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## Some results on one type of graph family with some special number sequences

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### ABSTRACT

In this study, we introduce a new graph family. Then, we calculate eigenvalues of the adjacency and the Laplacian matrix of this graph family. Moreover, we show that the perfect matching number of this graph family equals to special second order recurrence by hafnian method. For some special kinds of this family, we obtain that the perfect matching number of corresponding graphs equals to some famous number sequences such as Fibonacci, Pell and Jacobsthal numbers. Also, we find energies and obtain upper bounds for Laplacian energies of these graphs.

### KEYWORDS

Hafnian; eigenvalue; energy; Fibonacci number; Pell number

### 1. Introduction

In here, all graphs are considered as undirected, connected and without loops.

The adjacency matrix [12] of the graph  $G = (V, E)$  is an  $n \times n$  matrix  $A = (a_{ij})$ , where  $n$  is the number of vertices in  $G$  and  $a_{ij}$  = number of edges between  $v_i$  and  $v_j$ . Let the eigenvalues of  $A$  be  $\lambda_1, \lambda_2, \dots, \lambda_n$ . In [6], the energy of  $G$  is defined as

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

The degree of  $v_i$ ,  $\deg(v_i)$ , is the number of edges incident to  $v_i$ . Then, the degree matrix  $D$  is defined as  $D = \text{diag}(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$  in [5].

In [11], the Laplacian matrix of  $G$  is  $L = D - A$  and the Laplacian energy of  $G$  is defined as

$$LE(G) = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|,$$

where  $\mu_i$ 's are the eigenvalues of  $L$  and  $n$  and  $m$  are the number of vertices and edges, respectively [6].

Recently, many scientists study on relations of graph theory, chemistry and number theory. For example, Zhao and Li investigated the orderings of two classes of trees by their Fibonacci numbers. Using these orderings, they determined the unique tree with the second, and respectively the third smallest Fibonacci number among all trees with  $n$  vertices, where Fibonacci number is defined as  $F_n = F_{n-1} + F_{n-2}$  with initial conditions  $F_0 = 0$  and  $F_1 = 1$  [14]. In [3], the authors proved that the number of 1-factorizations of a generalized Petersen graph of the type  $GP(3k, k)$  is equal to the  $k$ -th Jacobsthal number  $J_k$  when  $k$  is odd, and equal to  $4J_k$  when  $k$  is even, where Jacobsthal number is defined as  $J_n = J_{n-1} + 2J_{n-2}$  with initial conditions  $J_0 = 0$  and  $J_1 = 1$ . In [8],

the authors showed that the perfect matching number of one type of graph family equals to the  $(k + 1)$ -th Pell number  $P_{k+1}$ , where Pell number is defined as  $P_n = 2P_{n-1} + P_{n-2}$  with initial conditions  $P_0 = 0$  and  $P_1 = 1$ . In [1], the authors considered different types of fullerene graphs. Then, they gave an overview of some graph invariants that can possibly correlate with the fullerene molecule stability, such as: the bipartite edge frustration, the independence number, the saturation number, the number of perfect matching, etc.

In this study, we define one type of graph family. Then, we explore some special properties of this graphs such as energy, Laplacian energy and perfect matching. Moreover, we show that the perfect matching numbers for some special cases of the family are equal to some special numbers.

Now, we introduce the following graph family: where  $a, b, c$  are the number of edges between  $v_i$  and  $v_j$  ( $1 \leq i, j \leq 2k$ ) (Figure 1).

So, we can easily write that the adjacency matrix of this graph family has the following form:

$$A = \begin{pmatrix} E_k & B_k \\ B_k & E_k \end{pmatrix},$$

where

$$E_k = \begin{pmatrix} 0 & b & & \\ b & 0 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} a & c & & \\ c & a & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}. \quad (1)$$

Let us consider zero diagonal and symmetric matrix  $T = [t_{ij}]$  with the order  $2k$  ( $k \geq 1$ ). The hafnian of  $T$  is

$$\text{haf}(T) = \sum_{\left\{ \begin{matrix} 1 & 2 & \dots & 2k-1 & 2k \\ i_1 & j_1 & \dots & i_k & j_k \end{matrix} \right\}} t_{i_1 j_1} t_{i_2 j_2} \dots t_{i_k j_k},$$

where  $1 \leq i_s \leq j_s \leq k$  and  $1 \leq s \leq k$  [2]. Further, from Laplace expansion along with last column for hafnian, we can write

$$\text{haf}(T) = \sum_{r=1}^{2k-1} t_{r,2k} \text{haf}(T_r),$$

where  $T_r$  matrix is obtained by deleting  $r$ -th and  $2k$ -th rows and columns of  $T$  [2].

**Theorem 1.** [10] Let  $X$  and  $Y$  are  $k \times k$  real matrices and  $S$  is block matrix as below:

$$S = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}.$$

Then, the eigenvalues of  $X + Y$  and  $X - Y$  matrices are the eigenvalues of  $S$ .

**Lemma 2.** [8] Let us consider  $Q_n$  block matrix as below:

$$Q_n = \begin{pmatrix} I_k & I_k \\ I_k & -I_k \end{pmatrix},$$

where  $I_k$  is identity matrix. So,

$$Q_n^{-1} = \frac{1}{2} \begin{pmatrix} I_k & I_k \\ I_k & -I_k \end{pmatrix}.$$

**Lemma 3.** Let  $H$  be block matrix as in the following:

$$H = \left( \begin{array}{c|c} F_k & 0 \\ \hline 0 & S_k \end{array} \right),$$

where

$$F_k = \begin{pmatrix} a & b+c & & \\ b+c & a & \ddots & \\ & \ddots & \ddots & \\ & & & \ddots \end{pmatrix}, S_k = \begin{pmatrix} -a & b-c & & \\ b-c & -a & \ddots & \\ & \ddots & \ddots & \\ & & & \ddots \end{pmatrix}.$$

Then,  $H$  and  $A$  matrices are similar matrices.

*Proof.* From matrix multiplication, we have  $H = Q_n A Q_n^{-1}$ , where  $Q_n$  and  $Q_n^{-1}$  are in Lemma 2. So which is desired.  $\square$

Therefore, we have following result.

**Corollary 4.** Let  $A$  be the adjacency matrix of the graph  $G$  as in Figure 1 of order  $2k$ . For  $k \geq 1$ , i)  $\det A = x_k y_k$  where  $n = 2k$  and  $x_k = ax_{k-1} - (b+c)^2 x_{k-2}$  and  $y_k = -ay_{k-1} - (b-c)^2 y_{k-2}$  with initial conditions  $x_0 = 1, x_1 = a, y_0 = 1$  and  $y_1 = -a$ . ii) The eigenvalues of  $A$  are (for  $t = 1, 2, \dots, k$ )

$$\eta_t = a - 2(b+c) \cos\left(\frac{t\pi}{k+1}\right)$$

and

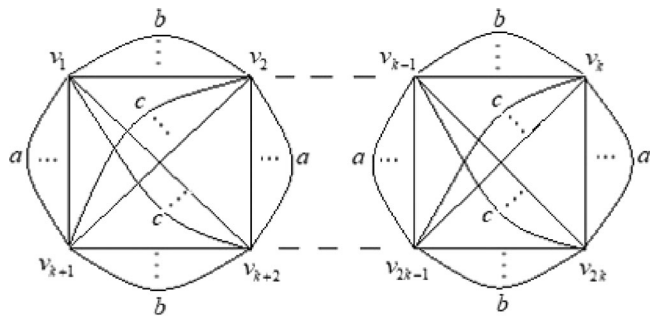


Figure 1. One type of graph family.

$$\gamma_t = -a - 2(b-c) \cos\left(\frac{t\pi}{k+1}\right).$$

*Proof.* i) From [7], we have  $\det F_k = x_k$  and  $\det S_k = y_k$ , where  $x_k = ax_{k-1} - (b+c)^2 x_{k-2}$  and  $v_k = -av_{k-1} - (b-c)^2 v_{k-2}$  with initial conditions  $x_0 = 1, x_1 = a, y_0 = 1$  and  $y_1 = -a$ . So,  $\det H = \det A = x_k y_k$ . ii) From Theorem 1, the set of eigenvalue of  $A$  are union of the set of eigenvalues of  $F_k = E_k + B_k$  and  $S_k = E_k - B_k$  matrices. From [9], the eigenvalues of  $A$  are (for  $t = 1, 2, \dots, k$ )

$$\eta_t = a - 2(b+c) \cos\left(\frac{t\pi}{k+1}\right)$$

and

$$\gamma_t = -a - 2(b-c) \cos\left(\frac{t\pi}{k+1}\right). \quad \square$$

## 2. Main results

In this section, we calculate eigenvalues of the Laplacian matrix of the graph  $G$  in Figure 1. Also, we show that the perfect matching number of the graph equals to special second order recurrence by hafnian method. Finally, for some special kinds of the graph  $G$  in Figure 1, we obtain that the perfect matching number of corresponding graphs equals to some famous number sequences such as Fibonacci, Pell and Jacobsthal numbers. Moreover, we find energies and obtain upper bounds for Laplacian energies of these special kinds graphs.

**Lemma 5.** The eigenvalues of Laplacian matrix  $L$  of the graph  $G$  in Figure 1 are

$$\mu_t = 2(b+c) \left(1 + \cos\frac{t\pi}{k}\right)$$

and

$$\mu_t^* = 2(a+b+c) + 2(c-b) \cos \theta_t,$$

for  $t = 1, 2, \dots, k$ , where  $\theta_t$ 's are the roots of

$$\begin{aligned} &(c-b)^2 \sin(n+1)\theta + 2(c^2-b^2) \sin n\theta \\ &+ (b+c)^2 \sin(n-1)\theta \\ &= 0, \end{aligned}$$

for  $\sin \theta \neq 0$ .



$$\mathcal{R}_{2t} = \begin{pmatrix} & & & & 0 & & & & & & 0 \\ & & & & 0 & & & & & & 0 \\ & & E_{t-1} & & \vdots & & B_{t-1} & & \vdots & & \\ & & & & 0 & & & & & & 0 \\ & & & & b & & & & & & 0 \\ 0 & 0 & \dots & 0 & b & 0 & 0 & 0 & \dots & 0 & c & b \\ & & & & 0 & & & & & & 0 \\ & & & & 0 & & & & & & 0 \\ & & B_{t-1} & & \vdots & & E_{t-1} & & \vdots & & \\ & & & & 0 & & & & & & 0 \\ & & & & c & & & & & & 0 \\ 0 & 0 & \dots & 0 & c & b & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \quad (4)$$

and

$$\mathcal{J}_{2t} = \begin{pmatrix} \begin{array}{cc|cc} & & & \\ & E_{t-1} & & B_{t-1} \\ & & & \\ & & & \end{array} & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ c \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \\ 0 \\ c \end{array} \\ \hline \begin{array}{cc|cc} & & & \\ & B_{t-1} & & E_{t-1} \\ & & & \\ & & & \end{array} & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ c \end{array} \end{pmatrix} \quad (5)$$

So, we have  $\mathbf{haf}(\mathcal{Y}_{2t}) = \mathbf{haf}(\mathcal{R}_{2t})$  and  $\mathbf{haf}(\mathcal{Z}_{2t}) = \mathbf{haf}(\mathcal{J}_{2t})$ . Using expansion method along with last columns of (4) and (5), we obtain

$$\mathbf{haf}(\mathcal{R}_{2t}) = b\mathbf{haf}(A_{2t-2}) = bz_t,$$

and

$$\mathbf{haf}(\mathcal{J}_{2t}) = c\mathbf{haf}(A_{2t-2}) = cz_t.$$

As a consequence, we have

$$\mathbf{haf}(A_{2t+2}) = az_{t+1} + (b^2 + c^2)z_t = z_{t+2}.$$

Furthermore, the number of perfect matching of this type graph family with  $2k$  vertices is equal to  $z_{k+1}$ . □

### 2.1. Special cases

**Case 1:** For  $a = b = 1$  and  $c = 0$ , the graph family in Figure 1 turns into ladder graph [4]. That is, this graph becomes Figure 2.

It is clear that  $\mathbf{haf}(A_{2k}) = F_{k+1}$ , that is, the number of perfect matching of ladder graph equals to  $F_{k+1}$ , where  $F_{k+1}$  is  $(k + 1)$  th Fibonacci number. For this  $a, b$  and  $c$  values, the eigenvalues of  $A_{2k}$  and  $L$  are

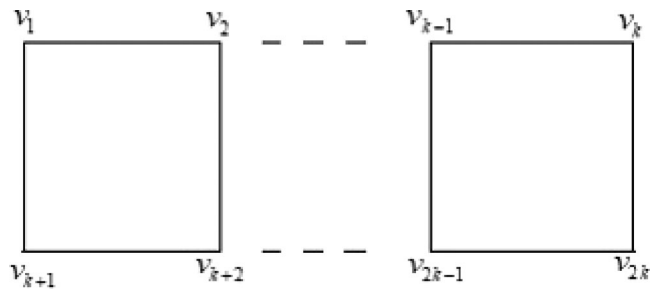


Figure 2. Ladder graph.

$$\eta_t = 1 - 2 \cos\left(\frac{t\pi}{k+1}\right)$$

$$\gamma_t = -1 - 2 \cos\left(\frac{t\pi}{k+1}\right)$$

and

$$\mu_t = 2 + 2 \cos\left(\frac{t\pi}{k}\right)$$

$$\mu_t^* = 4 + 2 \cos\left(\frac{t\pi}{k}\right),$$

for  $t = 1, 2, \dots, k$ , respectively. For this graph,

$$E(G) = \sum_{t=1}^k |\eta_t| + \sum_{t=1}^k |\gamma_t|$$

$$= \sum_{t=1}^k \left| 1 - 2 \cos\left(\frac{t\pi}{k+1}\right) \right| + \sum_{t=1}^k \left| -1 - 2 \cos\left(\frac{t\pi}{k+1}\right) \right|.$$

For  $k \equiv 0, 1 \pmod{3}$  and  $1 \leq x \leq \lfloor \frac{k}{3} \rfloor$ , we have  $1 - 2 \cos\left(\frac{x\pi}{k+1}\right) \leq 0$ . So, for  $\lfloor \frac{k}{3} \rfloor < t \leq k$ , we can see that  $1 - 2 \cos\left(\frac{t\pi}{k+1}\right) > 0$ . Also, for  $z = 1, 2, \dots, k - \lfloor \frac{k}{3} \rfloor$ , we have  $\frac{z\pi}{k+1} \leq \frac{2\pi}{3}$  and  $-1 - 2 \cos\left(\frac{z\pi}{k+1}\right) \leq 0$ . Then, for  $k - \lfloor \frac{k}{3} \rfloor < t \leq k$ , we can write that  $-1 - 2 \cos\left(\frac{t\pi}{k+1}\right) > 0$ . Consequently,

$$E(G) = \sum_{t=1}^{\lfloor \frac{k}{3} \rfloor} \left[ -1 + 2 \cos\left(\frac{t\pi}{k+1}\right) \right] + \sum_{t=\lfloor \frac{k}{3} \rfloor + 1}^k \left[ 1 - 2 \cos\left(\frac{t\pi}{k+1}\right) \right]$$

$$+ \sum_{t=1}^{\lfloor \frac{k}{3} \rfloor} \left[ 1 + 2 \cos\left(\frac{t\pi}{k+1}\right) \right] + \sum_{t=k - \lfloor \frac{k}{3} \rfloor + 1}^k \left[ -1 - 2 \cos\left(\frac{t\pi}{k+1}\right) \right]$$

$$= 2k - 4 - 4 \left\lfloor \frac{k}{3} \right\rfloor + 4 \left[ \sin\left(\frac{\lfloor \frac{k}{3} \rfloor \pi}{k+1}\right) \cot \frac{\pi}{2k+2} + \cos\left(\frac{\lfloor \frac{k}{3} \rfloor \pi}{k+1}\right) \right]$$

$$- 4 \left[ \cos\left(\frac{\left(\left\lfloor \frac{k}{3} \right\rfloor + 1\right) \pi}{k+1}\right) + \cos\left(\frac{\left(\left\lfloor \frac{k}{3} \right\rfloor + 2\right) \pi}{k+1}\right) + \dots + \cos\left(\frac{\left(k - \left\lfloor \frac{k}{3} \right\rfloor\right) \pi}{k+1}\right) \right]$$

$$= 2k - 4 - 4 \left\lfloor \frac{k}{3} \right\rfloor + 4 \left[ \sin\left(\frac{\lfloor \frac{k}{3} \rfloor \pi}{k+1}\right) \cot \frac{\pi}{2k+2} + \cos\left(\frac{\lfloor \frac{k}{3} \rfloor \pi}{k+1}\right) \right].$$

For  $k \equiv 2 \pmod{3}$  and  $1 \leq x \leq \frac{k+1}{3}$ , we have  $1 - 2 \cos\left(\frac{x\pi}{k+1}\right) \leq 0$ . So, for  $\frac{k+1}{3} < t \leq k$ , we can see that  $1 - 2 \cos\left(\frac{t\pi}{k+1}\right) > 0$ . Also, for  $z = 1, 2, \dots, k + 1 - \frac{k+1}{3}$ , we have

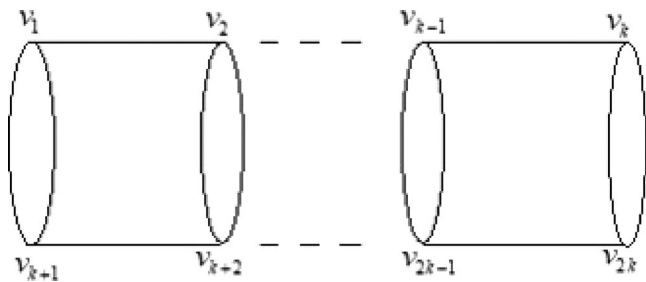


Figure 3. Cylinder type graph.

$\frac{z\pi}{k+1} \leq \frac{2\pi}{3}$  and  $-1 - 2 \cos\left(\frac{z\pi}{k+1}\right) \leq 0$ . Then, we can write that  $-1 - 2 \cos\left(\frac{t\pi}{k+1}\right) > 0$ , for  $\frac{2k+2}{3} < t \leq k$ . Consequently,

$$\begin{aligned} E(G) &= \sum_{i=1}^{k+1} \left[ -1 + 2 \cos\left(\frac{t\pi}{k+1}\right) \right] + \sum_{t=\frac{k+4}{3}}^k \left[ 1 - 2 \cos\left(\frac{t\pi}{k+1}\right) \right] \\ &+ \sum_{i=1}^3 \left[ 1 + 2 \cos\left(\frac{t\pi}{k+1}\right) \right] + \sum_{t=\frac{2k+5}{3}}^k \left[ -1 - 2 \cos\left(\frac{t\pi}{k+1}\right) \right] \\ &= \frac{2k-10}{3} + 2\sqrt{3} \cot \frac{\pi}{2k+2}. \end{aligned}$$

It is clear that the number of edges of this graph equals to  $3k - 2$  for  $n = 2k$ . So, we have

$$LE(G) = \sum_{i=1}^k \left| \mu_i - \frac{3k-2}{k} \right| + \sum_{i=1}^k \left| \mu_i^* - \frac{3k-2}{k} \right|.$$

It is easy to see  $LE(G) = 2$ , for  $k = 1$ . If  $k \geq 2$ , we obtain

$$\begin{aligned} LE(G) &= \sum_{i=1}^k \left| \mu_i - \frac{3k-2}{k} \right| + \sum_{i=1}^k \left| \mu_i^* - \frac{3k-2}{k} \right| \\ &= \sum_{i=1}^k \left| 2 + 2 \cos\left(\frac{i\pi}{k}\right) - \frac{3k-2}{k} \right| + \sum_{i=1}^k \left| 4 + 2 \cos\left(\frac{i\pi}{k}\right) - \frac{3k-2}{k} \right| \\ &= \sum_{i=1}^k \left| -1 + \frac{2}{k} + 2 \cos\left(\frac{i\pi}{k}\right) \right| + \sum_{i=1}^k \left| 1 + \frac{2}{k} + 2 \cos\left(\frac{i\pi}{k}\right) \right| \\ &\leq \sum_{i=1}^k \left| -1 + \frac{2}{k} \right| + 2 \sum_{i=1}^k \left| \cos\left(\frac{i\pi}{k}\right) \right| \\ &+ \sum_{i=1}^k \left( 1 + \frac{2}{k} \right) + 2 \sum_{i=1}^k \left| \cos\left(\frac{i\pi}{k}\right) \right|. \end{aligned}$$

Consequently, we get following inequality:

$$\begin{aligned} LE(G) &\leq 2k + 4 \sum_{i=1}^k \left| \cos\left(\frac{i\pi}{k}\right) \right| \\ &= 2k + 4 + 8 \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \cos\left(\frac{i\pi}{k}\right) \\ &= 2k + 4 \left( \sin \frac{\lfloor \frac{k}{2} \rfloor \pi}{k} \cot \frac{\pi}{2k} + \cos \frac{\lfloor \frac{k}{2} \rfloor \pi}{k} \right). \end{aligned}$$

**Case 2:** For  $a = 2, b = 1$  and  $c = 0$ , the graph family in Figure 1 turns into cylinder type graph [8]. That is, this graph is Figure 3.

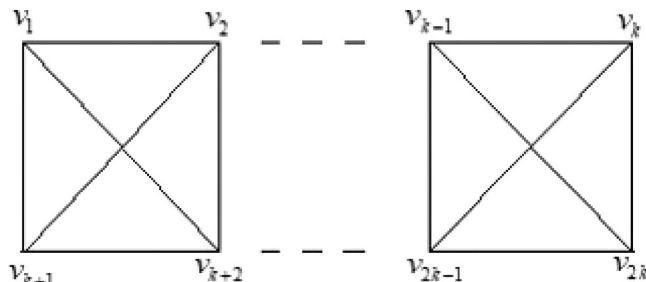


Figure 4. Special subgraph of graph family in Figure 1.

It is clear that  $\mathbf{haf}(A_{2k}) = P_{k+1}$  and the number of perfect matching of the cylinder type graph equals to  $P_{k+1}$ , where  $P_{k+1}$  is  $(k + 1)$  th Pell number [8]. For this  $a, b$  and  $c$  values, the authors obtained eigenvalues of  $A_{2k}$  and  $L$  and energy and upper bound for Laplacian energy of this graph [8].

**Case 3:** For  $a = b = c = 1$ , the graph family in Figure 1 turns into as below (Figure 4):

It is clear that  $\mathbf{haf}(A_{2k}) = J_{k+1}$  and the number of perfect matching of this graph equals to  $J_{k+1}$ , where  $J_{k+1}$  is  $(k + 1)$  th Jacobsthal number. For this  $a, b$  and  $c$  values, the eigenvalues of  $A_{2k}$  and  $L$  are

$$\begin{aligned} \eta_t &= 1 - 4 \cos\left(\frac{t\pi}{k+1}\right), \\ \gamma_t &= -1 - 4 \cos\left(\frac{t\pi}{k+1}\right) \end{aligned}$$

and

$$\begin{aligned} \mu_t &= 4 + 4 \cos\left(\frac{t\pi}{k}\right) \\ \mu_t^* &= 4 \text{ and } \mu_{k-1}^* = \mu_k^* = 6 \quad (1 \leq t \leq k-2), \end{aligned}$$

for  $t = 1, 2, \dots, k$ , respectively. For this graph,

$$\begin{aligned} E(G) &= \sum_{i=1}^k |\eta_i| + \sum_{i=1}^k |\gamma_i| \\ &= \sum_{i=1}^k \left| 1 - 4 \cos\left(\frac{t\pi}{k+1}\right) \right| + \sum_{i=1}^k \left| -1 - 4 \cos\left(\frac{t\pi}{k+1}\right) \right|. \end{aligned}$$

For  $x = 1, 2, \dots, t < k$  and  $\cos\left(\frac{x\pi}{k+1}\right) \geq \frac{1}{4}$ , we can write  $1 - 4 \cos\left(\frac{x\pi}{k+1}\right) \leq 0$  and  $1 - 4 \cos\left(\frac{z\pi}{k+1}\right) \geq 0$ , where  $t < z \leq k$ .

Also, for  $y = 1, 2, \dots, q < k$  and  $\cos\left(\frac{y\pi}{k+1}\right) \geq -\frac{1}{4}$ , we have  $-1 - 4 \cos\left(\frac{y\pi}{k+1}\right) \leq 0$  and  $-1 - 4 \cos\left(\frac{p\pi}{k+1}\right) \geq 0$ , where  $q < p \leq k$ . Then, we obtain

$$\begin{aligned} E(G) &= \sum_{i=1}^t \left[ -1 + 4 \cos\left(\frac{i\pi}{k+1}\right) \right] + \sum_{i=t+1}^k \left[ 1 - 4 \cos\left(\frac{i\pi}{k+1}\right) \right] \\ &+ \sum_{i=1}^q \left[ 1 + 4 \cos\left(\frac{i\pi}{k+1}\right) \right] + \sum_{i=q+1}^k \left[ -1 - 4 \cos\left(\frac{i\pi}{k+1}\right) \right] \\ &= 2 \left\{ q - t - 4 + 2 \left( \cot\left(\frac{\pi}{2k+2}\right) \left[ \sin\left(\frac{x\pi}{k+1}\right) + \sin\left(\frac{(k-y)\pi}{k+1}\right) \right] \right. \right. \\ &\left. \left. + \cos\left(\frac{x\pi}{k+1}\right) + \cos\left(\frac{(k-y)\pi}{k+1}\right) \right) \right\}. \end{aligned}$$

For example, we obtain  $q=3$  and  $t=2$ , for  $k=5$ . Therefore,

$$\begin{aligned} E(G) &= 2 \left[ 3 - 2 - 4 + 4 \left( \cot \frac{\pi}{12} \sin \frac{\pi}{3} + \cos \frac{\pi}{3} \right) \right] \\ &= 10 + 4\sqrt{3}. \end{aligned}$$

It is clear that the number of edges of this graph equals to  $5k-4$  for  $n=2k$ . So,

$$\begin{aligned} LE(G) &= \sum_{i=1}^k \left| \mu_i - \frac{5k-4}{k} \right| + \sum_{i=1}^k \left| \mu_i^* - \frac{5k-4}{k} \right| \\ &= \sum_{i=1}^k \left| 4 + 4 \cos \left( \frac{i\pi}{k} \right) - \frac{5k-4}{k} \right| + \sum_{i=1}^k \left| \mu_i^* - \frac{5k-4}{k} \right| \\ &= 2 + \frac{8}{k} + \sum_{i=1}^k \left| -1 + \frac{4}{k} + 4 \cos \left( \frac{i\pi}{k} \right) \right| + \sum_{i=1}^{k/2} \left| -1 + \frac{4}{k} \right| \\ &\leq 2 + \frac{8}{k} + \sum_{i=1}^k \left| -1 + \frac{4}{k} \right| + \sum_{i=1}^{k/2} \left| -1 + \frac{4}{k} \right| + 4 \sum_{i=1}^k \left| \cos \left( \frac{i\pi}{k} \right) \right| \end{aligned}$$

For  $k \leq 3$ , we can easily calculate  $LE(G)$ . If  $k \geq 4$ , we have

$$\begin{aligned} LE(G) &\leq 2 + \frac{8}{k} + \sum_{i=1}^k \left| -1 + \frac{4}{k} \right| + \sum_{i=1}^{k/2} \left| -1 + \frac{4}{k} \right| + 4 \sum_{i=1}^k \left| \cos \left( \frac{i\pi}{k} \right) \right| \\ &= 8 + 2 \sum_{i=1}^{k/2} \left( 1 - \frac{4}{k} \right) + 8 \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \cos \left( \frac{i\pi}{k} \right) \\ &= 2k - 8 + \frac{16}{k} + 4 \left( \sin \frac{\lfloor \frac{k}{2} \rfloor \pi}{k} \cot \frac{\pi}{2k} + \cos \frac{\lfloor \frac{k}{2} \rfloor \pi}{k} \right). \end{aligned}$$

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