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ABSTRACT
Let $G = (V, E)$ be a simple connected graph. An ordered subset W of V is said to be a resolving set of G if every vertex is uniquely determined by its vector of distances to the vertices in W. The minimum cardinality of a resolving set is called the resolving number of G and is denoted by $r(G)$. Total resolving number is the minimum cardinality taken over all resolving sets in which $\langle W \rangle$ has no isolates and it is denoted by $tr(G)$. In this paper, we determine the exact values of total resolving number of $K_{1,n-1}(C_k), B_{s,t}(C_k)$, $C_n(C_k)$, $P_n(C_k)$ and $K_n(C_k)$. Also, we obtain bounds for the total resolving number of $G(C_k)$ when G is an arbitrary graph and characterize the extremal graphs.

KEYWORDS
Resolving number; total resolving number; edge cycle graph

SUBJECT CLASSIFICATION
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1. Introduction

Let $G = (V, E)$ be a finite, simple, connected and undirected graph. The *degree* of a vertex v in a graph G is the number of edges incident with v and it is denoted by $d(v)$. The maximum degree in a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest u-v path in G. The maximum value of distance between vertices of G is called its *diameter*. Let P_n denote any *path* on n vertices, C_n denote any *cycle* on n vertices and K_n denote any *complete graph* on n vertices. A complete bipartite graph is denoted by $K_{s,t}$. $K_{1,n-1}$ is called a *star*. A tree containing exactly two vertices that are not end vertices is called a *bistar* and it is denoted by $B_{s,t}$. The *join* $G + H$ consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H. Let P denote the set of all pendent edges of G and $|P| = p$. Vertices which are adjacent to pendent vertices are called *support vertices*.

A graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph F of a graph G is called an *induced subgraph* $\langle F \rangle$ of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. For a non empty set X of edges, the subgraph $\langle X \rangle$ induced by X has edge set X and consists of all vertices that are incident with at least one edge in X. This subgraph is called an *edge induced subgraph* of G. A set of edges in a graph is *independent* if no two edges in the set are adjacent. The *edge independence number* $\beta_1(G)$ of a graph G is the maximum cardinality taken over all maximal independent set of edges.

If $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ is an ordered set, then the ordered k-tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the representation of v with respect to W and it is denoted by $r(v|W)$. Since the representation for each $w_i \in W$ contains exactly one 0 in the i^{th} position, all the vertices of W have distinct representations. W is called a *resolving set* for G if all the vertices of $V \setminus W$ also have distinct representations. The minimum cardinality of a resolving set is called the *resolving number* of G and it is denoted by $r(G)$.

In 1975, Slater [5] introduced these ideas and used *locating set* for what we have called resolving set. He referred to the cardinality of a minimum resolving set in G as its *location number*. In 1976, Harary and Melter [1] discovered these concepts independently as well but used the term *metric dimension* rather than location number. In 2003, Ping Zhang and Varaporn Saenpholphat [6, 7] studied *connected resolving number* and in 2015, we introduced and studied *total resolving number* in [3]. In this paper, we use the term *resolving number* to maintain uniformity in the current literature.

If W is a resolving set and the induced subgraph $\langle W \rangle$ has no isolates, then W is called a *total resolving set* of G. The minimum cardinality taken over all total resolving sets of G is called the *total resolving number* of G and is denoted by $tr(G)$. We introduced edge cycle graph in [2] and studied the resolving number of edge cycle graph $G(C_k)$ and we studied the total resolving number of edge cycle graph $G(C_3)$ in [4].

In this paper, we investigate the total resolving number of the edge cycle graph $G(C_k)$, $k \geq 4$.

Theorem 1.1. [3] *Let $\{w_1, w_2\} \subset V(G)$ be a total resolving set in G. Then the degrees of w_1 and w_2 are at most 3.*

Lemma 1.2. [3] *For $n \geq 3$, $tr(P_n) = 2$ and $tr(C_n) = 2$.*

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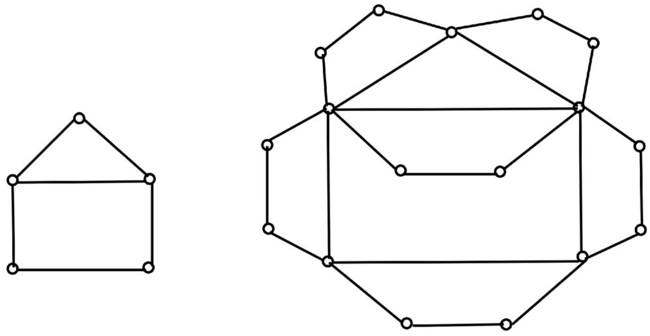

Figure 1. A graph G and its edge cycle graph.

Observation 1.3. [3] *For any graph G,* $2 \le tr(G) \le n-1$.

Theorem 1.4. [3] *For* $n \ge 3$, $tr(G) = n-1$ *if and only if* $G = K_n$ *or* $K_{1,n-1}$.

Definition 1.5. A block of G containing exactly one cut vertex of G is called an end block of G.

Lemma 1.6. [2] *Let G be a 1-connected graph with* $\delta(G) \ge 2$. *Then every resolving set contains at least one non cut vertex of each end block.*

Definition 1.7. A cycle C_r is called an end cycle if C_r contains exactly one vertex of degree at *least 3.*

Notation 1.8. Let e_c denote the number of end cycles of the graph G.

Lemma 1.9. *Let G be a graph with* $e_c \ge 1$ *and each end cycle of size at least 4. Then* $tr(G) \ge 2e_c$.

Proof. Let W be a total resolving set of G. Let $C_1, C_2, \ldots, C_{e_c}$ be the end cycles of G. By Lemma 1.6, $W \cap (V(C_i) \setminus \{v\}) \ne \emptyset$ for all $1 \le i \le e_c$. Let v be the common vertex of some end cycles $C_1, C_2, \ldots, C_r$. If $v \notin W$, then clearly, $W \cap (V(C_i) \setminus \{v\}) \ge 2$, for each $1 \le i \le r$. If $v \in W$, then we claim that $|W \cap X| \ge 2r$, where $X = \cup_{i=1}^{r} V(C_i)$. Suppose $|W \cap X| \le 2r-1$. Clearly, exactly two neighbors of v in $V(C_1) \cup V(C_2)$ belongs to W. Without loss of generality, let $v_1, v_2 \in W$. Then $r(v_3|W) = r(v_4|W)$, which is a contradiction. Thus $|W \cap X| \ge 2r$. Consequently, $tr(G) \ge 2e_c$. □

2. Resolving number of edge cycle graphs $G(C_k)$, $k \ge 4$

The following results are proved in [2].

Definition 2.1. An edge cycle graph of a graph G is the graph $G(C_k)$ formed from one copy of G and $|E(G)|$ copies of P_k, where the ends of the i$^{\text{th}}$ edge are identified with the ends of i$^{\text{th}}$ copy of P_k. A graph G and its edge cycle graph $G(C_k)$ are shown in Figure 1.

Lemma 2.2. *Let* v *be a vertex of degree* r *in* G *and* $e_1, e_2, \ldots, e_r$ *be edges incident with* v *and* C_i *be the edge cycle of* e_i, $1 \le i \le r$. *Then every resolving set of* $G(C_k)$ *contains at least one vertex of degree 2 from* C_i *for all* $1 \le i \le r$ *with at most one exception.*

Lemma 2.3. *Let* e *be an edge of degree* s *and* $e_1, e_2, \ldots, e_{s-2}$ *be the edges adjacent to* e *in* G. *If any resolving set* W *of* $G(C_k)$ *does not contain any internal vertex of the edge cycle of* e, *then* W *contains at least one internal vertex from each edge cycle of* e_i, $1 \le i \le s-2$.

Theorem 2.4. *Let* $E_1 = \{e_1, e_2, \ldots, e_t\}$ *be a subset of edges of* G *and* W *be a resolving set of* $G(C_k)$. *If* W *does not contain any internal vertex of edge cycle of* e_i, *then* E_1 *is independent.*

Lemma 2.5. *Let* G *be a graph of order* $n \ge 3$ *and* $\delta(G) \ge 2$. *Then* $r(G(C_k)) \ge m - \beta_1(G)$.

Lemma 2.6. *Let* G *be a graph of order* $n \ge 3$ *and* $\delta(G) = 1$. *Then* $r(G(C_k)) \ge m - \beta_1(G \setminus P)$.

Theorem 2.7. *Let* G *be a graph of order* $n \ge 5$ *and size* m. *If* k *is odd and* $\delta(G) \ge 2$, *then* $r(G(C_k)) = m - \beta_1(G)$.

Theorem 2.8. *Let* G *be a graph of order* $n \ge 5$, *size* m *and* $\delta(G) = 1$. *If* k *is odd, then* $r(G(C_k)) = m - \beta_1(G \setminus P)$.

3. Total resolving number of $G(C_k)$, $k \ge 4$.

In this section, we determine the exact values of total resolving number of $K_{1,n-1}(C_k), B_{s,t}(C_k)$, $C_n(C_k)$, $P_n(C_k)$ and $K_n(C_k)$.

Definition 3.1. A vertex cover in a graph G is a set of vertices that covers all edges of G. The minimum cardinality taken over all minimal vertex covers of G is the vertex covering number $\alpha(G)$ of G.

Theorem 3.2. *Let* G *be a graph of order* $n \ge 4$ *and size* m. *Let* $M_1, M_2, \ldots, M_r$ *be the collection of all maximum edge independent sets of* G *and* $G_i = \langle G \setminus M_i \rangle$, $1 \le i \le r$. *If* $\delta(G) \ge 2$, *then* $tr(G(C_k)) \ge m - \beta_1(G) + t$, *where* $t = min\{\alpha(G_1), \alpha(G_2), \ldots, \alpha(G_r)\}$.

Proof. Let $Y = \{v_1, v_2, \ldots, v_t\}$ be the minimum vertex covering of G_i for some i and W_1 be any total resolving set of $G(C_k)$. By Theorem 2.5, $r(G(C_k)) \ge m - \beta_1(G)$. Let W' be a minimum resolving set of $G(C_k)$. Using Lemmas 2.2, 2.3 and 2.5, $\langle W' \rangle$ is $\bar{K}_{m-\beta_1(G)}$. Thus $|W_1| \ge m - \beta_1(G) + |Y| = m - \beta_1(G) + t$. □

Theorem 3.3. *Let* G *be a graph of order* $n \ge 4$, *size* m *and* $\delta(G) = 1$. *Let* $M_1, M_2, \ldots, M_r$ *be the collection of all maximum edge independent sets of* G *and* $G_i = \langle G \setminus (M_i \cup P) \rangle$, $1 \le i \le r$. *Then* $tr(G(C_k)) \ge m - \beta_1(G \setminus P) + t + p$, *where* $t = min\{\alpha(G_1), \alpha(G_2), \ldots, \alpha(G_r)\}$.

Proof. Let $Y = \{v_1, v_2, \ldots, v_t\}$ be the minimum vertex covering of G_i for some i and W_1 be any total resolving set of $G(C_k)$. By Lemma 2.6, $r(G(C_k)) \ge m - \beta_1(G \setminus P)$. Let W' be a minimum resolving set of $G(C_k)$. Using Lemmas 1.6, 2.2, 2.3 and 2.6, $\langle W' \rangle$

is $\bar{K}_{m-\beta_1(G\setminus P)}$. Thus $|W_1| \geq m - \beta_1(G \setminus P) + |Y| + |P| = m - \beta_1(G \setminus P) + t + p$. □

Theorem 3.4. *Let G be a graph of order $n \geq 3$, size m and $k \geq 4$. If $\delta(G) \geq 2$, then $tr(G(C_k)) \leq 2[m - \beta_1(G)]$.*

Proof. Let $E(G) = \{e_1, e_2, ..., e_m\}$. Let $\beta_1(G) = s$ and $M = \{e_1, e_2, ..., e_s\}$ be the maximum edge independent set of G. Let C_i be the edge cycle of e_i. Let $V(C_i) = \{v_{i1}, v_{i2}, ..., v_{ik}\}$ and $e_i = v_{i1}v_{ik}$. If $n = 3$, then we can easily verify that $tr(G(C_k)) \leq 4$. So we may assume that $n \geq 4$. By Theorem 2.7, if k is odd, then $r(G(C_k)) \leq m - \beta_1(G)$. Therefore, if k is odd, then $tr(G(C_k)) \leq 2r(G(C_k)) \leq 2[m - \beta_1(G)]$. Now we claim that if k is even, then $tr(G(C_k)) \leq 2[m - \beta_1(G)]$. Let $W = \{v_{i\frac{k}{2}}, v_{i(\frac{k}{2}+1)} / s+1 \leq i \leq m\}$. We claim that W is a resolving set of $G(C_k)$. Let x, y be two distinct vertices of $V(G(C_k)) \setminus W$. We consider the following two cases.

Case 1: $x \in V(C_i)$ for some $1 \leq i \leq s$.

Without loss of generality, let $x \in V(C_1)$. Then x lies on either v_{11}-$v_{1\frac{k}{2}}$ path or $v_{1(\frac{k}{2}+1)}$-v_{1k} path. Without loss of generality, let x lie on v_{11}-$v_{1\frac{k}{2}}$ path. Since $\delta(G) \geq 2$, there exist two distinct edges $e_r, e_{r'} \in E(G) \setminus \{e_1\}$ such that e_r is incident with v_1 and $e_{r'}$ is incident with v_2. Since $e_1 \in M$, by Lemma 2.3, $e_r, e_{r'} \notin M$. Without loss of generality, let $e_r = e_m$ and $e_{r'} = e_{m-1}$. Let $S = \{v_{m\frac{k}{2}}, v_{m(\frac{k}{2}+1)}, v_{(m-1)\frac{k}{2}}, v_{(m-1)(\frac{k}{2}+1)}\}$. If $y \in V(C_1 \cup C_{m-1} \cup C_m)$, then $r(x|S) \neq r(y|S)$. It follows that $r(x|W) \neq r(y|W)$. So we assume that $y \in V(C_i)$ for some $2 \leq i \leq m-2$. If $d(x, s) \neq d(y, s)$ for some $s \in S$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, s) = d(y, s)$ for all $s \in S$. Therefore y-$v_{m\frac{k}{2}}$ path and y-$v_{m(\frac{k}{2}+1)}$ path passes through v_{11} and y-$v_{(m-1)\frac{k}{2}}$ path and y-$v_{(m-1)(\frac{k}{2}+1)}$ path passes through v_{11} and v_{1k}.

If $y \in V(C_i)$ for some $s+1 \leq i \leq m-2$, then without loss of generality, let $y \in V(C_{m-2})$. Thus either $d(x, v_{(m-2)\frac{k}{2}}) > d(y, v_{(m-2)\frac{k}{2}})$ or $d(x, v_{(m-2)(\frac{k}{2}+1)}) > d(y, v_{(m-2)(\frac{k}{2}+1)})$. If $y \in V(C_i)$ for some $2 \leq i \leq s$, then without loss of generality, let $y \in V(C_2)$. Since $\delta(G) \geq 2$, there exist two distinct edges $e_t, e_{t'} \in E(G) \setminus \{e_1, e_2\}$ such that e_t is incident with v_{21} and $e_{t'}$ is incident with v_{2k}. Since $e_1 \in M$, by Lemma 2.3, $e_t, e_{t'} \notin M$. If either $e_t \neq e_m$ or $e_{t'} \neq e_{m-1}$, then without loss of generality, let $e_t \neq e_m$. Let $e_t = e_{m-2}$ and $v_{(m-2)1} = v_{11}$. Then $d(x, v_{(m-2)\frac{k}{2}}) > d(y, v_{(m-2)\frac{k}{2}})$. If $e_t = e_m, e_{t'} = e_{m-1}$, then $r(x|S) / = r(y|S)$. It follows that $r(x|W) \neq r(y|W)$.

Case 2: $x \notin V(C_i)$ for all $1 \leq i \leq s$.

Then without loss of generality, let $x \in V(C_m)$. Let $X = \{v_{m\frac{k}{2}}, v_{m(\frac{k}{2}+1)}\}$. If $y \in V(C_m)$, then $r(x|X) \neq r(y|X)$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that $x \notin V(C_m)$. If $d(x, v_{m\frac{k}{2}}) \neq d(y, v_{m\frac{k}{2}})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{m\frac{k}{2}}) = d(y, v_{m\frac{k}{2}})$. Therefore $x = v_{mk}$ and y is the neighbor of v_{m1}. Thus $d(x, v_{m(\frac{k}{2}+1)}) = d(y, v_{m(\frac{k}{2}+1)}) - 2$. It follows that $r(x|W) \neq r(y|W)$.

Hence $tr(G(C_k)) \leq 2[m - \beta_1(G)]$. □

Open problem 3.5. *Let G be a graph of order $n \geq 3$, size m, $k \geq 4$. If $\delta(G) \geq 2$, then characterize G for which $tr(G(C_k)) = 2[m - \beta_1(G)]$.*

Theorem 3.6. *Let G be a graph of order $n \geq 3$, size m. If $\delta(G) = 1$, then $tr(G(C_k)) \leq 2[m - \beta_1(G \setminus P)]$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$, $E(G) = \{e_1, e_2, ..., e_m\}$. Let $\beta_1(G \setminus P) = s$. Let $M = \{e_1, e_2, ..., e_s\}$ be the maximum edge independent set of $G \setminus P$ and $P = \{e_{s+1}, e_{s+2}, ..., e_{s+p}\}$. Let C_i be the edge cycle of e_i and $V(C_i) = \{v_{i1}, v_{i2}, ..., v_{ik}\}$, $1 \leq i \leq m$. Let $e_i = v_{i1}v_{ik}, 1 \leq i \leq m$. If $n = 3$, then we can easily verify that $tr(G(C_k)) \leq 4$. So we may assume that $n \geq 4$. By Theorem 2.8, if k is odd, then $r(G(C_k)) \leq m - \beta_1(G \setminus P)$. Therfore $tr(G(C_k)) \leq 2r(G(C_k)) \leq 2[m - \beta_1(G \setminus P)]$. Now we claim that if k is even, then $tr(G(C_k)) \leq 2[m - \beta_1(G \setminus P)]$.

Let $W_i = \{v_{i\frac{k}{2}}, v_{i\frac{k}{2}+1}\}, s+1 \leq i \leq m$ and $W = \cup_{i=s+1}^{m} W_i$. We claim that W is a resolving set of $G(C_k)$. Let x, y be two distinct vertices of $V(G(C_k)) \setminus W$.

If $x, y \in V(C_i)$ for some $s+1 \leq i \leq s+p$, then without loss of generality, let $x, y \in V(C_{s+1})$. Let $d(v_{(s+1)1}) = 2$ in $G(C_k)$. If $d(x, v_{(s+1)\frac{k}{2}}) \neq d(y, v_{(s+1)\frac{k}{2}})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{(s+1)\frac{k}{2}}) = d(y, v_{(s+1)\frac{k}{2}})$. Then x lies on $v_{(s+1)\frac{k}{2}}$-$v_{(s+1)1}$ path and y lies on $v_{(s+1)\frac{k}{2}}$-$v_{(s+1)k}$ path. Therefore $d(x, w) = d(y, w) + 1$ for all $w \in W \setminus \{v_{(s+1)\frac{k}{2}}\}$. It follows that $r(x|W) \neq r(y|W)$. If $x \in V(C_{s+1})$, $y \notin V(C_{s+1})$, then $d(x, v_{(s+1)\frac{k}{2}}) < d(y, v_{(s+1)\frac{k}{2}})$. It follows that $r(x|W) \neq r(y|W)$. If $x, y \notin V(C_i)$ for all $s+1 \leq i \leq s+p$, then the proof is similar to Case 1 and Case 2 of Theorem 3.4. Thus $tr(G(C_k)) \leq 2[m - \beta_1(G \setminus P)]$. □

Open problem 3.7. *Let G be a graph order of $n \geq 3$, size m and $\delta(G) = 1$. Then characterize G for which $tr(G(C_k)) = 2[m - \beta_1(G \setminus P)]$.*

Theorem 3.8. *For $n \geq 2$, $tr(K_{1,n-1}(C_k)) = 2(n-1)$.*

Proof. By Theorem 3.6, $tr(K_{1,n-1}(C_k)) \leq 2(n-1)$ and by Lemma 1.9,

$$tr(K_{1,n-1}(C_k)) \geq 2(n-1). \text{ Hence } tr(K_{1,n-1}(C_k)) = 2(n-1).$$

□

Theorem 3.9. *For $s, t \geq 1$, $tr(B_{s,t}(C_k)) = 2(s+t)$.*

Proof. By Lemma 1.9, $tr(B_{s,t}(C_k)) \geq 2(s+t)$ and by Theorem 3.6,

$$tr(B_{s,t}(C_k)) \leq 2(s+t).$$

□

Theorem 3.10. *Let G be a graph of order $n \geq 3$ and $\delta(G) = 1$. Then $tr(G(C_k)) = 2p$ if and only if G is either a star or bistar.*

Proof. Let $tr(G(C_k)) = 2p$ and W be a total resolving set of $G(C_k)$. Let $E(G) = \{e_1, e_2, ..., e_m\}$, $P = \{e_1, e_2, ..., e_p\}$ be the set of pendent edges of G and C_i be the edge cycle of e_i. Let $V(C_i)=\{v_{i1},v_{i2},\ldots,v_{ik}\}$ and $V(A_i)=\{v_{i2},v_{i3},\ldots,v_{i(k-1)}\}$. By proof of Lemma 1.9, $|W\cap(\cup_{i=1}^{p}V(C_i))|\geq 2p$. Therefore $W\cap V(A_i)=\emptyset$ for all $p+1\leq i\leq m$. Let $E_1=\{e_{p+1},e_{p+2},\ldots,e_m\}$. By Theorem 2.4, E_1 is independent, which is a contradiction to $G(C_k)\setminus\cup_{i=1}^{p}V(C_i)$ is connected and hence $|E_1|\leq 1$. If $|E_1|=0$, then $G\cong K_{1,n-1}$. If $|E_1|=1$, then $G\cong B_{s,t}$.

The proof of the converse part follows from Theorems 3.8 and 3.9. □

Theorem 3.11. *For $n\geq 3$,*

$$tr(C_n(C_k))=\begin{cases} n+1 & \text{if } n \text{ is odd and } k\geq 6 \text{ or } n=3\\ n & \text{otherwise.}\end{cases}$$

Proof. Let $V(C_n)=\{v_1,v_2,\ldots,v_n\}$ and

$$E(C_n)=\{v_iv_{i+1}\ /\ 1\leq i\leq n-1\}\cup\{v_nv_1\}.$$

Let $M=\{e_1=v_1v_2,e_2=v_3v_4,e_3=v_5v_6,\ldots,e_{\lfloor\frac{n}{2}\rfloor}=v_{n-2}v_{n-1}\}$ and $e_n=v_nv_1$, $e_{n-1}=v_{n-1}v_n$, $e_{\lfloor\frac{n}{2}\rfloor+1}=v_2v_3$, $e_{\lfloor\frac{n}{2}\rfloor+2}=v_4v_5,\ldots$, $e_{n-2}=v_{n-3}v_{n-2}$. Let W be a total resolving set of $C_n(C_k)$ and A_i be the edge cycle of e_i. Let $V(A_i)=\{v_{i1},v_{i2},\ldots,v_{ik}\}$, $v_{i1}v_{ik}=e_i$. If n is even, then the proof follows from Theorems 3.2 and 3.4. So we may assume that n is odd. If $n=3$, then we can easily verify that $tr(C_n(C_k))=4$. So we may assume that $n\geq 5$. We consider the following two cases.

Case 1: $k=4$ or 5.

Let $v_n=v_{(n-1)k}=v_{n1}$, $v_{11}=v_1$, $v_{n1}=v_n$, $v_{(n-1)1}=v_{n-1}$ and $v_{\lfloor\frac{n}{2}\rfloor 1}=v_{n-2}$. Let $W_i=\{v_{i2},v_{i3}/\lfloor\frac{n}{2}\rfloor+1\leq i\leq n-2\}$, $W'=\cup_{\lfloor\frac{n}{2}\rfloor+1}^{n-2}W_i$ and $W''=\{v_{(n-1)(k-1)},v_{n1},v_{n2}\}$ and $W=W'\cup W''$. We claim that W is a resolving set of $C_n(C_k)$. Let x, y be two distinct vertices of $V(C_n(C_k))\setminus W$. Let $B=V\big(A_1\cup A_{\lfloor\frac{n}{2}\rfloor}\cup A_{n-1}\cup A_n\big)$. If either $x,y\in V(C_n(C_k))\setminus B$ or $x\in V(C_n(C_k))\setminus B$ and $y\in B$, then $r(x|W')\neq r(y|W')$. It follows that $r(x|W)\neq r(y|W)$. So we may assume that $x,y\in B$. If $d(x,w)\neq d(y,w)$ for some $w\in W'$, then $r(x|W)\neq r(y|W)$. So we may assume that $d(x,w)=d(y,w)$ for all $w\in W'$. Therefore either $x=v_{n(k-1)}$ and $y=v_{12}$ or $x=v_{(n-1)2}$ and $y=v_{\lfloor\frac{n}{2}\rfloor(k-1)}$. Without loss of generality, let $x=v_{n(k-1)}$ and $y=v_{12}$ or $x=v_{(n-1)2}$. Then $d(y,v_{n2})=3$ and $d(x,v_{n2})=\begin{cases}1 & \text{if } k=4\\ 2 & \text{if } k=5.\end{cases}$ It follows that $r(x|W)\neq r(y|W)$. Thus W is a resolving set of $C_n(C_k)$ and hence $tr(C_n(C_k))\leq n$. By Theorem 3.2, $tr(C_n(C_k))\geq n$ and hence $tr(C_n(C_k))=n$.

Case 2: $k\geq 6$.

By Theorem 3.2, $tr(C_n(C_k))\geq n$. But we claim that $tr(C_n(C_k))\geq n+1$. Suppose $tr(C_n(C_k))=n$. Since W is a total resolving set, $\langle W\rangle$ contain at most $\lfloor\frac{n}{2}\rfloor$ components. Then $W\cap\Big(\{v_{i2},v_{i3},\ldots,v_{i\lceil\frac{k}{2}\rceil}\}\cup\{v_{j2},v_{j3},\ldots,v_{j\lceil\frac{k}{2}\rceil}\}\Big)=\emptyset$ since some A_i and A_j meet the common vertex v. Let $v_{i1}=v_{j1}=v$ and $v_{i1}v_{ik}=e_i,v_{j1}v_{jk}=e_j$. Then $r(v_{i2}|W)=r(v_{j2}|W)$, which is a contradiction. Hence $tr(C_n(C_k))\geq n+1$. By Theorem 3.4, $tr(C_n(C_k))\leq 2[n-\lfloor\frac{n}{2}\rfloor]=n+1$ and hence $tr(C_n(C_k))=n+1$. □

Theorem 3.12. *For $n\geq 3$,*

$$tr(P_n(C_k))=\begin{cases} n+1 & \text{if } n \text{ is odd and } k\geq 6 \text{ or } n=3\\ n & \text{otherwise.}\end{cases}$$

Proof. Let $V(P_n)=\{v_1,v_2,\ldots,v_n\}$ and

$$E(P_n)=\{v_iv_{i+1}\ /\ 1\leq i\leq n-1\}.$$

Let $M=\{e_1=v_2v_3,e_2=v_4v_5,\ldots,e_{\lfloor\frac{n-2}{2}\rfloor}=v_{n-3}v_{n-2}\}$, $e_{n-2}=v_{n-2}v_{n-1},e_{n-1}=v_{n-1}v_n$. Let W be a total resolving set of $P_n(C_k)$ and C_i be the edge cycle of e_i. Let $V(C_i)=\{v_{i1},v_{i2},\ldots,v_{ik}\}$, $v_{i1}v_{ik}=e_i$. If n is even, then the proof follows from Theorems 3.3 and 3.6. So we may assume that n is odd. If $n=3$, then we can easily verify that $tr(P_n(C_k))=4$. So we may assume that $n\geq 5$. We consider the following two cases.

Case 1: $k=4$ or 5.

Let $v_{(n-2)1}=v_{n-2}$, $v_{(n-1)1}=v_{n-1}$ and $v_{\lfloor\frac{n-2}{2}\rfloor 1}=v_{n-3}$. Let $W_i=\{v_{i2},v_{i3}\ /\ \lfloor\frac{n-2}{2}\rfloor+1\leq i\leq n-3\}$, $W'=\cup_{\lfloor\frac{n-2}{2}\rfloor+1}^{n-3}W_i$ and $W''=\{v_{(n-2)(k-1)},v_{(n-1)1},v_{(n-1)2}\}$ and $W=W'\cup W''$. We claim that W is a resolving set of $P_n(C_k)$. Let x, y be two distinct vertices of $V(P_n(C_k))\setminus W$. Let $B=V\big(C_{\lfloor\frac{n-2}{2}\rfloor}\cup C_{n-2}\cup C_{n-1}\big)$. If either $x,y\in V(P_n(C_k))\setminus B$ or $x\in V(P_n(C_k))\setminus B$ and $y\in B$, then $r(x|W')\neq r(y|W')$. It follows that $r(x|W)\neq r(y|W)$. So we may assume that $x,y\in B$. If $d(x,w)\neq d(y,w)$ for some $w\in W'$, then $r(x|W)\neq r(y|W)$. So we may assume that $d(x,w)=d(y,w)$ for all $w\in W'$. Therefore $x=v_{\lfloor\frac{n-2}{2}\rfloor(k-1)}$ and $y=v_{(n-2)2}$. Then $d(x,v_{(n-2)(k-1)})=3$ and $d(y,v_{(n-2)(k-1)})=\begin{cases}1 & \text{if } k=4\\ 2 & \text{if } k=5.\end{cases}$ It follows that $r(x|W)\neq r(y|W)$. Thus W is a resolving set of $P_n(C_k)$ and hence $tr(P_n(C_k))\leq n$. By Theorem 3.3, $tr(P_n(C_k))\geq n$ and hence $tr(P_n(C_k))=n$.

Case 2: $k\geq 6$.

By Theorem 3.3, $tr(P_n(C_k))\geq n$. But we claim that $tr(P_n(C_k))\geq n+1$. Suppose $tr(P_n(C_k))=n$. Since W is a total resolving set, $\langle W\rangle$ contain at most $\lfloor\frac{n}{2}\rfloor$ components. Then $W\cap\Big(\{v_{i2},v_{i3},\ldots,v_{i\lceil\frac{k}{2}\rceil}\}\cup\{v_{j2},v_{j3},\ldots,v_{j\lceil\frac{k}{2}\rceil}\}\Big)=\emptyset$ since some C_i and C_j meet the common vertex v. Let $v_{i1}=v_{j1}=v$ and $v_{i1}v_{ik}=e_i,v_{j1}v_{jk}=e_j$. Then $r(v_{i2}|W)=r(v_{j2}|W)$, which is a contradiction. Hence $tr(P_n(C_k))\geq n+1$. By Theorem 3.6, $tr(P_n(C_k))\leq n+1$ and hence $tr(P_n(C_k))=n+1$. □

Theorem 3.13. *For $n\geq 3$ and $k\geq 6$,*

$$tr(K_n(C_k))=\begin{cases} n^2-2n & \text{if } n \text{ is even}\\ n^2-2n+1 & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Let $V(K_n)=\{v_1,v_2,\ldots,v_n\}$ and $E(K_n)=\{e_1,e_2,\ldots,e_m\}$. Let $C_1,C_2,C_3,\ldots,C_m$ be the edge cycles of $e_1,e_2,e_3,\ldots,e_m$ respectively. Let $V(C_i)=\{v_{i1},v_{i2},\ldots,v_{ik}\}$, $1\leq i\leq m$. Let W be a total resolving set of $K_n(C_k)$. First we claim that

$tr(K_n(C_k)) \geq 2[m - \lfloor \frac{n}{2} \rfloor]$. Suppose that $tr(K_n(C_k)) \leq 2[m - \lfloor \frac{n}{2} \rfloor] - 1$. Since $K_n(C_k)$ contain $\frac{n(n-1)}{2}$ edge cycles, $\langle W \rangle$ contain union of at most $[\frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor] - 1$ components. Then $W \cap \left(\{v_{i2}, v_{i3}, \ldots, v_{i\lceil \frac{k}{2} \rceil}\} \cup \{v_{j2}, v_{j3}, \ldots, v_{j\lceil \frac{k}{2} \rceil}\} \right) = \emptyset$ since C_i and C_j meet the common vertex and hence we have $r(v_{i2}|W) = r(v_{j2}|W)$, which is a contradiction. Thus $tr(K_n(C_k)) \geq 2[m - \lfloor \frac{n}{2} \rfloor]$. By Theorem 3.4, $tr(K_n(C_k)) \leq 2[m - \lfloor \frac{n}{2} \rfloor]$ and hence $tr(K_n(C_k)) = \begin{cases} n^2 - 2n & \text{if } n \text{ is even} \\ n^2 - 2n + 1 & \text{if } n \text{ is odd.} \end{cases}$ □

4. General bounds and extremal graphs

In this section, we obtain bounds for the total resolving number of $G(C_k)$ and characterize the extremal graphs.

Theorem 4.1. *Let G be a graph of order* $n \geq 3$ and $k \geq 6$. Then $4 \leq tr(G(C_k)) \leq n^2 - 2n + 1$.

Proof. Let W be a total resolving set of $G(C_k)$.

Claim 1: $tr(G(C_k)) \geq 4$.

If G is a tree, then G has at least two pendent edges. Therefore by Lemma 1.9, $tr(G(C_k)) \geq 4$. If G contains a cycle, then we claim that $tr(G(C_k)) \geq 4$. Suppose $tr(G(C_k)) \leq 3$. Then $\langle W \rangle$ is connected. Then $W \cap \left(\{v_{i2}, v_{i3}, \ldots, v_{i\lceil \frac{k}{2} \rceil}\} \cup \{v_{j2}, v_{j3}, \ldots, v_{j\lceil \frac{k}{2} \rceil}\} \right) = \emptyset$ since some C_i and C_j meet the common vertex v. Let $v_{i1} = v_{j1} = v$ and $v_{i1}v_{ik} = e_i, v_{j1}v_{jk} = e_j$. Then $r(v_{i2}|W) = r(v_{j2}|W)$, which is a contradiction. Hence $tr(G(C_k)) \geq 4$.

Claim 2: $tr(G(C_k)) \leq n^2 - 2n + 1$.

The upper bound depends on the number of edges and by Theorem 3.4, $tr(G(C_k)) \leq 2[\frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor]$. Thus

$$tr(G(C_k)) \leq \begin{cases} n^2 - 2n & \text{if } n \text{ is even} \\ n^2 - 2n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

□

Theorem 4.2. *Let G be a graph of order* $n \geq 3$ and $k \geq 6$. Then $tr(G(C_k)) = 4$ *if and only if* $G \cong P_3$, C_3, P_4 or C_4.

Proof. Let $tr(G(C_k)) = 4$ and W be a total resolving set of $G(C_k)$. If G is a tree, then G has at least two pendent edges. By Lemma 1.9, G has exactly two pendent edges. Therefore $G \cong P_n$. By Theorem 3.12, $G \cong P_3$ or P_4.

If G contains a cycle, then we claim that $n = 3$ or 4. Suppose $n \geq 5$. Then $m \geq 5$ and $m \geq n$. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$ and $M = \{e_1, e_2, \ldots, e_s\}$ be the maximum edge independent set of G. Let C_i be the edge cycle of e_i, $1 \leq i \leq m$. Since $tr((C_k)) = 4$, $\langle W \rangle$ is either connected or $2K_2$. Then all the vertices of W belong to union of two edge cycles of $G(C_k)$. Without loss of generality, let $W \subset (V(C_m) \cup V(C_{m-1}))$. Then $W \cap V(C_i) = \emptyset$ for all $1 \leq i \leq m - 2$. Since $m - 2 > \lfloor \frac{n}{2} \rfloor$ and $s \leq \lfloor \frac{n}{2} \rfloor$, $m - 2 > s$. Then we have $W \cap (V(C_i) \cup V(C_j)) = \emptyset$ since some C_i and C_j meet the common vertex v. Let $v_{i1} = v_{j1} = v$ and $v_{i1}v_{ik} = e_i$, $v_{j1}v_{jk} = e_j$. Then $r(v_{i2}|W) = r(v_{j2}|W)$, which is a contradiction. Hence $n = 3$ or 4. If $n = 3$, then $G \cong C_3$. If $n = 4$, then $G \cong C_4$ or $K_1 + (K_2 \cup K_1)$ or $K_4 - e$ or K_4. If G is $K_1 + (K_2 \cup K_1)$ or $K_4 - e$ or K_4, then we can easily verify that $tr(G(C_k)) > 4$, which is a contradiction and hence $G \cong C_4$.

Conversely, let $G \cong P_3$, C_3, P_4 or C_4. Then by Theorems 3.11 and 3.12, $tr(G(C_k)) = 4$. □

Theorem 4.3. *Let G be a graph of order* $n \geq 3$ and $k \geq 6$. Then $tr(G(C_k)) = n^2 - 2n + 1$ *if and only if n is odd and* $G \cong K_n$.

Proof. Let $tr(G(C_k)) = n^2 - 2n + 1$. If n is even, by Theorem 3.6, $tr(G(C_k)) \leq n^2 - 2n$, which is a contradiction. Therefore n is odd. Now we claim that $G \cong K_n$. It is enough to prove that $m = \frac{n(n-1)}{2}$. Suppose $m < \frac{n(n-1)}{2}$. If $\delta(G) \geq 2$, then by Theorem 3.4, $tr(G(C_k)) \leq 2[m - \beta_1(G)]$. Since $m < \frac{n(n-1)}{2}$ and $\beta_1(G) \leq \lfloor \frac{n}{2} \rfloor$, $tr(G(C_k)) < 2[\frac{n(n-1)}{2} - \lfloor \frac{n}{2} \rfloor] = n^2 - 2n - 1$, which is a contradiction. If $\delta(G) = 1$, then we can similarly prove that $tr(G(C_k)) < n^2 - 2n - 1$. Hence $G \cong K_n$. Conversely, let $G \cong K_n$, n is odd. Then by Theorem 3.13, $tr(G(C_k)) = n^2 - 2n + 1$. □

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