AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

A closed $(2,3)$-knight's tour on some cylinder chessboards

Sirirat Singhun, Parichat Ruengmanee & Adthasit Sinna

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 1, 296–302
https://doi.org/10.1016/j.akcej.2019.06.001

Taylor & Francis
Taylor & Francis Group

AKCE International Journal of Graphs and Combinatorics

A closed (2, 3)-knight's tour on some cylinder chessboards

Sirirat Singhun*, Parichat Ruengmanee, Adthasit Sinna

Department of Mathematics, Faculty of Science, Ramkhamhaeng University, Bangkok 10240, Thailand

Received 3 August 2018; received in revised form 1 June 2019; accepted 3 June 2019

Abstract

A $(2, 3)$-knight's move on the $m \times n$ cylinder chessboard is the move of the knight 2 squares vertically or 2 squares horizontally and then 3 squares perpendicular to it. In this paper, we show a closed $(2, 3)$-knight's tour on the $5k \times n$ cylinder chessboard for all positive integers k and n, and a closed $(2, 3)$-knight's tour on the $9k \times n$ cylinder chessboard for all positive integers k and $n \in \{4, 5, 7, 8, 9, 10, 11, 12, 13\}$. Moreover, we show that there is no closed $(2, 3)$-knight's tours on the $m \times n$ cylinder chessboard where (i) $m \in \{1, 2, 3, 4, 6, 7, 8\}$ and (ii) $m = 9$ and $n \in \{1, 2, 3, 6\}$.

Keywords: Closed (a, b)-knight's tour; Cylinder chessboard; Hamiltonian cycle

1. Introduction and preliminaries

The $m \times n$ chessboard is an array with m rows and n columns. A legal knight's move is the moves from one square vertically or one square horizontally and then two squares perpendicular to it. A question of the $m \times n$ chessboard is "which chessboard that the knight can move to all squares of the chessboard exactly once and return to the starting square?" The knight's moves that move to all squares of the chessboard exactly once and return to the starting square is called a *closed knight's tour*. The answer of this question was obtained by Schwenk [1] in 1991 as follows.

Theorem 1 ([1]). *The $m \times n$ chessboard with $m \leq n$ admits a closed knight's tour unless one or more of the following conditions hold :*
(i) m and n are both odd;
(ii) $m = 1, 2,$ or 4; or
(iii) $m = 3$ and $n = 4, 6$ or 8.

In 2005, Chia et al. [2] generalized the legal knight's move to an (a, b)-knight's move. That is, the knight can move a squares vertically or a squares horizontally and then b squares perpendicular to it. Then, the legal knight's move is a $(1, 2)$-knight's move. The problem is similar to the legal knight's move. The (a, b)-knight's moves that move to all squares of the chessboard exactly once and return to the starting square is called a *closed (a, b)-knight's tour*. The authors in [2] obtained the following result for the $(2, 3)$-knight's move.

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: sin_sirirat@ru.ac.th (S. Singhun), ru.parichat@gmail.com (P. Ruengmanee), adthasit@ru.ac.th (A. Sinna).

https://doi.org/10.1016/j.akcej.2019.06.001

Please cite this article as: S. Singhun, P. Ruengmanee and A. Sinna, A closed (2, 3)-knight's tour on some cylinder chessboards, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.06.001.

Theorem 2 *([2]). The* $5k \times n$ *chessboard where* $(5k, n) \neq (5, 18)$ *admits a closed* $(2, 3)$*-knight's tour if and only if*

(i) $k = 1$ *and* $n \geq 16$ *is even; or*
(ii) $k = 2$ *and* $n \geq 10$ *and* $n \neq 12$*; or*
(iii) $k \geq 3$ *is odd and* $n \geq 10$ *is even and* $n \neq 12$*; or*
(iv) $k \geq 4$ *is even and* $n = 5, 9, 10, 11$ *or* $n \geq 13$.

The knight's tour problem was discussed in the $m \times n$ cylinder chessboard. We can think of creating the $m \times n$ cylinder chessboard by starting with an $m \times n$ chessboard and then joining the left end with the right end. In [3], Watkins obtained the following result.

Theorem 3 *([3]). An* $m \times n$ *cylinder chessboard has a closed knight's tour unless one of the following two conditions holds: (a)* $m = 1$ *and* $n > 1$*; or (b)* $m = 2$ *or* 4 *and* n *is even.*

In this paper, we consider a $(2, 3)$-knight's move on the cylinder chessboard. A closed $(2, 3)$-knight's tour on the $5k \times n$ cylinder chessboard for all positive integers k and n, and a closed $(2, 3)$-knight's tour on the $9k \times n$ cylinder chessboard for all positive integers k and $n \in \{4, 5, 7, 8, 9, 10, 11, 12, 13\}$ are shown in Theorems 6 and 8, respectively. Moreover, we show that the $m \times n$ cylinder chessboard, where $m \in \{1, 2, 3, 4, 6, 7, 8\}$ contains no closed $(2, 3)$-knight's tours (in Theorem 5) and the $9 \times n$ cylinder chessboard, where $n \in \{1, 2, 3, 6\}$, also contains no closed $(2, 3)$-knight's tours (in Lemma 2).

2. Main results

For an $m \times n$ cylinder chessboard, we start with an $m \times n$ chessboard. When we consider the $m \times n$ cylinder chessboard with an $m \times n$ chessboard, let each square be (i, j) where (i, j) is counting in the matrix fashion. If the knight stands at (i, j), then the knight can move to at most eight squares : $(i \pm 2, j \pm 3)$ and $(i \pm 3, j \pm 2)$ where $j \pm 3$ and $j \pm 2$ are in modulo n.

The knight's tour problem can be considered by using a graph. Let G be a graph with $V(G) = \{(i, j) | i \in \{1, 2, 3, \ldots, m\}$ and $j \in \{1, 2, 3, \ldots, n\}\}$. Two vertices, (i, j) and (k, l), are adjacent if the knight can move from (i, j) to (k, l). Then, a closed $(2, 3)$-knight's tour is a Hamiltonian cycle of a graph G, a cycle of G that contains all vertices of G.

Before proving the main results, we need the following well-known result concerning Hamiltonian cycle (see for example [4]).

Theorem 4. *Let* S *be a proper subset of the vertex set of a graph* G*. If* G *contains a Hamiltonian cycle, then* $G - S$ *contains at most* $|S|$ *components.*

If G is a graph and A is a subset of $V(G)$, let $G[A]$ denote a subgraph of G induced by A.

Theorem 5. *Suppose that* $m \leq 8$ *and* $m \neq 5$*. Then, the* $m \times n$ *cylinder chessboard contains no closed* $(2, 3)$*-knight's tours for any positive integer* n*.*

Proof. Let G be a graph representing the $m \times n$ cylinder chessboard.

Case 1: $m \in \{1, 2, 3, 4\}$. If $m \in \{1, 2, 3\}$, then G is a disconnected graph. Thus, it has no closed $(2, 3)$-knight's tours. Next, let $m = 4$.

If $n = 1, 2, 3$ or $n = 6$, then the degree of vertices $(2, 1)$ is 1. Then, G contains no Hamiltonian cycle.

If $n \geq 4$ and $n \neq 6$, then we consider vertices in the 2nd row. Since, for $j \in \{1, 2, 3, \ldots, n\}$, vertex $(2, j)$ in the 2nd row is adjacent to vertices $(4, t_1)$ and $(4, t_2)$ where $t_1 \equiv j + 3 \pmod n$ and $t_2 \equiv j - 3 \pmod n$, the degree of each vertex in the 2nd row is 2. If G contains a Hamiltonian cycle, then such two edges that incident with each vertex in the 2nd row must be in the Hamiltonian cycle. We see that edges incident with vertices in the 2nd row form cycle which is not a Hamiltonian cycle. The cycle starts at $(2, 1)$, goes on the right side of the chessboard and then returns to the left side of the chessboard until we end at the right side.

Case (i): $n \equiv 0 \pmod 3$. Let $n = 3t$ for some positive integer $t \geq 3$. The cycle is in the following form.

$(2, 1), (4, 4), (2, 7), (4, 10), (2, 13), (4, 16), \ldots, (4, 3t - 5), (2, 3t - 2),$
$(4, 1), (2, 4), (4, 7), (2, 10), (4, 13), (2, 16), \ldots, (2, 3t - 5), (4, 3t - 2), (2, 1).$

Please cite this article as: S. Singhun, P. Ruengmanee and A. Sinna, A closed (2, 3)-knight's tour on some cylinder chessboards, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.06.001.

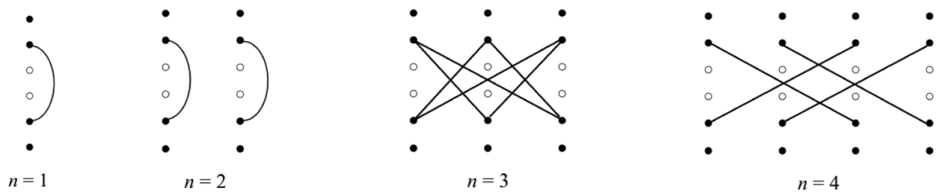

Fig. 1. $G - S$ where the white vertices are in S.

Case (ii): $n \equiv 1 \pmod 3$. Let $n = 3t + 1$ for some positive integer t. The cycle is in the following form.

$(2, 1), (4, 4), (2, 7), (4, 10), (2, 13), (4, 16), \ldots, (2, 3t - 2), (4, 3t + 1),$
$(2, 3), (4, 6), (2, 9), (4, 12), (2, 15), (4, 18), \ldots, (4, 3t - 3), (2, 3t),$
$(4, 2), (2, 5), (4, 8), (2, 11), (4, 14), (2, 17), \ldots, (2, 3t - 4), (4, 3t - 1), (2, 1).$

Case (iii): $n \equiv 2 \pmod 3$. Let $n = 3t + 2$ for some positive integer t. The cycle is in the following form.

$(2, 1), (4, 4), (2, 7), (4, 10), (2, 13), (4, 16), \ldots, (2, 3t - 2), (4, 3t + 1),$
$(2, 2), (4, 5), (2, 8), (4, 11), (2, 14), (4, 17), \ldots, (2, 3t - 1), (4, 3t + 2),$
$(2, 3), (4, 6), (2, 9), (4, 12), (2, 15), (4, 18), \ldots, (4, 3t - 3), (2, 3t)$
$(4, 1), (2, 4), (4, 7), (2, 10), (4, 13), (2, 16), \ldots, (4, 3t - 2), (2, 3t + 1),$
$(4, 2), (2, 5), (4, 8), (2, 11), (4, 14), (2, 17), \ldots, (4, 3t - 1), (2, 3t + 2),$
$(4, 3), (2, 6), (4, 9), (2, 12), (4, 15), (2, 18), \ldots, (2, 3t - 3), (4, 3t), (2, 1).$

Case 2: $m \in \{6, 7, 8\}$.

Case 2.1: $m = 6$. Let $S = \{(3, j), (4, j) | j \in \{1, 2, 3, \ldots, n\}\}$. Then, $|S| = 2n$.

If $n = 1, 3$ or $n \geq 5$, let G_1 be a subgraph of G induced by the set of vertices $(2, j)$ and $(4, j)$ where $j \in \{1, 2, 3, \ldots, n\}$. Then, $G - S$ contains G_1 and $2n$ isolated vertices from the 1st and 6th rows. By Theorem 4, G contains no Hamiltonian cycles. (See Fig. 1.)

If $n = 2$ (respectively $n = 4$), then $G - S$ contains 6 (respectively 12) components. Fig. 1 shows the graph $G - S$ where the white vertices are in S. By Theorem 4, G contains no Hamiltonian cycles.

Case 2.2: $m = 7$. Let $S = \{(3, j), (4, j), (5, j) | j \in \{1, 2, 3, \ldots, n\}\}$. Then, $|S| = 3n$ and $G - S$ contains $4n$ isolated vertices from the $1^{st}, 2^{nd}, 6^{th}$ and 7th rows. By Theorem 4, G contains no Hamiltonian cycles.

Case 2.3: $m = 8$. Let $S = \{(4, j), (5, j) | j \in \{1, 2, 3, \ldots, n\}\}$. Then, $|S| = 2n$.

If n is odd, let G_1 be a subgraph of G induced by the set of vertices $(1, j), (3, j), (6, j)$ and $(8, j)$ where $j \in \{1, 2, 3, \ldots, n\}$. Then, $G - S$ contains G_1 and $2n$ isolated vertices from the 2nd and 7th rows. By Theorem 4, G contains no Hamiltonian cycles.

If n is even, let $A_1 = \{(1, j), (3, j + 1), (6, j + 1), (8, j) | j \in \{1, 3, 5, \ldots, n - 1\}\}$ and $A_2 = \{(1, j), (3, j - 1), (6, j - 1), (8, j) | j \in \{2, 4, 6, \ldots, n\}\}$. Then, $G - S$ contains $G[A_1], G[A_2]$ and $2n$ isolated vertices from 2nd and 7th rows. By Theorem 4, G contains no Hamiltonian cycles.

This completes the proof. □

Lemma 1. *The $5 \times n$ cylinder chessboard admits a closed $(2, 3)$-knight's tour for any positive integer n.*

Proof. Let n be a positive integer. We give two patterns to construct a closed $(2, 3)$-knight's tour for the $5 \times n$ cylinder chessboard. Since the total number of squares of the $5 \times n$ cylinder chessboard is $5n$, for each pattern, we start the first position at $(1, 1)$ labeled with number 1. We define four directions of the knight's move as follows.

a means the knight moves two squares downward and then goes three squares to the right,
b means the knight moves two squares downward and then goes three squares to the left,
c means the knight moves three squares upward and then goes two squares to the right, and
d means the knight moves three squares upward and then goes two squares to the left.

Then, we obtain two patterns of a closed $(2, 3)$-knight's tour as follows.

For the 1st pattern, the first, second, third, fourth and fifth moves are a, b, c, b and c, respectively. We repeat this algorithm for n times (see Fig. 2(a)).

For the 2nd pattern, the first, second, third, fourth and fifth moves are b, a, d, a and d, respectively. We repeat this algorithm for n times (see Fig. 2(b)).

1	6	11	16	21	26	…	5n-19	5n-14	5n-9	5n-4
5n-6	5n-1	4	9	14	19	…	5n-26	5n-21	5n-16	5n-11
5n-13	5n-8	5n-3	2	7	12	…	5n-33	5n-28	5n-23	5n-18
10	15	20	25	30	35	…	5n-10	5n-5	5n	5
3	8	13	18	23	28	…	5n-17	5n-12	5n-7	5n-2

(a) The 1st pattern

1	5n-4	5n-9	5n-14	5n-19	5n-24	…	21	16	11	6
5n-6	5n-11	5n-16	5n-21	5n-26	5n-31	…	14	9	4	5n-1
5n-13	5n-18	5n-23	5n-28	5n-33	5n-38	…	7	2	5n-3	5n-8
10	5	5n	5n-5	5n-10	5n-15	…	30	25	20	15
3	5n-2	5n-7	5n-12	5n-17	5n-22	…	23	18	13	8

(b) The 2nd pattern

Fig. 2. Two patterns of a closed (2, 3)-knight's tour on the $5 \times n$ cylinder chessboard.

We obtain some observations as follows.
(i) For each loop, the starting point is on the 1st row,
(ii) the 1st number of each loop is $5k - 4$ where $k \in \{1, 2, 3, \ldots, n\}$,
(iii) squares labeled with the five consecutive numbers are on different rows, for example, squares labeled with $1, 2, 3, 4$ and 5 stand in the 1st, 3rd, 5th, 2nd and 4th rows, respectively,
(iv) if X is the first number of a loop and stands at $(1, j)$, then $X + 5$ is the first number of the next loop and stands at $(1, j + 1)$ if the knight move with the first pattern or at $(1, j - 1(\text{mod})\ n)$ if the knight move with the second pattern.

By the observation, the 1st number of each loop stands in the different squares. The 2nd, 3rd, 4th and 5th numbers of each loop are considered in similar way. Then, the knight moves to all squares exactly once and the number $5n$ is on $(4, n - 1)$ if the knight move with the first pattern, or $(4, 3)$ if the knight move with the second pattern. Thus, the knight can move to the first position $(1, 1)$. □

Theorem 6. *The $5k \times n$ cylinder chessboard admits a closed $(2, 3)$-knight's tour for any positive integers k and n.*

Proof. If $k = 1$, then a closed (2, 3)-knight's tour is obtained by Lemma 1. Suppose that $k \geq 2$. The $5k \times n$ cylinder chessboard is obtained from k copies of the $5 \times n$ cylinder chessboards by putting each copy from the top to the bottom. We count all copies from the top to the bottom. Since each copy contains a closed (2, 3)-knight's tour, we combine each tour of each $5 \times n$ cylinder chessboard to a single tour by deleting one edge from the top and the bottom and two edges from the ith cylinder chessboard for all $i \in \{2, 3, 4, \ldots, k - 1\}$ and then joining each piece to a single cycle.

Case 1 : $n \leq 5$. We give another pattern for the $5 \times n$ cylinder chessboard which is different from the two patterns of Lemma 1 (shown in Fig. 3).

If $n = 1$, then delete edge 2–3 of the ith copy, and edge 1–2 of $(i + 1)$th copy and then join 2 (respectively 3) of the ith copy to 1 (respectively 2) of the $(i + 1)$th copy for all $i \in \{1, 2, 3, \ldots, k - 1\}$.

If $n = 2, 3$ or 4, then delete edge 2–3 of the ith copy, and edge 6-7 of $(i + 1)$th copy and then join 2 (respectively 3) of the ith copy to 6 (respectively 7) of the $(i + 1)$th copy for all $i \in \{1, 2, 3, \ldots, k - 1\}$.

If $n = 5$, then delete edge 2–3 of the ith copy, and edge 11–12 of $(i + 1)$th copy and then join 2 (respectively 3) of the ith copy to 11 (respectively 12) of the $(i + 1)$th copy for all $i \in \{1, 2, 3, \ldots, k - 1\}$.

Case 2 : $n > 5$. If $k = 2$, the $10 \times n$ cylinder chessboard is obtained from 2 copies of the $5 \times n$ cylinder chessboards by putting each copy from the top to the bottom. The top copy is assigned by the 1st pattern of a

Please cite this article as: S. Singhun, P. Ruengmanee and A. Sinna, A closed (2, 3)-knight's tour on some cylinder chessboards, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.06.001.

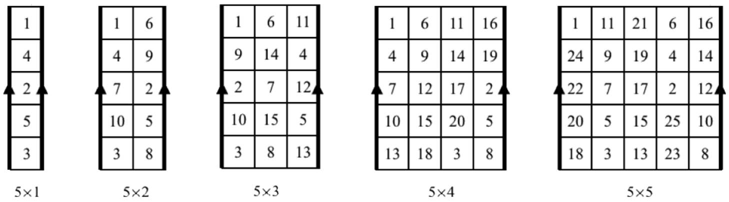

Fig. 3. Closed (2, 3)-knight's tours on the $5 \times n$ where $n \leq 5$.

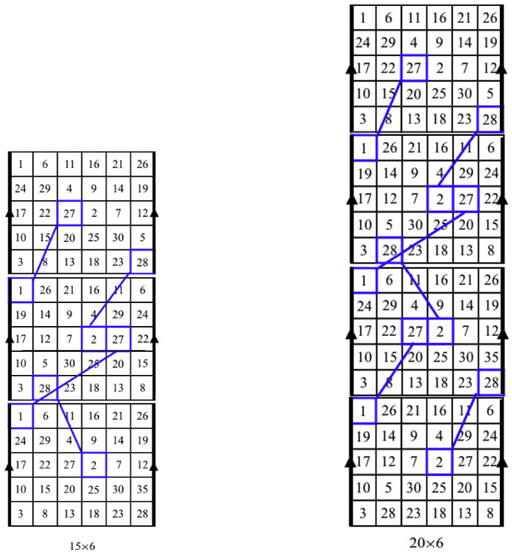

Fig. 4. Illustrated closed (2, 3)-knight's tours on 15×6 and 20×6 cylinder chessboards.

closed (2, 3)-knight's tour and the bottom copy is assigned by the 2nd pattern of a closed (2, 3)-knight's tour from Lemma 1. Thus, we combine each tour of each copy to a single tour by deleting edge $(5n-2)-(5n-3)$ from the top and edge $1-2$ from the bottom and then joining $5n-3$ to 1 and $5n-2$ to 2.

Next, suppose that $k \geq 3$. The $5k \times n$ cylinder chessboard is obtained from k copies of the $5 \times n$ cylinder chessboards by putting each copy from the top to the bottom and counting all copies from the top to the bottom. If i is odd, then the ith copy is assigned by the 1st pattern of a closed (2, 3)-knight's tour from Lemma 1. If i is even, then the ith copy is assigned by the 2nd pattern of a closed (2, 3)-knight's tour from Lemma 1. Thus, we combine each tour of each copy to a single tour as follows.

1. Delete edge $(5n-2)-(5n-3)$ from the 1st copy and edge $1-2$ from the kth copy.
2. For $i \in \{2, 3, 4, \ldots, k-1\}$, delete edges $1-2$ and $(5n-2)-(5n-3)$ from the ith copy.
3. For $i \in \{2, 3, 4, \ldots, k-1\}$, join 1 and 2 of the ith copy to $5n-3$ and $5n-2$ of the $(i-1)$th copy, respectively.

This completes the proof. □

Fig. 4 illustrates closed (2, 3)-knight's tours on 15×6 and 20×6 cylinder chessboards.

For the $9 \times n$ cylinder chessboard, we consider $n \leq 13$.

Lemma 2. *Let $n \in \{1, 2, 3, 6\}$. Then, the $9 \times n$ cylinder chessboard admits no closed (2, 3)-knight's tours.*

Proof. Let $n \in \{1, 2, 3, 6\}$ and G be a graph representing the $9 \times n$ cylinder chessboard. For $j \in \{1, 2, 3, \ldots, n\}$, let $B_j = \{(1, j), (3, t) | t \equiv j+3 \pmod{n}\}$ and $C_j = \{(7, j), (9, t) | t \equiv j+3 \pmod{n}\}$n. Let $S = \{(4, j), (5, j), (6, j) | j \in \{1, 2, 3, \ldots, n\}\}$. Then, $|S| = 3n$ and $G - S$ contains $2n$ isolated from the 2nd and 8th rows, $G[B_j]$ and $G[C_j]$ where $j \in \{1, 2, 3, \ldots, n\}$. By Theorem 4, G contains no Hamiltonian cycles. □

1	10	3	12
8	33	6	35
11	4	13	2
34	7	36	9
5	14	23	32
18	27	16	25
31	20	29	22
24	17	26	15
21	30	19	28

9×4

1	3	5	7	9
44	37	39	32	25
6	8	10	2	4
40	33	26	45	38
11	31	24	43	36
27	29	22	41	34
16	18	20	12	14
23	42	35	28	30
21	13	15	17	19

9×5

1	13	11	9	7	5	3
40	35	62	60	55	50	45
8	6	4	2	14	12	10
56	51	46	41	36	63	61
15	59	54	49	44	39	34
37	32	30	57	52	47	42
22	20	18	16	28	26	24
53	48	43	38	33	31	58
29	27	25	23	21	19	17

9×7

1	16	7	20	5	26	3	24
12	69	10	67	14	65	18	71
27	4	25	2	23	8	21	6
66	19	72	17	70	11	68	15
9	22	13	28	47	32	41	64
34	51	30	53	36	55	38	49
61	40	63	42	57	44	59	46
54	37	48	33	50	29	52	31
43	58	45	60	39	62	35	56

9×8

1	14	21	5	12	19	3	10	23
17	53	8	73	80	64	71	55	62
4	11	24	2	15	22	6	13	20
72	56	63	18	54	9	74	81	65
7	43	79	34	70	25	61	16	52
75	59	66	50	57	41	77	32	68
39	46	30	37	44	28	35	48	26
78	33	69	76	60	67	51	58	42
36	49	27	40	47	31	38	45	29

9×9

1	22	5	16	9	20	3	14	7	18
80	65	12	51	70	89	82	75	60	53
15	8	19	2	23	6	17	10	21	4
76	61	54	81	66	13	52	71	90	83
11	50	69	88	37	74	59	24	79	64
72	57	84	77	62	55	48	67	86	35
25	46	29	40	33	44	27	38	31	42
68	87	36	73	58	85	78	63	56	49
39	32	43	26	47	30	41	34	45	28

9×10

1	5	9	13	17	21	3	7	11	15	19
54	71	88	76	93	81	98	86	61	49	66
12	16	20	2	6	10	14	18	22	4	8
62	50	67	55	72	89	77	94	82	99	87
23	75	92	80	97	85	60	48	65	53	70
83	58	46	63	51	68	56	73	90	78	95
34	38	42	24	28	32	36	40	44	26	30
91	79	96	84	59	47	64	52	69	57	74
45	27	31	35	39	43	25	29	33	37	41

9×11

Fig. 5. Closed (2, 3)-knight's tours on the $9 \times n$ cylinder chessboard where $n \in \{4, 5, 7, 8, 9, 10, 11\}$.

1	10	19	28	37	46	3	12	21	30	39	48
44	107	6	15	24	33	42	105	8	17	26	35
31	40	49	2	11	20	29	38	47	4	13	22
18	27	36	45	106	7	16	25	34	43	108	9
5	14	23	32	41	104	75	66	57	102	93	84
88	79	68	59	50	95	86	77	70	61	52	97
101	92	83	72	63	54	99	90	81	74	65	56
60	51	96	87	76	67	58	103	94	85	78	69
73	64	55	100	91	82	71	62	53	98	89	80

9×12

1	23	19	15	11	7	3	25	21	17	13	9	5
89	62	96	111	84	57	72	106	79	94	67	101	116
14	10	6	2	24	20	16	12	8	4	26	22	18
68	102	117	90	63	97	112	85	58	73	107	80	95
27	110	83	56	71	105	78	93	66	100	115	88	61
108	81	54	69	103	76	91	64	98	113	86	59	74
40	36	32	28	50	46	42	38	34	30	52	48	44
87	60	75	109	82	55	70	104	77	92	65	99	114
53	49	45	41	37	33	29	51	47	43	39	35	31

9×13

Fig. 6. Closed (2, 3)-knight's tours on the $9 \times n$ cylinder chessboard where $n \in \{12, 13\}$.

Theorem 7. *Let n be an integer such that $n \leq 13$. Then, the $9 \times n$ cylinder chessboard admits a closed (2, 3)-knight's tour if and only if $n \notin \{1, 2, 3, 6\}$.*

Proof. If $n \in \{1, 2, 3, 6\}$, then by Lemma 2, the $9 \times n$ cylinder chessboard admits no closed (2, 3)-knight's tours. Closed (2, 3)-knight's tours on the $9 \times n$ cylinder chessboard where $n \in \{4, 5, 7, 8, 9, 10, 11, 12, 13\}$ are shown in Figs. 5 and 6. □

Theorem 8. *Suppose that $n \leq 13$ and $n \notin \{1, 2, 3, 6\}$. Then, the $9k \times n$ cylinder chessboard admits a closed $(2, 3)$-knight's tour for any positive integer k.*

Proof. If $k = 1$, then a closed $(2, 3)$-knight's tour is obtained by Theorem 7. Suppose that $k \geq 2$. The $9k \times n$ cylinder chessboard is obtained from the k copies of $9 \times n$ cylinder chessboards by putting each copy from the top to the bottom. We count all copies from the top to the bottom. Since each copy contains a closed $(2, 3)$-knight's tour, we combine each tour of each $9 \times n$ cylinder chessboard to a single tour by deleting one edge from the top and the bottom and two edges from the ith copy where $i \in \{2, 3, 4, \ldots, k - 1\}$ and then joining each piece to a single cycle.

Please cite this article as: S. Singhun, P. Ruengmanee and A. Sinna, A closed (2, 3)-knight's tour on some cylinder chessboards, AKCE International Journal of Graphs and Combinatorics (2019), https://doi.org/10.1016/j.akcej.2019.06.001.

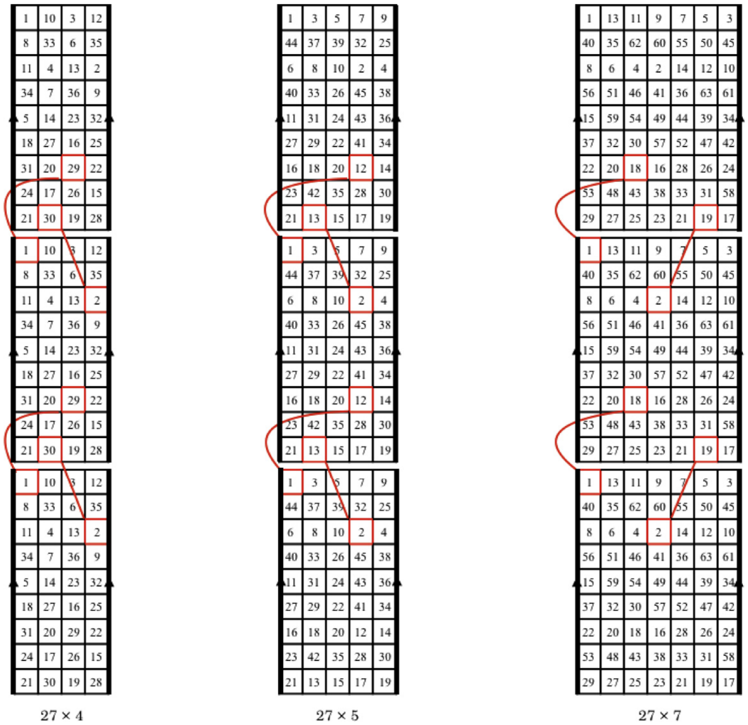

Fig. 7. Illustrated closed (2, 3)-knight's tours on the $27 \times 4, 27 \times 5$ and 27×7 cylinder chessboards.

Case 1: $n = 4$. Delete edge 29–30 of the ith copy and edge 1–2 of the $(i+1)$th copy and then join 29 (respectively 30) of the ith copy to 1 (respectively 2) of the $(i+1)$th copy for all $i \in \{1, 2, 3, \dots, k-1\}$.

Case 2: $n = 5$. Delete edge 12–13 of the ith copy and edge 1–2 of the $(i+1)$th copy and then join 12 (respectively 13) of the ith copy to 1 (respectively 2) of the $(i+1)$th copy for all $i \in \{1, 2, 3, \dots, k-1\}$.

Case 3 : $n \geq 7$. Let A be a number X of $(n-2, 3)$. Then, for each n, the number $X+1$ or $X-1$ is on $(n, 6)$. Assume that $B \in \{X+1, X-1\}$. Delete edge $A - B$ of the ith copy and edge 1–2 of the $(i+1)$th copy and then join A (respectively B) of the ith copy to 1 (respectively 2) of the $(i+1)$th copy for all $i \in \{1, 2, 3, \dots, k-1\}$. □

Fig. 7 illustrates closed (2,3)-knight's tours on the 27×4, 27×5 and 27×7 cylinder chessboards. In future research, we will try to see whether the $9k \times n$ cylinder chessboard has a closed (2, 3)-knight's tour or not.

Acknowledgments

I would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] A.L. Schwenk, Which rectangular chessboards have a knight's tour, Math. Mag. 64 (1991) 325–332.
[2] G.L. Chia, Siew-Hui Ong, Generalized knight's tours on rectangular chessboards, Discrete Appl. Math. 159 (2005) 80–98.
[3] J.J. Watkins, Knight's tours on cylinders and other surfaces, Congr. Numer. 143 (2000) 117–127.
[4] G. Chartrand, P. Zhang, A First Course in Graph Theory, Mc Graw-Hill Higher Education, Boston, 2005.