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$\mathscr{H}$-panchromatic digraphs

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Abstract

Let H and D be two digraphs; D without loops or multiple arcs. An $H-$coloring of D is a function $\rho : A(D) \to V(H)$. We say that D is an $(H, \rho)-$colored digraph. For an arc (x, y) of D, we say that $\rho(x, y)$ is the color of (x, y) over the $H-$coloring ρ.

A directed path $(x_1, \ldots, x_n)$ in D is an $(H, \rho)-$path if $(\rho(x_1, x_2), \ldots, \rho(x_{n-1}, x_n))$ is a directed walk in H. An $(H, \rho)-$kernel in an $(H, \rho)-$colored digraph is a subset of vertices of D, say S, such that for every pair of different vertices in S there is no $(H, \rho)-$path between them and every vertex outside S can reach S by an $(H, \rho)-$path. A digraph D is an $\mathscr{H}$-panchromatic digraph if D has an $(H, \rho)-$kernel for every digraph H and every $H-$coloring ρ of D. In this paper we show that $\mathscr{H}$-panchromatic digraphs cannot be characterized by means of certain forbidden subdigraphs. Also we will show $\mathscr{H}$-panchromaticity of some classes of digraphs and we show that $\mathscr{H}$-panchromaticity can be hereditary in some operations of digraphs.

Keywords: Kernel; Coloring; Panchromatic; H-kernel

1. Introduction

For general concepts and notation we refer the reader to [1]. A walk (path) considered in this paper always means a *directed walk* (*directed path*). If D is a digraph, a *block* of D is a maximal induced subdigraph of D whose underlying graph is $2-$connected. A vertex set $S \subseteq V(D)$ is *path-independent* if there is no two different vertices in S joined by a path. If U and V are disjoint subsets of vertices of D, denote by $A[U; V]$ the set $\{(x, z) \in A(D) : x \in U, z \in V\}$. The complement of a digraph H (not necessarily without loops), denoted by $\overline{H}$, is the digraph whose vertex set is $V(H)$ and arc set is $(V(H) \times V(H)) \setminus A(H)$. We observe that either H or $\overline{H}$ may contain loops. If D_1 and D_2 are (not necessarily vertex disjoint) digraphs, the *union of* D_1 *and* D_2, denoted by $D_1 \cup D_2$, is the digraph with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$. If $\{D_i : i \in \{1, \ldots, n\}\}$ is a family of (not necessarily pairwise vertex disjoint) digraphs, the *union* of such digraphs, denoted by $\cup_{i=1}^{n} D_i$, is the digraph whose set of vertices is $\cup_{i=1}^{n} V(D_i)$ and arc set $\cup_{i=1}^{n} A(D_i)$. If G is a digraph with vertex set $\{v_1, \ldots, v_n\}$ and $\mathscr{D} = \{D_1, \ldots, D_n\}$ is a family of pairwise vertex disjoint digraphs, the *composition of*

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$\mathscr{D}$ *over* G, denoted by $G[D_1, \ldots, D_n]$ or simply $G[\mathscr{D}]$, is the digraph whose vertex set is $\cup_{i=1}^{n} V(D_i)$ and arc set $\{(x, z) : x \in V(D_i),\ z \in V(D_j)$ and $(v_i, v_j) \in A(G)\} \cup (\cup_{i=1}^{n} A(D_i))$. If G is a path of order 2, say $G = (v_1, v_2)$, and $\mathscr{D} = \{D_1, D_2\}$, we denote by $D_1 + D_2$ the composition $G[\mathscr{D}]$. If D and G are digraphs, we say that D *is an extension of* G if $D = G[\mathscr{D}]$ for some family $\mathscr{D}$ of digraphs such that every digraph in $\mathscr{D}$ has no arcs.

A subset S of vertices of D is a *kernel of* D if S is both independent, that is, there is no arc between two distinct vertices in S, and absorbent, that is, every vertex not in S has at least one out-neighbor in S. The concept of kernel was introduced by von Neumann and Morgenstern in [2]. It is not difficult to see that some digraphs such as odd cycles have no kernels. Several sufficient conditions for the existence of a kernel have been proved. The following lemma is a well known result of existence of kernels.

Lemma 1.1 (*[2–4]*)**.** *Let D be a digraph. If D satisfies one of the following conditions, then it has a kernel.*
(i) D is a transitive digraph.
(ii) D is an acyclic digraph.
(iii) D is an even cycle.

An m-*coloring* of a digraph D is a function $\rho : A(D) \to \{1, \ldots, m\}$. We say that D is m-*colored* if the arcs of D are colored with the set $\{1, \ldots, m\}$. Let D be a $\rho-$colored digraph. A path $(x_1, \ldots, x_n)$ in D is a $\rho-$*monochromatic path* if $\rho(x_i, x_{i+1}) = \rho(x_{i+1}, x_{i+2})$ for every $i \in \{1, \ldots, n-2\}$. A set $S \subseteq V(D)$ is said to be a $\rho-$*monochromatic kernel* if for every two different vertices S there is no $\rho-$monochromatic path between them (i.e. S is $\rho-$*independent*) and every vertex not in S reaches a vertex in S by a $\rho-$monochromatic path (i.e. S is $\rho-$*absorbent*).

A digraph D is *panchromatic* if D contains a $\rho-$kernel for every m-coloring ρ of D. The concept of panchromatic digraphs was introduced by Galeana-Sánchez and Strausz in [5] and some kinds of operations that preserve panchromaticity were studied by Galeana-Sánchez and Toledo in [6]. An easy proof shows that:

Lemma 1.2. *Every panchromatic digraph has a kernel.*

Proof. Let us suppose that D has size q. Let $\rho : A(D) \to \{1, \ldots, q\}$ be a $q-$coloration of D such that $\rho(e) \neq \rho(a)$ whenever $e \neq a$. Since D is panchromatic, then D has a $\rho-$kernel, say S. It is straightforward to see that S is a kernel of D. □

Let H and D be two digraphs; D without loops or multiple arcs. An $H-$coloring of D is a function $\rho : A(D) \to V(H)$. We say that D is an $(H, \rho)-$colored digraph, or simply $H-$*colored* if ρ is understood. For an arc (x, y) of D, we say that $\rho(x, y)$ is the color of (x, y) over the $H-$coloring ρ. A directed walk (respectively directed path) $(x_1, \ldots, x_n)$ in D is an $(H, \rho)-$*walk* (respectively $(H, \rho)-$*path*), or simply $H-$*walk* ($H-$*path*) if ρ is understood, if $(\rho(x_1, x_2), \ldots, \rho(x_{n-1}, x_n))$ is a walk in H. If $\{u, v\} \subseteq V(D)$, an $uv-(H, \rho)-$*path* ($uv-(H, \rho)-$*walk*) or simply $uv-H-$*path* ($uv-H-$*walk*) if ρ is understood, is an $H-$path ($H-$walk) whose initial vertex is u and final vertex is v. An $(H, \rho)-$colored digraph D is an $(H, \rho)-$*digraph*, or similarly an $H-$*digraph*, if every path in D is an $(H, \rho)-$path.

A subset S of vertices of D is an $(H, \rho)-$*independent set*, or simply $H-$*independent* if ρ is understood, if no two vertices of S are connected by an $(H, \rho)-$path. We say that S is an $(H, \rho)-$*absorbent set*, or simply $H-$*absorbent* if ρ is understood, if every vertex outside S can reach some vertex in S by an $(H, \rho)-$path. We say that S is an $(H, \rho)-$*kernel*, or similarly an $H-$*kernel*, if S is both $(H, \rho)-$independent and $(H, \rho)-$absorbent.

Lemma 1.3. *Let D be an $H-$colored digraph. If D is an $\overline{H}-$digraph, then D contains an $H-$kernel if and only if D contains a kernel.*

Proof. Notice that if D is an $\overline{H}-$digraph then a path in D is an $H-$path if and only if it has length 1.

For the sufficiency of Lemma 1.3, if S is an $H-$kernel of D, then S is both $H-$absorbent and $H-$independent, and we conclude that S is a kernel.

For the necessity, let S be a kernel of D. Clearly S is $H-$absorbent. On the other hand, if there is an $uv-H-$path for some $\{u, v\} \subseteq S$, we conclude that $(u, v) \in A(D)$ which is no possible since S is an independent set, so S is $H-$ independent. Therefore S is an $H-$kernel. □

If ρ is an $H-$coloring of D and $T = (x_1, x_2, \ldots, x_n)$ is a path in D, we say that T is a *maximal* $\overline{H}$*−path* if T is an $\overline{H}-$path not contained properly in another $\overline{H}-$path. It is straightforward to see the following observation.

Observation 1.4. *Every* $\overline{H}-$*path is contained in some maximal* $\overline{H}-$*path and subpaths of* $\overline{H}-$*paths are* $\overline{H}-$*paths.*

Let D be an $(H, \rho)-$colored digraph. The $(H, \rho)-$*closure of* D, denoted by $\mathscr{C}_H^\rho(D)$, is the digraph whose vertex set is $V(D)$ and whose arc set is $\{(u, v) :$ there exists an $uv-(H, \rho)-$ path in $D\}$. It is not difficult to see that the following simple result holds.

Observation 1.5. *An* $(H, \rho)-$*colored digraph* D *has an* $(H, \rho)-$*kernel if and only if* $\mathscr{C}_H^\rho(D)$ *has a kernel.*

The concept of $H-$path was introduced by Linked and Sands in [7] and was later used by Arpin and Linek [8] to work on three classes of digraphs: $\mathscr{B}_3$, the class of all digraphs H such that any $H-$colored multidigraph D has an $H-$kernel by walks, that is, a set S of vertices that is both $H-$independent by walks (i.e., no two vertices in S are joined by an $H-$walk) and $H-$absorbent by walks (i.e., every vertex not in S can reach some vertex in S by an $H-$walk). The class $\mathscr{B}_2$ which contains all digraphs H such that every $H-$colored multidigraph has an independent set of vertices that is $H-$absorbent by walks, and the class $\mathscr{B}_1$ of all digraphs H such that any $H-$colored tournament has a single vertex that is $H-$absorbent by walks.

In [7] Arpin and Linek proved that $\mathscr{B}_3 \subsetneq \mathscr{B}_2 \subsetneq \mathscr{B}_1$. Also they gave a characterization of $\mathscr{B}_2$ and made inroads in the classification of $\mathscr{B}_3$ and $\mathscr{B}_1$. Galeana-Sanchez and Strausz [9] continue the research of Arpin and Linek in order to characterize those digraphs in $\mathscr{B}_3$. The class $\mathscr{B}_1$ remains unclassified.

In the same spirit of the definitions given by Arpin and Linek [8], Delgado-Escalante and Galeana-Sánchez [10] defined a new class of digraphs $\widehat{\mathscr{B}_3}$ of all digraphs H such that any $H-$colored digraph has an $H-$kernel. Such digraphs remain unclassified.

In a similar way we will define a new class of digraphs. For a digraph D, we say that D is $\mathscr{H}$*−pan-chromatic* ($\mathscr{H}$*-panchromatic by walks*) if D has an $(H, \rho)-$kernel (respectively an $(H, \rho)-$kernel by walks) for every digraph H and every $H-$coloring ρ of D. In this paper we show that such digraphs cannot be characterized by means of forbidden subdigraphs. So, it is of interest to find some large classes of digraphs that are $\mathscr{H}$-panchromatic and operations that preserve $\mathscr{H}$-panchromaticity. In particular we show that transitive digraphs and acyclic digraphs are $\mathscr{H}$-panchromatic. In the same way, we will characterize quasi-transitive digraphs and tournaments which are $\mathscr{H}$-panchromatic.

In the last section we will show that some operations on digraphs, such as the union of (non necessarily vertex disjoint) digraphs and the composition of digraphs, preserve $\mathscr{H}$-panchromaticity.

2. $\mathscr{H}$-panchromatic digraphs

Lemma 2.1. *Every* $\mathscr{H}$*-panchromatic digraph is panchromatic. In particular, every* $\mathscr{H}$*-panchromatic digraph has a kernel.*

Proof. Consider $\rho : V(D) \to \{1, \ldots, m\}$ an arbitrary m-coloring of D and let H_1 be the digraph with vertex set $\{1, \ldots, m\}$ and arc set $\{(n, n) : n \in \{1, \ldots, m\}\}$. Clearly ρ is an H_1-coloring of D and since D is $\mathscr{H}$-panchromatic then D has an $(H_1, \rho)-$kernel, say S. It is not difficult to see that S is a $\rho-$monochromatic kernel, concluding that D is panchromatic. On the other hand, it follows from Lemma 1.2 that every $\mathscr{H}$-panchromatic digraph has a kernel. □

Theorem 2.2. *An* $H-$*colored cycle* C *has an* $H-$*kernel if and only if* C *is not an* $\overline{H}-$*cycle of odd length.*

Proof. The sufficiency of Theorem 2.2 follows from Lemma 1.3 and the fact that each odd cycle has no kernel.

For the converse, let $C = (x_1, x_2, \ldots, x_n, x_1)$ be an $(H, \rho)-$colored cycle that is not an $\overline{H}-$cycle of odd length (subscripts are taken modulo n). Notice that if C is an $H-$cycle, then each vertex of C is an $H-$kernel. So we will assume that C is not an $H-$cycle.

If C is an $\overline{H}-$cycle then by assumption C is not an odd cycle. In that case C has a kernel and by Lemma 1.3 we conclude that C has an $H-$kernel.

Now we can assume that C is not an $\overline{H}-$cycle. Consider the following cases.

Case 1. C contains a spanning $\overline{H}-$path.

Suppose (w.l.o.g.) that $T = (x_1, \dots, x_n)$ is a spanning $\overline{H}-$path.

Since C is not a $\overline{H}-$cycle, then $(\rho(x_{n-1}, x_n), \rho(x_n, x_1)) \in A(H)$ or $(\rho(x_n, x_1), \rho(x_1, x_2)) \in A(H)$. We consider the following subcases.

Subcase 1.1 n is even.

If both arcs $(\rho(x_{n-1}, x_n), \rho(x_n, x_1))$ and $(\rho(x_n, x_1), \rho(x_1, x_2))$ belong to H, then $S = \{x_k : k \in \{1, \dots, n-1\}$ with k even$\}$ is an $H-$kernel.

If $(\rho(x_{n-1}, x_n), \rho(x_n, x_1)) \notin A(H)$, then $S = \{x_k : k \in \{1, \dots, n\}$ with k odd$\}$ is an $H-$kernel.

If $(\rho(x_n, x_1), \rho(x_1, x_2)) \notin A(H)$ then $S = \{x_k : k \in \{1, \dots, n\}$ with k even$\}$ is an $H-$kernel.

Subcase 1.2 n is odd.

If both arcs $(\rho(x_{n-1}, x_n), \rho(x_n, x_1))$ and $(\rho(x_n, x_1), \rho(x_1, x_2))$ belong to H, then $S = \{x_k : k \in \{1, \dots, n-1\}$ with k odd$\}$ is an $H-$kernel.

If $(\rho(x_n, x_1), \rho(x_1, x_2)) \notin A(H)$, then $S = \{x_k : k \in \{1, \dots, n-1\}$ with k odd$\}$ is an $H-$kernel.

If $(\rho(x_{n-1}, x_n), \rho(x_n, x_1)) \notin A(H)$, then $S = \{x_k : k \in \{1, \dots, n\}$ with k even$\}$ is an $H-$kernel.

Case 2. C contains no spanning $\overline{H}-$path.

Consider the family $\mathscr{T}$ of maximal $\overline{H}-$paths of C. Since C is not an $H-$cycle, $\mathscr{T}$ is a nonempty family and by Observation 1.4 every $\overline{H}-$path is contained in some member of the family $\mathscr{T}$. Suppose (w.l.o.g reordering the vertices of C) that $\mathscr{T} = \{T_1, \dots, T_k\}$, $T_i = (x_{\alpha_i}, x_{\alpha_i+1}, \dots, x_{\beta_i})$ for every $i \in \{1, \dots, k\}$, $\beta_i \leq \alpha_{i+1}$ for every $i \in \{1, \dots, k-1\}$ and $\beta_k = n$.

- **Claim 1.** For every $i \in \{1, \dots, k\}$, if $T_i' = x_{\beta_i} C x_{\alpha_{i+1}}$ is nontrivial, then T_i' is an $H-$path of C. (subscripts are taken modulo k).
  If there exists $i \in \{1, \dots, k\}$ such that $T_i' = x_{\beta_i} C x_{\alpha_{i+1}}$ is not an $H-$path of C, then there exists $l \in \{\beta_i, \dots, \alpha_{i+1} - 2\}$ such that $(\rho(x_l, x_{l+1}), \rho(x_{l+1}, x_{l+2})) \notin A(H)$. In that case, by Observation 1.4 we conclude that there exists $j \in \{1, \dots, k\} \setminus \{i, i+1\}$ such that T_j contains (x_l, x_{l+1}) and (x_{l+1}, x_{l+2}) which is no possible by the ordering of $\mathscr{T}$.
- **Claim 2.** For every $i \in \{1, \dots, k\}$, $T_i'' = x_{\beta_i - 1} C x_{\alpha_{i+1}+1}$ is an $H-$path of C.
  Since T_i is a maximal $\overline{H}-$path, then $x_{\alpha_i} C x_{\beta_i+1}$ is not in $\mathscr{T}$. In that case, since C has no spanning $\overline{H}-$paths, then $x_{\alpha_i} C x_{\beta_i+1}$ is a path that is not an $\overline{H}-$path, concluding that

  $$(\rho(x_{\beta_i - 1}, x_{\beta_i}), \rho(x_{\beta_i}, x_{\beta_i+1})) \in A(H).$$

  A similar proof shows that $(\rho(x_{\alpha_{i+1}-1}, x_{\alpha_{i+1}}), \rho(x_{\alpha_{i+1}}, x_{\alpha_{i+1}+1})) \in A(H)$.
  If $x_{\beta_i} = x_{\alpha_{i+1}}$ then the claim holds. On the other hand, if $x_{\beta_i} \neq x_{\alpha_{i+1}}$ then by claim 1 we have that $x_{\beta_i} C x_{\alpha_{i+1}}$ is an $H-$path. Therefore, $T_i'' = x_{\beta_i - 1} C x_{\alpha_{i+1}+1}$ is an $H-$path.

For every $i \in \{1, \dots, k\}$, we define t_i as the maximum natural number such that $\beta_i - 2t_i \geq \alpha_i$. Consider $S_i = \{x_{\beta_i - 2}, x_{\beta_i - 4}, \dots, x_{\beta_i - 2t_i}\}$ and let

$$S = \bigcup_{i=1}^{k} S_i.$$

We claim that S is an $H-$kernel of C.

We will show that S is an $H-$absorbent set in C. Consider any vertex x_m in $V(C) \setminus S$. Suppose that x_m does not belong to a member of $\mathscr{T}$, in that case, let $j = \min\{i \in \{1, \dots, k\} : m \leq \alpha_i\}$. According to Claim 1, there exists an $x_m x_{\alpha_j+1} - H-$path. So, x_m is $H-$absorbed by both x_{α_j} and x_{α_j+1}. Notice that either $x_{\alpha_j} \in S$ or $x_{\alpha_j+1} \in S$. Hence x_m is $H-$absorbed by S.

Now we suppose that $x \in V(T_i)$ for some $i \in \{1, \dots, k\}$ and consider the following cases. If $m \leq \beta_i - 2$, then x_m is $H-$absorbed by x_{m+1}. Now we suppose that $m \in \{\beta_i, \beta_i - 1\}$. According to Claim 2, there exists an $x_{\beta_i - 1} x_{\alpha_{i+1}+1} - H-$path. It is straightforward to see that $\{x_{\beta_i - 1}, x_{\beta_i}\}$ is $H-$absorbed by both $x_{\alpha_{i+1}}$ and $x_{\alpha_{i+1}+1}$. Notice that either $x_{\alpha_{i+1}} \in S$ or $x_{\alpha_{i+1}+1} \in S$. Hence, x_m is $H-$absorbed by S. Therefore, S is an $H-$absorbent set in C.

In order to prove that S is $H-$independent notice that by definition of S_i, if $x_r \in S$, then (x_r, x_{r+1}, x_{r+2}) is a subpath of some member in $\mathscr{T}$, so (x_r, x_{r+1}, x_{r+2}) is not an $H-$path. In that case, for every pair of vertices x_r and x_t in S, there is no $x_r x_s - H-$path in C. Therefore, S is an $H-$ independent set in C.

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By the above, we conclude that S is an $H-$kernel of C. □

Corollary 2.3. *Every even cycle is $\mathscr{H}$-panchromatic.*

Proposition 2.4. *Every acyclic digraph is $\mathscr{H}$-panchromatic.*

Proof. Let D be an acyclic $(H, \rho)-$colored digraph. We assert that $\mathscr{C}_H^\rho(D)$ is an acyclic digraph. Otherwise, suppose that $\mathscr{C}_H^\rho(D)$ has a cycle $(x_1, \ldots, x_n, x_1)$. By definition, for every $i \in \{1, \ldots, n\}$ there exists an $x_i x_{i+1} - (H, \rho)-$path in D, say T_i. Clearly, the concatenation of such paths is a closed walk in D, which contains a cycle, a contradiction. Hence, $\mathscr{C}_H^\rho(D)$ is an acyclic digraph. By Lemma 1.1 it follows that $\mathscr{C}_H^\rho(D)$ has a kernel and by Observation 1.5 D has an $(H, \rho)-$kernel. Therefore D is $\mathscr{H}$-panchromatic. □

Corollary 2.5. *If D is an asymmetric digraph such that every block of D is a transitive digraph, then D is $\mathscr{H}$-panchromatic.*

Proof. First, proceeding by contradiction, we will prove that D is an acyclic digraph. Suppose that C is a cycle in D, then C is contained in some block of D, say B. By assumption, B is a transitive digraph, concluding that C contains a symmetric arc, which is no possible. Therefore, D is an acyclic digraph and by Proposition 2.4 it follows that D is $\mathscr{H}$-panchromatic. □

Let D be a digraph and S be a subset of vertices of D. We say that S is an *$\mathscr{H}$-panchromatic set* if S is an $(H, \rho)-$kernel of D for every digraph H and every $H-$coloring ρ of D. A digraph D is *weakly $\mathscr{H}$-panchromatic* if D contains an $\mathscr{H}$-panchromatic set.

Lemma 2.6. *A subset S of vertices of a digraph D is an $\mathscr{H}$-panchromatic set if and only if S is both absorbent and path-independent.*

Proof. Suppose that S is both absorbent and path-independent. Let H be an arbitrary digraph and ρ an $H-$coloring of D. Clearly S is $(H, \rho)-$absorbent and since S is path-independent then S is $(H, \rho)-$independent, concluding that S is an $(H, \rho)-$kernel of D. Therefore, S is an $\mathscr{H}$-panchromatic set.

Now suppose that S is an $\mathscr{H}$-panchromatic set. Consider H_1 a digraph with vertex set $\{c_1\}$ and no arcs and let ρ_1 be an H_1-coloring of D. Notice that the only $(H_1, \rho_1)-$paths are the arcs of D. Since S is an $\mathscr{H}$-panchromatic set, then S is an $(H_1, \rho_1)-$kernel, particularly S is an $(H_1, \rho_1)-$absorbent set, which implies that S is an absorbent set.

On the other hand, consider H_2 the digraph with vertex set $\{c_2\}$ and arc set $\{(c_2, c_2)\}$ and let ρ_2 be an H_2-coloring of D. Notice that every path in D is an $(H_2, \rho_2)-$path. Since S is an $\mathscr{H}$-panchromatic set, then S is an $(H_2, \rho_2)-$kernel, particularly S is an $(H_2, \rho_2)-$independent set, which implies that S is path-independent. □

Observation 2.7. *Every weakly $\mathscr{H}$-panchromatic digraph is $\mathscr{H}$-panchromatic.*

The converse of Observation 2.7 is not true. An even cycle C of order at least 4 is $\mathscr{H}$-panchromatic, nevertheless every absorbent set in C contains at least two vertices. In that case, since C is strongly connected, the only path-independent sets are the single sets which are no absorbent sets, concluding that C is not a weakly $\mathscr{H}$-panchromatic digraph.

Theorem 2.8. *Every transitive digraph is $\mathscr{H}$-panchromatic.*

Proof. We will show that every transitive digraph is weakly $\mathscr{H}$-panchromatic. According to Lemma 1.1 every transitive digraph D has a kernel S. Clearly S is an absorbent set. On the other hand, if there exists an $uv-$path for some vertices u and v in S, then $(u, v) \in A(D)$, which is no possible, concluding that S is path-independent. By Lemma 2.6 S is an $\mathscr{H}$-panchromatic set. Hence D is weakly $\mathscr{H}$-panchromatic and by Observation 2.7, D is $\mathscr{H}$-panchromatic. □

The following lemma will be a useful result to characterize quasi-transitive digraphs that are $\mathscr{H}$-pan-chromatic.

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Lemma 2.9 ([1]). *Let D be a quasi-transitive digraph. If there exists an $uv-$path for some vertices u and v in $V(D)$, then either u and v are adjacent or there exist vertices w and z such that (u, w), (w, z), (z, v), (z, u) and (v, w) are all arcs of D.*

Theorem 2.10. *A quasi-transitive digraph is $\mathscr{H}$-panchromatic if and only if it has a kernel.*

Proof. The sufficiency of Theorem 2.10 follows from Lemma 2.1.

For the converse, we will show that every quasi-transitive digraph that contains a kernel is weakly $\mathscr{H}$-panchromatic. Suppose that S is a kernel of D. Clearly S is an absorbent set. On the other hand, suppose that there exists an $uv-$path for some vertices u and v in S. According to Lemma 2.9 there exist vertices w and z such that (u, w), (w, z), (z, v), (z, u) and (v, w) are all arcs of D. Clearly $w \notin S$. In that case there exists a vertex $x \in S$ such that $(w, x) \in A(D)$. Notice that $x \neq v$, otherwise u and v must be adjacent, which is no possible. So, $x \neq v$ and it follows that x and v must be adjacent, a contradiction. Therefore, S is a path-independent set. By Lemma 2.6 S is an $\mathscr{H}$-panchromatic set, concluding that D is weakly $\mathscr{H}$-panchromatic and by Observation 2.7 D is $\mathscr{H}$-panchromatic. □

Theorem 2.11. *A tournament is $\mathscr{H}$-panchromatic if and only if it has a kernel.*

Proof. The sufficiency of Theorem 2.11 follows from Lemma 2.1.

For the converse, notice that a kernel S of T consists of a single vertex. In that case, S is both absorbent and path-independent, concluding that S is an $\mathscr{H}$-panchromatic set. Therefore, T is weakly $\mathscr{H}$-panchromatic and by Observation 2.7 T is $\mathscr{H}$-panchromatic. □

Lemma 2.12. *Every digraph of order p with a vertex of in-degree $p - 1$ is $\mathscr{H}$-panchromatic.*

Proof. If D is a digraph of order p with a vertex x of in-degree $p - 1$, then $\{x\}$ is both absorbent and path-independent, concluding that $\{x\}$ is an $\mathscr{H}$-panchromatic set. Therefore, D is weakly $\mathscr{H}$-panchromatic and by Observation 2.7 D is $\mathscr{H}$-panchromatic. □

Corollary 2.13. *For any digraph D there exists an $\mathscr{H}$-panchromatic digraph D' containing D as an induced subdigraph.*

Proof. Let D be any digraph and consider D' obtained from D by adding a new vertex x and joining every vertex from $V(D)$ toward x. Clearly D is an induced subdigraph of D' and by Lemma 2.12 D' is $\mathscr{H}$-panchromatic. □

Notice that Corollary 2.13 implies that $\mathscr{H}$-panchromatic digraphs cannot be characterized by forbidden induced subdigraphs.

Proposition 2.14. *Let D be an $\mathscr{H}$-panchromatic digraph. If D_0 is an induced subdigraph of D such that $A[V(D_0); V(D) \setminus V(D_0)] = \emptyset$, then D_0 is $\mathscr{H}$-panchromatic.*

Proof. Let D be an $\mathscr{H}$-panchromatic digraph and D_0 an induced digraph of D such that $A[V(D_0); V(D) \setminus V(D_0)] = \emptyset$. Notice that every path in D with initial vertex in D_0 must be entirely contained in D_0, otherwise, $A[V(D_0); V(D) \setminus V(D_0)] \neq \emptyset$.

Consider an arbitrary digraph H_0 and ρ_0 an H_0-coloring of D_0. We define H obtained from H_0 by adding an isolated vertex c. Define $\rho : A(D) \to V(H)$ as follows:

$$\rho(e) = \begin{cases} \rho_0(e) \text{ if } e \in A(D_0). \\ \quad c \text{ In another case.} \end{cases}$$

Notice that every $(H, \rho)-$path in D contained in D_0 is an $(H_0, \rho_0)-$path in D_0. Since ρ is an $H-$coloring of D, then D has an $(H, \rho)-$kernel, say K. We claim that $K_0 = K \cap V(D_0)$ is an $(H_0, \rho_0)-$kernel of D_0. Consider $v \in V(D_0) \setminus K_0$. Clearly $v \notin K$, so there exists an $vx - (H, \rho)-$path in D, say T, where $x \in K$. As $v \in V(D_0)$ it

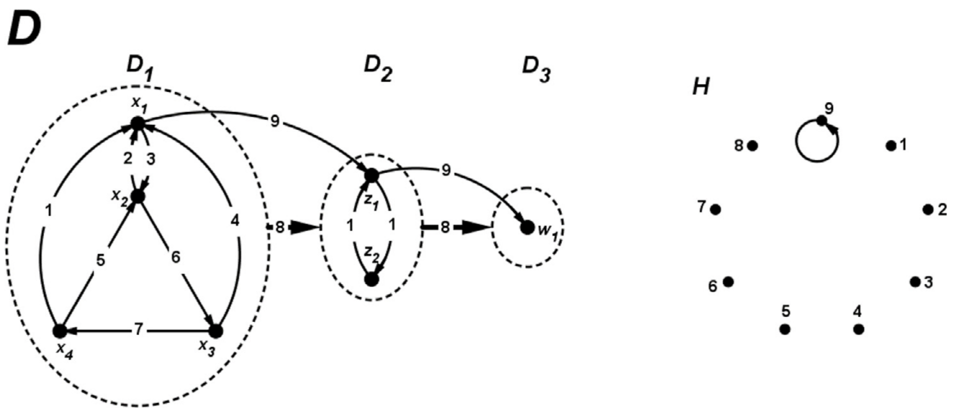

Fig. 2.1. A digraph D and an $H-$coloring of D without $H-$kernel.

follows that T is contained in D_0 concluding T is an $vx-(H_0,\rho_0)-$path in D_0, where $x\in K_0$. Hence, K_0 is an $(H_0,\rho_0)-$absorbent set in D_0.

On the other hand, we will show by contradiction that K_0 is an $(H_0,\rho_0)-$independent set in D_0. Assume that T is an $uv-(H_0,\rho_0)-$path in D_0, where $\{u,v\}\subseteq K_0$. It is straightforward to see that T is an $uv-(H,\rho)-$path, which is no possible since K is $(H,\rho)-$independent. Therefore, D_0 has an $(H,\rho)-$kernel, concluding that D_0 is an $\mathscr{H}$-panchromatic digraph. □

Corollary 2.15. *Every terminal strongly component of an $\mathscr{H}$-panchromatic digraph is $\mathscr{H}$-panchromatic.*

Corollary 2.16. *Let D be an $\mathscr{H}$-panchromatic digraph. If $D_1,\dots,D_n$ are the initial strongly components of D and $V(D)\neq\cup_{i=1}^{n}V(D_i)$, then $D\setminus(\cup_{i=1}^{n}V(D_i))$ is an $\mathscr{H}$-panchromatic digraph.*

Converse of Corollaries 2.15 and 2.16 is not true. Consider the digraph $D=P_3[D_1,D_2,D_3]$ shown in Fig. 2.1. Notice that by Lemma 2.12 D_1, D_2 and D_3 are $\mathscr{H}$-panchromatic digraphs. On the other hand, we claim that D is not $\mathscr{H}$-panchromatic.

Consider the $H-$coloring ρ defined as follows: every two different arcs in D_1 have different color, every two arcs in D_2 have the same color, every arc from $V(D_1)$ toward $V(D_2)$ has color 8 and every arc from $V(D_2)$ toward $V(D_3)$ has color 8, unless $\rho(x_1,z_1)=\rho(z_1,w_1)=9$.

Proceeding by contradiction we will prove that D contains no $H-$kernel. Suppose that S is an $H-$kernel of D. Clearly $w_1\in S$, so $(V(D_2)\cup\{x_1\})\cap S=\emptyset$. Now consider the following cases about the vertex x_3.

Case 1. $x_3\notin S$.

In this case x_3 must be $H-$absorbed by some vertex in S, say z. Since there is no x_3w_1-H-path and $V(D_2)\cap S=\emptyset$, then $z\in V(D_1)$, so $z=x_4$. That implies that $x_2\notin S$. Notice that the only vertices that x_2 can reach by an $H-$path are the vertices in $V(D_2)$, x_1 and x_3 in which no one of them belongs to S, concluding that x_2 is not $H-$absorbed by S, a contradiction.

Case 2. $x_3\in S$.

In this case we have that $x_4\notin S$. Notice that the only vertices that x_4 can reach by an $H-$path are the vertices in $V(D_2)$, x_1 and x_2. That implies that $x_2\in S$ which is no possible since $x_3\in S$.

We conclude that D has no $H-$kernel. Particularly, D is not an $\mathscr{H}$-panchromatic digraph.

On the other hand, notice that the strongly connected components of D are D_1, D_2 and D_3 which are $\mathscr{H}$-panchromatic digraphs. It follows that a digraph whose strongly components are $\mathscr{H}$-panchromatic digraphs is not necessarily an $\mathscr{H}$-panchromatic digraph. Particularly, converse of Corollary 2.15 is not true.

On the other hand, notice that by Lemma 2.12 we have that $D\langle V(D_2)\cup V(D_3)\rangle$ is $\mathscr{H}$-panchromatic. So, $D\setminus V(D_1)$ is an $\mathscr{H}$-panchromatic digraph but D is not. Hence, converse of Corollary 2.16 is not true.

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3. $\mathscr{H}$-panchromatic digraphs and operations on digraphs

Lemma 3.1. *Let D_1 and D_2 be two $\mathscr{H}$-panchromatic digraphs. If every vertex in $V(D_1) \cap V(D_2)$ has out-degree 0 in $D_1 \cup D_2$, then $D_1 \cup D_2$ is $\mathscr{H}$-panchromatic.*

Proof. Notice that every path in $D = D_1 \cup D_2$, say T, satisfies either $V(T) \subseteq V(D_1)$ or $V(T) \subseteq V(D_2)$. Otherwise, if there exists a path $T = (x_1, \ldots, x_n)$ such that $V(T) \cap (V(D_1) \setminus V(D_2)) \neq \emptyset$ and $V(T) \cap (V(D_2) \setminus V(D_1)) \neq \emptyset$ then there exists a vertex $x_i \in V(D_1) \cap V(D_2)$ with out-degree at least one, which is no possible.

Suppose that D is $H-$colored for some digraph H. Since D_1 is $\mathscr{H}$-panchromatic, then D_1 has an $H-$kernel, say K_1. In the same way, D_2 has an $H-$kernel, say K_2. We claim that $K = K_1 \cup K_2$ is an $H-$kernel of D.

It is straightforward to see that K is an $H-$absorbent set in D. It only remains to show that K is an $H-$independent set. Suppose that u and v are vertices in K such that there exists an $uv - H-$path in D, say T. It follows that T is contained either in D_1 or D_2. Suppose (w.l.o.g) that T is contained in D_1. So, T is an $uv - H-$path in D_1 where u and v belongs to K_1, which is no possible. Hence, K is an $H-$independent set, concluding that K is an $H-$kernel in D. Therefore $D_1 \cup D_2$ is an $\mathscr{H}$-panchromatic digraph. □

By applying an inductive argument we get the following proposition.

Proposition 3.2. *Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be a family of $\mathscr{H}$-panchromatic digraphs and let $D = \cup_{i=1}^{n} D_i$. If $\delta_D^+(x) = 0$ for every vertex $x \in V(D_i) \cap V(D_j)$ with $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$, then D is $\mathscr{H}$-panchromatic.*

Lemma 3.3. *Let D_1 and D_2 be two $\mathscr{H}$-panchromatic digraphs. If every vertex in $V(D_1) \cap V(D_2)$ has in-degree 0 in $D_1 \cup D_2$, then $D_1 \cup D_2$ is $\mathscr{H}$-panchromatic.*

Proof. Notice that every path in $D = D_1 \cup D_2$, say T, satisfies either $V(T) \subseteq V(D_1)$ or $V(T) \subseteq V(D_2)$. Otherwise, if there exists a path $T = (x_1, \ldots, x_n)$ such that $V(T) \cap (V(D_1) \setminus V(D_2)) \neq \emptyset$ and $V(T) \cap (V(D_2) \setminus V(D_1)) \neq \emptyset$ then there exists a vertex $x_i \in V(D_1) \cap V(D_2)$ with in-degree at least one, which is no possible.

Suppose that D is $H-$colored for some digraph H. Since D_1 is $\mathscr{H}$-panchromatic, then D_1 has an $H-$kernel, say K_1. In the same way, D_2 has an $H-$kernel, say K_2. We claim that $K = [(K_1 \cup K_2) \setminus (V(D_1) \cap V(D_2))] \cup (K_1 \cap K_2)$ is an $H-$kernel of D.

We will prove that K is $H-$absorbent in D. Let $x \in V(D) \setminus K$ and suppose (w.l.o.g.) that $x \in V(D_1)$. Consider the following cases.

Case 1 $x \notin V(D_2)$.

In this case we have that $x \notin K_1$. So, there exists a vertex $u \in K_1$ such that there exists an $xu - H-$path in D_1. Clearly u has in-degree at least 1, so $u \notin V(D_1) \cap V(D_2)$, concluding that $u \in K$. Hence, x is $H-$absorbed by K in D.

Case 2 $x \in V(D_2)$.

If $x \notin K_2$, then there exists a vertex $v \in K_2$ such that there exists an $xu - H-$path. Clearly v has in-degree at least 1, so $v \notin V(D_1) \cap V(D_2)$, concluding that $v \in K$. Hence x is $H-$absorbed by K in D.

If $x \in K_2$ we have that $x \notin K_1$, otherwise $x \in K$. It follows that there exists an $xu - H-$path in D_1 for some vertex $u \in K_1$. Clearly u has in-degree at least 1, so $u \notin V(D_1) \cap V(D_2)$, concluding that $u \in K$. Hence x is $H-$absorbed by K in D.

Now we will prove that K is $H-$independent. Proceeding by contradiction suppose that there exist u and v in K such that there is an $uv - H-$path in D, say T. It follows that T is contained either in D_1 or D_2. We can assume that T is contained in D_1. So, T is an $uv - H-$path in D_1 with u and v in K_1, which is no possible since K_1 is an $H-$independent set in D_1.

By the above, K is an $H-$independent set in D, concluding that K is an $H-$kernel in D. Therefore, $D_1 \cup D_2$ is an $\mathscr{H}$-panchromatic digraph. □

By applying an inductive argument we get the following proposition.

Proposition 3.4. *Let $\mathscr{D} = \{D_1, \ldots, D_n\}$ be a family of $\mathscr{H}$-panchromatic digraphs and let $D = \cup_{i=1}^{n} D_i$. If $\delta_D^-(x) = 0$ for every vertex $x \in V(D_i) \cap V(D_j)$ with $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$, then D is $\mathscr{H}$-panchromatic.*

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Proposition 3.5. *Let G be a weakly $\mathscr{H}$-panchromatic digraph without symmetric arcs and vertex set $\{v_1, \ldots, v_n\}$. Suppose that N is an $\mathscr{H}$-panchromatic set of G and let $J = \{i \in \{1, \ldots, n\} : v_i \in N\}$. If $\mathscr{D} = \{D_1, \ldots, D_n\}$ is a family of pairwise vertex disjoint digraphs such that D_j is an $\mathscr{H}$-panchromatic digraph for every $j \in J$, then $G[\mathscr{D}]$ is $\mathscr{H}$-panchromatic.*

Proof. Suppose that $D = G[\mathscr{D}]$ is an $H-$colored digraph and for every $j \in J$ consider an $H-$kernel of D_j, say K_j. We claim that $K = \cup_{j \in J} K_j$ is an $H-$kernel in D.

In order to prove that K is an $H-$absorbent set, consider a vertex $u \in V(D) \setminus K$. If $u \in V(D_j)$ for some $j \in J$, then there exists an $ux - H-$path in D_j for some $x \in K_j$, concluding that u is $H-$absorbed by K in D. Now suppose that $u \in V(D_k)$ for some $k \in \{1, \ldots, n\} \setminus J$. Since N is an absorbent set of G, then $(v_k, v_j) \in A(G)$ for some $j \in J$. So, if $x \in K_j$, we have that $(u, x) \in A(D)$, which implies that u is $H-$absorbed by K. Hence, K is an $H-$absorbent set in D.

Proceeding by contradiction we will prove that K is an $H-$independent set in D. Suppose that there exists an $uv - H-$path in D for some pair of different vertices u and v in K. We may assume that $u \in K_i$ and $v \in K_j$ for some $i \in J$ and $j \in J$. Since there exists an $uv-$path in D, we have that there exists an $v_i v_j-$path in G. In view of the fact that N is a path-independent set in G, we conclude that $i = j$.

Clearly T is not contained in D_i, otherwise K_i is not an $H-$independent set in D_i. It follows that v_i has an out-neighbor in G, say v. Clearly $v \notin N$, which implies that v is absorbed by some vertex in N, say z. Since N is path-independent, we have that $z = v_i$, so (v_i, v) is a symmetric arc in G, which is no possible by assumption. Hence, K is an $H-$independent set in D. Therefore, K is an $H-$kernel in D, concluding that D is $\mathscr{H}$-panchromatic. □

Proposition 3.5 cannot be generalized to composition over $\mathscr{H}$-panchromatic digraphs. Notice that the digraph shown in Fig. 2.1 is not $\mathscr{H}$-panchromatic. Nevertheless D is the composition of $\mathscr{H}$-panchromatic digraphs over an $\mathscr{H}$-panchromatic digraph.

Corollary 3.6. *Composition of $\mathscr{H}$-panchromatic digraphs over an asymmetric weakly $\mathscr{H}$-panchromatic digraph is $\mathscr{H}$-panchromatic.*

Corollary 3.7. *If D_1 and D_2 are two vertex disjoint digraphs and D_2 is $\mathscr{H}$-panchromatic, then $D_1 + D_2$ is $\mathscr{H}$-panchromatic.*

Lemma 3.8. *Let D be an $\mathscr{H}$-panchromatic digraph by walks. If u and v are vertices in D such that $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$, then $D \setminus \{u\}$ is $\mathscr{H}$-panchromatic by walks.*

Proof. Suppose that ρ is an $H-$coloring of $D_0 = D \setminus \{u\}$ and consider the $H-$coloring ρ' of D defined as follows:

$$\rho'(e) = \begin{cases} \rho(e) \text{ if } e \in A(D_0). \\ \rho(x, v) \text{ if } e = (x, u) \text{ for some } x \in N^-(u) \\ \rho(v, x) \text{ if } e = (u, x) \text{ for some } x \in N^+(u) \end{cases}$$

Clearly ρ' is an $H-$coloring of D. Notice that every $(H, \rho')-$walk in D which does not contain u is an $(H, \rho)-$walk in D_0.

Since D is an $\mathscr{H}$-panchromatic digraph by walks, then D contains an $(H, \rho')-$kernel by walks, say S. Consider the following cases:

- **Case 1.** $u \notin S$.
  In this case we claim that S is an $(H, \rho)-$kernel by walks of D_0. In order to prove that S is an $(H, \rho)-$absorbent set by walks in D_0, consider a vertex $x \in V(D_0) \setminus S$. It follows that there exists an $xz - (H, \rho')-$walk in D, say $T = (x = x_0, \ldots, x_n = z)$, for some $z \in S$. If $u \notin V(T)$ then T is an $xz - (H, \rho)-$walk in D_0. Now we suppose that $u = x_r$ for some $r \in \{1, \ldots, n-1\}$. Since $N^-(u) = N^-(v)$ and $N^+(u) = N^+(v)$, then $T' = (x = x_0, \ldots, x_{r-1}, v, x_{r+1}, \ldots, x_n = z)$ is an $xz-$walk in D_0. Moreover, since $\rho(x_{r-1}, v) = \rho'(x_{r-1}, u)$ and $\rho(v, x_{r+1}) = \rho'(u, x_{r+1})$ then T' is an $xz - (H, \rho)-$walk in D_0, concluding that S is an $(H, \rho)-$absorbent set by walks in D_0.

On the other hand, suppose that S is not an $(H,\rho)-$independent set by walks in D_0, so, there exists an $xz-(H,\rho)-$walk in D_0, say T, for some $\{x,z\}\subseteq S$. It is straightforward to see that T is an $xz-(H,\rho')-$walk in D, which is no possible. Therefore, S is an $(H,\rho)-$kernel by walks of D_0.

- Case 2. $u\in S$.
  In this case we claim that $S'=(S\setminus\{u\})\cup\{v\}$ is an $(H,\rho)-$kernel by walks of D_0. First we will show that S' is an $(H,\rho)-$absorbent set by walks. Let x_0 be a vertex in $V(D_0)\setminus S'$. Since S is an $(H,\rho')-$absorbent set by walks in D, there exists an $x_0z-(H,\rho')-$walk in D, say $T=(x_0,x_1,\dots,x_n=z)$, for some $z\in S$. If $z\neq u$ then T is an $x_0z-(H,\rho)-$walk in D_0. Now we assume that $z=u$. Since $N^-(u)=N^-(v)$ then $(x_{n-1},v)\in A(D_0)$, which implies that $T'=(x_0,\dots,x_{n-1},v)$ is an $xv-$walk in D_0. Moreover, since $\rho(x_{n-1},v)=\rho'(x_{n-1},u)$ then T' is an $x_0v-(H,\rho)-$walk in D_0. Hence, S' is an $(H,\rho)-$absorbent set by walks in D_0.
  On the other hand, suppose that S' is not an $(H,\rho)-$independent set by walks in D_0. So, there exists an $xz-(H,\rho)-$walk in D_0, say $P=(x=x_0,\dots,x_n=z)$, for some $\{x,z\}\subseteq S'$. Notice that P is an $xz-(H,\rho')-$walk in D, it follows that either $x=v$ or $z=v$, otherwise S is not an $(H,\rho')-$independent set by walks. If $x=v$ then by assumption $x_1\in N^+(u)$ and $\rho'(u,x_1)=\rho(v,x_1)$ concluding that $P'=(u,x_1,\dots,x_n=z)$ is an $uz-(H,\rho')-$walk in D_0 which is no possible. In the same way, if $z=v$ an analogous proof shows that S is not an $(H,\rho')-$independent set by walks in D_0, which is no possible. It follows that S' is an $(H,\rho)-$independent set by walks in D_0.
  Therefore, S' is an $(H,\rho)-$kernel by walks in D_0, which implies that D_0 is an $\mathscr{H}$-panchromatic digraph by walks. □

Proposition 3.9. *Let D be an extension of some digraph D_0. If D is an $\mathscr{H}$-panchromatic digraph by walks, then D_0 is an $\mathscr{H}$-panchromatic digraph by walks.*

Proof. We will show that Proposition 3.9 holds by induction on the order of D.

If D is an extension of D_0 such that D is $\mathscr{H}$-panchromatic and $|V(D)|=|V(D_0)|$ then D is isomorphic to D_0, concluding that D_0 is $\mathscr{H}$-panchromatic by walks.

Now we assume that if D' is an extension of D_0 such that D' is an $\mathscr{H}$-panchromatic digraph by walks and $|V(D_0)|\leq|V(D')|=n$, then D_0 is an $\mathscr{H}$-panchromatic digraph by walks.

For the inductive step suppose that D is an extension of D_0 such that D is an $\mathscr{H}$-panchromatic digraph by walks and $|V(D_0)|\leq|V(D)|=n+1$. By assumption, $D=D_0[V_1,V_2,\dots,V_k]$ where V_i is an independent set for every $i\in\{1,\dots,k\}$. We can assume that $|V(D_0)|\neq|V(D)|$, so there exists $i\in\{1,\dots,k\}$ such that $|V_i|\geq 2$. Consider u and v two different vertices in V_i. Clearly $N_D^+(u)=N_D^+(v)$ and $N_D^-(u)=N_D^-(v)$ and by Lemma 3.8 we conclude that $D'=D\setminus\{u\}$ is an $\mathscr{H}$-panchromatic digraph by walks. On the other hand, notice that $D'=D_0[V_1,\dots,V_i\setminus\{u\},\dots,V_k]$ and by induction hypothesis we conclude that D_0 is $\mathscr{H}$-panchromatic by walks. □

Converse of Proposition 3.9 is not true. Galeana-Sánchez and Toledo [6] proved that the digraph D shown in Fig. 3.2 is not panchromatic. We claim that D contains no $H-$kernel by walks with the $H-$coloration shown in Fig. 3.2.

Suppose that D contains an $H-$kernel by walks, say S. Notice that a path T is an $H-$path if and only if T is monochromatic. If $x_1\in S$, then $v_1\notin S$ and $v_2\notin S$ since (v_1,u_2,x_1) is a v_1x_1-H-path and (v_2,u_1,x_1) is a v_2x_1-H-path. Hence, neither w_1 and w_2 are $H-$absorbed by walks by S, which is no possible. Therefore, $x_1\notin S$.

An analogous proof shows that $x_2\notin S$. So, we have that either $w_1\in S$ or $w_2\in S$. If $w_1\in S$ we conclude that $u_1\notin S$, since (w_1,v_1,u_1) is an w_1u_1-H-path. It follows that u_1 is not $H-$absorbed by S, which is no possible. That implies that $w_1\notin S$ concluding that $w_2\in S$. An analogous proof shows that u_1 is not $H-$absorbed by S, a contradiction. Therefore, D has no $H-$kernel by walks.

Notice that D is an extension of C_4. In particular, from Corollary 2.3 D is an extension of an $\mathscr{H}$-panchromatic digraph, showing that an extension of an $\mathscr{H}$-panchromatic digraph is not necessarily $\mathscr{H}$-panchromatic.

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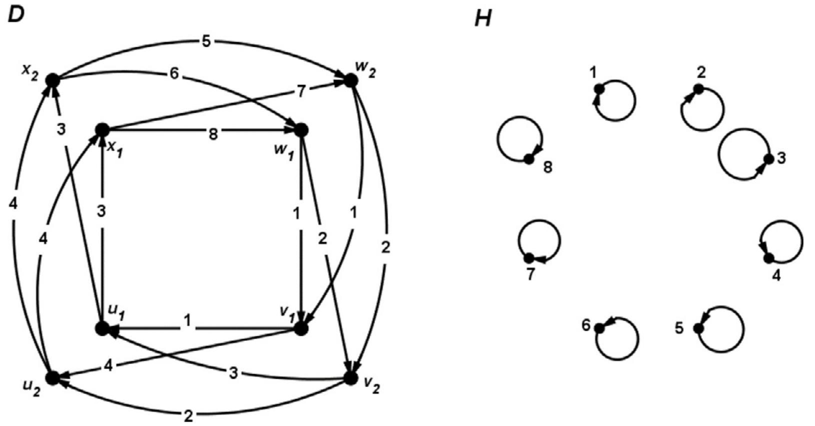

Fig. 3.2. An extension of an $\mathscr{H}$-panchromatic digraph by walks that is not $\mathscr{H}$-panchromatic by walks.

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