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Further results on Erdős–Faber–Lovász conjecture

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Abstract

In 1972, Erdős–Faber–Lovász (EFL) conjectured that, if **H** is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices with n colors so that no two vertices with the same color are in the same edge. In 1978, Deza, Erdös and Frankl had given an equivalent version of the same for graphs: Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $A_1, A_2, \ldots, A_n$, each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n.

The clique degree $d^K(v)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \le i \le n\}|$. In this paper we give a method for assigning colors to the graphs satisfying the hypothesis of the Erdős–Faber–Lovász conjecture and every A_i $(1 \le i \le n)$ has atmost $\frac{n}{2}$ vertices of clique degree greater than one using Symmetric latin Squares and clique degrees of the vertices of G.

Keywords: Chromatic number; Erdős–Faber–Lovász conjecture; Symmetric latin square

1. Introduction

One of the famous conjectures in graph theory is Erdős–Faber–Lovász conjecture. It states that, if **H** is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices of **H** with n colors so that no two vertices with the same color are in the same edge [1]. Erdős, in 1975, offered 50 USD [2,3] and in 1981, offered 500 USD [3,4] for the proof or disproof of the conjecture.

Vance Faber quoted: "In 1972, Paul Erdös, László Lovász and I got together at a tea party in Colorado. This was a continuation of the discussions we had a few weeks before in Columbus, Ohio, at a conference on hypergraphs. We talked about various conjectures for linear hypergraphs analogous to Vizing's theorem for graphs. Finding tight bounds in general seemed difficult, so we created an elementary conjecture that we thought that it would be easy to prove. We called this the n sets problem: given n sets, no two of which meet more than once and each with n elements, color the elements with n colors so that each set contains all the colors. In fact, we agreed to meet the next day to write down the solution. Thirty-Eight years later, this problem is still unsolved in general".

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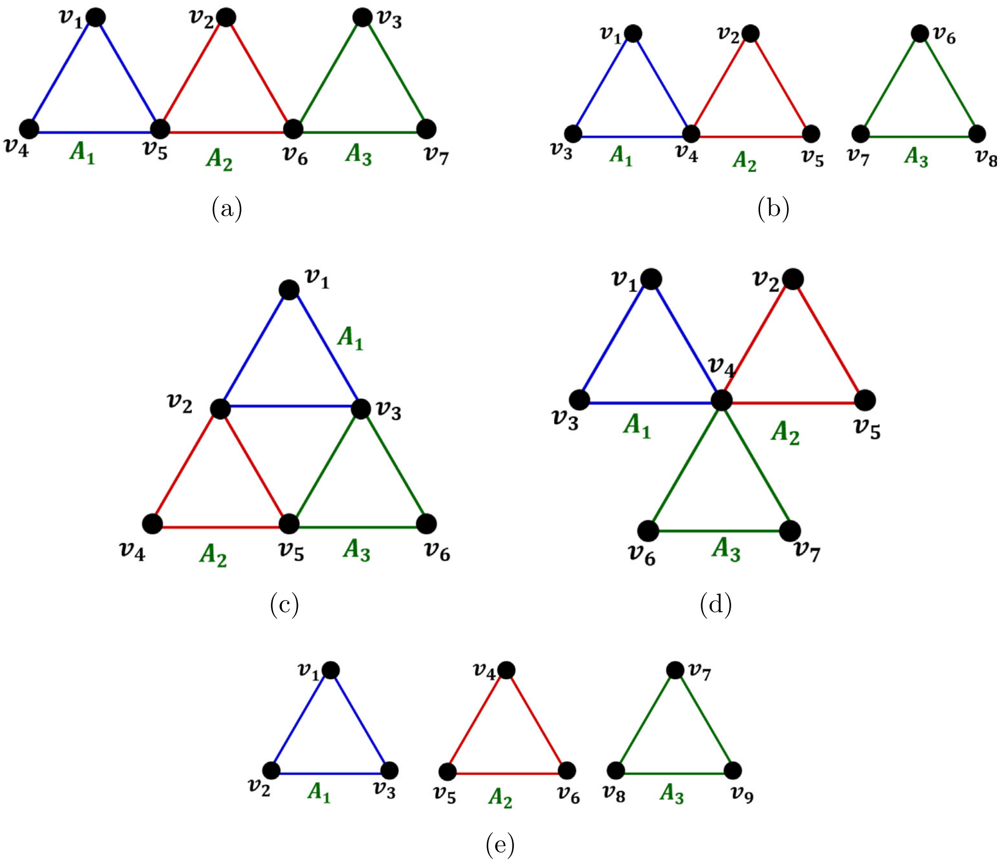

Fig. 1. All graphs satisfying the hypothesis of the conjecture for n = 3.

Chang and Lawler [5] presented a simple proof that the edges of a simple hypergraph on n vertices can be colored with at most [1.5n-2] colors. Kahn [6] showed that the chromatic number of **H** is at most $n+o(n)$. Jackson et al. [7] proved that the conjecture is true when the partial hypergraph S of **H** determined by the edges of size at least three can be Δ_S-edge-colored and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$. Viji Paul and Germina [8] established the truth of the conjecture for all linear hypergraphs on n vertices with $\Delta(\mathbf{H}) \leq \sqrt{n+\sqrt{n}+1}$. Sanchez-Arroyo [9] proved the conjecture for dense hypergraphs. We consider the equivalent version of the conjecture for graphs given by Deza, Erdős and Frankl in 1978 [4,9–11].

Conjecture 1.1. *Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $(A_1, A_2, \ldots, A_n)$, each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n.*

Example 1.2. Fig. 1 shows all the graphs for $n = 3$ which are satisfying the hypothesis of Conjecture 1.1.

Haddad & Tardif (2004) [12] introduced the problem with a story about seating assignment in committees: suppose that, in a university department, there are k committees, each consisting of k faculty members, and that all committees meet in the same room, which has k chairs. Suppose also that at most one person belongs to the intersection of any two committees. Is it possible to assign the committee members to chairs in such a way that each member sits in the same chair for all the different committees to which he or she belongs? In this model of the problem, the faculty members correspond to graph vertices, committees correspond to complete graphs, and chairs correspond to vertex colors.

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Definition 1.3. Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $A_1, A_2, \ldots, A_n$, each having exactly n vertices and the property that every pair of complete graphs has at most one common vertex. The clique degree $d^K(v)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \le i \le n\}|$. The maximum clique degree $\Delta^K(G)$ of the graph G is given by $\Delta^K(G) = max_{v \in V(G)} d^K(v)$.

From the above definition, one can observe that degree of a vertex in hypergraph is same as the clique degree of a vertex in a graph.

Definition 1.4. Let G_1 and G_2 be two vertex disjoint graphs, and let x_1, x_2 be two vertices of G_1, G_2 respectively. Then, the graph $G(x_1x_2)$ obtained by merging the vertices x_1 and x_2 into a single vertex is called the concatenation of G_1 and G_2 at the points x_1 and x_2 (see [13]).

Definition 1.5. A Latin Square is an $n \times n$ array containing n different symbols such that each symbol appears exactly once in each row and once in each column. Moreover, a Latin Square of order n is an $n \times n$ matrix $M = [m_{ij}]$ with entries from an n-set $V = \{1, 2, \ldots, n\}$, where every row and every column is a permutation of V (see [14]). If the matrix M is symmetric, then the Latin Square is called Symmetric Latin Square.

In this paper we give a method of coloring to the graph G satisfying the hypothesis of the conjecture using a Symmetric Latin Square in the following steps:

- Construct the graph H_n having the minimum number of vertices among the graphs satisfying the hypothesis of the conjecture for given n
- Construct any other graph satisfying the hypothesis of the conjecture for the same n.
- We give a coloring to the graph H_n with n colors using a Symmetric Latin Square.
- Extend the n-coloring of H_n to the other graphs satisfying the hypothesis of Conjecture 1.1 for any given n.

2. Construction of H_n

We know that a symmetric $n \times n$ matrix is determined by $\frac{n(n+1)}{2}$ scalars. Using symmetric latin squares we give an n-coloring of H_n constructed below.

Construction of H_n:

Let n be a positive integer and $B_1, B_2, \ldots, B_n$ be n copies of K_n. Let the vertex set $V(B_i) = \{a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,n}\}$, $1 \le i \le n$.

Step 1. Let $H^1 = B_1$.

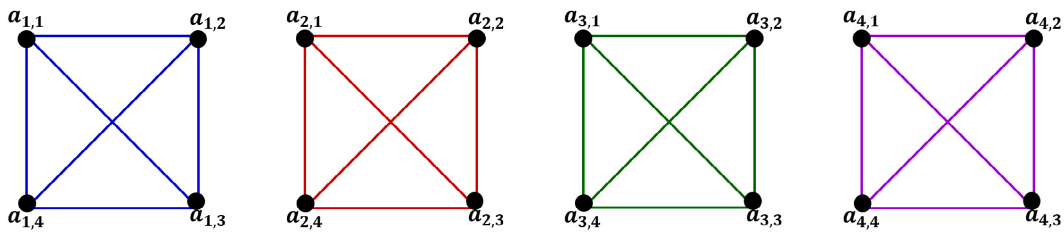
Step 2. Consider the vertices $a_{1,2}$ of H^1 and $a_{2,1}$ of B_2. Let $b_{1,2}$ be the vertex obtained by the concatenation of the vertices $a_{1,2}$ and $a_{2,1}$. Let the resultant graph be H^2.

Step 3. Consider the vertices $a_{1,3}$, $a_{2,3}$ of H^2 and $a_{3,1}$, $a_{3,2}$ of B_3. Let $b_{1,3}$ be the vertex obtained by the concatenation of vertices $a_{1,3}$, $a_{3,1}$ and let $b_{2,3}$ be the vertex obtained by the concatenation of vertices $a_{2,3}$, $a_{3,2}$. Let the resultant graph be H^3.

Continuing in the similar way, at the nth step we obtain the graph $H^n = H_n$ (for the sake of convenience we take H^n as H_n).

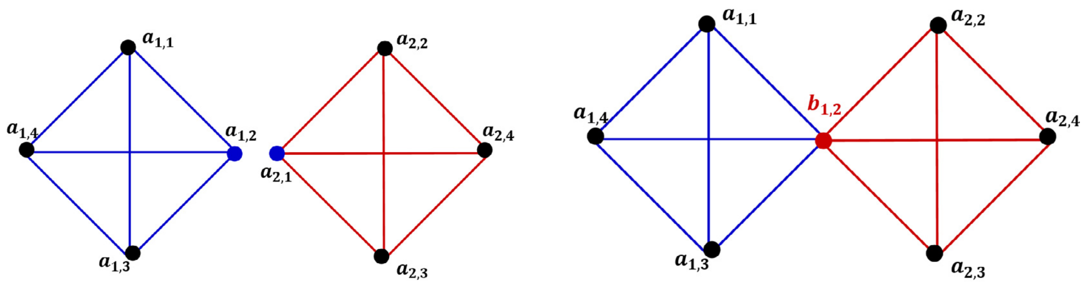
By the construction of H_n one can observe the following:

1. H_n is a connected graph and also it is satisfying the hypothesis of Conjecture 1.1.
2. H_n has exactly n vertices of clique degree one and $\frac{n(n-1)}{2}$ vertices of clique degree 2 (each B_i has exactly $(n-1)$ vertices of clique degree 2 and one vertex of clique degree one, $1 \le i \le n$).
3. $H_n = \bigcup_{i=1}^{n} B_i$, where $B_i = A_i$ and B_i, B_j have exactly one common vertex for $1 \le i < j \le n$.
4. H_n has exactly $\frac{n(n+1)}{2}$ vertices.



(a) B_1 (b) B_2 (c) B_3 (d) B_4

Fig. 2. 4 copies of K_4.



(a) H^1, B_2 (b) H^2

Fig. 3. Construction of H^2 from H^1, B_2.

5. One can observe that in a connected graph G if clique degree increases the number of vertices also increases. From this it follows that, H_n is the graph with minimum number of vertices satisfying the hypothesis of Conjecture 1.1. If all the vertices of G are of clique degree one, then G will have n^2 vertices. Thus, $\frac{n(n+1)}{2} \leq |V(G)| \leq n^2$.

Following example is an illustration of the graph H_n for $n = 4$

Example 2.1. Let $n = 4$ and B_1, B_2, B_3, B_4 be the 4 copies of K_4. Let the vertex set $V(B_i) = \{a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\}$, $1 \leq i \leq 4$.

Step 1: Let $H^1 = B_1$. The graph H^1 is shown in Fig. 2a.

Step 2: Consider the vertices $a_{1,2}$ of H^1 and $a_{2,1}$ of B_2. Let $b_{1,2}$ be the vertex obtained by the concatenation of vertices $a_{1,2}$, $a_{2,1}$. Let the resultant graph be H^2 as shown in Fig. 3b.

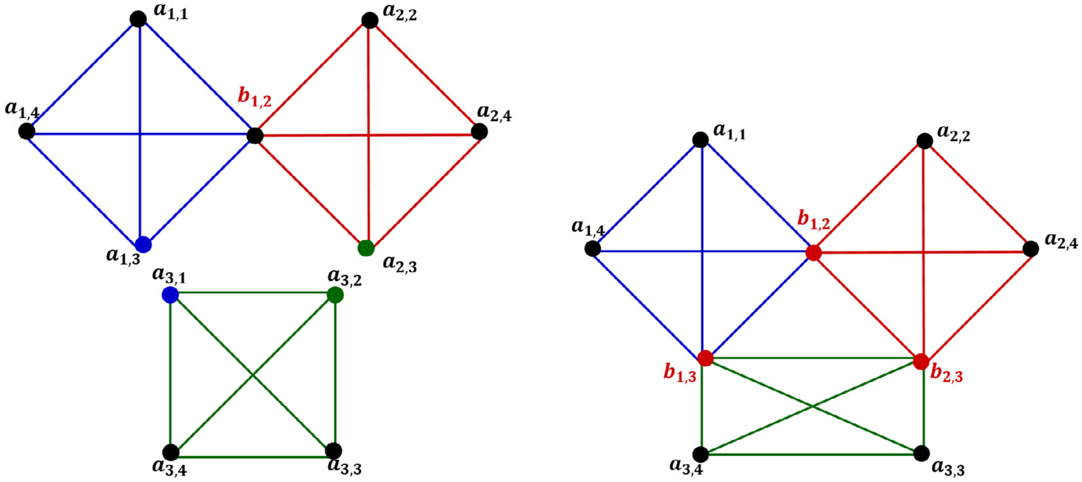
Step 3: Consider the vertices $a_{1,3}$, $a_{2,3}$ of H^2 and $a_{3,1}$, $a_{3,2}$ of B_3. Let $b_{1,3}$ be the vertex obtained by the concatenation of vertices $a_{1,3}$, $a_{3,1}$ and let $b_{2,3}$ be the vertex obtained by the concatenation of vertices $a_{2,3}$, $a_{3,2}$. Let the resultant graph be H^3 as shown in Fig. 4b.

Step 4: Consider the vertices $a_{1,4}$, $a_{2,4}$, $a_{3,4}$ of H^3 and $a_{4,1}$, $a_{4,2}$, $a_{4,3}$ of B_4. Let $b_{1,4}$ be the vertex obtained by the concatenation of vertices $a_{1,4}$, $a_{4,1}$, let $b_{2,4}$ be the vertex obtained by the concatenation of vertices $a_{2,4}$, $a_{4,2}$ and let $b_{3,4}$ be the vertex obtained by the concatenation of vertices $a_{3,4}$, $a_{4,3}$. Let the resultant graph be H^4 as shown in Fig. 5b.

Example 2.2. For $n = 6$, the graph H_6 is shown in Fig. 6.

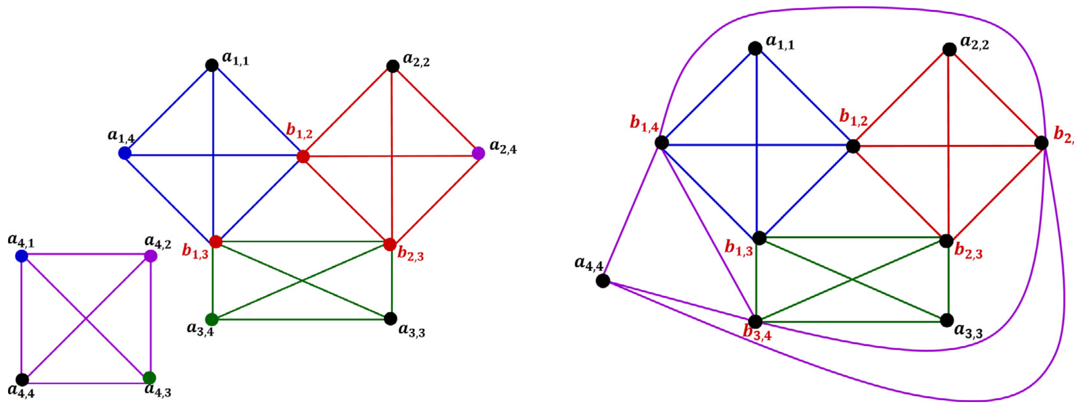
Lemma 2.3. *If G is a graph satisfying the hypothesis of Conjecture 1.1, then G can be obtained from H_n, n in $\mathbb{N}$.*

Proof. Let G be a graph satisfying the hypothesis of Conjecture 1.1. Let b_X be the new labeling to the vertices v of clique degree greater than 1 in G, where $x = \{i : \text{vertex } v \text{ is in } A_i\}$. Define $N_i = \{b_X : |X| = i\}$ for $i = 2, 3, \ldots, n$. Then the graph G is constructed from H_n as given below:



(a) H^2, B_3 (b) H^3

Fig. 4. Construction of H^3 from H^2, B_3.



(a) H^3, B_4 (b) $H_4 = H^4$

Fig. 5. Construction of H^4 from H^3, B_4.

Step 1: For every common vertex $b_{i,j}$ in H_n which is not in N_2, split the vertex $b_{i,j}$ into two vertices $u_{i,j}, u_{j,i}$ such that vertex $u_{i,j}$ is adjacent only to the vertices of B_i and the vertex $u_{j,i}$ is adjacent only to the vertices of B_j in H_n.

Step 2: For every vertex b_X in N_i where $i = 3, 4, \ldots, n$, merge the vertices $u_{l_1,l_2}, u_{l_2,l_3}, \ldots, u_{l_{m-1},l_m}, u_{l_m,l_1}$ into a single vertex u_X in H_n where $l_i \in X$ and $l_i < l_j$ for $i < j$.

Let G' be the graph obtained in Step 2. Let $V(B_i')$, $V(A_i')$ be the set of all clique degree 1 vertices of B_i of G', A_i of G respectively, $1 \le i \le n$. Thus, by splitting all the common vertices of H_n which are not in N_2 and merging the vertices of H_n corresponding to the vertices in $N_i, i \ge 3$, we get the graph G'. One can observe that $|V(A_i')| = |V(B_i')|$, $1 \le i \le n$. Define a function $f : V(G) \to V(G')$ by

$$\begin{aligned} f(b_{i,j}) &= b_{i,j} && \text{for } b_{i,j} \in N_2 \\ f(b_{i_1,i_2,\ldots i_k}) &= u_{i_1,i_2,\ldots i_k} && \text{for } b_{i_1,i_2,\ldots i_k} \in \cup_{i=3}^{n} N_i \\ f|_{V(A_i')} &= g_i && (\text{any 1–1 map } g_i : V(A_i') \to V(B_i')), \text{ for } 1 \le i \le n \end{aligned}$$

One can observe that f is an isomorphism from G to G'. ■

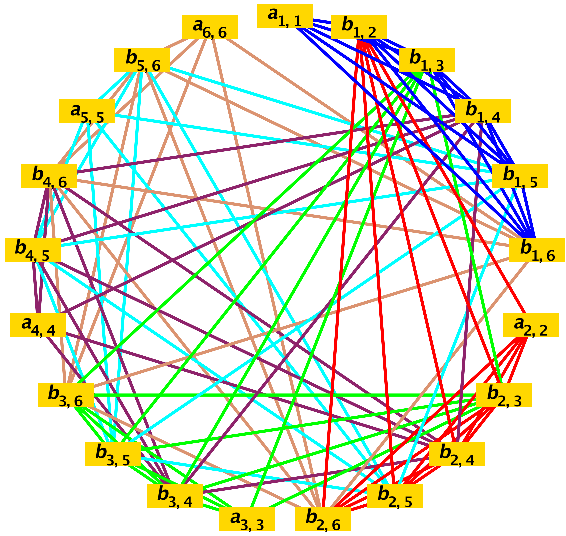
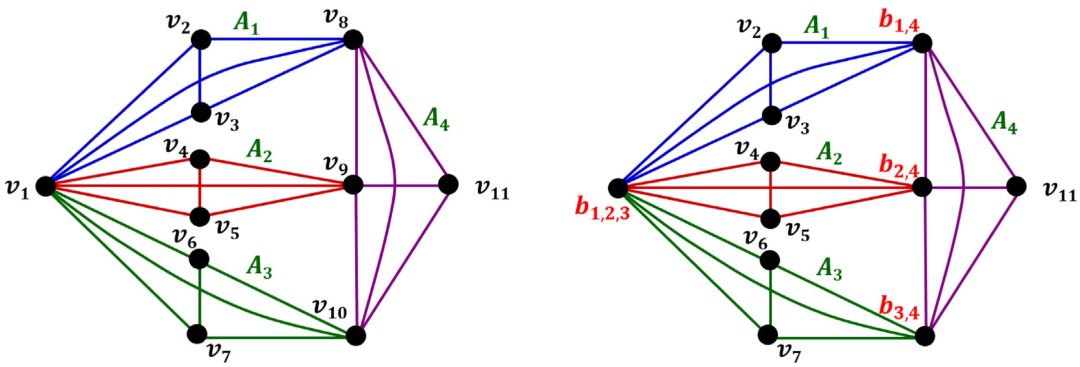


Fig. 6. H_6.



(a) Graph G (b) Graph G after relabeling the vertices

Fig. 7. Graph G, before and after relabeling the vertices.

From Lemma 2.3, one can observe that in G there are at most $\frac{n(n-1)}{2}$ common vertices.
Following example is an illustration of the graph G obtained from H_n for $n = 4$.

Example 2.4. Let G be the graph shown in Fig. 7a.

Relabel the vertices of clique degree greater than one in G by b_A where $A = \{i : v \in A_i \text{ for } 1 \leq i \leq 4\}$. The labeled graph is shown in Fig. 7.

Let $N_i = \{b_x : |x| = i\}$ for $i = 2, 3, 4$, then $N_2 = \{b_{1,4}, b_{2,4}, b_{3,4}\}$, $N_3 = \{b_{1,2,3}\}$.

Consider the graph H_4 as shown in Fig. 5b, then $V(H_4) = \{a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\}$ and common vertices of H_4 are $\{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\} = A(say)$. Then $A \setminus N_2 = \{b_{1,2}, b_{1,3}, b_{2,3}\}$. By the construction given in the proof of Lemma 2.3 we get,

Step 1: Since $A \setminus N_2 \neq \emptyset$, split the common vertices of H_4 which are not in N_2, as shown in Fig. 8.

Step 2: Since $\cup_{i=2}^{4} N_i = \{b_{1,2,3}\} \neq \emptyset$, merge the vertices $u_{1,2}, u_{2,3}, u_{3,1}$ into a single vertex $u_{1,2,3}$, as shown in Fig. 9. Let the resultant graph be G'.

The isomorphism $f : V(G) \to V(G')$ is given below.

$$f(v_2) = a_{11} \quad f(v_3) = u_{13} \quad f(v_4) = u_{21}$$
$$f(v_5) = a_{22} \quad f(v_6) = u_{32} \quad f(v_7) = a_{33}$$

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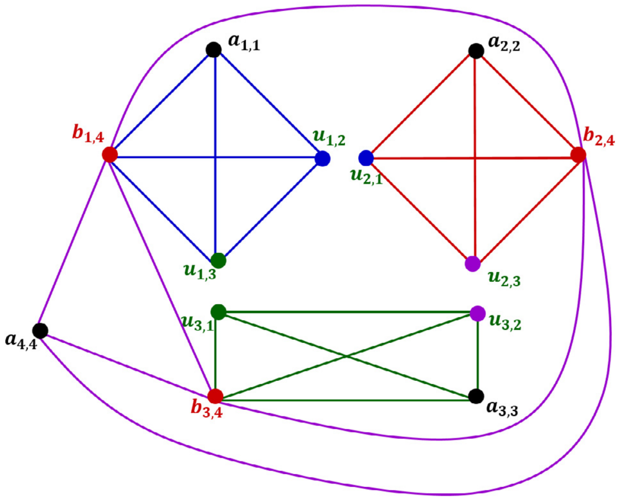

Fig. 8. Splitting the common vertices of H_4 which are not in N_2.

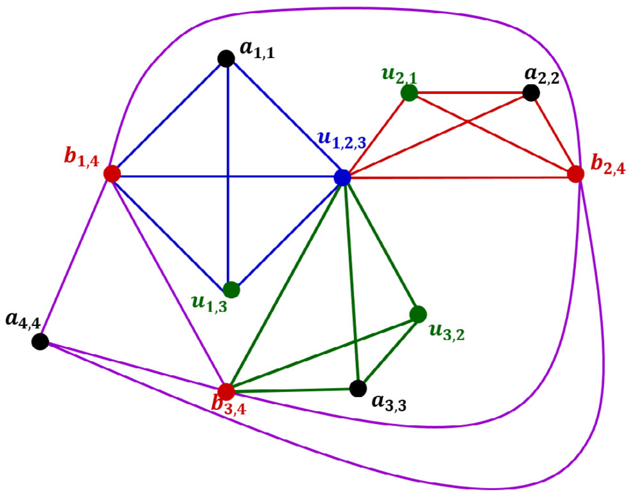

Fig. 9. Graph G'.

$f(v_1 1) = a_{44}$ $f(b_{14}) = b_{14}$ $f(b_{24}) = b_{24}$
$f(b_{34}) = b_{34}$ $f(b_{123}) = u_{123}$

3. Coloring of H_n

Lemma 3.1. *The chromatic number of H_n is n.*

Proof. Let H_n be the graph defined as above. Let M (given below) be an $n \times n$ matrix in which an entry $m_{ij} = b_{ij}$, be a vertex of H_n, belongs to both B_i, B_j for $i \neq j$ and $m_{ii} = a_{ii}$ be the vertex of H_n which belongs to B_i. i.e.,

$$\mathrm{M} = \begin{pmatrix} a_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{12} & a_{22} & b_{23} & \dots & b_{2n} \\ b_{13} & b_{23} & a_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & b_{3n} & \dots & a_{nn} \end{pmatrix}.$$

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Clearly M is a symmetric matrix. We know that, for every n in $\mathbb{N}$ there is a Symmetric Latin Square (see [15]) of order $n \times n$. Bryant and Rodger [16] gave a necessary and sufficient condition for the existence of an $(n-1)$-edge coloring of K_n (n even), and n-edge coloring of K_n (n odd) using Symmetric Latin Squares. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and e_{ij} be the edge joining the vertices v_i and v_j of K_n, where $i < j$, then arrange the edges of K_n in the matrix form $A = [a_{ij}]$ where $a_{ij} = e_{ij}$, $a_{ji} = e_{ij}$ for $i < j$ and $a_{ii} = 0$ for $1 \le i \le n$, we have $A = \begin{pmatrix} 0 & e_{12} & e_{13} & \ldots & e_{1n} \\ e_{12} & 0 & e_{23} & \ldots & e_{2n} \\ e_{13} & e_{23} & 0 & \ldots & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1n} & e_{2n} & e_{3n} & \ldots & 0 \end{pmatrix}$ and let V be a matrix given by $V = \begin{pmatrix} v_1 & 0 & 0 & \ldots & 0 \\ 0 & v_2 & 0 & \ldots & 0 \\ 0 & 0 & v_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & v_n \end{pmatrix}$. Then, define a matrix A' as

$$A' = A + V = \begin{pmatrix} v_1 & e_{12} & e_{13} & \ldots & e_{1n} \\ e_{12} & v_2 & e_{23} & \ldots & e_{2n} \\ e_{13} & e_{23} & v_3 & \ldots & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1n} & e_{2n} & e_{3n} & \ldots & v_n \end{pmatrix}.$$

Let $C = [c_{ij}]$ be a matrix where c_{ij} $(i \ne j)$, is the color of e_{ij} (i.e., $c_{ij} = c(e_{ij})$) and c_{ii} is the color of v_i. We call C as the color matrix of A'. Then C is the Symmetric Latin Square (see [16]). As the elements of M are the vertices of H_n, one can assign the colors to the vertices of H_n from the color matrix C, by the color c_{ij}, for $i, j = 1, 2, \ldots, n$ and $i \ne j$ to the vertex b_{ij} in H_n and the color c_{ii}, for $i = 1, 2, \ldots n$ to the vertex a_{ii} in H_n. Hence H_n is n colorable. ■

H_n is the graph satisfying the hypothesis of Conjecture 1.1. By using the coloring of H_n which is the graph satisfying the hypothesis of Conjecture 1.1 we extend the n-coloring of all possible graphs G satisfying the hypothesis of Conjecture 1.1.

The following example is an illustration of assigning colors to the graph H_n for $n = 6$.

Example 3.2. Consider the graph H_6 as shown in Fig. 6. The corresponding Symmetric Latin Square C of order 6×6 is given below:

$$C = \begin{pmatrix} 6 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 6 & 2 & 4 \\ 2 & 5 & 4 & 1 & 6 & 3 \\ 3 & 6 & 1 & 4 & 5 & 2 \\ 4 & 2 & 6 & 5 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}.$$

Assign the six colors to the graph H_6 using the above Symmetric Latin Square as follows:

Assign the color $c_{i,j}$ (where $c_{i,j}$ denotes the value at the (i, j)th entry in the color matrix C) for $i \ne j$ and $i, j = 1, 2, \ldots, 6$ to the vertex $b_{i,j}$ in H_6, and assign the color $c_{i,i}$ (where $c_{i,i}$ denotes the value at the (i, i)th entry in the color matrix C) for $i = 1, 2, \ldots, 6$ to the vertex a_{ii} in H_6. The colors Red, Green, Cyan, Blue, Tan, Maroon in Fig. 10 corresponds to the numbers 1, 2, 3, 4, 5, 6 respectively in the matrix C.

Then one can verify that the resultant graph is 6 colorable as shown in Fig. 10.

4. Coloring of G

Let G be the graph satisfying the hypothesis of Conjecture 1.1. Let $\hat{H}$ be the graph obtained by removing the vertices of clique degree one from graph G. i.e. $\hat{H}$ is the induced subgraph of G having all the common vertices of G.

Method for assigning colors to graph G satisfying the hypothesis of Conjecture 1.1 and every A_i $(1 \le i \le n)$ has atmost $\frac{n}{2}$ vertices of clique degree greater than one:

Let G be a graph satisfying the hypothesis of Conjecture 1.1 and every A_i $(1 \le i \le n)$ has atmost $\frac{n}{2}$ vertices of clique degree greater than one. Let $\hat{H}$ be the induced subgraph of G consisting of the vertices of clique degree

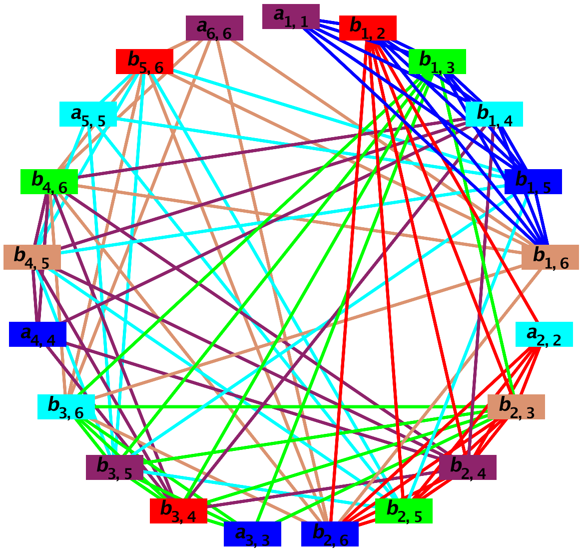

Fig. 10. A coloring of H_6 with six colors . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

greater than one in G. For every vertex v of clique degree greater than one in G, label the vertex v by u_A where $A = \{i : v \in A_i; i = 1, 2, \ldots, n\}$. Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \ldots, m$.

Let $1, 2, \ldots, n$ be the n-colors and C be the color matrix (of size $n \times n$) as defined in the proof of Lemma 3.1. The following construction applied on the color matrix C, gives a modified color matrix C_M, using which we assign the colors to the graph $\hat{H}$. Then this coloring can be extended to the graph G. Construct a new color matrix C_1 by putting $c_{i,j} = 0, c_{j,i} = 0$ for every $b_{i,j}$ in X. Also, let $c_{i,i} = 0$ for each $i = 1, 2, \ldots, n$.

Let $T = \cup_{i=3}^{n} X_i$, $P = \emptyset$, $T'' = X_2$ and $P'' = \emptyset$.

Step 1: If $T = \emptyset$, let C_m be the color matrix obtained in Step 4 and go to Step 5. Otherwise, choose a vertex $u_{i_1,i_2,\ldots,i_m}$ from T, where $i_1 < i_2 < \cdots < i_m$, and then choose $\binom{m}{2}$ vertices $b_{i_1,i_2}, b_{i_1,i_3}, \ldots, b_{i_1,i_m}, b_{i_2,i_3}, \ldots, b_{i_{m-1},i_m}$ from $V(H_n)$ corresponding to the set $\{i_1, i_2, \ldots, i_m\}$. Take $T' = \{b_{i_1,i_2}, b_{i_1,i_3}, \ldots, b_{i_1,i_m}, b_{i_2,i_3}, \ldots, b_{i_{m-1},i_m}\}$ and $P' = \emptyset$. Let $T'_1 = \{b_{i,j} : b_{i,j} \in T', c(b_{i,j})$ appear more than once in the ith row or jth column in $C\}$ and $T'_2 = \{b_{i,j} : b_{i,j} \in T', c(b_{i,j})$ appear exactly once in the ith row and jth column in $C\}$. If $T'_1 \neq \emptyset$, choose a vertex $b_{s,t}$ from T'_1, otherwise choose a vertex $b_{s,t}$ from T'_2. Then add the vertex $b_{s,t}$ to P' and remove it from T'. Go to Step 2.

Step 2: If $T'_2 \neq \emptyset$, go to Step 3. Otherwise, choose a vertex b_{i_{m-1},i_m} from T'_1. Let $A = \{c_{i,j} : c_{i,j} \neq 0; i = i_{m-1}, 1 \leq j \leq n\}$, $B = \{c_{i,j} : c_{i,j} \neq 0; j = i_m, 1 \leq i \leq n\}$. If $|A \cap B| < n$, then construct a new color matrix C_2, replacing c_{i_{m-1},i_m}, $c_{i_m,i_{m-1}}$ by x, where $x \in \{1, 2, \ldots, n\} \setminus A \cup B$. Then add the vertex b_{i_{m-1},i_m} to T'_2 and remove it from T'_1. Go to Step 3. Otherwise choose a color x which appears exactly once either in i_{m-1}th row or in i_mth column of the color matrix and construct a new color matrix C_2 replacing c_{i_{m-1},i_m}, $c_{i_m,i_{m-1}}$ by x. Then add the vertex b_{i_{m-1},i_m} to T'_2 and remove it from T'_1. Go to Step 3.

Step 3: If $T' = \emptyset$, then add the vertex $u_{i_1,i_2,\ldots,i_m}$ to P and remove it from T, go to Step 1. Otherwise, if $T' \cap T'_1 \neq \emptyset$ choose a vertex $b_{i,j}$ from $T' \cap T'_1$, if not choose a vertex $b_{i,j}$ from $T' \cap T'_2$. Go to Step 4.

Step 4: Let $c(b_{i,j}) = x,\ c(b_{s,t}) = y$. If $c(b_{i,j}) = c(b_{s,t})$, then add the vertex $b_{i,j}$ to P' and remove it from T'. Go to Step 3. Otherwise, let $A = \{c_{l,m} : c_{l,m} = x\}$, $B = \{c_{l,m} : c_{l,m} = y\} \setminus \{c_{l,m}, c_{m,l} : b_{l,m} \in P', l < m\}$. Construct a new color matrix C_3 by putting $c_{l,m} = y$ for every $c_{l,m}$ in A and $c_{l,m} = x$ for every $c_{l,m}$ in B. Then add the vertex $b_{i,j}$ to P' and remove it from T'. Go to Step 3.

Step 5: If $T'' = \emptyset$, consider $C_M = C_{m_1}$ stop the process. Otherwise, choose a vertex $u_{i,j}$ from T'' and go to Step 6.

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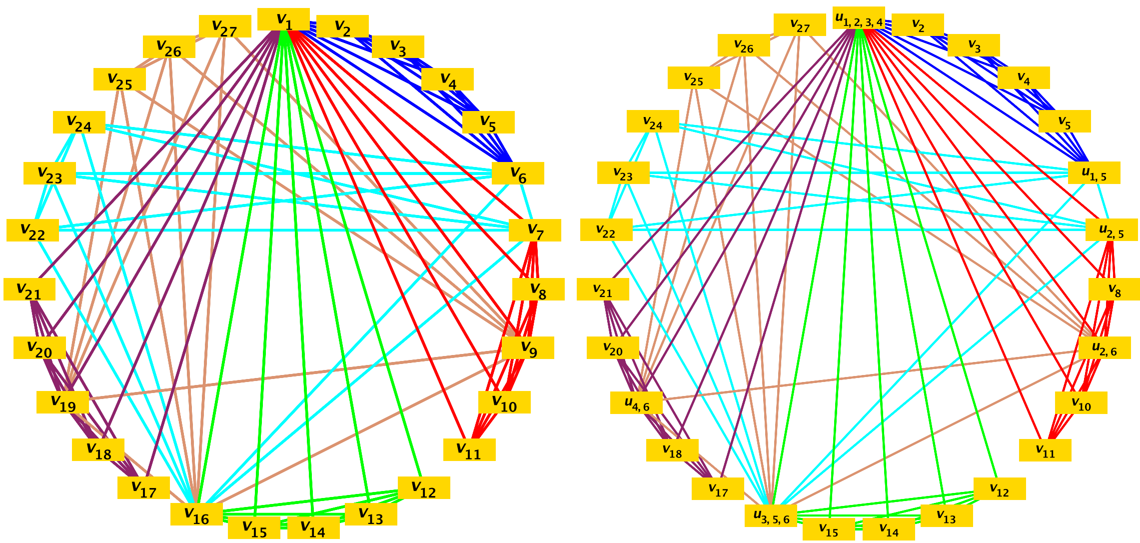

(a) Graph G

(b) Graph G after relabeling the vertices of clique degree greater than one

Fig. 11. Graph G: before and after relabeling the vertices.

Step 6: If $c_{i,j}$ appears exactly once in both ith row and jth column of the color matrix C_m, then add the vertex $b_{i,j}$ to P'' and remove it from T'', go to Step 5. Otherwise, let $A = \{c_{i,j} : c_{i,j} \neq 0; 1 \leq j \leq n\}$, $B = \{c_{i,j} : c_{i,j} \neq 0; 1 \leq i \leq n\}$. Construct a new color matrix C_{m_1} by putting x in $c_{i,j}$, $c_{j,i}$ where $x \in \{1, 2, \ldots, n\} \setminus A \cup B$ (since every A_i $(1 \leq i \leq n)$ has atmost $\frac{n}{2}$ vertices of clique degree greater than one, $|A \cup B| < n$). Then add the vertex $u_{i,j}$ to P'' and remove it from T'', go to Step 5.

Thus, in step 6, we get the modified color matrix C_M. Then, color the vertex v of $\hat{H}$ by $c_{i,j}$ of C_M, whenever $v \in A_i \cap A_j$. Then, extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of n-colors, to the vertices of clique degree one in A_i, $1 \leq i \leq n$. Thus G is n-colorable.

Remark 4.1. One can see that the above method covers the following cases:

1. G has no clique degree 2 vertices.
2. G has atmost $\frac{n}{2}$ vertices of clique degree greater than one in each A_i, $1 \leq i \leq n$.

Following is an example illustrating the above method.

Example 4.2. Let G be the graph shown in Fig. 11a.
Let $V(A_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V(A_2) = \{v_1, v_7, v_8, v_9, v_{10}, v_{11}\}$,
$V(A_3) = \{v_1, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, $V(A_4) = \{v_1, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}\}$,
$V(A_5) = \{v_6, v_7, v_{16}, v_{22}, v_{23}, v_{24}\}$, $V(A_6) = \{v_9, v_{16}, v_{19}, v_{25}, v_{26}, v_{27}\}$.
Relabel the vertices of clique degree greater than one in G by u_A where $A = \{i : v \in A_i$ for $1 \leq i \leq 6\}$. The labeled graph is shown in Fig. 11b. Fig. 12 is the graph $\hat{H}$, where $\hat{H}$ is obtained by removing the vertices of clique degree 1 from G.

Let $X = \{b_{ij} : A_i \cap A_j = \emptyset\} = \{b_{1,6}, b_{4,5}\}$,

$$X_1 = \{v \in G : d^K(v) = 1\} = \{v_2, v_3, v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{17}, v_{18}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\},$$

$$X_2 = \{v \in G : d^K(v) = 2\} = \{v_6, v_7, v_9, v_{19}\} = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\},$$

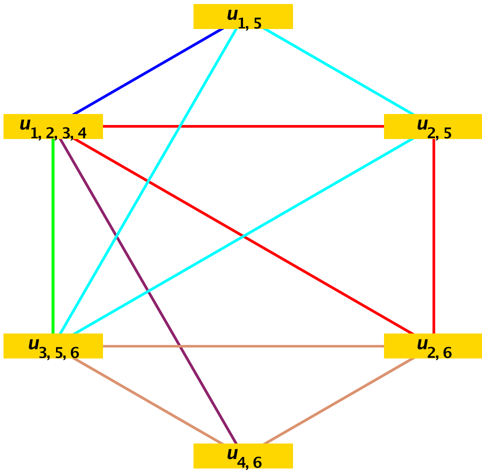

Fig. 12. Graph $\hat{H}$.

$X_3 = \{v \in G : d^K(v) = 3\} = \{v_{16}\} = \{u_{3,5,6}\}$,

and $X_4 = \{v \in G : d^K(v) = 4\} = \{v_1\} = \{u_{1,2,3,4}\}$.

Let 1, 2, …, 6 be the six colors and $C = \begin{pmatrix} 6 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 6 & 2 & 4 \\ 2 & 5 & 4 & 1 & 6 & 3 \\ 3 & 6 & 1 & 4 & 5 & 2 \\ 4 & 2 & 6 & 5 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$

be the color matrix (as well as symmetric latin square) of order 6×6.

Consider the sets $T = X_3 \cup X_4 = \{u_{3,5,6}, u_{1,2,3,4}\}$, $T'' = X_2 = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$, $P = \emptyset$ and $P'' = \emptyset$. Then, by applying the aforementioned method we get a new color matrix C_1 by putting $c_{i,j} = 0$, $c_{j,i} = 0$ for every $b_{i,j}$ in X and $c_{i,i} = 0$ for each $i = 1, 2, \ldots, 6$ and go to Step 1.

$$C_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 \\ 1 & 0 & 5 & 6 & 2 & 4 \\ 2 & 5 & 0 & 1 & 6 & 3 \\ 3 & 6 & 1 & 0 & 0 & 2 \\ 4 & 2 & 6 & 0 & 0 & 1 \\ 0 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Step 1: Since $T \neq \emptyset$, choose the vertex $u_{1,2,3,4}$ from T. Let $T' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\}$ and $P' = \emptyset$, then $T'_1 = \emptyset$ and $T'_2 = T'$. Since $T'_1 = \emptyset$, choose the vertex $b_{2,4}$ from T'_2, add it to P' and remove it from T'. Then $T' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{3,4}\}$ and $P' = \{b_{2,4}\}$. Go to step 2.

Step 2: Since $T'_2 \neq \emptyset$, go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{1,2}$ from $T' \cap T'_2$ and go to step 4.

Step 4: Since $c(b_{1,2}) = 1$, $c(b_{2,4}) = 6$ and $c(b_{1,2}) \neq c(b_{2,4})$, interchange 1, 6 in the matrix C_1 except the color of $b_{2,4}$. Add the vertex $b_{1,2}$ to P' and remove it from T'. Then

$$C_2 = \begin{pmatrix} 0 & 6 & 2 & 3 & 4 & 0 \\ 6 & 0 & 5 & 6 & 2 & 4 \\ 2 & 5 & 0 & 6 & 1 & 3 \\ 3 & 6 & 6 & 0 & 0 & 2 \\ 4 & 2 & 1 & 0 & 0 & 6 \\ 0 & 4 & 3 & 2 & 6 & 0 \end{pmatrix},$$

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$T' = \{b_{1,3}, b_{1,4}, b_{2,3}, b_{3,4}\}$ and $P' = \{b_{1,2}, b_{2,4}\}$. Go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T_1' = \emptyset$, choose the vertex $b_{1,3}$ from $T' \cap T_2'$ and go to step 4.

Step 4: Since $c(b_{1,3}) = 2$, $c(b_{2,4}) = 6$ and $c(b_{1,3}) \neq c(b_{2,4})$, interchange 2, 6 in the matrix C_2 except the color of $b_{1,2}, b_{2,4}$. Add the vertex $b_{1,3}$ to P' and remove it from T'. Then

$$C_3 = \begin{pmatrix} 0 & 6 & 6 & 3 & 4 & 0 \\ 6 & 0 & 5 & 6 & 6 & 4 \\ 6 & 5 & 0 & 2 & 1 & 3 \\ 3 & 6 & 2 & 0 & 0 & 6 \\ 4 & 6 & 1 & 0 & 0 & 2 \\ 0 & 4 & 3 & 6 & 2 & 0 \end{pmatrix},$$

$T' = \{b_{1,4}, b_{2,3}, b_{3,4}\}$ and $P' = \{b_{1,2}, b_{1,3}, b_{2,4}\}$. Go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T_1' = \emptyset$, choose the vertex $b_{1,4}$ from $T' \cap T_2'$ and go to step 4.

Step 4: Since $c(b_{1,4}) = 3$, $c(b_{2,4}) = 6$ and $c(b_{1,4}) \neq c(b_{2,4})$, interchange 3, 6 in the matrix C_3 except the color of $b_{1,2}, b_{1,3}, b_{2,4}$. Add the vertex $b_{1,4}$ to P' and remove it from T'. Then

$$C_4 = \begin{pmatrix} 0 & 6 & 6 & 6 & 4 & 0 \\ 6 & 0 & 5 & 6 & 3 & 4 \\ 6 & 5 & 0 & 2 & 1 & 6 \\ 6 & 6 & 2 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 2 \\ 0 & 4 & 6 & 3 & 2 & 0 \end{pmatrix},$$

$T' = \{b_{2,3}, b_{3,4}\}$ and $P' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,4}\}$. Go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T_1' = \emptyset$, choose the vertex $b_{2,3}$ from $T' \cap T_2'$ and go to step 4.

Step 4: Since $c(b_{2,3}) = 5$, $c(b_{2,4}) = 6$ and $c(b_{2,3}) \neq c(b_{2,4})$, interchange 5, 6 in the matrix C_4 except the color of $b_{1,2}, b_{1,3}, b_{1,4}, b_{2,4}$. Add the vertex $b_{2,3}$ to P' and remove it from T'. Then

$$C_5 = \begin{pmatrix} 0 & 6 & 6 & 6 & 4 & 0 \\ 6 & 0 & 6 & 6 & 3 & 4 \\ 6 & 6 & 0 & 2 & 1 & 5 \\ 6 & 6 & 2 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 2 \\ 0 & 4 & 5 & 3 & 2 & 0 \end{pmatrix},$$

$T' = \{b_{3,4}\}$ and $P' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}\}$. Go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T_1' = \emptyset$, choose the vertex $b_{3,4}$ from $T' \cap T_2'$ and go to step 4.

Step 4: Since $c(b_{3,4}) = 2$, $c(b_{2,4}) = 6$ and $c(b_{3,4}) \neq c(b_{2,4})$, interchange 2, 6 in the matrix C_5 except the color of $b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}$. Add the vertex $b_{3,4}$ to P' and remove it from T'. Then

$$C_6 = \begin{pmatrix} 0 & 6 & 6 & 6 & 4 & 0 \\ 6 & 0 & 6 & 6 & 3 & 4 \\ 6 & 6 & 0 & 6 & 1 & 5 \\ 6 & 6 & 6 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 6 \\ 0 & 4 & 5 & 3 & 6 & 0 \end{pmatrix},$$

$T' = \emptyset$ and $P' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\}$. Go to step 3.

Step 3: Since $T' = \emptyset$, add the vertex $u_{1,2,3,4}$ to P and remove it from T, then $T = \{u_{3,5,6}\}$ and $P = \{u_{1,2,3,4}\}$. Go to step 1.

Step 1: Since $T \neq \emptyset$, choose the vertex $u_{3,5,6}$ from T. Let $T' = \{b_{3,5}, b_{3,6}, b_{5,6}\}$ and $P' = \emptyset$, then $T_1' = \emptyset$ and $T_2' = T'$. Since $T_1' = \emptyset$, choose the vertex $b_{5,6}$ from T_2', add it to P' and remove it from T'. Then $T' = \{b_{3,5}, b_{3,6}\}$ and $P' = \{b_{5,6}\}$. Go to step 2.

Step 2: Since $T_2' \neq \emptyset$, go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T_1' = \emptyset$, choose the vertex $b_{3,6}$ from $T' \cap T_2'$ and go to step 4.

Step 4: Since $c(b_{3,6}) = 5$, $c(b_{5,6}) = 6$ and $c(b_{3,6}) \neq c(b_{5,6})$, interchange 5, 6 in the matrix C_6 except the color of $b_{5,6}$. Add the vertex $b_{3,6}$ to P' and remove it from T'. Then

$$C_7 = \begin{pmatrix} 0 & 5 & 5 & 5 & 4 & 0 \\ 5 & 0 & 5 & 5 & 3 & 4 \\ 5 & 5 & 0 & 5 & 1 & 6 \\ 5 & 5 & 5 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 6 \\ 0 & 4 & 6 & 3 & 6 & 0 \end{pmatrix},$$

$T' = \{b_{3,5}\}$ and $P' = \{b_{3,6}, b_{5,6}\}$. Go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T_1' = \emptyset$, choose the vertex $b_{3,5}$ from $T' \cap T_2'$ and go to step 4.

Step 4: Since $c(b_{3,5}) = 1$, $c(b_{5,6}) = 6$ and $c(b_{3,5}) \neq c(b_{5,6})$, interchange 1, 6 in the matrix C_7 except the color of $b_{3,6}, b_{5,6}$. Add the vertex $b_{3,5}$ to P' and remove it from T'. Then

$$C_8 = \begin{pmatrix} 0 & 5 & 5 & 5 & 4 & 0 \\ 5 & 0 & 5 & 5 & 3 & 4 \\ 5 & 5 & 0 & 5 & 6 & 6 \\ 5 & 5 & 5 & 0 & 0 & 3 \\ 4 & 3 & 6 & 0 & 0 & 6 \\ 0 & 4 & 6 & 3 & 6 & 0 \end{pmatrix},$$

$T' = \emptyset$ and $P' = \{b_{3,5}, b_{3,6}, b_{5,6}\}$. Go to step 3.

Step 3: Since $T' = \emptyset$, add the vertex $u_{3,5,6}$ to P and remove it from T, then $T = \emptyset$ and $P = \{u_{3,5,6}, u_{1,2,3,4}\}$. Go to step 1.

Step 1: Since $T = \emptyset$ consider $C_m = C_8$, go to step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{1,5}$ from T''. Go to step 6.

Step 6: Since $c_{1,5} = 4$ appears exactly once in both 1st row and 5th column of the color matrix C_m. Add the vertex $u_{1,5}$ to P'' and remove it from T''. Then $T'' = \{u_{2,5}, u_{2,6}, u_{4,6}\}$ and $P'' = \{u_{1,5}\}$. Go to Step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{2,5}$ from T''. Go to step 6.

Step 6: Since $c_{2,5} = 3$ appears exactly once in both 2nd row and 5th column of the color matrix C_m. Add the vertex $u_{2,5}$ to P'' and remove it from T''. Then $T'' = \{u_{2,6}, u_{4,6}\}$ and $P'' = \{u_{1,5}, u_{2,5}\}$. Go to Step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{2,6}$ from T''. Go to step 6.

Step 6: Since $c_{2,6} = 4$ appears exactly once in both 2nd row and 6th column of the color matrix C_m. Add the vertex $u_{2,6}$ to P'' and remove it from T''. Then $T'' = \{u_{4,6}\}$ and $P'' = \{u_{1,5}, u_{2,5,u_{2,6}}\}$. Go to Step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{4,6}$ from T''. Go to step 6.

Step 6: Since $c_{4,6} = 3$ appears exactly once in both 4th row and 6th column of the color matrix C_m. Add the vertex $u_{4,6}$ to P'' and remove it from T''. Then $T'' = \emptyset$ and $P'' = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$. Go to Step 5.

Step 5: Since $T'' = \emptyset$, consider $C_M = C_m$.

Stop the process.

Assign the colors to the graph $\hat{H}$ using the matrix C_M, i.e., color the vertex v by the (i, j)th entry $c_{i,j}$ of the matrix C_M, whenever $A_i \cap A_j \neq \emptyset$ (see Fig. 13a), where the numbers 1, 2, 3, 4, 5, 6 correspond to the colors Green, Cyan, Blue, Maroon, Tan, Red respectively. Extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of 6-colors to the vertices of clique degree one in each A_i, $1 \leq i \leq 6$. The colored graph G is shown in Fig. 13b.

Following example shows that the aforementioned method does not work, if the graph G has more than $\frac{n}{2}$ vertices of clique degree greater than one in some A_i, $1 \leq i \leq n$.

Example 4.3. Let G be the graph shown in Fig. 14a.

Let $V(A_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V(A_2) = \{v_2, v_7, v_8, v_9, v_{10}, v_{11}\}$,
$V(A_3) = \{v_3, v_8, v_{12}, v_{13}, v_{14}, v_{15}\}$, $V(A_4) = \{v_4, v_9, v_{16}, v_{17}, v_{18}, v_{20}, v_{21}\}$,
$V(A_5) = \{v_5, v_{10}, v_{14}, v_{18}, v_{20}, v_{21}\}$, $V(A_6) = \{v_6, v_{10}, v_{15}, v_{19}, v_{22}, v_{23}\}$.

Relabel the vertices of clique degree greater than one in G by u_A where $A = \{i : v \in A_i \text{ for } 1 \leq i \leq 6\}$. The labeled graph is shown in Fig. 14b. Fig. 15 is the graph $\hat{H}$, where $\hat{H}$ is obtained by removing the vertices of clique degree 1 from G.

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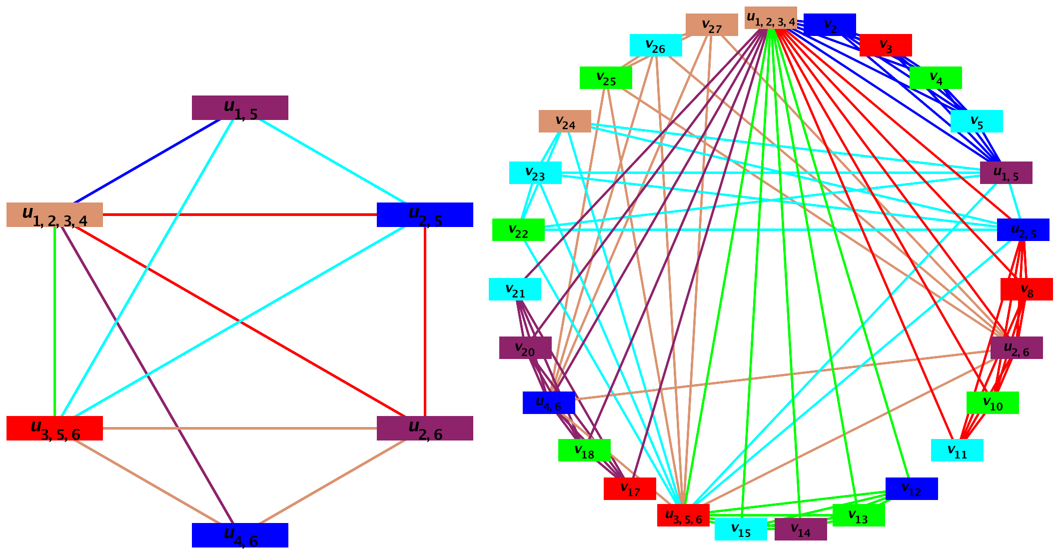
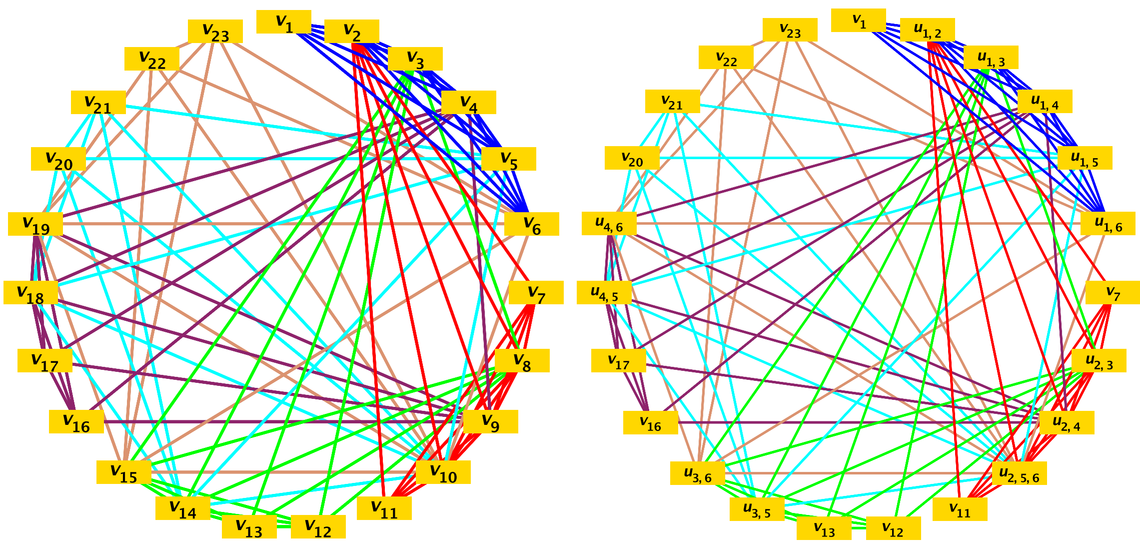


(a) Graph $\hat{H}$

(b) A 6 coloring of graph G

Fig. 13. The graphs $\hat{H}$ and G, after colors have been assigned to their vertices . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)



(a) Graph G

(b) Graph G after relabeling the vertices of clique degree greater than one

Fig. 14. Graph G: before and after relabeling the vertices.

Let $X = \{b_{ij} : A_i \cap A_j = \emptyset\} = \{b_{3,4}\}$,

$$X_1 = \{v \in G : d^K(v) = 1\} = \{v_1, v_7, v_{11}, v_{12}, v_{13}, v_{16}, v_{17}, v_{20}, v_{21}, v_{22}, v_{23}\},$$

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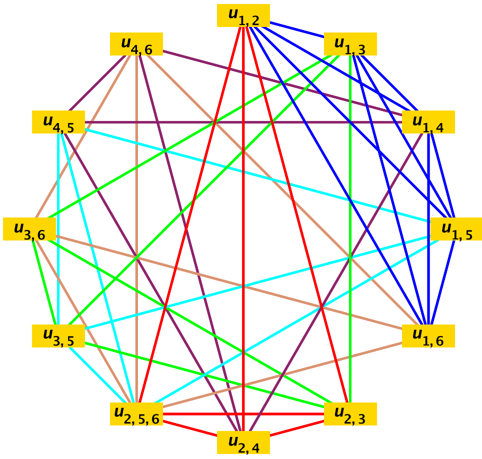

Fig. 15. Graph $\hat{H}$.

$$X_2 = \{v \in G : d^K(v) = 2\} = \{v_2, v_3, v_4, v_5, v_6, v_8, v_9, v_{14}, v_{15}, v_{18}, v_{19}\}$$
$$= \{u_{1,2}, u_{1,3}, u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\},$$

and $X_3 = \{v \in G : d^K(v) = 3\} = \{v_{10}\} = \{u_{2,5,6}\}$.

Let $1, 2, \ldots, 6$ be the six colors and $C = \begin{pmatrix} 6 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 6 & 2 & 4 \\ 2 & 5 & 4 & 1 & 6 & 3 \\ 3 & 6 & 1 & 4 & 5 & 2 \\ 4 & 2 & 6 & 5 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$ be the color matrix (as well as symmetric latin square) of order 6×6.

Consider the sets $T = X_3 = \{u_{2,5,6}\}$,

$T'' = X_2 = \{u_{1,2}, u_{1,3}, u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$, $P = \emptyset$ and $P'' = \emptyset$. Then by applying the aforementioned method we get a new color matrix C_1 by putting $c_{i,j} = 0$, $c_{j,i} = 0$ for every $b_{i,j}$ in X and $c_{i,i} = 0$ for each $i = 1, 2, \ldots, 6$ and go to Step 1.

$$C_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 6 & 2 & 4 \\ 2 & 5 & 0 & 0 & 6 & 3 \\ 3 & 6 & 0 & 0 & 5 & 2 \\ 4 & 2 & 6 & 5 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Step 1: Since $T \neq \emptyset$, choose the vertex $u_{2,5,6}$ from T. Let $T' = \{b_{2,5}, b_{2,6}, b_{5,6}\}$ and $P' = \emptyset$, then $T'_1 = \emptyset$ and $T'_2 = T'$. Since $T'_1 = \emptyset$, choose the vertex $b_{5,6}$ from T'_2, add it to P' and remove it from T'. Then $T' = \{b_{2,5}, b_{2,6}\}$ and $P' = \{b_{5,6}\}$. Go to step 2.

Step 2: Since $T'_2 \neq \emptyset$, go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{2,5}$ from $T' \cap T'_2$ and go to step 4.

Step 4: Since $c(b_{2,5}) = 2$, $c(b_{5,6}) = 1$ and $c(b_{2,5}) \neq c(b_{5,6})$, interchange 2, 1 in the matrix C_1 except the color of $b_{5,6}$. Add the vertex $b_{2,5}$ to P' and remove it from T'. Then

$$C_2 = \begin{pmatrix} 0 & 2 & 1 & 3 & 4 & 5 \\ 2 & 0 & 5 & 6 & 1 & 4 \\ 1 & 5 & 0 & 0 & 6 & 3 \\ 3 & 6 & 0 & 0 & 5 & 2 \\ 4 & 1 & 6 & 5 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix},$$

$T' = \{b_{2,6}\}$ and $P' = \{b_{2,5}, b_{5,6}\}$. Go to step 3.

Step 3: Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{2,6}$ from $T' \cap T'_2$ and go to step 4.

Step 4: Since $c(b_{2,6}) = 4$, $c(b_{5,6}) = 1$ and $c(b_{2,6}) \neq c(b_{5,6})$, interchange 4, 1 in the matrix C_2 except the color of $b_{2,5}, b_{5,6}$. Add the vertex $b_{2,6}$ to P' and remove it from T'. Then

$$C_3 = \begin{pmatrix} 0 & 2 & 4 & 3 & 1 & 5 \\ 2 & 0 & 5 & 6 & 1 & 1 \\ 4 & 5 & 0 & 0 & 6 & 3 \\ 3 & 6 & 0 & 0 & 5 & 2 \\ 1 & 1 & 6 & 5 & 0 & 1 \\ 5 & 1 & 3 & 2 & 1 & 0 \end{pmatrix},$$

$T' = \emptyset$ and $P' = \{b_{2,5}, b_{2,6}, b_{5,6}\}$. Go to step 3.

Step 3: Since $T' = \emptyset$, add the vertex $u_{2,5,6}$ to P and remove it from T, then $T = \emptyset$ and $P = \{u_{2,5,6}\}$. Go to step 1.

Step 1: Since $T = \emptyset$ consider $C_m = C_3$, go to step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{1,2}$ from T''. Go to step 6.

Step 6: Since $c_{1,2} = 2$ appears exactly once in both 1st row and 2nd column of the color matrix C_m. Add the vertex $u_{1,2}$ to P'' and remove it from T''. Then $T'' = \{u_{1,3}, u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$ and $P'' = \{u_{1,2}\}$. Go to Step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{1,3}$ from T''. Go to step 6.

Step 6: Since $c_{1,3} = 4$ appears exactly once in both 1st row and 3rd column of the color matrix C_m. Add the vertex $u_{1,3}$ to P'' and remove it from T''. Then $T'' = \{u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$ and $P'' = \{u_{1,2}, u_{1,3}\}$. Go to Step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{1,4}$ from T''. Go to step 6.

Step 6: Since $c_{1,4} = 3$ appears exactly once in both 1st row and 4th column of the color matrix C_m. Add the vertex $u_{1,4}$ to P'' and remove it from T''. Then $T'' = \{u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$ and $P'' = \{u_{1,2}, u_{1,3}, u_{1,4}\}$. Go to Step 5.

Step 5: Since $T'' \neq \emptyset$, choose the vertex $u_{1,5}$ from T''. Go to step 6.

Step 6: Since $c_{1,5} = 1$ and it appears more than once in the 5th column of the color matrix C_m. Let $A = \{c_{1,j} : c_{1,j} \neq 0; 1 \leq j \leq 6\} = \{1, 2, 3, 4, 5\}$, $B = \{c_{i,5} : c_{i,5} \neq 0; 1 \leq i \leq 6\} = \{1, 5, 6\}$, then $A \cup B$ = $\{1, 2, 3, 4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 6\} \setminus A \cup B = \emptyset$.

It cannot be go further.

In the illustration of Example 4.3, if we choose the color matrix (symmetric latin square) given below, then exists an n-coloring of G.

$$\text{Let } C' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Applying the method in Example 4.3, we get

$$C'_M = \begin{pmatrix} 0 & 2 & 3 & 6 & 5 & 1 \\ 2 & 0 & 6 & 5 & 4 & 4 \\ 3 & 6 & 0 & 0 & 1 & 2 \\ 6 & 5 & 0 & 0 & 2 & 3 \\ 5 & 4 & 1 & 2 & 0 & 4 \\ 1 & 4 & 2 & 3 & 4 & 0 \end{pmatrix}.$$

Color the vertex v by the (i, j)th entry $c_{i,j}$ of the matrix C'_M, whenever $A_i \cap A_j \neq \emptyset$ (see Fig. 16a), where the numbers 1, 2, 3, 4, 5, 6 correspond to the colors Blue, Red, Green, Maroon, Tan, Cyan respectively. Extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of 6-colors to the vertices of clique degree one in each A_i, $1 \leq i \leq 6$. The colored graph G is shown in Fig. 16b.

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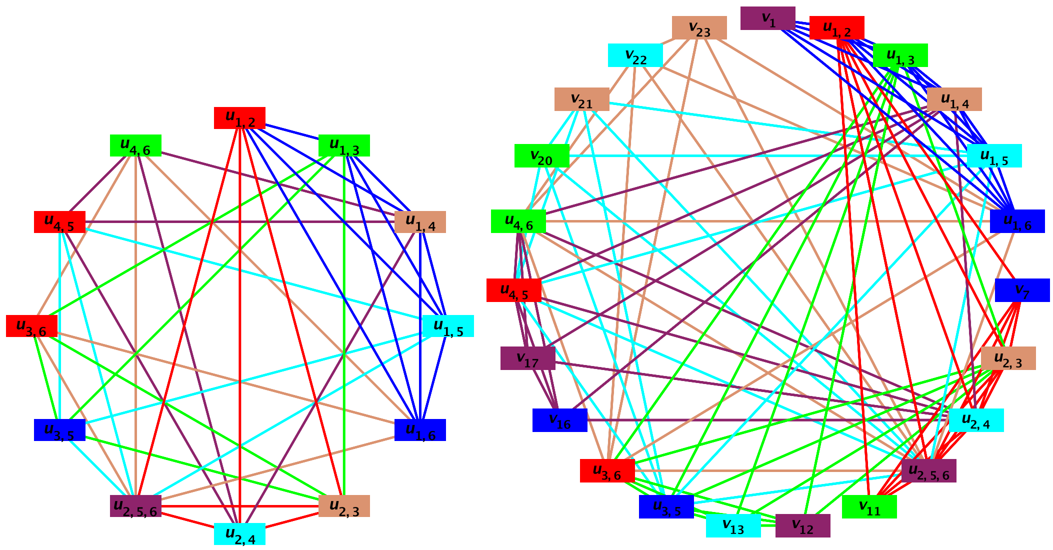

(a) Graph $\hat{H}$ (b) A 6 coloring of graph G

Fig. 16. The graphs $\hat{H}$ and G, after colors have been assigned to their vertices . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Remark 4.4. From the above example, one can see that the method will work for some symmetric latin squares and will not work for some other, for the graphs having more than $\frac{n}{2}$ vertices of clique degree greater than one in some A_i $(1 \leq i \leq n)$ in G.

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