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Approximations of *e* and π: an exploration

Philip R. Brown

Department of Liberal Studies, Texas A&M University at Galveston, Galveston, Texas

ABSTRACT
Fractional approximations of e and π are discovered by searching for repetitions or partial repetitions of digit strings in their expansions in different number bases. The discovery of such fractional approximations is suggested for students and teachers as an entry point into mathematics research.

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1. Introduction

The exploration of the fundamental constants e (the natural base of the logarithm) and π (the ratio of the circumference of a circle to its diameter) provides an access point for discovery, research and scholarship in mathematics for students and mathematicians at all levels, from the high school or college student up to the professional. The history of these constants also provides a gateway into the history of mathematics.

We aim to demonstrate in this article how mathematical discovery can begin with simple curiosity. Some theory and techniques, including the theory of continued fractions and the conversion of repeating decimals to fractions, that are accessible to advanced high school or beginning college students are learned along the way.

The number e has the decimal expansion $2.71828182845904\ldots$. Hand-held calculators typically provide 10 significant digits of a number. Therefore, the evaluation of e on a calculator would usually display 2.718281828. A person who is unaware of the irrational nature of e might suspect that $e = 2.7\overline{1828}$. We are thus led to curiosity about the repetition of the digits '1828'. Some mathematicians remember 1828 as the birth date of the famous Russian author Leo Tolstoy. We now consider the following questions:

- What is the number $2.7\overline{1828}$ and is it a good approximation of e?
- How do we find better approximations of e?
- What repetitions can we find for π in its decimal or other number expansions?
- What repetitions can we find in the decimal or other number expansions of $\frac{1}{e}$ or $\frac{1}{\pi}$?

We will explore some of these questions below.

CONTACT Philip R. Brown brownp@tamug.edu

It is a basic fact about real numbers that a repeating decimal corresponds to a rational number, which can always be expressed as a fraction. Therefore, we need to learn something about rational approximations of irrational numbers before we begin our discovery relating to the questions we have asked.

2. Continued fractions

In [1, p.66], the authors Pierre Eymard and Jean-Pierre Lafon complain that although continued fractions have played a major role in the history of mathematics, they have almost completely disappeared from teaching, even at universities, despite the fact that they are essential to an understanding of the approximation of real numbers by rationals. The reason for this, perhaps, is that they are particularly ill-adapted to computational uses.

In [2, p.3], it is stated that algorithms equivalent to the modern use of continued fractions were in use for many centuries before their real discovery. The best known example is Euclid's algorithm for the greatest common divisor of two integers, which leads to a terminating continued fraction. The approximate computation of square roots led to some numerical methods which can be viewed as the ancestors of continued fractions. Certain algorithms discovered by Indian mathematicians, beginning with Brahmagupta in the seventh century, for the solution of Pell's equation,[1] a type of Diophantine equation, are also analogous to continued fractions.

A first attempt for a general definition of continued fractions was made by Leonardo of Pisa (Fibonacci), who described ascending continued fractions in his book 'Liber Abaci' that he wrote in 1202 (see [2, p.51–52]).

The real discoverer of continued fractions was the Italian mathematician and astronomer Pietro Antonio Cataldi, who developed a symbolism for continued fractions and derived some of their properties in a book published in 1613. His work was based on the earlier work on the extraction of square roots by his older contemporary, Rafael Bombelli (the founder of imaginary numbers) [2, p.61–70].

We now introduce continued fractions and explain their relationship to the approximation of irrational numbers.

If x is a real number, an irreducible fraction $\frac{p}{q}$ with $q > 0$ is said to be a *best rational approximation* for x if, for any integer q' with $0 < q' \leq q$ and any irreducible fraction $\frac{p'}{q'} \neq \frac{p}{q}$, we have

$$\left| x - \frac{p}{q} \right| < \left| x - \frac{p'}{q'} \right| .$$

Furthermore, $\frac{p}{q}$ is said to be a best rational approximation in the *strong sense* if, under the same conditions,

$$\left| qx - p \right| < \left| q'x - p' \right| .$$

It follows that, if $\frac{p}{q}$ is a best rational approximation in the strong sense, then it is a best rational approximation.

It is known [1, p.73] that the best rational approximations of an irrational number x in the strong sense are exactly the *convergents* of the *regular continued fraction* for x. We use the notation

$$[b_0, b_1, \ldots, b_n, \ldots] = \lim_{n\to\infty} [b_0, b_1, \ldots, b_n],$$

where the b_n are natural numbers, possibly with the exception of $b_0 = 0$, and

$$[b_0, b_1, \ldots, b_n] = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + b_{n-1} + \cfrac{1}{b_n}}}}$$

are the convergents, for a regular continued fraction.

Any irrational number is the limit of a regular continued fraction, and of only one, which can be derived by means of the following formula (see [1, p.72]). We use the notation $\lfloor x \rfloor$ for the *floor* of x, i.e. the largest integer less than or equal to x.

Let $b_0 = \lfloor x \rfloor$, $x = b_0 + \frac{1}{x_1}$; $b_1 = \lfloor x_1 \rfloor$, $x_1 = b_1 + \frac{1}{x_2}$; ...; $b_n = \lfloor x_n \rfloor$, $x_n = b_n + \frac{1}{x_{n+1}}$; ....

The expansion of the number e as a regular continued fraction is known explicitly [1, p.78]:

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots, 1, 1, 2n, \ldots],$$

with the following sequence of convergents:

$$2, 3, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}, \frac{1264}{465}, \frac{1457}{536}, \frac{2721}{1001}, \frac{23225}{8544}, \ldots. \tag{1}$$

In the case of the number π, however, the regular continued fraction is not known explicitly. Using the formula above, with the known decimal digits for π, we obtain [1, p.78]

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, \ldots],$$

with the following sequence of convergents:

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \frac{312689}{99532}, \ldots. \tag{2}$$

3. Fractional approximations of *e*

Fractional approximations of e are rarely mentioned in the mathematics literature. Eli Maor mentions in [3, p.37] that the closest rational approximation of e using integers below 1000 is $\frac{878}{323}$. The French mathematician, Charles Hermite, who published the first proof of the transcendence of e in 1873 gave as a sequel to his proof the approximations $\frac{58\,291}{21\,444}$ and $\frac{158\,452}{21\,444}$ of e and e^2, respectively (see [3, p.189]).

An approximate value of e can easily be obtained by calculating a partial sum of the series

$$e = \sum_{n=1}^{\infty} \frac{1}{n!} .$$

For example, the first 22 terms of the series add up to $\frac{61107015069852259209 7}{22480014555552153600 0}$, which evaluates to a decimal giving correctly the first 20 digits of e:

$$e \approx 2.71828182845904523536 . \tag{3}$$

As mentioned earlier, we can approximate e by the repeating decimal fraction $2.7\overline{1828}$. The method that is usually taught to students to convert a recurring decimal to a fraction is the following:

$$\begin{aligned} x &= 2.7\overline{1828} \\ y &= 10x = 27.\overline{1828} \\ 10000y &= 271828.\overline{1828} \\ 99990x &= 9999y = 271801 \\ x &= \frac{271801}{99990} . \end{aligned} \tag{4}$$

An alternative method of converting the repeating decimal to a fraction is to express it as the geometric series

$$\begin{aligned} 10e &= 27 + 1828 \sum_{k=1}^{\infty} \left(\frac{1}{10^4} \right)^k \\ &= 27 + \frac{1828}{10^4 - 1} \\ &= 27 + \frac{1828}{9999} \\ e &= \frac{271801}{99990} . \end{aligned} \tag{5}$$

A mentioned in [4, p.15], the fraction $\frac{271801}{99990}$ cannot be reduced to a simpler fraction, however, the slightly smaller fraction $\frac{271800}{99990}$ reduces to $\frac{3020}{1111} = 2.7\overline{1827}$, which differs from the decimal expression for e by one unit in the fifth decimal position.

We can improve the estimate by approximating e by $2.7\overline{182818}$. By means of either of the methods above the corresponding fraction is found to be $\frac{27182791}{9999990}$, which cannot be reduced; however, the smaller fraction $\frac{27182790}{9999990}$ reduces to the eleventh convergent $\frac{2721}{1001} = 2.\overline{718281}$ of the regular continued fraction expansion for e in Equation (1). It is also the closest rational approximation of e with integers less than 10,000, as can be verified by means of a computer search of all fractions that have no more than four digits in the numerator and the denominator.

4. Fractional approximations of π

The value of π with 20 correct decimal digits is

$$\pi \approx 3.14159265358979323846. \tag{6}$$

We will begin a short historical overview of the calculation of π.

4.1. Historical overview

The letter π was introduced as the notation for the ratio of the circumference to the diameter of a circle by the English mathematician William Jones in 1706 in his publication *A New Introduction to Mathematics*. He used the letter π as an abbreviation for 'periphery' (of a circle with unit diameter); see [5, p.145].

The approximation $\pi > 3\frac{1}{8} = 3.125$ was known to the Babylonians in 2000 B.C. (see [5, p.21–22]) and the Egyptians found the approximate value $4\times(\frac{8}{9})^2=\frac{258}{81}=3.\overline{185}$ by approximating the area of a disk by the area of an octagon [5, p.24–25].

The famous Greek mathematician and engineer Archimedes determined $3.14085 \approx \frac{223}{71} < \pi < \frac{22}{7} \approx 3.14286$ by approximating the area of a disk by the areas of inscribed and circumscribed polygons [5, p.67]. The Alexandrian astronomer Ptolemy who lived in the second century A.D. used the approximation $\pi < 3\frac{17}{120} = \frac{377}{120} \approx 3.14167$, which he had probably inherited from an earlier Greek mathematician Appollonius of Perga who was about 30 years younger than Archimedes [5, p.74].

Early Hindu knowledge of mathematics was summarized by Aryabhatiya in 499 A.D. He provided the the following recipe for approximating the value of π: *Add 4 to 100, multiply by 8 and add 62000. The result is approximately the circumference of a circle of which the diameter is 20, 000* (see [5, p.26]). This translates as $\pi \approx \frac{3927}{1250} = 3.1416$. Most likely this fraction was also obtained by means of polygonal approximation. The Hindus at this time had at their disposal the superior decimal number system which facilitated more complicated calculations than the Greeks could handle.

Fibonacci (the author of Liber Abaci) calculated the approximation $\frac{864}{275} = 3.14\overline{18}$ of π using a 96 sided polygon, as Archimedes had done. However, he had the advantage of being able to compute approximations of square roots using the new decimal arithmetic [5, p.84].

The invention of decimal fractions and logarithms facilitated numerical calculations in Europe from the 1500s onward. The approximation $\frac{355}{311} \approx 3.141592920$ (correct to six decimal digits) known as the Metius approximation (after the Dutch Mathematician Adriaen Metius) was discovered by several mathematicians during the sixteenth century using the Euclidean algorithm and continued fractions [2, p.71]. It is the fourth convergent in the sequence of convergents of the regular continued fraction for π given in Section 2. This fraction was also reported in 480 A.D. by the Chinese Tsu Ch'ung [1, p.78].

After Isaac Newton discovered the binomial formula for fractional exponents, he was able to compute π accurately to 16 decimal places by using an integration formula, introduced by Pierre de Fermat and Blaise Pascal, to calculate the area of a sector of a circle [5, p.142].

In 1706, the astronomer Johan Machin derived the formula

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

[5, p.144–145] and then calculated π accurately to 100 decimal places by substituting the fractions on the right hand side into the Gregor series $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$. The same formula (Machin's formula) was used in 1949 by programmers of one of the earliest computers, the ENIAC (Electronic Numerical Integrator and Computer) at Ballistic Research Labs to calculate 2037 decimal places of π in 70 hours [5, p.184]. With the advance of computational power and sophisticated programming, the number of digits of π that can currently be calculated is over 22 trillion! (see [6]).

In the 1990s the mathematicians David Bailey, Peter Borwein and Simon Plouffe discovered the formula

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{4}{8k+6}\right)$$

(now known as the BBP formula) which can be used to calculate any specified *hexadecimal* digit of π [1, p.79]. For example [7], an international network of computers was organized by Colin Percival to calculate the quadrillionth binary digit of π.

4.2. A few approximations of pi

An interesting fact about the decimal expansion of π is that the first zero digit does not occur until the 32nd position after the decimal point. Many poets have written poems using the digits of π to determine the number of letters in each word. In the following poem, which appears in the children's book *Somewhen* by David Saul and Danielle Mathieson, the word 'nothing' is used in the 32nd position.

It's a fact
A ratio immutable
Of circle round and width
Produces geometry's deepest conundrum
For as the numerals stay random
No repeat lets out its presence
Yet it forever stretches forth
Nothing to eternity

Other poets have used words containing 10 letters to represent the zero digit.

However, in some expansions of π in different bases, zero digits do appear near the beginning of the expansion. Here are the expansions of π in some other bases:

(binary)	11.00100 10000 11111 10110 10101 00010 00100 00101
(ternary	10.01021 10122 22010 21100 21111 10221 22222 01112
(quarternary)	3.02100 33312 22202 02011 22030 02031 03010 30121
(quintary)	3.03232 21430 33432 41124 12240 41402 31421 11430
(sextary)	3.05033 00514 15124 10523 44140 53125 32110 23012

(heptary)	3.06636 51432 03613 41102 63402 24465 22266 43520
(octal)	3.11037 55242 10264 30215 14230 63050 56006 70163
(nonal)	3.12418 81240 74427 88645 17776 17310 35828 51654
(decimal)	3.14159 26535 89793 23846 26433 83279 50288 41971
(duodecimal)	3.18480 9493*B* 91866 4573*A* 6211*B* *B*1515 51*A*05 72929
(hexadecimal)	3.243*F*6 *A*8885 *A*308*D* 31319 8*A*2*E*0 37073 44*A*40 93822

which can be verified at the website [8]. The two zero digits appearing in the fourth and fifth positions after the point in the quarternary expansion suggest the approximation π by 3.021 (in base 4). This corresponds to the decimal fraction $3+\frac{2}{16}+\frac{1}{64}=\frac{201}{64}=3.140625$.

The heptary (base 7) approximation $3.0\dot{6} = 3.1$ equals the famous approximation $\frac{22}{7}$ in decimal arithmetic and the quintary approximation of π $3.0\overline{32}$ corresponds to the decimal fraction

$$3+\frac{1}{5}(3\times 5+2)\sum_{k=1}^{\infty}\left(\frac{1}{5^2}\right)^k = 3+\frac{17}{5}\times\frac{1}{24}=\frac{377}{120}$$

that was given by Ptolemy.

Approximating π in base 9 by $3.\overline{124188}$ produces the decimal fraction

$$3+(9^5+2\times 9^4+4\times 9^3+1\times 9^2+8\times 9+8)\sum_{k=1}^{\infty}\left(\frac{1}{9^6}\right)^k = 3+\frac{75248}{531440}=\frac{104348}{33215}$$

which is the sixth convergent in the continued fraction of π mentioned earlier.

We can also examine expansions of $\frac{1}{\pi}$ in the bases above.

(binary)	0.01010 00101 11110 01100 00011 01101
(ternary	0.02212 10010 21220 22120 21110 12121
(quarternary)	0.11011 33030 01231 30213 02020 02221
(quintary)	0.12434 32434 44234 24132 34230
(sextary)	0.15243 10221 33341 41411 31214
(heptary)	0.21411 55610 23405 33404 21346
(octal)	0.24276 301556 23442 02512 37604
(nonal)	0.27703 78276 74177 47214 17757
(decimal)	0.318309886183790671537767526745
(duodecimal)	0.39*A*05 82886 *B*3742 17852 28778
(hexadecimal)	0.517*CC* 1*B*727 220*A*9 4*FE*13 *ABE*8*F*

The quintary expansion is interesting because the approximation $0.1\overline{24343}$ corresponds to the decimal fraction

$$\frac{1}{5}+\frac{1}{5}(2\times 5^4+4\times 5^3+3\times 5^2+4\times 5+3)\sum_{k=1}^{\infty}\left(\frac{1}{5^5}\right)^k=\frac{1}{5}\left(1+\frac{1848}{3124}\right)=\frac{113}{355}.$$

The reciprocal $\frac{355}{113}$ is the Metius approximation.

4.3. Approximating π by a sum of fractions

Instead of approximating π by one fraction, we can look for approximations of π by a sum of fractions. For example, the repeating decimal fraction $3.14\overline{159}$ is $\frac{313835}{99900}$. The slightly smaller fraction $\frac{313834}{99900}$ simplifies to $\frac{4241}{1350}$ and we find that $\pi-\frac{4241}{1350}\approx 0.0001111721$. Therefore, π is approximated by $\frac{4241}{1350}+\frac{1}{9000}=\frac{84823}{27000}=3.141\overline{592}$. Furthermore, we have

$$84823=271\times 313=270\times 313+313=270\times 313+27\times 11+16$$

so π is approximated by $\frac{313}{100}+\frac{11}{1000}+\frac{16}{27000}$. The last fraction simplifies to $\frac{2}{3375}$. If we replace it by $\frac{2}{3374}=\frac{1}{1687}$ we find that

$$\pi\approx\frac{3141}{1000}+\frac{1}{1687}\approx 3.141592768 \tag{7}$$

which differs from π by one unit in the seventh digit after the decimal point. This is an approximation for π that can be easily remembered because 1687 is the year in which Isaac Newton published his *Principia*!

Using the quarternary expansion for π, we have the following calculation in base 4 arithmetic:

$$\begin{aligned}3.\dot{3}-\pi &= 0.31233000211113131322113033\ldots\\ &= 0.31233+\frac{1}{4^9}\left(2.\dot{1}+\frac{1}{4^4}0.2020211001322\ldots\right)\\ &\approx 0.31233+\frac{1}{4^9}\left(2.\dot{1}+\frac{1}{4^4}0.\overline{20}\right).\end{aligned}$$

In decimal arithmetic, we, therefore, find that π can be approximated by

$$4-\left(\frac{879}{4^5}+\frac{7}{3\times 4^9}+\frac{8}{15\times 4^{13}}\right).$$

That is

$$\pi\approx 3+\frac{145}{4^5}-\frac{7}{3\times 4^9}-\frac{8}{15\times 4^{13}}=\frac{395303839}{125829120}\approx 3.141592653\mathbf{5}924$$

which differs from the decimal expansion of π by one unit in the 11th decimal position after the decimal point.

Another way to approximate π by a sum of fractions is to subtract a fraction from π and then to find a fraction that approximates the result. For example,

$$\pi - \frac{355}{113} \approx 0.266764189 \times 10^{-7}.$$

The corresponding value of the decimal string in base 14 is $0.3A40027C$ which we can approximate by $0.3A4$. This is the decimal fraction $\frac{183}{686}$. Therefore,

$$\pi - \frac{355}{113} - \frac{1}{10^6}\frac{183}{686} = 0.34066592 \times 10^{-12}.$$

The decimal string is equivalent to the base 14 string $0.4AAB044$, which can be approximated by $0.4\dot{A}$. This is the decimal fraction $\frac{31}{91}$. Consequently, we have

$$\pi \approx \frac{355}{113} - \frac{1}{10^6}\left(\frac{183}{686}\right) - \frac{1}{10^{12}}\left(\frac{31}{91}\right) \tag{8}$$

which differs from π by one unit in the 17th position after the decimal point.

Double precision arithmetic on a computer can represent a number with at most 17 significant digits [9]. Therefore, the expression in Equation (8) can be used as the value for π for computer calculations.

5. A suggestion for teachers

We suggest that teachers challenge their students with the following sequence of questions:

- find the best approximation for e or π among fractions having at most one digit in the numerator and the denominator
- find the best approximation for e or π among fractions having at most two digits in the numerator and the denominator
- find the best approximation for e or π among fractions having at most three digits in the numerator and the denominator
- find the best approximation for e or π among fractions having at most four digits in the numerator and the denominator

Depending on the teacher's goals and the level of the students, the best approximating fractions above may be discovered by means of a computer search, as a programming exercise, by the calculation of continued fraction convergents (which would require the teacher to introduce continued fractions), or by the discovery of partial digit repetitions in the number expansions of e and π (or their reciprocals) in different bases, as we have done in this article.

Note

1. As mentioned in [10, p.43], the Diophantine equation $x^2 - Dy^2 = 1$, where D is a non-square integer, is known as Pell's equation because Euler mistakenly attributed a solution of it to the

seventeenth century English mathematician Pell. This is an example of European mathematicians mis-naming parts of mathematics. Brahmagupta's method of finding solutions of this equation was the first major advance in number theory since Diophantus (see [10, p.72]), who lived in Alexandria in the third century AD.

Disclosure statement

No potential conflict of interest was reported by the author.

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