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On the 2-token graph of a graph

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Abstract

Let $G = (V, E)$ be a graph and let k be a positive integer. Let $P_k(V)$=$\{S : S \subseteq V$ and $|S| = k\}$. The k-token graph $F_k(G)$ is the graph with vertex set $P_k(V)$ and two vertices A and B are adjacent if $A \Delta B = \{a, b\}$ and $ab \in E(G)$, where Δ denotes the symmetric difference. In this paper we present several basic results on 2-token graphs.

Keywords: k-token graph; Line graph; Chordal graph; Independence number

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Zhang [1].

For any set V we denote by $P_k(V)$ the set of all k-element subsets of V. Monray et al. [2] introduced the notion of k-token graph of a graph G.

Definition 1.1 (*[2]*)**.** Let $G = (V, E)$ be a graph and let $k \geq 1$ be an integer. The k-token graph $F_k(G)$ of G is the graph with vertex set $P_k(V)$ and two vertices A and B are adjacent if $A \Delta B = \{a, b\}$ where $ab \in E(G)$.

Clearly $|V(F_k(G))| = \binom{n}{k}$ and $|E(F_k(G))| = \binom{n-2}{k-1}|E(G)|$.

We need the following theorems.

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Theorem 1.2 ([2]). *If $F_k(G)$ is bipartite for some $k \geq 1$, then $F_l(G)$ is bipartite for all $l \geq 1$.*

Theorem 1.3 ([3]). *Let $G = (V, E)$ be a graph of order n and let $v_i, v_j \in V$. Then*

$$deg_{F_2(G)}(\{v_i, v_j\}) = \begin{cases} deg(v_i) + deg(v_j) & \text{if } v_i v_j \notin E \\ deg(v_i) + deg(v_j) - 2 & \text{if } v_i v_j \in E \end{cases}$$

In this paper we investigate the 2-token graph $F_2(G)$. We repeatedly use the following observation.

Observation 1.4. *Two vertices x and y of $F_2(G)$ are adjacent if and only if $x = \{a, b\}$ and $y = \{a, c\}$ with $bc \in E(G)$.*

2. Main results

We first prove that for complete graphs the 2-token graph is its line graph.

Theorem 2.1. *Let G be a graph of order $n \geq 2$. Then $F_2(G)$ is isomorphic to the line graph $L(G)$ if and only if $G = K_n$.*

Proof. Suppose $G = K_n$. Let $\{v_i, v_j\} \in V(F_2(G))$. Since $v_i v_j \in E(G)$, it follows that $\{v_i, v_j\}$ is adjacent to $\{v_i, v_k\}$ and $\{v_j, v_k\}$ for all $k \neq i, j$. Thus $N_{F_2(G)}(\{v_i, v_j\}) = N_{L(G)}(v_i v_j)$. Hence $F_2(G)$ is isomorphic to $L(G)$. Conversely, suppose that $F_2(G) = L(G)$. Then $|E(G)| = |V(L(G))| = |V(F_2(G))| = \binom{n}{2}$ and hence it follows that $G = K_n$.

Lemma 2.2. *If a graph G contains two vertex disjoint paths $P_2 = (v_1, v_2)$ and $P_r = (w_1, w_2, \ldots, w_r)$, then $F_2(G)$ contains a cycle of length $2r$.*

Proof. Let $X_i = \{v_1, w_i\}$ and $Y_i = \{v_2, w_i\}$ where $1 \leq i \leq r$. Then $(X_1, X_2, \ldots, X_r, Y_r, Y_{r-1}, \ldots, Y_2, Y_1, X_1\}$ is a cycle of length $2r$ in $F_2(G)$.

Observation 2.3. *If $v_1 w_1$ and $v_2 w_2$ are two nonadjacent edges in G, then the corresponding cycle C_4 given in Lemma 2.2 is an induced cycle.*

Lemma 2.4. *Let $G = (V, E)$ be a graph. Then G is triangle free if and only if $F_2(G)$ is triangle free.*

Proof. Suppose G is triangle free. If $F_2(G)$ contains a triangle say $(\{v_1, v_2\}, \{v_1, v_3\}, \{v_3, v_2\}, \{v_1, v_2\})$, then (v_1, v_2, v_3, v_1) is a triangle in G which is a contradiction. The proof of the converse is similar.

In the following theorems we obtain a characterization of graph G for which $F_2(G)$ is a tree or chordal graph.

Theorem 2.5. *Let G be a graph of order $n \geq 2$. Then $F_2(G)$ is a tree if and only if $G = P_2$ or P_3.*

Proof. If $G = P_2$, then $F_2(G) = K_1$. If $G = P_3$, then $F_2(G) = P_3$. Conversely, suppose $F_2(G)$ is a tree. If there exist two nonadjacent edges $v_1 v_2$ and $v_3 v_4$ in G, then $(\{v_3, v_1\}, \{v_3, v_2\}, \{v_2, v_4\}, \{v_1, v_4\}, \{v_3, v_1\})$ is a cycle in $F_2(G)$, which is a contradiction. Hence any two edges in G are adjacent. Thus $G = K_3$ or $K_{1,n}$. If $G = K_3$, then $F_2(G) = K_3$. Now suppose $G = K_{1,n}$ where $n \geq 3$. Let v_1 be the centre and let $v_2, v_3, \ldots, v_n$ be the pendent vertices of G. Then $(\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_1, v_2\})$ is the cycle C_6 in $F_2(G)$, which is a contradiction. Hence $G = K_{1,n}$ where $n \leq 2$. Thus $G = P_2$ or P_3.

Theorem 2.6. *Let G be a connected graph of order n and let $n \geq 2$. Then $F_2(G)$ is a chordal graph if and only if G is isomorphic to one of the graphs P_2, P_3 or K_3.*

Proof. Suppose $F_2(G)$ is a chordal graph. It follows from Observation 2.3 that any two edges of G are adjacent. Now proceeding as in the proof of Theorem 2.5 we get $G = P_2$, P_3 or K_3. The converse is obvious.

In the following theorem we obtain a lower bound for the independence number of $F_2(G)$.

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Theorem 2.7. *Let G be a connected graph of order n with $\beta_0(G) \geq 2$. Then $\beta_0(F_2(G)) \geq \binom{\beta_0(G)}{2} + \left\lfloor \frac{n-\beta_0(G)}{2} \right\rfloor$ and the bound is sharp.*

Proof. Let S be a β_0-set of G. Let $S_1 = \{\{u_i, u_j\} : u_i, u_j \in S\}$ and let S_2 be a collection of disjoint 2-element subsets of $V - S$. Clearly $|S_2| = \left\lfloor \frac{n-\beta_0(G)}{2} \right\rfloor$. Let $T = S_1 \cup S_2$. Let $\{u_i, u_j\}, \{u_i, u_k\} \in S_1$. Since $u_j u_k \notin E(G)$, it follows that $\{u_i, u_j\}$ is not adjacent to $\{u_i, u_k\}$ in $F_2(G)$. Obviously no element x of S_2 is adjacent with any element of $T - \{x\}$. Hence T is an independent set of $F_2(G)$. Thus $\beta_0(F_2(G)) \geq |T| = \binom{\beta_0(G)}{2} + \left\lfloor \frac{n-\beta_0(G)}{2} \right\rfloor$. We observe that if $G = K_4 - e$, then $\beta_0(G) = 2$ and $\beta_0(F_2(G)) = 2$, which shows that the above bound is sharp.

Theorem 2.8. *Let G be a graph of order n. If there exists a vertex $v_1 \in V(G)$ such that deg $v_1 = 2$, then G is isomorphic to a subgraph of $F_2(G)$.*

Proof. Let $N(v_1) = \{v_2, v_3\}$. Let $S = \{\{v_1, v_i\} : i \geq 2\}$. Clearly the subgraph of $F_2(G)$ induced by S is isomorphic $G - \{v_1\}$. Now $\{v_2, v_3\}$ is adjacent to $\{v_1, v_2\}$ and $\{v_1, v_3\}$ in $F_2(G)$. Hence the subgraph of $F_2(G)$ induced by the set $S \cup \{\{v_2, v_3\}\}$ is isomorphic to G.

Observation 2.9. *The graph $F_2(K_4)$ does not contain an induced subgraph isomorphic to K_4. This shows that Theorem 2.8 is not true if G has no vertex of degree 2.*

Theorem 2.10. *A connected graph H is isomorphic to $F_2(K_{1,n-1})$ if and only if the following conditions are satisfied.*

(i) H is a bipartite graph of order $\binom{n}{2}$ with bipartition V_1, V_2 where $|V_1| = n - 1$ and $|V_2| = \binom{n-1}{2}$.
(ii) Every vertex of V_1 has degree $n - 2$.
(iii) Every vertex of V_2 has degree 2.
(iv) Any two vertices of V_1 have exactly one common neighbour.

Proof. Let $H = F_2(G)$ where $G = K_{1,n-1}$. Let v_1 be the centre of the star. Let $\{v_2, v_3, \ldots, v_n\}$ be the set of pendent vertices of G.

(i) Let $V_1 = \{\{v_1, v_i\} : 2 \leq i \leq n\}$ and let $V_2 = P_2(V) - V_1$ where $P_2(V)$ is the set of all 2-element subsets of V. Clearly $|V_1| = n - 1$ and $|V_2| = \binom{n}{2} - (n - 1) = \binom{n-1}{2}$. Since $v_i, v_j \notin E(G)$ if $i, j \neq 1$, it follows that $\{v_1, v_i\}$ is not adjacent to $\{v_1, v_j\}$ in H. Hence V_1 is independent.

Now, let $\{v_i, v_j\}$ and $\{v_k, v_i\} \in V_2$ since $i, k, j \neq 1$, it follows that $v_j v_k \notin E(G)$. Hence $\{v_i, v_j\}$ is not adjacent to $\{v_k, v_i\}$ in H. Hence V_2 is independent. This proves (i).

(ii) Let $\{v_1, v_i\} \in V_1$. The vertices adjacent to $\{v_1, v_i\}$ in H are given by $\{v_k, v_i\}$ for any $k \neq 1, i$. Hence every vertex of V_1 has degree $n - 1$.

(iii) Let $\{v_r, v_s\} \in V_2$. Hence $s, r \neq 1$. The vertices adjacent to $\{v_r, v_s\}$ are $\{v_1, v_r\}$ and $\{v_1, v_s\}$. Hence any vertex of V_2 has degree 2.

(iv) Let $\{v_i, v_j\}, \{v_1, v_j\} \in V_1$. Clearly $\{v_i, v_j\}$ is the unique common neighbour.

Conversely, suppose H satisfies the conditions (i), (ii), (iii) and (iv). Suppose $H = F_2(G)$ for some G. Since $|V(H)| = n - 1 + \binom{n-1}{2} = \binom{n}{2}$, it follows that $|V(G)| = n$. Now, let $m = |E(G)|$. Hence the number of edges in H is $(n - 2)m$. But $|E(H)| = 2\binom{n-1}{2} = (n - 1)(n - 2)$. Hence it follows that $m = n - 1$, so that G is a tree.

Suppose G has 2 nonadjacent edges say $v_1 v_2$ and $v_3 v_4$. Then $(\{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_1, v_4\}, \{v_4, v_3\})$ is a cycle in H. Hence $\{v_1, v_3\}$ and $\{v_2, v_4\}$ have $\{v_2, v_3\}$ and $\{v_1, v_4\}$ as common neighbours which contradicts (iv).

Hence it follows that G is star $K_{1,n-1}$.

Theorem 2.11. *Let $G = (V, E)$ be a connected graph of order n. Then G is bipartite if and only if $F_2(G)$ is bipartite.*

Proof. Suppose G is bipartite. Let V_1, V_2 be the bipartition of $V(G)$. If $|V_1| = 1$, then $G = K_{1,n-1}$ and hence it follows from Theorem 2.10 that $F_2(G)$ is bipartite.

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Suppose $|V_1| \geq 2$ and $|V_2| \geq 2$. Let $X = P_2(V_1) \cup P_2(V_2)$ and $Y = V(F_2(G)) - X$. We claim that X, Y is a bipartition of $F_2(G)$. Since V_1 and V_2 are independent sets in G, it follows from Theorem 2.7 that $P_2(V_1)$ and $P_2(V_2)$ are independent sets in $F_2(G)$. Further any element of $P_2(V_1)$ is not adjacent to any element of $P_2(V_2)$. Hence X is independent.

Theorem 2.12. *The cycle C_r is $F_2(G)$ for some graph G if and only if $r = 3$ or 6.*

Proof. Obviously $F_2(C_3) = C_3$ and $F_2(K_{1,3}) = C_6$. Conversely, suppose $C_r = F_2(G)$ for some graph G. Let $|V(G)| = n$. Now since $F_2(G)$ is C_4 free, it follows from Lemma 2.2 that any two edges in G are adjacent. Hence $G = K_{1,n}$ or K_3. If $n \geq 4$, then $deg(v_1, v_5) \geq 3$ where v_1 is the centre of $K_{1,n}$ which is a contradiction. Hence $n = 3$. Thus $G = K_{1,3}$ or C_3. Hence $F_2(G) = C_3$ or C_6.

3. Conclusion and scope

We observe that if $H = F_2(G)$ for some graph G of order n then $|V(H)| = \binom{n}{2}$. The following fundamental problem arises naturally.

Problem 3.1. If H is a graph of order $\binom{n}{2}$, obtain a necessary and sufficient condition for the existence of a graph G of order n such that $H = F_2(G)$.

Theorem 2.7 gives a bound for $\beta_0(F_2(G))$ and leads to the following problem.

Problem 3.2. Characterize graphs G for which $\beta_0(F_2(G)) = \binom{\beta_0(G)}{2} + \left\lfloor \frac{n-\beta_0(G)}{2} \right\rfloor$.

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