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Some identities for generalized Fibonacci and Lucas numbers

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Abstract

In this paper we study one parameter generalization of the Fibonacci numbers, Lucas numbers which generalizes the Jacobsthal numbers, Jacobsthal–Lucas numbers simultaneously. We present some their properties and interpretations also in graphs. Presented results generalize well-known results for Fibonacci numbers, Lucas numbers, Jacobsthal numbers and Jacobsthal–Lucas numbers.

Keywords: Fibonacci numbers; Jacobsthal numbers; Recurrence relations

1. Introduction and preliminary results

The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with $F_0 = 0$, $F_1 = 1$.

The nth Lucas number L_n is defined recursively by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with initial terms $L_0 = 2$, $L_1 = 1$.

Apart from the Fibonacci numbers and the Lucas numbers the well-known are the Jacobsthal numbers and the Jacobsthal–Lucas numbers.

For an integer $n \geq 0$ the nth Jacobsthal number J_n is defined recursively by $J_n = J_{n-1} + 2J_{n-2}$, for $n \geq 2$ with $J_0 = 0$, $J_1 = 1$. The nth Jacobsthal–Lucas number j_n is defined by $j_n = j_{n-1} + 2j_{n-2}$, for $n \geq 2$ with $j_0 = 2$, $j_1 = 1$.

Let us consider one parameter generalization of the Fibonacci numbers.

Let $n \geq 0$, $t \geq 1$ be integers. The nth generalized Fibonacci number $J_{t,n}$ is defined recursively as follows

$$J_{t,n} = J_{t,n-1} + t \cdot J_{t,n-2}, \tag{1}$$

for $n \geq 2$ with initial conditions $J_{t,0} = 0$ and $J_{t,1} = 1$.

It is interesting to note that $J_{t,n}$ generalizes the Fibonacci numbers and the Jacobsthal numbers. If $t = 1$ then $J_{1,n} = F_n$ and for $t = 2$ holds $J_{2,n} = J_n$. For these reasons numbers $J_{t,n}$ also are named as generalized Jacobsthal numbers.

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In the same way we can define the generalized Lucas numbers, which are a generalization of Lucas numbers and Jacobsthal–Lucas numbers.

Let $n \geq 0$, $t \geq 1$ be integers. The nth generalized Lucas number $j_{t,n}$ is defined recursively as follows

$$j_{t,n} = j_{t,n-1} + t \cdot j_{t,n-2}, \tag{2}$$

for $n \geq 2$ with initial conditions $j_{t,0} = 2$ and $j_{t,1} = 1$.

If $t = 1$ then $j_{1,n} = L_n$ and for $t = 2$ holds $j_{2,n} = j_n$.

Since the characteristic equation of relations (1) and (2) is $r^2 - r - t = 0$, so roots of it are $\alpha = \frac{1+\sqrt{1+4t}}{2}$, $\beta = \frac{1-\sqrt{1+4t}}{2}$. Consequently for $n \geq 0$ the direct formulas (named also as Binet formulas) for $J_{t,n}$ and $j_{t,n}$ have the forms

$$J_{t,n} = \frac{1}{\sqrt{1+4t}}\left[\left(\frac{1+\sqrt{1+4t}}{2}\right)^n - \left(\frac{1-\sqrt{1+4t}}{2}\right)^n\right], \tag{3}$$

$$j_{t,n} = \left(\frac{1+\sqrt{1+4t}}{2}\right)^n + \left(\frac{1-\sqrt{1+4t}}{2}\right)^n. \tag{4}$$

The Fibonacci numbers have many interpretations also in graph theory, see [1–6]. It was shown that the generalized Fibonacci numbers $J_{t,n}$ are related to the Merrifield–Simmons index of a special graph product, see for details [7]. In this paper we show another graph interpretation of $J_{t,n}$.

Only undirected, connected simple graphs are considered. A subset $S \subseteq V(G)$ is independent if for each $x, y \in S$, there is no edge between them. Moreover the empty set and every subset containing exactly one vertex are independent. Let $n \geq 1$ be an integer. Let us consider n copies of edgeless graph N_t of order t, $t \geq 1$, denoted by N_t^i, with $V(N_t^i) = \{x_1^i, \ldots, x_t^i\}$ for $i = 1, \ldots, n$. Then G_n is a graph such that $V(G_n) = \bigcup_{i=1}^n V(N_t^i)$ and $E(G_n) = \bigcup_{i=1}^n \{x_j^i x_k^{i+1};\, 1 \leq i \leq n, 1 \leq j,k \leq t\}$. Let $\sigma(G_n)$ be the number of all independent sets S of G_n such that $\left|S \cap V(N_t^i)\right| \leq 1$ for $i = 1, \ldots, n$.

Theorem 1. *Let $n \geq 1$, $t \geq 1$ be integers. Then*

$$\sigma(G_n) = J_{t,n+2}. \tag{5}$$

Proof (*By Induction on n*)**.** Let n, t be as in the statement of the theorem. If $n = 1, 2$ then the result is obvious. Suppose that $n \geq 3$ and let $\sigma(G_k) = J_{t,k+2}$ for $k < n$. We shall show that $\sigma(G_n) = J_{t,n+2}$. Let $S \subseteq V(G_n)$ be an arbitrary independent set such that $\left|S \cap V(N_t^i)\right| \leq 1$ for $i = 1, \ldots, n$. We consider the following cases.

1. $S \cap V(N_t^n) = \emptyset$.
   Then S is an arbitrary independent set of a graph G_{n-1} with the condition $\left|S \cap V(N_t^j)\right| \leq 1$ for $j = 1, \ldots, n-1$. By the induction hypothesis there are $\sigma(G_{n-1}) = J_{t,n+1}$ subsets S in this case.
2. $S \cap V(N_t^n) \neq \emptyset$.
   Clearly $\left|S \cap V(N_t^n)\right| = 1$ and by the definition of the graph G_n holds $S \cap V(N_t^{n-1}) = \emptyset$. Since the unique vertex belonging to $S \cap V(N_t^n)$ can be chosen in t ways and by induction hypothesis there are $t \cdot \sigma(G_{n-2}) = t \cdot J_{t,n}$ subsets S in this case.

Finally $\sigma(G_n) = \sigma(G_{n-1}) + t \cdot \sigma(G_{n-2}) = J_{t,n+1} + t \cdot J_{t,n} = J_{t,n+2}$ which ends the proof. □

2. Identities for generalized Fibonacci and Lucas numbers

In this section we give some properties of generalized Fibonacci numbers and generalized Lucas numbers.

Theorem 2. *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$J_{t,n+2} + t \cdot J_{t,n} = j_{t,n+1}. \tag{6}$$

Proof (*By Induction on n*)**.** For $n = 0$ we have $J_{t,2} + t \cdot J_{t,0} = 1 + t \cdot 0 = 1 = j_{t,1}$.

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Let $k \geq 0$ be given and suppose that (6) is true for all $n = 0, 1, 2, \ldots, k$. We shall show that (6) holds for $n = k + 1$. Using induction's assumption for $n = k$ and $n = k - 1$ and (1), (2) we have

$$\begin{aligned}
&J_{t,(k+1)+2} + t \cdot J_{t,k+1} = \\
&= J_{t,k+3} + t \cdot J_{t,k+1} = \\
&= (J_{t,k+2} + t \cdot J_{t,k+1}) + t \cdot \left(J_{t,k} + t \cdot J_{t,k-1}\right) = \\
&= (J_{t,k+2} + t \cdot J_{t,k}) + t \cdot \left(J_{t,k+1} + t \cdot J_{t,k-1}\right) = \\
&= j_{t,k+1} + t \cdot j_{t,k} = j_{t,k+2} = j_{t,(k+1)+1}.
\end{aligned}$$

Thus, (6) holds for $n = k + 1$, and the proof of the induction step is complete. □

Theorem 3. *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$J_{t,n} + j_{t,n} = 2 \cdot J_{t,n+1}. \tag{7}$$

Proof (*By Induction on n*)**.** For $n = 0$ we have $J_{t,0} + j_{t,0} = 0 + 2 = 2 \cdot 1 = 2 \cdot J_{t,1}$.

Let $k \geq 0$ be given and suppose that (7) is true for all $n = 0, 1, 2, \ldots, k$. We shall show that (7) holds for $n = k + 1$. Using induction's assumption for $n = k$ and $n = k - 1$ and (1), (2) we have

$$\begin{aligned}
&J_{t,k+1} + j_{t,k+1} = \\
&= (J_{t,k} + t \cdot J_{t,k-1}) + (j_{t,k} + t \cdot j_{t,k-1}) = \\
&= (J_{t,k} + j_{t,k}) + t \cdot (J_{t,k-1} + j_{t,k-1}) = \\
&= 2 \cdot J_{t,k+1} + t \cdot 2 \cdot J_{t,k} = \\
&= 2 \cdot \left(J_{t,k+1} + t \cdot J_{t,k}\right) = 2 \cdot J_{t,k+2} = 2 \cdot J_{t,(k+1)+1}.
\end{aligned}$$

Thus, (7) holds for $n = k + 1$, and the proof of the induction step is complete. □

Theorem 4. *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$\sum_{i=0}^{n} J_{t,i} = \frac{J_{t,n+2} - J_{t,1}}{t}. \tag{8}$$

Proof. Using (1) we have $J_{t,n} = \frac{1}{t}(J_{t,n+2} - J_{t,n+1})$.

For integers $0, 1, \ldots, n$ we obtain

$$\begin{aligned}
J_{t,0} &= \frac{1}{t}(J_{t,2} - J_{t,1}) \\
J_{t,1} &= \frac{1}{t}(J_{t,3} - J_{t,2}) \\
J_{t,2} &= \frac{1}{t}(J_{t,4} - J_{t,3}) \\
&\vdots \\
J_{t,n-1} &= \frac{1}{t}(J_{t,n+1} - J_{t,n}) \\
J_{t,n} &= \frac{1}{t}(J_{t,n+2} - J_{t,n+1}).
\end{aligned}$$

Adding these equalities we obtain (8). □

In the same way one can easily prove the next theorem.

Theorem 5. *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$\sum_{i=0}^{n} j_{t,i} = \frac{j_{t,n+2} - j_{t,1}}{t}. \tag{9}$$

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From the above theorems we obtain the well-known identities for Fibonacci numbers, Lucas numbers, Jacobsthal numbers and Jacobsthal–Lucas numbers.

Corollary 6. *Let $n \geq 0$ be an integer. Then*

$$\sum_{i=0}^{n} F_i = F_{n+2} - F_1 = F_{n+2} - 1,$$
$$\sum_{i=0}^{n} L_i = L_{n+2} - L_1 = L_{n+2} - 1,$$
$$\sum_{i=0}^{n} J_i = \frac{J_{n+2} - J_1}{2} = \frac{J_{n+2} - 1}{2},$$
$$\sum_{i=0}^{n} j_i = \frac{j_{n+2} - j_1}{2} = \frac{j_{n+2} - 1}{2}.$$

Some identities for numbers $J_{t,n}$ and $j_{t,n}$ can be found using their matrix generators.

For integers $n \geq 1$ and $t \geq 1$ let $J(t,n) = \begin{bmatrix} J_{t,n} & J_{t,n-1} \\ J_{t,n+1} & J_{t,n} \end{bmatrix}$ be a matrix with entries being generalized Fibonacci numbers.

Theorem 7. *Let $n \geq 1$, $t \geq 1$ be integers. Then*

$$\begin{bmatrix} J_{t,n} & J_{t,n-1} \\ J_{t,n+1} & J_{t,n} \end{bmatrix} = \begin{bmatrix} J_{t,1} & J_{t,0} \\ J_{t,2} & J_{t,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ t & 0 \end{bmatrix}^{n-1}.$$

Proof (*By Induction on n*)**.** If $n = 1$ then the result is obvious. Assume that

$$\begin{bmatrix} J_{t,n} & J_{t,n-1} \\ J_{t,n+1} & J_{t,n} \end{bmatrix} = \begin{bmatrix} J_{t,1} & J_{t,0} \\ J_{t,2} & J_{t,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ t & 0 \end{bmatrix}^{n-1}.$$

We shall show that

$$\begin{bmatrix} J_{t,n+1} & J_{t,n} \\ J_{t,n+2} & J_{t,n+1} \end{bmatrix} = \begin{bmatrix} J_{t,1} & J_{t,0} \\ J_{t,2} & J_{t,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ t & 0 \end{bmatrix}^{n}.$$

By simple calculation using induction's hypothesis we have

$$\begin{bmatrix} J_{t,1} & J_{t,0} \\ J_{t,2} & J_{t,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ t & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ t & 0 \end{bmatrix} = \begin{bmatrix} J_{t,n} & J_{t,n-1} \\ J_{t,n+1} & J_{t,n} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ t & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} J_{t,n} + t \cdot J_{t,n-1} & J_{t,n} \\ J_{t,n+1} + t \cdot J_{t,n} & J_{t,n+1} \end{bmatrix} = \begin{bmatrix} J_{t,n+1} & J_{t,n} \\ J_{t,n+2} & J_{t,n+1} \end{bmatrix},$$

which ends the proof. □

This generator immediately gives the Cassini formula for the generalized Fibonacci numbers.

Corollary 8. *Let $n \geq 1$, $t \geq 1$ be integers. Then*

$$J_{t,n}^2 - J_{t,n-1} \cdot J_{t,n+1} = \left(J_{t,1}^2 - J_{t,0} \cdot J_{t,2}\right) \cdot (-t)^{n-1} = (-t)^{n-1}.$$

If $t = 1$ and $t = 2$ then we obtain the well-known Cassini formulas for the Fibonacci numbers and the Jacobsthal numbers, respectively.

Corollary 9. *Let $n \geq 1$ be an integer. Then*

$$F_n^2 - F_{n-1} \cdot F_{n+1} = (-1)^{n-1}$$
$$J_n^2 - J_{n-1} \cdot J_{n+1} = (-2)^{n-1}.$$

Analogously we can define the matrix generator and the Cassini formula for the generalized Lucas numbers.

For integers $n \geq 1$ and $t \geq 1$ let $j(t,n) = \begin{bmatrix} j_{t,n} & j_{t,n-1} \\ j_{t,n+1} & j_{t,n} \end{bmatrix}$ be a matrix with entries being generalized Lucas numbers.

Theorem 10. *Let $n \geq 1$, $t \geq 1$ be integers. Then*

$$\begin{bmatrix} j_{t,n} & j_{t,n-1} \\ j_{t,n+1} & j_{t,n} \end{bmatrix} = \begin{bmatrix} j_{t,1} & j_{t,0} \\ j_{t,2} & j_{t,1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ t & 0 \end{bmatrix}^{n-1}.$$

Corollary 11. *Let $n \geq 1$, $t \geq 1$ be integers. Then*

$$j_{t,n}^2 - j_{t,n-1} \cdot j_{t,n+1} = (-1-4t) \cdot (-t)^{n-1}.$$

If $t = 1$ and $t = 2$ then we have the well-known Cassini formulas for the Lucas numbers and the Jacobsthal–Lucas numbers, respectively.

Corollary 12. *Let $n \geq 1$ be an integer. Then*

$$L_n^2 - L_{n-1} \cdot L_{n+1} = -5 \cdot (-1)^{n-1},$$
$$j_n^2 - j_{n-1} \cdot j_{n+1} = -9 \cdot (-2)^{n-1}.$$

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