AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: Héctor Castañeda-López, Pablo C. Palomino, Andrea B. Ramos-Tort, Christian Rubio-Montiel & Claudia Silva-Ruiz (2020) The 6-girth-thickness of the complete graph, AKCE International Journal of Graphs and Combinatorics, 17:3, 856-861, DOI: 10.1016/j.akcej.2019.05.004

To link to this article: https://doi.org/10.1016/j.akcej.2019.05.004

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 856–861
https://doi.org/10.1016/j.akcej.2019.05.004

The 6-girth-thickness of the complete graph

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ABSTRACT

The g-girth-thickness $\theta(g, G)$ of a graph G is the minimum number of planar subgraphs of girth at least g whose union is G. In this paper, we determine the 6-girth-thickness $\theta(6, K_n)$ of the complete graph K_n in almost all cases. And also, we calculate by computer the missing value of $\theta(4, K_n)$.

KEYWORDS
Thickness; planar decomposition; complete graph; girth

2010 MATHEMATICS SUBJECT CLASSIFICATION
05C10

1. Introduction

In this paper, all graphs are finite and simple. A graph in which any two vertices are adjacent is called a *complete graph* and it is denoted by K_n if it has n vertices. If a graph can be drawn in the Euclidean plane such that no inner point of its edges is a vertex or lies on another edge, then the graph G is called *planar*. The *girth* of a graph is the size of its shortest cycle or ∞ if it is acyclic. It is known that an acyclic graph of order n has size at most $n-1$ and a planar graph of order n and finite girth g has size at most $\frac{g}{g-2}(n-2)$, see [8].

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G. Equivalently, it is the minimum number of colors used in any edge coloring of G such that each set of edges in the same chromatic class induces a planar subgraph.

The concept of the thickness was introduced by Tutte [19]. The problem to determine the thickness of a graph G is NP-hard [15], and only a few of exact results are known, for instance, when G is a complete graph [2, 5, 6], a complete multipartite graph [7, 11, 18, 21, 22] or a hypercube [14].

Generalizations of the thickness for the complete graphs also have been studied such that the outerthickness θ_o, defined similarly but with outerplanar instead of planar [12], and the S-thickness θ_S, considering the thickness on a surface S instead of the plane [4]. The thickness has many applications, for example, in the design of circuits [1], in the Ringel's earth-moon problem [13], or to bound the achromatic numbers of planar graphs [3]. See also [16].

In [17], the *g-girth-thickness* $\theta(g, G)$ of a graph G was defined as the minimum number of planar subgraphs of girth at least g whose union is G. Indeed, the g-girth thickness generalizes the thickness when $g=3$ and the *arboricity number* when $g=\infty$.

This paper is organized as follows. In Section 2, we obtain the 6-girth-thickness $\theta(6, K_n)$ of the complete graph K_n getting that $\theta(6, K_n)$ equals $\lceil\frac{n+2}{3}\rceil$, except for $n=3t+1, t\geq 4$ and $n\neq 2$, for which $\theta(6, K_2)=1$. In Section 3, we show that there exists a set of 3 planar triangle-free subgraphs of K_{10} whose union is K_{10}. The decomposition was found by computer and, as a consequence, we disproved the conjecture that appears in [17] about the missing case of the 4-girth-thickness of the complete graph.

2. Determining $\theta(6, K_n)$

A planar graph of n vertices with girth at least 6 has size at most $3(n-2)/2$ for $n\geq 6$ and size at most $n-1$ for $1\leq n\leq 5$, therefore, the 6-girth-thickness $\theta(6, K_n)$ of the complete graph K_n is at least

$$\left\lceil\frac{n(n-1)}{3(n-2)}\right\rceil = \left\lceil\frac{n+1}{3}+\frac{2}{3n-6}\right\rceil = \left\lceil\frac{n+2}{3}\right\rceil$$

for $n\geq 6$, as well as, $\lceil\frac{n+2}{3}\rceil$ for $n\in\{1,3,4,5\}$. We have the following theorem.

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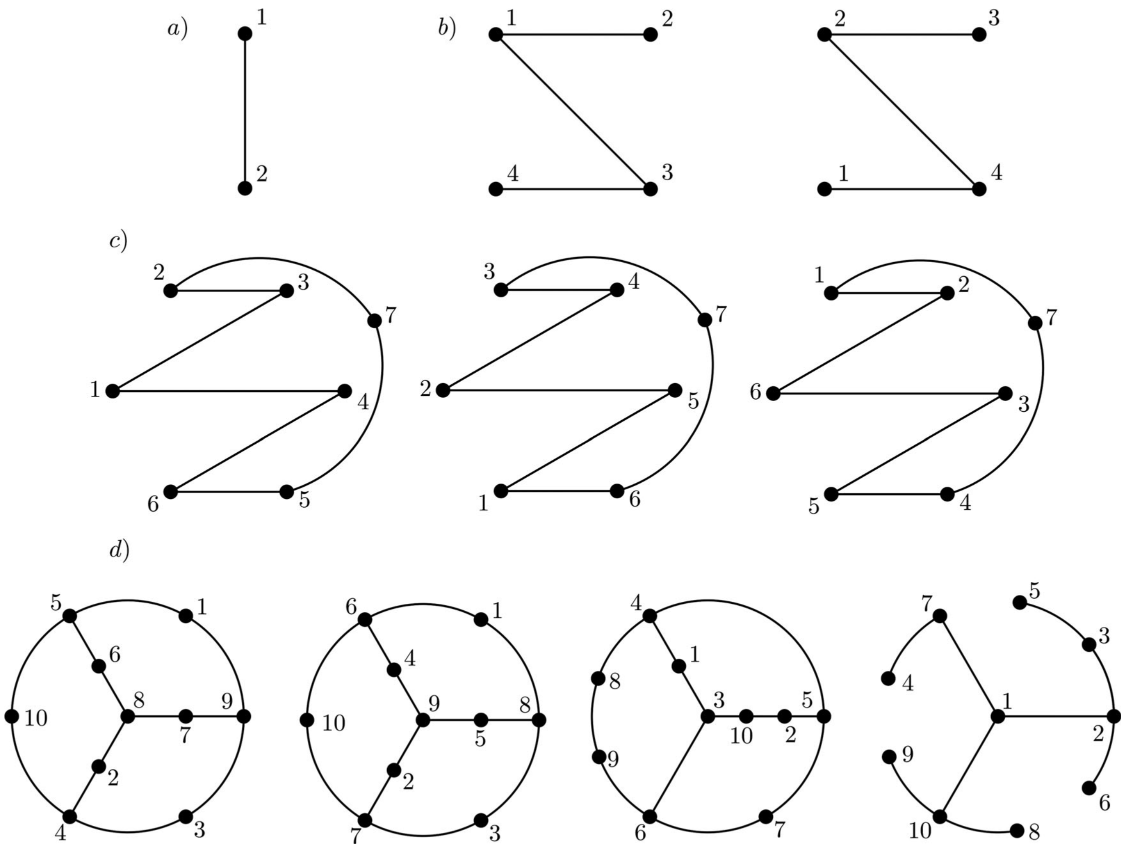

Figure 1. A decomposition of K_n into $\theta(6, K_n)$ planar subgraphs of girth at least 6: (a) for $n = 2$, (b) for $n = 4$, (c) for $n = 7$ and (d) for $n = 10$.

Theorem 2.1. *The 6-girth-thickness $\theta(6, K_n)$ of K_n is equal to $\lceil \frac{n+2}{3} \rceil$ except possibly when $n = 3t + 1$, for $t \geq 4$, and $n \neq 2$ for which $\theta(6, K_2) = 1$.*

Proof. To begin with, Figure 1 displays equality for $n = 2, 4, 7, 10$ with $\theta(6, K_n) = 1, 2, 3, 4,$ respectively. The rest of the cases for $1 \leq n \leq 10$ are obtained by the hereditary property of the induced subgraphs. We remark that the decomposition of K_{10} was found by computer using the database of the connected planar graphs of order 10 that appears in [9].

Now, we need to distinguish two main cases, namely, when t is even or t is odd for $n = 3t$, that is, when $n = 6k$ and $n = 6k + 3$ for $k \geq 2$. The cases $n = 6k - 1$ and $n = 6k + 2$, i.e., for $n = 3t + 1$, are obtained by the hereditary property of the induced subgraphs, that is, since $K_{6k-1} \subset K_{6k}$ and $K_{6k+2} \subset K_{6k+3}$, we have

$$2k + 1 \leq \theta(6, K_{6k-1}) \leq \theta(6, K_{6k}) \text{ and}$$
$$2k + 2 \leq \theta(6, K_{6k+2}) \leq \theta(6, K_{6k+3}), \text{respectively.}$$

Therefore, the case of $n = 6k$ shows a decomposition of K_{6k} into $2k + 1$ planar subgraphs of girth at least 6, while the case of $n = 6k + 3$ shows a decomposition of K_{6k+3} into $2k + 2$ planar subgraphs of girth at least 6. Both constructions are based on the planar decomposition of K_{6k} of Beineke and Harary [5] (see also [2, 6, 20]) but we use the combinatorial approach given in [3]. Then, for the sake of completeness, we give a decomposition of K_{6k} in order to obtain its usual thickness. In the remainder of this proof, all sums are taken modulo $2k$.

We recall that complete graphs of even order $2k$ are decomposable into a cyclic factorization of Hamiltonian paths, see [10]. Let G^x be a complete graph of order $2k$, label its vertex set $V(G^x)$ as $\{x_1, x_2, ..., x_{2k}\}$ and let $\mathcal{F}^x_i$ be the Hamiltonian path with edges

$$x_i x_{i+1}, x_{i+1} x_{i-1}, x_{i-1} x_{i+2}, x_{i+2} x_{i-2}, ..., x_{i+k+1} x_{i+k},$$

for all $i \in \{1, 2, ..., k\}$. The partition $\{E(\mathcal{F}^x_1), E(\mathcal{F}^x_2), ..., E(\mathcal{F}^x_k)\}$ is such factorization of G^x. We remark that the center of $\mathcal{F}^x_i$ has the edge $e^x_i = x_{i+\lceil \frac{k}{2} \rceil} x_{i+\lceil \frac{3k}{2} \rceil}$, see Figure 2.

Let G^u, G^v and G^w be the complete subgraphs of K_{6k} having $2k$ vertices each of them and such that G^w is $K_{6k} \setminus (V(G^u) \cup V(G^v))$. The vertices of $V(G^u), V(G^v)$ and $V(G^w)$ are labeled as $\{u_1, u_2, ..., u_{2k}\}, \{v_1, v_2, ..., v_{2k}\}$ and $\{w_1, w_2, ..., w_{2k}\}$, respectively.

Let x be an element of $\{u, v, w\}$. Take the cyclic factorization $\{E(\mathcal{F}^x_1), E(\mathcal{F}^x_2), ..., E(\mathcal{F}^x_k)\}$ of G^x into Hamiltonian paths and denote as P_{x_i} and $P_{x_{i+k}}$ the subpaths of $\mathcal{F}^x_i$ containing k vertices and the leaves x_i and x_{i+k}, respectively. We define the other leaves of P_{x_i} and $P_{x_{i+k}}$ as $f(x_i)$ and $f(x_{i+k})$, respectively and according to the parity of k, that is (see Figure 2),

$$f(x_i) = \begin{cases} x_{i+\lceil \frac{3k}{2} \rceil} & \text{if } k \text{ is odd,} \\ x_{i+\lceil \frac{k}{2} \rceil} & \text{if } k \text{ is even.} \end{cases} \text{ and}$$

$$f(x_{i+k}) = \begin{cases} x_{i+\lceil \frac{k}{2} \rceil} & \text{if } k \text{ is odd,} \\ x_{i+\lceil \frac{3k}{2} \rceil} & \text{if } k \text{ is even.} \end{cases}$$

We remark that the set of edges $\{x_i x_{i+k} : 1 \leq i \leq k\}$ is the same set of edges that $\{f(x_i) f(x_{i+k}) : 1 \leq i \leq k\}$.

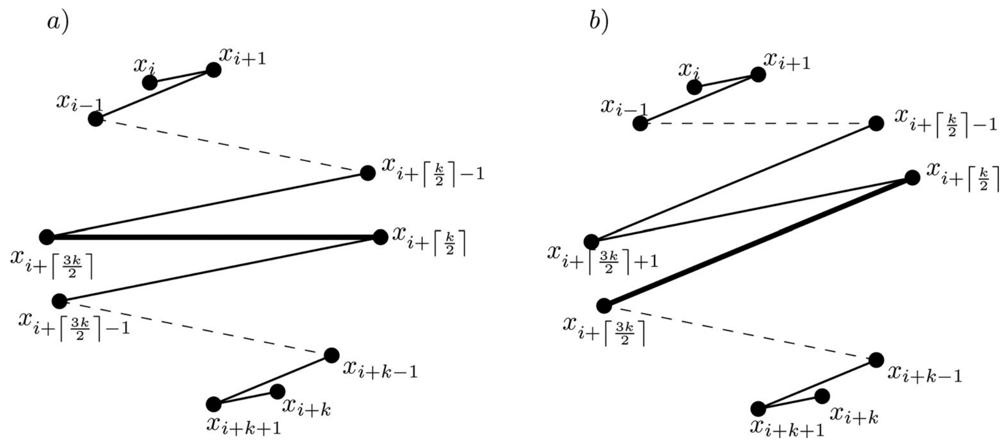

Figure 2. The Hamiltonian path $\mathcal{F}_i^x$: Left (a) The edge e_i^x in bold for k odd. Right (b) The edge e_i^x in bold for k even.

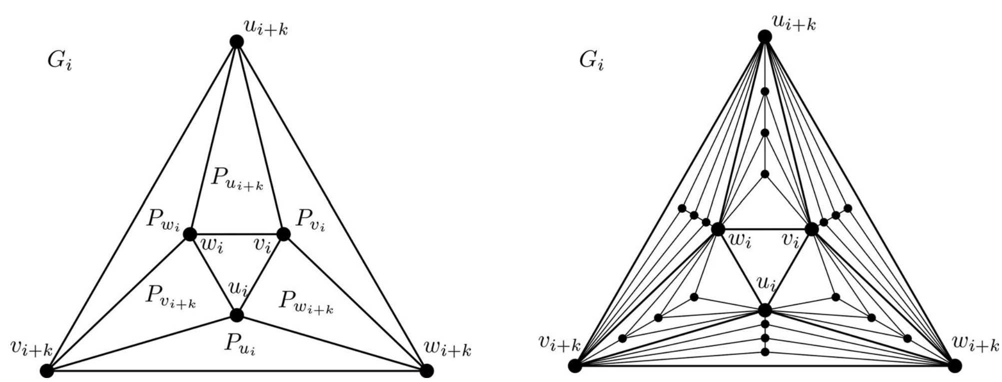

Figure 3. (Left) The octahedron subgraph of the graph G_i. (Right) The graph G_i.

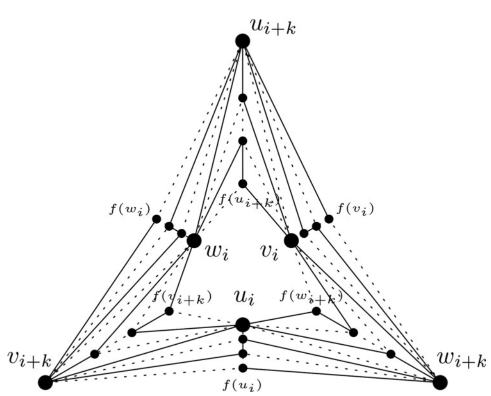

Figure 4. Partial modification of the subgraph G_i.

Now, we construct the maximal planar subgraphs G_1, $G_2, \dots, G_k$ and a matching G_{k+1} with $6k$ vertices each in the following way. Let G_{k+1} be the perfect matching with the edges u_ju_{j+k}, v_jv_{j+k} and w_jw_{j+k} for $j \in \{1, 2, ..., k\}$.

For each $i \in \{1, 2, ..., k\}$, let G_i be the spanning planar graph of K_{6k} whose adjacencies are given as follows: we take the 6 paths, $P_{u_i}, P_{u_{i+k}}, P_{v_i}, P_{v_{i+k}}, P_{w_i}$ and $P_{w_{i+k}}$ and insert them in the octahedron with the vertices $u_i, u_{i+k}, v_i, v_{i+k}, w_i$ and w_{i+k} as is shown in Figure 2 (Left). The vertex x_j of each path P_{x_j} is identified with the vertex x_j in the corresponding triangle face and join all the other vertices of the path with both of the other vertices of the triangle face, see Figure 3 (Right).

By construction of G_i, $K_{6k} = \cup_{i=1}{}^{k+1} G_i$, see [2, 5] to check a full proof. In consequence, the $k+1$ planar subgraphs G_i show that $\theta(3, K_{6k}) \leq k+1$ and then, $\theta(3, K_{6k}) = k+1$ owing to the fact that $\theta(3, K_{6k}) \geq \left\lceil \frac{\binom{6k}{2}}{3(6k-2)} \right\rceil = k+1$.

Now, we proceed to prove that $\theta(6, K_{6k}) \leq 2k+1$ in Case 1 and $\theta(6, K_{6k+3}) \leq 2k+2$ in Case 2. The main idea of both cases is divide each G_i into two subgraphs of girth 6 for any $i \in \{1, ..., k\}$.

1. Case $n = 6k$.
   Consider the set of planar subgraphs $\{G_1, G_2, ..., G_{k+1}\}$ of K_{6k} which is described above.
   Step 1. For each $i \in \{1, ..., k\}$, remove the six edges of the triangles $u_iv_iw_i$ and $u_{i+k}v_{i+k}w_{i+k}$.
   Step 2. For each $i \in \{1, ..., k\}$, divide the obtained subgraph into two subgraphs H_i^1 and H_i^2 as follows: The maximum matching of P_{x_i} incident to the vertex $f(x_i)$ belongs to H_i^1 (see dotted subgraph in Figure 4) while the maximum matching of $P_{x_{i+k}}$ incident to the vertex $f(x_{i+k})$ belongs to H_i^2.
   Next, the rest of the edges joined to the vertices of the paths P_{x_i} and $P_{x_{i+k}}$, in an alternative way from the exterior region to the region with the vertices $\{u_i, v_i, w_i\}$, belong to H_i^1 and H_i^2 respectively, such that the edges $f(w_i)u_{i+k}, f(v_i)w_{i+k}$ and $f(u_i)v_{i+k}$ belong to H_i^1 and the edges $f(w_i)v_{i+k}, f(v_i)u_{i+k}$ and $f(u_i)w_{i+k}$ belong to H_i^2, see Figure 4.

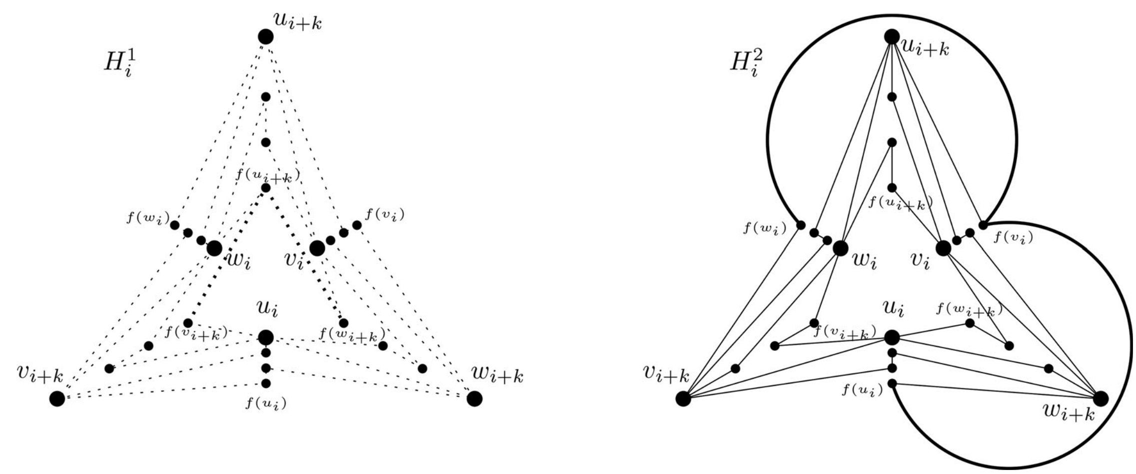

Figure 5. Subgraphs H_i^1 and H_i^2 for the Case 1.

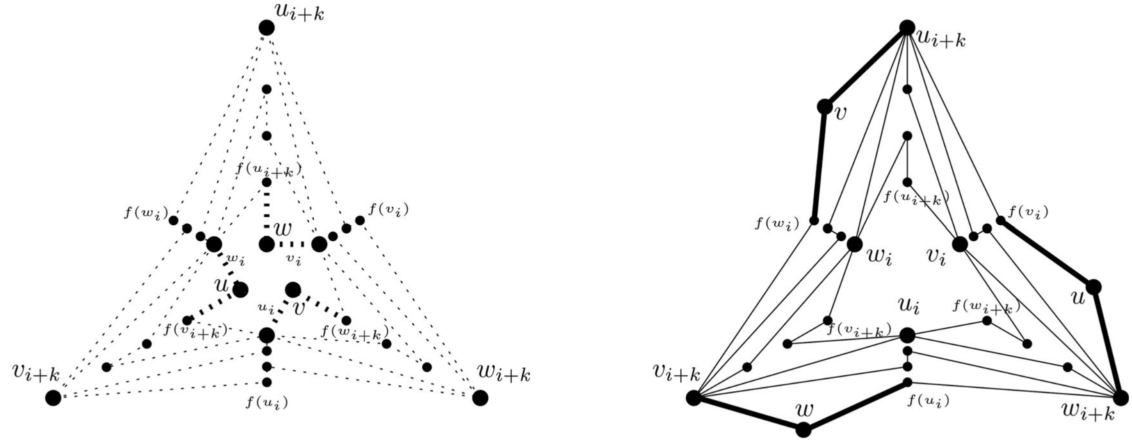

Figure 6. Subgraphs H_i^1 and H_i^2 for Case 2.

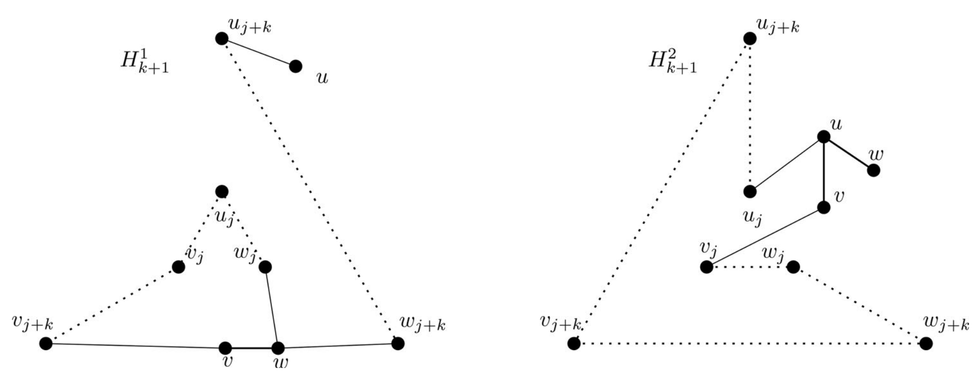

Figure 7. Partial subgraphs H_{k+1}^1 and H_{k+1}^2.

Step 3. Consider the removed edges in Step 1, add the edges $f(v_{i+k})f(u_{i+k})$ and $f(u_{i+k})f(w_{i+k})$ to H_i^1 and the edges $f(w_i)f(v_i)$ and $f(v_i)f(u_i)$ to H_i^2, see Figure 5. The rest of the edges removed in Step 1 are added to G_{k+1} getting the subgraph H_{k+1} which is the union of the paths $\{f(v_i), f(v_{i+k}), f(w_{i+k}), f(w_i), f(u_i), f(u_{i+k})\}$.

2. Case $n = 6k + 3$.
   Consider the set of planar subgraphs $\{G_1, G_2, \ldots, G_{k+1}\}$ of K_{6k} which is described above as well as Step 1 and 2 of the previous case.

Step 3. Add three vertices u, v and w in the subgraphs H_i^1 and H_i^2, for each $i \in \{1, \ldots, k\}$, and the edges uw_i, $uf(v_{i+k})$, vu_i, $vf(w_{i+k})$, wv_i, $wf(u_{i+k})$ into H_i^1 as well as the edges $uw_{i+k}, uf(v_i), vu_{i+k}, vf(w_i), wv_{i+k}, wf(u_i)$ into H_i^2, see Figure 6.
Step 4. On one hand, remains to define the adjacencies between u, v, w and all the adjacencies between u and u_i, v and v_i, w and w_i, for each $j \in \{1, \ldots, k\}$. On the other hand, the edges of the graph G_{k+1} together with the removed edges of the Step 1 form a set of triangle

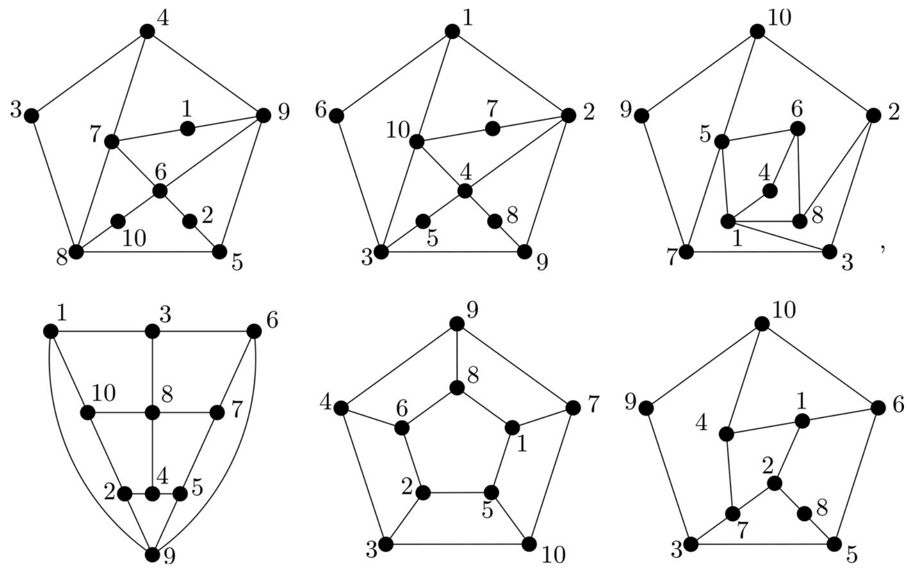

Figure 8. Two planar decompositions of K_{10} into three subgraphs of girth 4.

prisms which we split into two subgraphs called H^1_{k+1} and H^2_{k+1} in the following way:

(a) The adjacency vw is in H^1_{k+1} while the adjacencies uv and uw are in H^2_{k+1}, see Figure 7.

(b) The set of adjacencies vv_{j+k}, ww_j, ww_{j+k} and uu_{j+k} are in H^1_{k+1} while the set of adjacencies vv_j, and uu_j are in H^2_{k+1}, for each $j \in \{1, \ldots, k\}$, see Figure 7.

(c) The subgraph H^1_{k+1} contains the adjacencies $v_{j+k}v_j, v_ju_j, u_jw_j$ and $w_{j+k}u_{j+k}$ (a set of subgraphs $P_4 \cup K_2$) and the subgraph H^2_{k+1} contains the adjacencies $u_ju_{j+k}, u_{j+k}v_{j+k}, v_{j+k}w_{j+k}, w_{j+k}w_j$ and w_jv_j (a set of subgraphs P_6) for all $j \in \{1, \ldots, k\}$, see Figure 7.

By the small cases and the two main cases, the theorem follows. □

3. The 4-girth thickness of K_{10}

In [17], Rubio-Montiel gave a decomposition of K_n into $\theta(4, K_n) = \lceil \frac{n+2}{4} \rceil$ triangle-free planar subgraphs, except for $n = 10$. In that case, it was bounded by $3 \leq \theta(4, K_{10}) \leq 4$ and conjectured that the correct value was the upper bound. Using the database of the connected planar graphs of order 10 that appears in [9] and the SageMath program, we found two decompositions of K_{10} into 3 planar subgraphs of girth at least 4 illustrated in Figure 8. In summary, the correct value of $\theta(4, K_n)$ was the lower bound and then, we have the following theorem.

Theorem 3.1. *The 4-girth-thickness $\theta(4, K_n)$ of K_n equals $\lceil \frac{n+2}{4} \rceil$ for $n \neq 6$ and $\theta(4, K_6) = 3$.*

Acknowledgments

Part of the work was done during the IV Taller de Matemáticas Discretas, held at Campus-Juriquilla, Universidad Nacional Autónoma de México, Querétaro City, Mexico on June 11–16, 2017. We thank Miguel Raggi and Jessica Sánchez for their useful discussions and help with the SageMath program.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

C. Rubio-Montiel was partially supported by PAIDI grant 007/19 and PAPIIT grant IN107218.

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