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Construction of L-equienergetic graphs using some graph operations

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ABSTRACT

For a graph G with n vertices and m edges, the eigenvalues of its adjacency matrix $A(G)$ are known as eigenvalues of G . The sum of absolute values of eigenvalues of G is called the energy of G . The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix with $(i, i)^{th}$ entry is the degree of vertex v_i . The collection of eigenvalues of $L(G)$ with their multiplicities is called spectra of $L(G)$. If $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of $L(G)$ then the Laplacian energy $LE(G)$ of G is defined as $LE(G) = \sum_{i=1}^n |\mu_i - \frac{2m}{n}|$. It is always interesting and challenging as well to investigate the graphs which are L -equienergetic but L -noncopectral as L -cospectral graphs are obviously L -equienergetic. We have devised a method to construct L -equienergetic graphs which are L -noncospectral.

KEYWORDS

Eigenvalue; graph energy; spectrum; equienergetic

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05C50; 05C76

1. Introduction

All the graphs considered here are simple, finite, connected and with n vertices and m edges denoted as $G(n, m)$. We denote the complement of graph G by \bar{G} , the complete graph on p vertices by K_p , the null graph by \bar{K}_p . The average vertex degree denoted by \bar{d} , defined as $\bar{d} = \frac{2m}{n}$. For any undefined term in graph theory we rely upon Balakrishnan and Ranganathan [2] while for terminology related to matrix theory we refer to Horn and Johnson [11].

The adjacency matrix $A(G)$ of a graph G with vertices v_1, v_2, \dots, v_n is an $n \times n$ matrix $[a_{ij}]$ such that,

$$a_{ij} = 1, \text{ if } v_i \text{ is adjacent with } v_j \\ = 0, \text{ otherwise}$$

The spectra of adjacency matrix of graph G is called spectra of G . If $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of graph G then the energy of graph G is $E(G) = \sum_{i=1}^n |\lambda_i|$. The concept of graph energy was introduced by Gutman [9] in 1978. A brief account of graph energy can be found in Cvetković [6] and Li [12].

Let $D(G)$ be the diagonal matrix of whose $(i, i)^{th}$ entry is the degree of a vertex v_i . The matrix $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are called the Laplacian and Signless Laplacian matrices of G and their spectra are called Laplacian spectra (L -spectra) and signless Laplacian spectra (Q -spectra) of G . Let $0 = \mu_n \leq \mu_{n-1} \leq \dots \leq \mu_1$ be L -spectra of G . Fiedler [7] have prove that $\mu_n = 0$ with multiplicity equal to the number of connected components of G . It is easy to see that

$$tr(L(G)) = \sum_{i=1}^n \mu_i = 2m \quad tr(L^+(G)) = \sum_{i=1}^n \mu_i^+ = 2m$$

with tr is the trace of the matrix.

All Laplacian eigenvalues are nonnegative, and therefore their sum is non-zero. On the other hand,

$$\sum_{i=1}^n \left(\mu_i - \frac{2m}{n} \right) = 0$$

Gutman and Zhou [10] have pointed out that the equality $LE(G) = E(G)$ holds, if G is regular.

The multiplicity of μ_i is denoted by $m(\mu_i)$. The collection of all eigenvalues μ_i together their multiplicity is known as Laplacian spectra of G denoted by $spec_L(G)$. Hence,

$$spec_L(G) = \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n \\ m(\mu_1) & m(\mu_2) & \dots & m(\mu_n) \end{pmatrix}$$

The Laplacian energy of a graph G is defined by

$$LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|$$

Basic properties and other results on Laplacian energy can be found in Andriantiana [1]. Two graphs G_1 and G_2 of same order are said to be L -equienergetic if $LE(G_1) = LE(G_2)$. Two graphs are said to be L -cospectral if they have same Laplacian eigenvalues. Since L -cospectral graphs are always L -equienergetic, the problem of constructing L -equienergetic graphs is challenging for L -noncospectral graphs.

The join of G_1 and G_2 is the graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2 . The L -spectra of join of graphs is given by the following result.

Proposition 1.1. [5] If $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ are two graphs having L -spectra $\mu_1, \mu_2, \dots, \mu_{n_1-1}, \mu_{n_1} = 0$ and $\sigma_1, \sigma_2, \dots, \sigma_{n_2-1}, \sigma_{n_2} = 0$ respectively then,

$$\begin{aligned} & \text{spec}_L(G_1 \vee G_2) \\ &= \begin{pmatrix} n_1 + n_2 & n_1 + \sigma_1 & \cdots & n_1 + \sigma_{n_2-1} & n_2 + \mu_1 & \cdots & n_2 + \mu_{n_1-1} & 0 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \end{aligned}$$

The Kronecker product of G_1 and G_2 is the graph $G = G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if u_1 is adjacent to u_2 and v_1 is adjacent to v_2 in G_1 and G_2 respectively. The following result gives the L -spectra of the Kronecker product of graphs of $G \otimes K_2$.

Proposition 1.2. [3] Let $G(n, m)$ be a graph having L -spectra and Q -spectra respectively as $\mu_1, \mu_2, \dots, \mu_n$ and $\mu_1^+, \mu_2^+, \dots, \mu_n^+$ then,

$$\text{spec}_L(G \otimes K_2) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n & \mu_1^+ & \mu_2^+ & \cdots & \mu_n^+ \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

The m -shadow graph $D_m(G)$ of a connected graph G is graph obtained by taking m copies of G , say G_1, G_2, \dots, G_m , then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j , $1 \leq i, j \leq m$. For $m=2$, the graph is known as shadow (double) graph.

Proposition 1.3. [13] Let G be a graph with n vertices having degrees d_1, d_2, \dots, d_n and let $\mu_1, \mu_2, \dots, \mu_n$ be its Laplacian spectra. Then the Laplacian spectra of $D_m(G)$ is $m\mu_i, md_i$ for $1 \leq i \leq n$.

Proposition 1.4. [8] Let $D_2(G)$ be the shadow graph of the graph $G(n, m)$. Then, for $p \geq 2n + k$ and $m \leq \frac{k^2 + 2nk}{8}$, $k \geq 4$ we have

$$LE(D_2(G) \vee \overline{K_p}) = 4n + (p - 2n) \frac{2m'}{n'} + 8m$$

with $\frac{2m'}{n'} = \frac{4m + 2np}{n + 2p}$

The extended double cover [4] of the graph $G(n, m)$ with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ is a bipartite graph G^* with bipartition (X, Y) , $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ where two vertices x_i and y_j are adjacent if and only if $i=j$ or v_i adjacent v_j in G . It is easy to see that G^* is connected if and only if G is connected and a vertex v_i is of degree d_i in G if and only if it is of degree $d_i + 1$ in G^* . Following are some results associated with G^*

Proposition 1.5. [8] Let $G(n, m)$ be a graph with L -spectra and Q -spectra as $\mu_1, \mu_2, \dots, \mu_n$ and $\mu_1^+, \mu_2^+, \dots, \mu_n^+$ respectively, then

$$\text{spec}_L(G^*) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n & \mu_1^+ + 2 & \mu_2^+ + 2 & \cdots & \mu_n^+ + 2 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Proposition 1.6. [8] Let $G(n, m)$ be the graph then for $p \geq 2n + k$ and $m \leq \frac{(k-1)n}{2} + \frac{k^2}{4}$, $k \geq 3$, we have

$$LE(G^* \vee \overline{K_p}) = 6n + (p - 2n) \frac{2m'}{n'} + 4m$$

with $\frac{2m'}{n'} = \frac{4m + 2np + 2n}{p + 2n}$

Proposition 1.7. [11] Let

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric block matrix. Then the spectra of A is the union of $A_0 + A_1$ and $A_0 - A_1$.

2. Laplacian energy of extended shadow graph

Definition 2.1. The extended shadow graph $D_2^*(G)$ of a connected graph G is constructed by taking two copies of G say G' and G'' . Join each vertex u' in G' to the neighbours of the corresponding vertex u'' and with u'' in G'' .

Theorem 2.2. Let G be a graph with Laplacian eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ and degrees d_1, d_2, \dots, d_n then the Laplacian spectra of $D_2^*(G)$ is

$$\begin{aligned} & \text{spec}_L(D_2^*(G)) \\ &= \begin{pmatrix} 2\mu_1 & 2\mu_2 & \cdots & 2\mu_n & 2(d_1 + 1) & 2(d_2 + 1) & \cdots & 2(d_n + 1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \end{aligned}$$

Proof. Let v_1, v_2, \dots, v_n be the vertices of the graph G and $A(G), D(G)$ be the adjacency matrix and degree matrix of the graph G respectively. Then,

$$L(G) = D(G) - A(G)$$

Consider second a copy graph G with vertices $u_1, u_2, u_3, \dots, u_n$ to obtain $D_2^*(G)$, such that, $N(u_i) = N(v_i) \cup \{u_i\}$, $i = 1, 2, \dots, n$. Let $G_1 = D_2^*(G)$.

The adjacency matrix and degree matrix of G_1 are respectively given as

$$\begin{aligned} A(G_1) &= \begin{bmatrix} A(G) & A(G) + I \\ A(G) + I & A(G) \end{bmatrix} \\ D(G_1) &= \begin{bmatrix} 2D(G) + I & 0 \\ 0 & 2D(G) + I \end{bmatrix} \end{aligned}$$

Then,

$$\begin{aligned} L(G_1) &= D(G_1) - A(G_1) \\ &= \begin{bmatrix} 2D(G) - A(G) + I & -A(G) - I \\ -A(G) - I & 2D(G) - A(G) + I \end{bmatrix} \\ &= \begin{bmatrix} L(G) + D(G) + I & L(G) - D(G) - I \\ L(G) - D(G) - I & L(G) + D(G) + I \end{bmatrix} \end{aligned}$$

Hence, by Proposition 1.7, spectra of $L(G_1)$ is union of spectra of $2L(G)$ and $2(D(G) + I)$.

Hence,

$$\begin{aligned} & \text{spec}_L(D_2^*(G)) \\ &= \begin{pmatrix} 2\mu_1 & 2\mu_2 & \cdots & 2\mu_n & 2(d_1 + 1) & 2(d_2 + 1) & \cdots & 2(d_n + 1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \end{aligned}$$

Lemma 2.3. Let $G(n, m)$ be a graph then for $p \geq (2n + k)$ and $m \leq \frac{k^2+2nk}{4}$, we have

$$LE((G \otimes K_2) \vee \overline{K_p}) = 4n + (p - 2n) \frac{2m'}{n'} + 4m$$

Proof. Let $G(n, m)$ be an n -vertex graph having L -spectra and Q -spectra, as $\mu_1, \mu_2, \dots, \mu_n$ and $\mu_1^+, \mu_2^+, \dots, \mu_n^+$ respectively, then by Proposition 1.2

$$spec_L(G \otimes K_2) = \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n & \mu_1^+ & \mu_2^+ & \dots & \mu_n^+ \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

and so by Proposition 1.1

$$spec_L((G \otimes K_2) \vee \overline{K_p}) = \begin{pmatrix} p+2n & p+\mu_1 & \dots & p+\mu_{n-1} & p+\mu_1^+ & \dots & p+\mu_n^+ & 2n & 0 \\ 1 & 1 & \dots & 1 & 1 & \dots & 1 & p-1 & 1 \end{pmatrix}$$

Average vertex degree of $(G \otimes K_2) \vee \overline{K_p}$ is

$$\frac{2m'}{n'} = \frac{4m + 4np}{p + 2n}$$

Therefore,

$$LE((G \otimes K_2) \vee \overline{K_p}) = \left| p + 2n - \frac{2m'}{n'} \right| + \sum_{i=1}^{n-1} \left| p + \mu_i - \frac{2m'}{n'} \right| + \sum_{i=1}^n \left| p + \mu_i^+ - \frac{2m'}{n'} \right| + (p - 1) \left| 2n - \frac{2m'}{n'} \right| + \left| \frac{2m'}{n'} \right|$$

Now if, $p \geq 2n + k$ and $m \leq \frac{k^2+2nk}{4}$, we have for $i = 1, 2, \dots, n$

$$\begin{aligned} p + \mu_i - \frac{2m'}{n'} &= p + \mu_i - \frac{4m + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &\geq \frac{k(2n + k) - k(2n + k)}{p + 2n} = 0 \end{aligned}$$

Similarly,

$$p + \mu_i^+ - \frac{2m'}{n'} > 0$$

Therefore,

$$LE((G \otimes K_2) \vee \overline{K_p}) = \left| p + 2n - \frac{2m'}{n'} \right| + \sum_{i=1}^{n-1} \left| p + \mu_i - \frac{2m'}{n'} \right| + \sum_{i=1}^n \left| p + \mu_i^+ - \frac{2m'}{n'} \right|$$

$$\begin{aligned} &+ (p - 1) \left| 2n - \frac{2m'}{n'} \right| + \left| \frac{2m'}{n'} \right| \\ &= \left(p + 2n - \frac{2m'}{n'} \right) + \left(\sum_{i=1}^{n-1} \mu_i + 0 \right) \\ &+ \sum_{i=1}^n \mu_i^+ + (n - 1) \left(p - \frac{2m'}{n'} \right) \\ &+ n \left(p - \frac{2m'}{n'} \right) + (p - 1) \left(\frac{2m'}{n'} - 2n \right) \\ &+ \frac{2m'}{n'} \\ &= 4n + (p - 2n) \frac{2m'}{n'} + 4m \end{aligned}$$

Remark 2.4. We have considered the only case when $p \geq 2n + k$ and $m \leq \frac{k^2+2nk}{4}$. We discard the remaining possibilities for p and m due to following reasons.

Case(I) If, $p < 2n + k$ and $m \leq \frac{k^2+2nk}{4}$,

$$\begin{aligned} p + \mu_i - \frac{2m'}{n'} &= p + \mu_i - \frac{4m + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &\geq \frac{p(p - 2n) - 4m}{p + 2n} \\ &\geq \frac{p(p - 2n) - k^2 - 2nk}{p + 2n} \\ &= \frac{(p - k)(p + k) - 2n(p + k)}{p + 2n} \\ &= \frac{(p + k)(p - k - 2n)}{p + 2n} \\ &\geq \frac{(p - k - 2n)}{p + 2n} \end{aligned}$$

As, $p < 2n + k$ and $p + 2n > 0$, we have

$$\frac{(p - k - 2n)}{p + 2n} < 0$$

Hence, in this case we are not able to determine the sign of $p + \mu_i - \frac{2m'}{n'}$. In this situation the term on L.H.S. might be either positive or negative.

Case(II) If, $p \geq 2n + k$ and $m > \frac{k^2+2nk}{4}$,

$$\begin{aligned} p + \mu_i - \frac{2m'}{n'} &= p + \mu_i - \frac{4m + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &\geq \frac{p(p - 2n) - 4m}{p + 2n} \\ &\geq \frac{k(2n + k) - 4m}{p + 2n} \\ &= \frac{k^2 + 2nk - 4m}{p + 2n} \end{aligned}$$

Here, $m > \frac{k^2+2nk}{4}$ and $p + 2n > 0$, we have

$$\frac{k^2 + 2nk - 4m}{p + 2n} < 0$$

Again, in this case we are not able to determine the sign of $p + \mu_i - \frac{2m'}{n'}$.

Case(III) If, $p < 2n + k$ and $m > \frac{k^2+2nk}{4}$,

$$\begin{aligned} p + \mu_i - \frac{2m'}{n'} &= p + \mu_i - \frac{4m + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &< \frac{k(2n + k) + (p + 2n)\mu_i - 4m}{p + 2n} \\ &< \frac{k(2n + k) + (p + 2n)\mu_i - k^2 - 2nk}{p + 2n} \\ &= \frac{(p + 2n)\mu_i}{p + 2n} \\ &= \mu_i \end{aligned}$$

In this case also we are not able to decide the sign of $p + \mu_i - \frac{2m'}{n'}$ as $\mu_i \geq 0$.

Thus, in all the cases discussed above, it is not possible to determine the sign of the term $p + \mu_i - \frac{2m'}{n'}$ definitely.

3. Construction of L-equinergetic graphs

Theorem 3.1. Let $G_1(n, m)$ and $G_2(n, m)$ be two graphs having L-spectra as $\mu_1, \mu_2, \dots, \mu_n$ and $\gamma_1, \gamma_2, \dots, \gamma_n$ respectively, then for $p \geq 2n + k$ and $m \leq \frac{k^2+2n(k-1)}{8}$, $k \geq 4$ we have

$$LE(D_2^*(G_1) \vee \overline{K_p}) = LE(D_2^*(G_2) \vee \overline{K_p})$$

Proof. Let $D_2^*(G_1)$ be the extended shadow graph of G_1 . Then by Theorem 2.2,

$$\begin{aligned} \text{spec}_L(D_2^*(G_1)) \\ = \begin{pmatrix} 2\mu_1 & 2\mu_2 & \dots & 2\mu_n & 2(d_1 + 1) & 2(d_2 + 1) & \dots & 2(d_n + 1) \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \end{pmatrix} \end{aligned}$$

and so by Proposition 1.1,

$$\text{spec}_L(D_2^*(G_1) \vee \overline{K_p}) = \begin{pmatrix} p + 2n & p + 2\mu_1 & \dots & p + 2\mu_{n-1} & p + 2(d_1 + 1) & \dots & p + 2(d_n + 1) & 2n & 0 \\ 1 & 1 & \dots & 1 & 1 & \dots & 1 & p - 1 & 1 \end{pmatrix}$$

Average vertex degree of $D_2^*(G_1) \vee \overline{K_p}$ is

$$\frac{2m'_1}{n'} = \frac{8m + 2n + 4np}{p + 2n}$$

Therefore,

$$\begin{aligned} LE(D_2^*(G_1) \vee \overline{K_p}) &= \left| p + 2n - \frac{2m'}{n'} \right| \\ &+ \sum_{i=1}^{n-1} \left| p + 2\mu_i - \frac{2m'}{n'} \right| \\ &+ \sum_{i=1}^n \left| p + 2(d_i + 1) - \frac{2m'}{n'} \right| \\ &+ (p - 1) \left| 2n - \frac{2m'}{n'} \right| + \left| \frac{2m'}{n'} \right| \end{aligned}$$

Now if, $p \geq 2n + k$ and $m \leq \frac{k^2+2n(k-1)}{8}$, $k \geq 4$, we have for $i = 1, 2, \dots, n$

$$\begin{aligned} p + 2\mu_i - \frac{2m'}{n'} &= p + 2\mu_i - \frac{8m + 2n + 4np}{p + 2n} \\ &= \frac{p(p - 2n) + 2\mu_i(p + 2n) - 8m - 2n}{p + 2n} \\ &\geq \frac{k(2n + k) - k(2n + k) + 2n - 2n}{p + 2n} = 0 \end{aligned}$$

Similarly we see that,

$$p + 2(d_i + 1) - \frac{2m'}{n'} \geq 2 > 0$$

Therefore,

$$\begin{aligned} LE(D_2^*(G) \vee \overline{K_p}) &= \left| p + 2n - \frac{2m'}{n'} \right| + \sum_{i=1}^{n-1} \left| p + 2\mu_i - \frac{2m'}{n'} \right| \\ &+ \sum_{i=1}^n \left| p + 2(d_i + 1) - \frac{2m'}{n'} \right| \\ &+ (p - 1) \left| 2n - \frac{2m'}{n'} \right| + \left| \frac{2m'}{n'} \right| \\ &= \left(p + 2n - \frac{2m'}{n'} \right) + 2 \left(\sum_{i=1}^{n-1} \mu_i + 0 \right) \\ &+ (n - 1) \left(p - \frac{2m'}{n'} \right) \\ &+ 2 \sum_{i=1}^n d_i + n \left(p + 2 - \frac{2m'}{n'} \right) \\ &+ (p - 1) \left(\frac{2m'}{n'} - 2n \right) + \frac{2m'}{n'} \\ &= 6n + (p - 2n) \frac{2m'}{n'} + 8m \end{aligned}$$

Remark 3.2. We can prove the remaining cases by similar arguments as discussed in Remark 2.4.

Corollary 3.3. Let $G_1(n, m_1)$, $G_2(n, m_2)$, $G_3(n, m_3)$ and $G_4(n, m_4)$ be four graphs of order $n \equiv 0 \pmod{4}$ with $m_2 =$

$m_1 + \frac{n}{4}$, $m_3 = 2m_1$ and $m_4 = 2m_1 + \frac{n}{2}$. Then for $p \geq 2n + k$ and $m_1 \leq \frac{k^2 + 2n(k-1)}{8}$ we have

$$\begin{aligned} LE(D_2^*(G_1) \vee \overline{K_p}) &= LE(D_2(G_2) \vee \overline{K_p}) = LE(G_3^* \vee \overline{K_p}) \\ &= LE((G \otimes K_2) \vee \overline{K_p}) \end{aligned}$$

Proof. Let $D_2^*(G_1)$, $D_2(G_2)$ and G_3^* be the extended shadow graph of $G_1(n, m_1)$, shadow graph of $G_2(n, m_2)$ and extended double cover of $G_3(n, m_3)$, respectively. Average degrees of $D_2^*(G_1)$, $D_2(G_2)$, G_3^* and $(G \otimes K_2) \vee \overline{K_p}$ are respectively as,

$$\begin{aligned} \frac{2m'_1}{n'} &= \frac{8m_1 + 4np + 2n}{p + 2n}, & \frac{2m'_2}{n'} &= \frac{8m_2 + 4np}{p + 2n}, \\ \frac{2m'_3}{n'} &= \frac{4m_3 + 4np + 2n}{p + 2n}, & \frac{2m'_4}{n'} &= \frac{4m_4 + 4np}{p + 2n} \end{aligned}$$

Now, for $p \geq 2n + k$ and $m_1 \leq \frac{k^2 + 2n(k-1)}{8}$, we have by [Theorem 3.1](#)

$$LE(D_2^*(G) \vee \overline{K_p}) = 6n + (p - 2n) \frac{2m'_1}{n'} + 8m_1 \tag{1}$$

For $p \geq 2n + k$ and $m_2 \leq \frac{k^2 + 2nk}{8}$ we have by [Proposition 1.4](#)

$$LE(D_2(G) \vee \overline{K_p}) = 4n + (p - 2n) \frac{2m'_2}{n'} + 8m_2$$

If $m_2 = m_1 + \frac{n}{4}$ then

$$spec_L(D_2(D_2^*(G_1))) = \begin{pmatrix} 4\mu_1 & \cdots & 4\mu_n & 4(d_1 + 1) & \cdots & 4(d_n + 1) & 2(2d_1 + 1) & \cdots & 2(2d_n + 1) \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 \end{pmatrix}$$

$$LE(D_2(G) \vee \overline{K_p}) = 6n + (p - 2n) \frac{2m'_1}{n'} + 8m_1 \tag{2}$$

For $p \geq 2n + k$ and $m_3 \leq \frac{n(k-1)}{2} + \frac{k^2}{8}$, we have by [Proposition 1.6](#)

$$LE(G_3^* \vee \overline{K_p}) = 6n + (p - 2n) \frac{2m'_3}{n'} + 4m_3$$

and if we suppose that $m_3 = 2m_1$ then

$$LE(G_3^* \vee \overline{K_p}) = 6n + (p - 2n) \frac{2m'_1}{n'} + 8m_1 \tag{3}$$

Also for, For $p \geq 2n + k$ and $m_4 \leq \frac{n(k-1)}{2} + \frac{k^2}{8}$, we have by [Lemma 2.3](#)

$$LE((G \otimes K_2) \vee \overline{K_p}) = 4n + (p - 2n) \frac{2m'_4}{n'} + 4m_4$$

and if we suppose that $m_4 = 2m_1 + \frac{n}{2}$ then

$$LE((G \otimes K_2) \vee \overline{K_p}) = 6n + (p - 2n) \frac{2m'_1}{n'} + 8m_1 \tag{4}$$

Therefore, from (1)-(4) it is clear that

$$\begin{aligned} LE(D_2^*(G_1) \vee \overline{K_p}) &= LE(D_2(G_2) \vee \overline{K_p}) = LE(G_3^* \vee \overline{K_p}) \\ &= LE((G \otimes K_2) \vee \overline{K_p}) \end{aligned}$$

Theorem 3.4. Let $G_1(n, m_1)$ and $G_2(n, m_2)$ be two graphs having L -spectra respectively as $\mu_1, \mu_2, \dots, \mu_n$ and $\gamma_1, \gamma_2, \dots, \gamma_n$. Then with $n \equiv 0 \pmod{8}$ and $m_2 = m_1 + \frac{n}{4}$ for $p \geq 4n + k$ and $m_2 \leq \frac{k^2 + 4nk - 4n}{32}$ we have

$$LE(D_2(D_2^*(G_1)) \vee \overline{K_p}) = LE(D_2^*(D_2(G_2)) \vee \overline{K_p})$$

Proof. Let $D_2^*(G)$ and $D_2(G)$ be the shadow and extended shadow graphs of G , respectively. $D_2(D_2^*(G_1)) \vee \overline{K_p}$ and $D_2^*(D_2(G_2)) \vee \overline{K_p}$ are graphs with $p + 4n$ vertices and average degrees respectively as

$$\frac{2m'_1}{n'} = \frac{32m + 8n + 8np}{p + 4n}, \quad \frac{2m'_2}{n'} = \frac{32m + 4n + 8np}{p + 4n}.$$

By [Theorem 2.2](#),

$$\begin{aligned} spec_L(D_2^*(G_1)) &= \begin{pmatrix} 2\mu_1 & 2\mu_2 & \cdots & 2\mu_n & 2(d_1 + 1) & 2(d_2 + 1) & \cdots & 2(d_n + 1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \end{aligned}$$

by [Lemma 1.3](#),

and so by [Proposition 1.1](#), L -spectra of $D_2(D_2^*(G)) \vee \overline{K_p}$ is $p + 4n$, $p + 4\mu_i$ ($1 \leq i \leq n - 1$), $p + 4(d_i + 1)$, $p + 2(2d_i + 1)$ (2 times) ($1 \leq i \leq n$), $4n$ ($(p - 1)$ times), 0

So if $p \geq 4n + k$ and $m_1 \leq \frac{k(4n+k) - 8n}{32}$, $k \leq 4$, we have for $i = 1, 2, \dots, n$

$$\begin{aligned} p + 4\mu_i - \frac{2m'_1}{n'} &= p + 4\mu_i - \frac{32m + 8n + 8np}{p + 4n} \\ &= \frac{p(p - 4n) + 4(p + 4n)\mu_i - 32m - 8n}{p + 4n} \\ &\geq \frac{k(4n + k) - k(4n + k) + 8n - 8n}{p + 4n} = 0 \end{aligned}$$

Similarly we can show

$$p + 4(d_i + 1) - \frac{2m'_2}{n'} \geq 0, \quad p + 2(2d_i + 1) \geq 0$$

Therefore,

$$\begin{aligned}
LE(D_2(D_2^*(G)) \vee \overline{K_p}) &= \left| p + 4n - \frac{2m'_1}{n'} \right| \\
&+ \sum_{i=1}^{n-1} \left| p + 4\mu_i - \frac{2m'_1}{n'} \right| \\
&+ \sum_{i=1}^n \left| p + 4(d_i + 1) - \frac{2m'_1}{n'} \right| \\
&+ 2 \sum_{i=1}^n \left| p + 2(2d_i + 1) - \frac{2m'_1}{n'} \right| \\
&+ (p-1) \left| 4n - \frac{2m'_1}{n'} \right| + \left| \frac{2m'_1}{n'} \right| \\
&= 8n + (p-4n) \frac{2m'_1}{n'} + 16m_1
\end{aligned}$$

Similarly,

L -spectra of $D_2^*(D_2(G)) \vee \overline{K_p}$ is $p + 4n$, $p + 4\gamma_i$ ($1 \leq i \leq n-1$), $p + 4d'_i$, $p + 2(2d'_i + 1)$ (2 times) ($1 \leq i \leq n$), $4n$ ($(p-1)$ times), 0 and

$$LE(D_2(D_2^*(G)) \vee \overline{K_p}) = 4n + (p-4n) \frac{2m'_2}{n'} + 16m_2$$

Using $m_2 = m_1 + \frac{n}{4}$

$$LE(D_2^*(D_2(G_1)) \vee \overline{K_p}) = LE(D_2(D_2^*(G_2)) \vee \overline{K_p})$$

4. Concluding remarks

In most of the existing results only a pair of graphs are shown to be L -equienergetic while we have investigated four

graphs which are simultaneously L -equienergetic. Moreover, we have used the concept of extended shadow graph to construct L -equienergetic graphs from the given graphs.

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