned} wt_{\lambda_1}(z_{s,t}) = {} & \lambda_1((x_s, y_t)) + \lambda_1((x_{s+1}, y_t)) \\ & + \lambda_1((x_s, y_{t+1})) + \lambda_1((x_{s+1}, y_{t+1})) \\ & + \lambda_1((x_s, y_t)(x_{s+1}, y_t)) + \lambda_1((x_s, y_{t+1})(x_{s+1}, y_{t+1})) \end{aligned}$$

$$wt_{\lambda_1}(z_{s,t}) = \begin{cases} n(4m+2t+1)+6+2s, & \text{for } 0 \le s \le n-2,\ 0 \le t \le m-2 \\ n(4m+2t+1)+4, & \text{for } s = n-1,\ 0 \le t \le m-2. \end{cases} \tag{4}$$

This shows that for every t and s, $0 \le t \le m-2$, $0 \le s \le n-1$, the partial weights of 4-sided faces $z_{s,t}$ form the arithmetic sequence with difference 2 from $n(4m+1)+4$ up to $n(6m-1)+2$. For $1 \le s \le n-2$ the partial weights of $z_{s,m-1}$ are $5mn+5$ and $wt_{\lambda_1}(z_{0,m-1}) = wt_{\lambda_1}(z_{n-1,m-1}) = n(5m-1)+5$. □

Lemma 7. *Let*

$$\mu_1((x_s, y_t)(x_s, y_{t+1})) = \begin{cases} n(3m-t)-s-1, & \text{for } 0 \le s \le n-2,\ 0 \le t \le m-2 \\ n(3m-t), & \text{for } s = n-1,\ 0 \le t \le m-2 \end{cases}$$

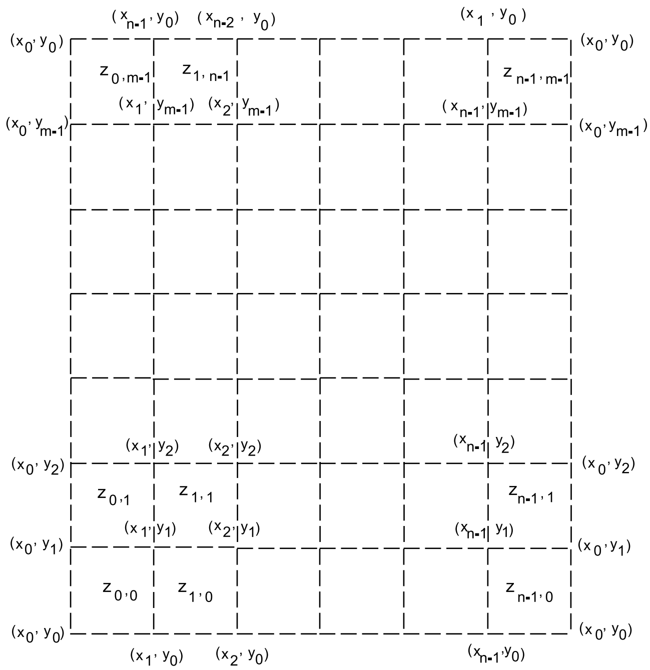

Fig. 4. Klein bottle grid identification.

$$\mu_1((x_s, y_{m-1})(x_{n-s}, y_0)) = \begin{cases} n(2m+1) - s - 1, & \text{for } 0 \le s \le n-2 \\ n(2m+1), & \text{for } s = n-1 \end{cases}$$

$$\mu_1(z_{s,t}) = \begin{cases} n(3m+t) + 3 + s, & \text{for } 0 \le s \le n-3,\ 0 \le t \le m-2 \\ n(3m+t-1) + 3 + s, & \text{for } n-2 \le s \le n-1,\ 0 \le t \le m-2 \end{cases}$$

$$\mu_1(z_{s,m-1}) = \begin{cases} n(4m-1) + 3 + s, & \text{for } 0 \le s \le n-3 \\ n(4m-2) + 3 + s, & \text{for } n-2 \le s \le n-1. \end{cases}$$

If $n \ge 3$ and $m \ge 3$, then the partial weights of $z_{s,t}$ under the labeling μ_1 for every t and s, $0 \le t \le m-1$, $0 \le s \le n-1$, constitute an arithmetic sequence of difference 1.

Proof. Under the given labeling μ_1, for every t, $0 \le t \le m-2$, the partial weights of the 4-sided faces, admit the following values

$$wt_{\mu_1}(z_{s,t}) = \mu_1((x_s, y_t)(x_s, y_{t+1})) + \mu_1(z_{s,t}) + \mu_1((x_{s+1}, y_t)(x_{s+1}, y_{t+1}))$$

$$wt_{\mu_1}(z_{s,t}) = \begin{cases} n(9m-t) + 1, & \text{for } s = n-1,\ 0 \le t \le m-2 \\ n(9m-t) - s, & \text{for } 0 \le s \le n-2,\ 0 \le t \le m-2 \end{cases} \quad (5)$$

$$wt_{\mu_1}(z_{s,m-1}) = \begin{cases} n(8m+1) + 1, & \text{for } s = n-1 \\ n(8m+1) - s, & \text{for } 0 \le s \le n-2. \end{cases} \quad (6)$$

This shows that the partial weights of 4-sided faces form the arithmetic progression with difference 1 with values from $8mn + 2$ up to $9mn + 1$. □

Lemma 8. *Let* $\mu_2((x_s, y_t)(x_s, y_{t+1})) =$

$$n(2m+t+1) + s + 1, \quad \text{for } 0 \le s \le n-1,\ 0 \le t \le m-2$$

Please cite this article in press as: S.I. Butt, et al., Face antimagic labelings of toroidal and Klein bottle grid graphs, AKCE International Journal of Graphs and Combinatorics (2018), https://doi.org/10.1016/j.akcej.2018.09.005.

$\mu_2(z_{s,t}) = n(4m - t) - s, \quad \text{for } 0 \le s \le n-1, \ 0 \le t \le m-2.$

If $n \ge 3$ and $m \ge 3$, then the partial weights of $z_{s,t}$ under the labeling μ_2, for every t, $0 \le t \le m-2$, constitute an arithmetic sequence of difference 1.

Proof. Under the labeling μ_2, for every t, $0 \le t \le m-2$, the partial weights of the 4-sided face attain values
$wt_{\mu_2}(z_{s,t}) = \mu_2((x_s, y_t)(x_s, y_{t+1})) + \mu_2(z_{s,t}) + \mu_2((x_{s+1}, y_t)(x_{s+1}, y_{t+1}))$.

$$wt_{\mu_2}(z_{s,t}) = \begin{cases} n(8m+2+t)+s+3, & \text{for } 0 \le s \le n-2, \ 0 \le t \le m-2 \\ n(8m+2+t)+2, & \text{for } s = n-1, \ 0 \le t \le m-2. \end{cases} \tag{7}$$

Thus partial weights of 4-sided faces under the labeling μ_2, constitute the arithmetic sequence of difference 1. □

Lemma 9. *Let* $\mu_2((x_s, y_{m-1})(x_{n-s}, y_0)) =$

$$\begin{cases} n(2m+1)-s+1, & \text{for } 1 \le s \le n-1, \ t = m-1 \\ 2mn+1, & \text{for } s = 0, \ t = m-1 \end{cases}$$

$\mu_2(z_{s,m-1}) =$

$n(3m+1) - s, \quad \text{for } 0 \le s \le n-1.$

If $n \ge 3$ and $m \ge 3$, then the partial weights of $z_{s,m-1}$ under the labeling μ_2, constitute an arithmetic sequence of difference 1.

Proof. Under the labeling μ_2 defined for $0 \le t \le m-2$ in Lemma 8 and for $t = m-1$ defined above, the partial weights of the 4-sided face attain values

$wt_{\mu_2}(z_{s,m-1}) =$

$\mu_2((x_s, y_{m-1})(x_{n-s}, y_0)) + \mu_2(z_{s,m-1}) + \mu_2((x_{s+1}, y_{m-1})(x_{n-(s+1)}, y_0)).$

$$wt_{\mu_2}(z_{s,m-1}) = \begin{cases} n(7m-3s+3)+1, & \text{for } 1 \le s \le n-1 \\ n(7m+2)+1, & \text{for } s = 0. \end{cases} \tag{8}$$

Thus weights of 4-sided faces under the labeling μ_2, constitute the arithmetic sequence of difference 1. □

Theorem 3. *For $n \ge 3$ and $m \ge 3$, the Klein bottle grid graph K_n^m admits a super* 1*-antimagic labeling and a super* 3*-antimagic labeling of type* (1, 1, 1).

Proof. *Case d=1.* It follows from Lemmas 6 and 7 that under the labeling λ_1 and μ_1 the weights of all 4-sided faces are

$$\begin{aligned} wt(z_{s,t}) &= wt_{\lambda_1}(z_{s,t}) + wt_{\mu_1}(z_{s,t}) \\ &= \begin{cases} n(13m+t+1)+s+6, & \text{for } 0 \le s \le n-2, \ 0 \le t \le m-2 \\ n(13m+t+1)+5, & \text{for } s = n-1, \ 0 \le t \le m-2 \\ n(13m+1)+5-s, & \text{for } 1 \le s \le n-1, \ t = m-1 \\ 13mn+5, & \text{for } s = 0, \ t = m-1. \end{cases} \end{aligned}$$

This shows that the weights of the 4-sided faces form an arithmetic sequence with difference 1 with values from $13mn + 5$ up to $14mn + 4$.

Case d=3.

Taking into account Lemmas 6, 8 and 9 along with the following swapping

- $\lambda_1((x_{n-3}, y_{m-1})) \longleftrightarrow \lambda_1((x_{n-1}, y_{m-1}))$
- $\mu_2((z_{n-1,m-1})) \longleftrightarrow \mu_2((z_{n-3,m-1}))$
- $\mu_2((z_{n-2,m-1})) \longleftrightarrow \mu_2((z_{4,m-1}))$
- $\mu_2((z_{n-2,m-2})) \longleftrightarrow \mu_2((z_{4,m-2}))$
- $\mu_2((z_{n-1,m-2})) \longleftrightarrow \mu_2((z_{n-3,m-2}))$

we can see that under the labeling λ_1 and μ_2 the weights of 4-sided faces are

$$wt(z_{s,t}) = wt_{\lambda_1}(z_{s,t}) + wt_{\mu_2}(z_{s,t})$$
$$= \begin{cases} n(12m+3t+3s+3)+9, & \text{for } 0 \le s \le n-2,\ 0 \le t \le m-2 \\ n(12m+3t+3)+6, & \text{for } s = n-1,\ 0 \le t \le m-2 \\ n(12m+3)-3s+6, & \text{for } 1 \le s \le n-2,\ t = m-1 \\ n(12m+1)+1, & \text{for } s = 0,\ t = m-1 \\ n(12m+1)-2, & \text{for } s = n-1,\ t = m-1. \end{cases}$$

Finally the weights of the all 4-sided faces of given graph form an arithmetic progression with the common difference 3, starting from $12mn+6$ up to $15mn+3$. □

5. Conclusion

In this paper we examine the existence of super d-antimagic labeling of type (1, 1, 1) for toroidal grid graph T_n^m and Klein bottle grid graph K_n^m. We show that T_n^m and K_n^m admit a super d-antimagic labeling of type (1, 1, 1) for $d = 1,3$, for all $n, m \ge 3$. However we tried to describe a super d-antimagic labeling of type (1, 1, 1) of graphs T_n^m and K_n^m for $d = 0, 2, 4$ but without success.

Therefore we conclude the paper with the following open problem.

Open problem 1. *For the toroidal grid T_n^m and Klein bottle grid $\mathbb{K}_n^m$, $n, m \ge 3$, determine whether there is a super d-antimagic labeling of type $(1, 1, 1)$ for $d = 0, 2, 4$.*

Acknowledgment

The research for this article was supported by APVV-15-0116 and by VEGA 1/0233/18.

References

[1] M. Bača, M. Miller, On d-antimagic labelings of type (1, 1, 1) for prisms, J. Combin. Math. Combin. Comput. 44 (2003) 199–207.
[2] M. Bača, S. Jendrol', M. Miller, J. Ryan, Antimagic labelings of generalized Petersen graphs that are plane, Ars Combin. 73 (2004) 115–128.
[3] M. Bača, E.T. Baskoro, S. Jendrol', M. Miller, Antimagic labelings of hexagonal planar maps, Util. Math. 66 (2004) 231–238.
[4] M. Bača, Y. Lin, M. Miller, Antimagic labelings of grids, Util. Math. 72 (2007) 65–75.
[5] Y. Lin, Slamin M. Bača, M. Miller, On d-antimagic labelings of prisms, Ars Combin. 72 (2004) 65–76.
[6] K.A. Sugeng, M. Miller, Y. Lin, M. Bača, Face antimagic labelings of prisms, Util. Math. 71 (2006) 269–286.
[7] K.W. Lih, On magic and consecutive labelings of plane graphs, Util. Math. 24 (1983) 165–197.
[8] K. Kathiresan, S. Gokulakrishnan, On magic labelings of type (1, 1, 1) for the special classes of plane graphs, Util. Math. 63 (2003) 25–32.
[9] M. Bača, F. Bashir, A. Semaničová, Face antimagic labelings of antiprisms, Util. Math. 84 (2011) 209–224.
[10] G. Ali, M. Bača, F. Bashir, A. Semaničová-Feňovčíková, On face antimagic labelings of disjoint union of prisms, Util. Math. 85 (2011) 97–112.
[11] M. Bača, L. Brankovic, A. Semaničová-Feňovčíková, Labelings of plane graphs containing Hamilton path, Acta Math. Sin. (Engl. Ser.) 27 (4) (2011) 701–714.
[12] M. Bača, M. Miller, O. Phanalasy, A. Semaničová-Feňovčíková, Super d-antimagic labelings of disconnected plane graphs, Acta Math. Sin. (Engl. Ser.) 26 (12) (2010) 2283–2294.
[13] J. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 17 (2016) #DS6.
[14] M. Deza, P.W. Fowler, A. Rassat, K.M. Rogers, Fullerenes as tilings of surfaces, J. Chem. Inf. Comput. Sci. 40 (2000) 550–558.
[15] M. Bača, M. Numan, A. Shabbir, Labelings of type (1, 1, 1) for toroidal fullerenes, Turk. J. Math. 37 (2013) 899–907.
[16] Tao-Ming Wang, Toroidal grids are anti-magic, in: COCOON 2005, pp. 671–679.
[17] S.I. Butt, M. Numan, I.A. Shah, S. Ali, Face labelings of type (1, 1, 1) for generalized prism, Ars Combin. 137 (2018) 41–52.