AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

Graphs whose line graphs are ring graphs

Mahdi Reza Khorsandi

To cite this article: Mahdi Reza Khorsandi (2020) Graphs whose line graphs are ring graphs, AKCE International Journal of Graphs and Combinatorics, 17:3, 801-803, DOI: 10.1016/j.akcej.2019.10.002

To link to this article: https://doi.org/10.1016/j.akcej.2019.10.002

Published online: 02 Jul 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 801–803
https://doi.org/10.1016/j.akcej.2019.10.002

Graphs whose line graphs are ring graphs

Mahdi Reza Khorsandi

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran

ABSTRACT
Given a graph H, a path $\mathcal{P}$ of length at least two is called an H-path if $\mathcal{P}$ meets H exactly in its ends. A graph G is a ring graph if each block of G which is not a bridge or a vertex can be constructed inductively by starting from a single cycle and then in each step adding an H-path that meets graph H in the previous step in two adjacent vertices. In this article, we classify all graphs whose line graphs and total graphs are ring graphs.

KEYWORDS
Ring graph; line graph; total graph

2010 MATHEMATICS SUBJECT CLASSIFICATION
05C10; 05C76

1. Introduction

Throughout this article, the graphs we consider are simple and finite. A cut-vertex (resp., a bridge) of a graph is a vertex (resp., edge) whose deletion increases the number of components. A maximal connected subgraph that has no cut-vertex is called a block.

Given a graph H, a path $\mathcal{P}$ of length at least two is called an H-path if $\mathcal{P}$ meets H exactly in its ends. A graph G is a ring graph if each block of G which is not a bridge or a vertex can be constructed inductively by starting from a single cycle and then in each step adding an H-path that meets graph H in the previous step in two adjacent vertices. Ring graphs were first introduced by Gitler et al. [5, 6]. For an application of ring graphs in commutative algebra, the reader should refer to Gitler et al. [6]. Examples of ring graphs include forests and cycles.

A cycle without chords is called *primitive*. Recall that a graph G has the *primitive cycle property* if any two primitive cycles intersect in at most one edge. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. Two graphs are said to be *homeomorphic* if both can be obtained from the same graph by subdivision. The complete graph with n vertices is denoted by K_n. For an integer $r \geq 2$, the complete r-partite graph whose partite sets having $n_1, \ldots, n_r$ vertices, respectively, is denoted by $K_{n_1,\ldots,n_r}$. Gitler et al. [6, Theorem 2.13], showed that:

Theorem 1.1. *The following conditions are equivalent for a graph G with n vertices and q edges:*

a. *G is a ring graph.*
b. *The number of primitive cycles is equal to $q-n+r$, where r is the number of components of G.*
c. *G satisfies the primitive cycle property and G does not contain a subdivision of K_4 as a subgraph.*

An immediate consequence of Theorem 1.1 is that ring graphs are planar. Another subclass of planar graphs is the class of the *outerplanar graphs*. A planar graph is outerplanar if it can be embedded in the plane such that all its vertices lie on the outer face. Also, Gitler et al. [6, Proposition 2.17] proved that outerplanar graphs are ring graphs. Note that the class of outerplanar graphs is a proper subclass of ring graphs (see Figure 1).

The *line graph* $L(G)$ of a graph G is the graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. The *total graph* $T(G)$ of a graph G is the graph whose vertices are the vertices and the edges of G and two vertices of $T(G)$ are adjacent if the corresponding elements of G are adjacent or incident. The line graph arose from the work of Whitney [9] while the total graph one introduced by Behzad [1]. The *maximum degree* of a graph G, denoted by $\Delta(G)$, is the maximum degree of its vertices. The main motivation for this work is a result of Sedláček [8], which classifies planar line graphs.

Theorem 1.2. *The line graph of a graph G is planar if and only if G is planar, $\Delta(G) \leq 4$ and if $deg(v) = 4$ for a vertex v of G, then v is a cut-vertex.*

A Similar result was proved by Chartrand et al. [3] for outerplanar graphs; it states that:

Theorem 1.3. *The line graph of a graph G is outerplanar if and only if $\Delta(G) \leq 3$ and if $deg(v) = 3$ for a vertex v of G, then v is a cut-vertex.*

Also, Ghebleh and Khatirinejad [4] and Lin et al. [7] studied the planarity and outerplanarity of the iterated line

CONTACT Mahdi Reza Khorsandi khorsandi@shahroodut.ac.ir Faculty of Mathematical Sciences, Shahrood University of Technology, P.O. Box 36199-95161, Shahrood, Iran.

graphs, respectively. In this article, we characterize all graphs whose line graphs and total graphs are ring graphs.

2. Line graphs which are ring graphs

In this section, we classify all graphs whose line graphs are ring graphs. First, we recall the following Lemma which uses frequently in this article:

Lemma 2.1. (see [6, Lemma 2.11]) *Let G be a ring graph. Then for any two non-adjacent vertices x, y of G there are at most two vertex disjoint paths joining x and y.*

Now, we are ready to prove the main theorem:

Theorem 2.2. *The following statements are equivalent for a graph G.*

a. *The line graph $L(G)$ of G is a ring graph.*
b. *The line graph $L(G)$ of G is outerplanar.*
c. *$\Delta(G) \leq 3$ and every vertex of degree 3 in G is a cut-vertex.*
d. *G has no subgraph homeomorphic to $K_{1,4}, K_{2,3}$ or $K_{1,1,2}$.*

Proof. (a) $\Rightarrow$ (c) If $\Delta(G) \geq 4$, then there exists a vertex of G incident with four edges. These four edges are mutually adjacent vertices in $L(G)$, i.e., the graph K_4, which contradicts Theorem 1.1. Now, suppose on the contrary there exists a vertex v of G such that $\deg(v) = 3$ but that v is not a cut-vertex. It follows that there exist three distinct vertices x, y and z of G that are adjacent to v. Since v is not a cut-vertex, there exists a path of the form

$$v, x = x_0, x_1, \ldots, x_k, y = x_{k+1}.$$

Now, we have two cases:

Case 1. For some $1 \leq j \leq k$, there exists a path of the form

$$v = z_{-1}, z = z_0, \ldots, z_l = x_j,$$

where z_i's, $-1 \leq i < l$, and x_t's are distinct.

Consider the following paths in $L(G)$:

$$\mathcal{P}_1 : (vx), (z_{-1}z_0), \ldots, (z_{l-1}z_l), (x_jx_{j+1}),$$
$$\mathcal{P}_2 : (vx), (x_0x_1), \ldots, (x_jx_{j+1}),$$
$$\mathcal{P}_3 : (vx), (vy), (x_{k+1}x_k), \ldots, (x_{j+1}x_j).$$

These paths are three vertex disjoint paths joining two non-adjacent vertices (vx) and (x_jx_{j+1}) of $L(G)$, which contradicts Lemma 2.1.

Case 2. There exists a path of the form

$$v = z_{-1}, z = z_0, \ldots, z_l,$$

where either $z_l = x$ or $z_l = y$ and also z_i's, $-1 \leq i < l$, and x_j's are distinct. Without loss of generality we can assume that $z_l = y$.

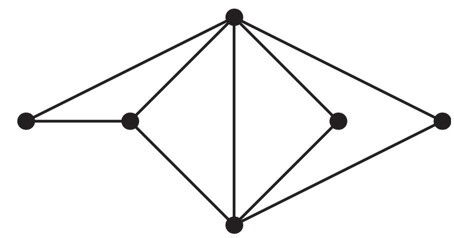

Figure 1. An example of a ring graph which is not outerplanar.

Consider the following paths in $L(G)$:

$$\mathcal{P}_1 : (zv), (vy), (x_kx_{k+1}),$$
$$\mathcal{P}_2 : (zv), (vx), (x_0x_1), \ldots, (x_kx_{k+1}),$$
$$\mathcal{P}_3 : (zv), (z_0z_1), \ldots, (z_{l-1}z_l), (x_kx_{k+1}).$$

These paths are three vertex disjoint paths joining two non-adjacent vertices (zv) and (x_kx_{k+1}) of $L(G)$, which contradicts Lemma 2.1.

(c) $\Rightarrow$ (b) It is an immediate consequence from Theorem 1.3.

(b) $\Rightarrow$ (a) Since by [6, Proposition 2.17], every outerplanar graph is a ring graph, $L(G)$ is a ring graph.

(b) $\iff$ (d) It follows from [7, Theorem 2.2]. □

3. Total graphs which are ring graphs

In this section, we classify all graphs whose total graphs are ring graphs. In [2, Theorem 3], Behzad proved the following theorem:

Theorem 3.1. *The total graph $T(G)$ of a graph G is planar if and only if $\Delta(G) \leq 3$, and if $\deg(v) = 3$ for a vertex v of G, then v is a cut-vertex.*

By combining Theorems 2.2 and 3.1, we obtain the following corollary immediately.

Corollary 3.2. *The total graph $T(G)$ of a graph G is planar if and only if its line graph $L(G)$ is a ring graph.*

Afterward, Chartrand et al. [3, Theorem 2] proved the following theorem for outerplanar graphs:

Theorem 3.3. *The total graph $T(G)$ of a graph G is outerplanar if and only if each component of G is a path.*

Analogous to these theorems, we state the following:

Theorem 3.4. *The following statements are equivalent for a graph G:*

a. *The total graph $T(G)$ of G is a ring graph.*
b. *The total graph $T(G)$ of G is outerplanar.*
c. *Each component of G is a path.*
d. *The line graph $L(G)$ of G is a forest.*

Proof. (a) $\Rightarrow$ (c) It is sufficient to show that $\Delta(G) \leq 2$ and every vertex of degree 2 in G is a cut-vertex. If $\Delta(G) \geq 3$, then there exists a vertex G that incidence with three edges. This vertex with these three edges is four mutually adjacent vertices in $T(G)$, i.e., the graph K_4, which contradicts Theorem 1.1.

Now, assume to the contrary there exists a vertex v such that $\deg(v) = 2$, but v is not a cut-vertex. It follows that v lies on a cycle of the form

$$v, v_1, v_2, \ldots, v_n, v.$$

Consider the following paths in $T(G)$:

$$\mathcal{P}_1 : v_1, (vv_1), (vv_n),$$
$$\mathcal{P}_2 : v_1, v_2, \ldots, v_n, (vv_n),$$
$$\mathcal{P}_3 : v_1, (v_1v_2), \ldots, (v_{n-1}v_n), (vv_n).$$

These paths are three vertex disjoint paths joining two non-adjacent vertices v_1 and (vv_n) of $T(G)$, which contradicts Lemma 2.1.

(c) $\Rightarrow$ (b) It is an immediate consequence from Theorem 3.3.

(b) $\Rightarrow$ (a) Since by [6, Proposition 2.17], every outerplanar graph is a ring graph, $T(G)$ is a ring graph.

(c) $\Longleftrightarrow$ (d) It follows from [3, Proposition 5]. □

Acknowledgments

The author would like to thank the referee for careful reading the article and valuable suggestions that improved the article.

Disclosure statement

No potential conflict of interest was reported by the author.

ORCID

Mahdi Reza Khorsandi http://orcid.org/0000-0001-5337-3570

References

[1] Behzad, M. (1965). *Graphs and Their Chromatic Numbers*. Ann Arbor, MI: ProQuest LLC.

[2] Behzad, M. (1967). A criterion for the planarity of the total graph of a graph. *Proc. Cambridge Philos. Soc.* 63:679–681.DOI: 10.1017/S0305004100041657

[3] Chartrand, G., Geller, D, Hedetniemi, S. (1971). Graphs with forbidden subgraphs. *J. Combin. Theory Ser. B* 10(1):12–41.

[4] Ghebleh, M, Khatirinejad, M. (2008). Planarity of iterated line graphs. *Discrete Math.* 308(1):144–147.

[5] Gitler, I., Reyes, E, Villarreal, R. (2007). Ring graphs and toric ideals. *Elect. Note. Discrete Math.* 28:393–400.

[6] Gitler, I., Reyes, E, Villarreal, R. (2010). Ring graphs and complete intersection toric ideals. *Discrete Math.* 310(3):430–441.

[7] Lin, H., Yang, W., Zhang, H, Shu, J. (2011). Outerplanarity of line graphs and iterated line graphs. *Appl. Math. Lett.* 24(7):1214–1217.

[8] Sedláček, J. (1963), (1964). Some properties of interchange graphs. *Theory of Graphs and its Applications: Proceedings of the Symposium held in Smolenice*. Cambridge, MA: Academic Press, pp. 145–150.

[9] Whitney, H. (1932). Congruent graphs and the connectivity of graphs. *Amer. J. Math.* 54(1):150–168.