AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

On eternal domination and Vizing-type inequalities

Keith Driscoll, William F. Klostermeyer, Elliot Krop, Colton Magnant & Patrick Taylor

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 708–712
https://doi.org/10.1016/j.akcej.2019.12.013

On eternal domination and Vizing-type inequalities

Keith Driscoll[a], William F. Klostermeyer[b], Elliot Krop[a], Colton Magnant[a,c], and Patrick Taylor[d]

[a]Department of Mathematics, Clayton State University, Morrow, GA, USA; [b]School of Computing, University of North Florida, Jacksonville, FL, USA; [c]Center for Mathematics and Interdisciplinary Sciences of Qinghai Province, Xining, Qinghai, China; [d]Shelton State Community College, Tuscaloosa, AL, USA

ABSTRACT
We show sharp Vizing-type inequalities for eternal domination. Namely, we prove that for any graphs G and H, $\gamma^\infty(G \boxtimes H) \geq \alpha(G)\gamma^\infty(H)$, where γ^∞ is the eternal domination function, α is the independence number, and $\boxtimes$ is the strong product of graphs. This addresses a question of Klostermeyer and Mynhardt. We also show some families of graphs attaining the strict inequality $\gamma^\infty(G\Box H) > \gamma^\infty(G)\gamma^\infty(H)$ where $\Box$ is the Cartesian product. For the eviction model of eternal domination, we show a sharp upper bound for $e^\infty(G \boxtimes H)$.

KEYWORDS
Domination; eternal domination; Vizing's conjecture

2010 MATHEMATICS SUBJECT CLASSIFICATION
05C69

1. Introduction

In this paper, we focus our attention on the relationship between the eternal domination numbers of two graphs and their Cartesian product. Questions of this type are related to the famous conjecture of V.G Vizing [12] which states that for any graphs G and H, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$, where $\Box$ is the Cartesian product of graphs and $\gamma(G)$ is the domination number. Many variations of this question exist and for more on the topic we recommend the survey [1].

For completeness, we define the *Cartesian product* of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1\Box G_2$, as a graph with vertex set $V_1 \times V_2$ and edge set $E(G_1\Box G_2) = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2 \text{ and } (u_1, u_2) \in E_1, \text{ or } u_1 = u_2 \text{ and } (v_1, v_2) \in E_2\}$.

As usual, the *independence number* of a graph G, $\alpha(G)$, is the maximum number of vertices which are pairwise non-adjacent. The *two-packing number* of G, $\rho(G)$, is the maximum number of vertices which are pairwise of distance at least three from each other. The *clique partition number*, $\theta(G)$, is the minimum number of cliques that partition the vertices of G. We also use the following more refined notion of the clique partition number.

Definition 1.1. Let $\mathcal{P}$ be the family of minimum clique partitions of G and for a fixed partition $\Theta \in \mathcal{P}$, let Θ_2 be the set of cliques in Θ of size at least 2 and Θ_1 be the set of cliques in Θ of size 1. We define $\theta_2(G) = \max_{\Theta\in\mathcal{P}}|\Theta_2|$ and $\theta_1(G) = \theta(G) - \theta_2(G)$.

In a dynamic version of the minimum dominating set problem, the first player (defender) can choose an initial subset of vertices in a graph G and place guards in those positions. The second player (attacker) chooses a vertex without a guard and attacks it. If the vertex is not at distance one from a guard, second player wins. Otherwise, the first player must move a guard from an adjacent (guarded) vertex to the vertex under attack. The second player may then choose another vertex to attack and the maneuvers can continue indefinitely, or until the second player wins. If the procedure continues forever, we say that our initial selection of vertices (and any subsequent configuration) is an *eternal dominating set*. The size of any minimum eternal dominating set of G is called the *eternal domination number* and is denoted by $\gamma^\infty(G)$.

In an alternate version of eternal domination, known as the *eviction model*, we begin as in the original model, however, the second player (attacker) attacks a vertex which is occupied by a guard. The first player (defender) must then move the guard to some adjacent vertex making sure that the set of vertices occupied by guards is a dominating set. If this procedure can continue indefinitely, then the initial selection of vertices for guards (and any subsequent configuration) is an *eternal eviction set*. The size of any minimum eternal eviction set of G is called the *eternal eviction number* and is denoted by $e^\infty(G)$. This model was first defined in [7].

Eternal dominating sets were first defined in [2]. There are many open questions in the field, references [4, 6, 8, 9], and [10] contain a list of some of these. We state here the previously posed question for Cartesian products, which is Question 7.9 in [8] and Problem 8 in [9].

Question 1.2. *Is it true for all graphs G and H, that* $\gamma^\infty(G\Box H) \geq \gamma^\infty(G)\gamma^\infty(H)$*?*

In this paper, we also consider the strong product. We note that strong products have been considered in the

CONTACT Keith Driscoll keithdriscoll@clayton.edu Department of Mathematics, Clayton State University, 2000 Clayton State Blvd, Morrow GA30260, USA.

context of eternal domination, albeit in the so-called *all-guards move* version of the problem, in [11].

Definition 1.3. The strong product $G \boxtimes H$ of graphs G and H is the graph with vertices $\{(a_i, b_j) : a_i \in V(G), b_j \in V(H)\}$. Two vertices are adjacent if and only if $a_i \simeq a_k$ and $b_j \simeq b_l$, where the symbol $\simeq$ means identical or adjacent.

Clearly, $\gamma(G \square H) \geq \gamma(G \boxtimes H)$, so any lower bound on the strong product addresses Question 1.2.

In Section 2, we show a bound related to Question 1.2,

$$\gamma^{\infty}(G \boxtimes H) \geq \alpha(G)\gamma^{\infty}(H)$$

Moreover, this inequality is sharp.

In particular, this means that Question 1.2 can be answered in the affirmative when $\gamma^{\infty}(G) = \alpha(G)$. However, this may not be the best possible bound for the Cartesian product since it is known that the eternal domination number cannot be bounded from above by a constant times the independence number for all graphs [5].

As a consequence of this result, we reduce the problem of solving Question 1.2 to graphs which contain an induced odd cycle of length at least 5 or the complement of such a cycle.

In Section 3, we show a series of strict Vizing-type inequalities for eternal domination functions. In particular, we show that $\gamma^{\infty}(G \square H) > \gamma^{\infty}(G)\gamma^{\infty}(H)$ if $\alpha(G) = \gamma^{\infty}(G)$ and $\alpha(H) = \gamma^{\infty}(H)$, that $\gamma^{\infty}(G \square P_{2k}) > \gamma^{\infty}(G)\gamma^{\infty}(P_{2k})$, and that $\gamma^{\infty}(G \square P_{2k+1}) > \gamma^{\infty}(G)\gamma^{\infty}(P_{2k+1})$, which requires a more involved proof.

In Section 4, we show an upper bound for the eviction model of the eternal domination number, that $e^{\infty}(G \boxtimes H) \leq \theta_2(G)\gamma(H) + \theta_1(G)e^{\infty}(H)$. We also conjecture the lower bound $e^{\infty}(G \boxtimes H) \geq \min\{\rho(G)e^{\infty}(H), \rho(H)e^{\infty}(G)\}$.

2. Eternal domination of product graphs

Theorem 2.1. *For any graphs G and H,*

$$\gamma^{\infty}(G \boxtimes H) \geq \alpha(G)\gamma^{\infty}(H).$$

Proof. Let $I = \{v_1, ..., v_{\alpha(G)}\}$ be a maximum independent set of G and D a minimum eternal dominating set of $G \boxtimes H$. We consider attacks on the vertices of the H-fiber, $H^{v_1} = \{(h, v_1) : h \in V(H)\}$.

By definition of $\gamma^{\infty}(H)$, there exists a series of attacks on H^{v_1} that necessitates $\gamma^{\infty}(H)$ guards to defend the vertices. Indeed, if H^{v_1} contains fewer than $\gamma^{\infty}(H)$ guards, then there exists a sequence of attacks on H^{v_1} so that some vertex (v_1, h) is not defended by a guard on H^{v_1}, and since D is an eternal dominating set, must be defended by a guard on a vertex (v, h) where $v \neq v_1, v \in V(G)$ and $h \in V(H)$. Again, if H^{v_1} contains fewer than $\gamma^{\infty}(H)$ vertices, we may attack the vertices of H^{v_1}, and move some guard not on H^{v_1} to H^{v_1}, increasing the number of guards on that H-fiber.

When the number of guards on H^{v_1} is at least $\gamma^{\infty}(H)$, repeat this procedure for each H-fiber, H^{v_i} where $1 < i \leq \alpha(G)$. Notice that since I is an independent set, no guard on any previously considered fiber, H^{v_i} may defend a vertex on a subsequent fiber H^{v_j} where $i < j$. Since every H-fiber under consideration eventually contains at least γ^{∞} guards and there are $\alpha(G)$ fibers, the inequality follows. □

An example of two graphs attaining equality in Theorem 2.1, is $G = C_4$ and $H = P_2$. Notice first that $\alpha(C_4) = 2$, and $\gamma^{\infty}(P_2) = 1$. To see that $\gamma^{\infty}(C_4 \boxtimes P_2) = 2$, call the vertices of P_2, u_1 and u_2. Notice that placing guards on any pair of vertices dominates $C_4 \boxtimes P_2$, unless that pair is of the form $((v, u_1), (v, u_2))$, which we call forbidden. Starting with any configuration of two guards other than this forbidden one, the forbidden configuration can always be avoided since for any sequence of attacks, there always exists a defense which avoids this forbidden configuration.

A *proper vertex coloring* of a graph G is an assignment of integers, representing colors, to the vertices of G so that no two vertices of the same color are adjacent. The minimum number of colors needed to color a graph G is called the *chromatic number* of G, written $\chi(G)$.

A graph G is called *perfect* if for every induced H of G, the size of a maximum clique in H, $\omega(H)$, is equal to $\chi(H)$.

Theorem 2.2 (Strong Perfect Graph Theorem [3]). *Every graph G is perfect if and only if no induced subgraph of G is an odd cycle of length at least 5 or the complement of such a graph.*

The following observation is well-known.

Proposition 2.3. *G is a perfect graph if and only if every induced subgraph H of G satisfies* $\alpha(H) = \theta(H)$.

Applying the fundamental inequality chain [10], $\alpha(G) \leq \gamma^{\infty}(G) \leq \theta(G)$, to the equality in Proposition 2.3 gives us the following result.

Corollary 2.4. *If G is a perfect graph, then* $\gamma^{\infty}(G) = \alpha(G)$.

3. Cartesian products

It will be useful to note in this section that for any graph G with $n > 1$ vertices and no isolated vertices, $\gamma^{\infty}(G) < n$. This is because $\theta(G) < n$ for such graphs. For paths of even order, we provide the following result.

Theorem 3.1. *For any graph G with $n > 1$ vertices and no isolated vertices and for any $\ell \geq 1$,*

$$\gamma^{\infty}(G \square P_{2\ell}) > \gamma^{\infty}(G)\gamma^{\infty}(P_{2\ell}).$$

Proof. Suppose, for a contradiction, that $G \square P_{2\ell}$ is eternally dominated by a set of $\gamma^{\infty}(G)\gamma^{\infty}(P_{2\ell})$ guards. Note that $\gamma^{\infty}(P_{2\ell}) = \ell$ for any integer ℓ. Denote by $G_1, G_2, ..., G_{2\ell}$ the horizontal fibers of $G \square P_{2\ell}$ that are each isomorphic to G. Attack vertices in G_{2i-1} until all the $\ell \times \gamma^{\infty}(G)$ guards are on vertices of G_{2i-1} for i with $1 \leq i \leq \ell$. Then since $\gamma^{\infty}(G) < n$, there must be a vertex in $G_{2\ell}$ with no adjacent guard a contradiction to the assumption that $\gamma^{\infty}(G)\gamma^{\infty}(P_{2\ell})$ was eternally dominated. □

For paths of odd order, we obtain the same conclusion. This next result also follows from the generalization stated after Theorem 3.4, however that is a different, and more complicated argument. We thus include the following for its simplicity and since the technique may prove useful in other cases.

Theorem 3.2. *For any graph G with $n > 1$ vertices, no isolated vertices, and such that $\gamma^\infty(G) \le n/2$ and for any $\ell \ge 1$,*

$$\gamma^\infty(G \Box P_{2\ell+1}) > \gamma^\infty(G)\gamma^\infty(P_{2\ell+1}).$$

Proof. Suppose, for a contradiction, that $G \Box P_{2\ell+1}$ is eternally dominated by a set of $\gamma^\infty(G)\gamma^\infty(P_{2\ell+1})$ guards. Denote by $G_1, G_2, \ldots, G_{2\ell+1}$ the horizontal fibers of $G \Box P_{2\ell}$ that are each isomorphic to G. Note that each vertex in G_2 is adjacent to exactly one vertex of G_1 and one vertex of G_3. Attack vertices in G_{2i-1} until all the guards are on vertices of G_{2i-1} for i with $1 \le i \le \ell + 1$. Then if $\gamma^\infty(G) < n/2$, there must be a vertex in G_2 with no adjacent guard. On the other hand, if $\gamma^\infty(G) = n/2$, then any attack of an unoccupied vertex in G_1 causes there to be a vertex in G_2 with no adjacent guard. □

Next, we show that the general strict inequality follows easily with added assumptions on G and H.

Theorem 3.3. *Let G and H be graphs with no isolated vertices so that $\alpha(G) = \gamma^\infty(G)$ and $\alpha(H) = \gamma^\infty(H)$. If G and H each have at least two vertices, then*

$$\gamma^\infty(G \Box H) > \gamma^\infty(G)\gamma^\infty(H).$$

Proof. Let A be a maximum independent set of size n in G and B be a maximum independent set of size m in H. Then there exists an independent set I of size nm in $G \Box H$ which is composed of the product of A and B. Since neither G nor H has isolated vertices, choose $v \in V(G)$ and $u \in V(H)$ so that $v \notin A$ and $u \notin B$. Notice that $(v, u) \cup I$ is an independent set in $G \Box H$, which means that $\gamma^\infty(G \Box H) \ge nm + 1 = \gamma^\infty(G)\gamma^\infty(H) + 1$. □

Note that Theorem 3.3 also follows from the inequality $\gamma^\infty(G) \ge \alpha(G)$ and the well-known fact that $\alpha(G \Box H) > \alpha(G)\alpha(H)$ when both G and H have edges.

Removing the additional assumptions that $\gamma^\infty(G) \le n/2$ or $\alpha(G) = \gamma^\infty(G)$ proves to be more difficult. We do this in the following theorem for $H = P_3$.

Let $G[X]$ denote the subgraph of G induced by vertex set X.

Theorem 3.4. *For any graph G with $n > 1$ vertices and no isolated vertices,*

$$\gamma^\infty(G \Box P_3) > \gamma^\infty(G)\gamma^\infty(P_3).$$

Proof. Let $h_1h_2h_3$ by the P_3 and G_1, G_2, G_3 the three horizontal fibers of $G \Box P_3$ that are each isomorphic to G, arranged in the obvious manner (i.e., each vertex in G_2 is adjacent to precisely its copy in G_1 and its copy in G_3).

Suppose to the contrary that $\gamma^\infty(G \Box P_3) \le \gamma^\infty(G)\gamma^\infty(P_3)$. Let $\gamma^\infty(G) = k$. From Theorem 3.2 above, we need only consider the case when $\gamma^\infty(G) > n/2$. We may assume that initially, k guards are in G_1 and k guards are in G_3, else we may carry out a sequence of attacks so that this is the case. Thus we may assume that $\gamma^\infty(G \Box P_3) = \gamma^\infty(G)\gamma^\infty(P_3)$. Since $\gamma^\infty(G) > n/2$, it is the case that G has at least three vertices; furthermore, we may assume that each vertex in G_2 is adjacent to either one or two guards (and each such guard is in G_1 or G_3). Partition the vertices of G (and therefore G_1 and G_3 in particular) into three sets:

- A is the set of vertices v with a guard on (v, h_1) but not on (v, h_3);
- B is the set of vertices v with a guard on (v, h_1) and on (v, h_3);
- C is the set of vertices v with a guard on (v, h_3) but not on (v, h_1).

Thus the vertices of G_1 are partitioned into sets A, B, C and the vertices of G_3 are likewise partitioned into three sets – and if v is in one set in G_1, the corresponding vertex in G_3 is in the corresponding set in G_3.

Note that $|A| + |B| = k = |B| + |C|$, and so it follows that $|A| = |C|$. Let us say that $|A| = q$. From this configuration of guards, if there was a sequence of attacks in G_1 that caused a guard in A, say on vertex v, to move, we would have a contradiction (since the corresponding vertex v in G_2 would no longer be protected). Hence, no guard in A ever moves and, by a symmetric argument, no guard in C ever moves in any sequence of attacks on $G_1 \cup G_3$. Therefore, $\gamma^\infty(G[V(G_1) - A]) = k - q$ and $\gamma^\infty(G[V(G_3) - C]) = k - q$.

We next claim that A and C are independent sets. To see this, suppose there were an edge in A. Since G – A can be eternally defended by just the guards initially on B, we need only to be able to protect the vertices in A by the guards on A. If A were not an independent set, this could be done with fewer than $q = |A|$ guards, which would contradict $\gamma^\infty(G) = k$.

Since C is an independent set in G_1 and there are no guards on C in G_1 (and since no attack in C is defended from A), there must be a matching of edges between B and C which covers all of C, say from vertices $\{b_1, \ldots, b_q\}$ to $\{c_1, \ldots, c_q\}$ in which guards on vertices of B defend attacks against matched vertices in C. We call such a matching a *defense matching*. Of course, by symmetry, there exists a defense matching between B and A, say from vertices $\{b'_1, \ldots, b'_q\}$ to $\{a_1, \ldots, a_q\}$, since G_1 is isomorphic to G_3. There must therefore be two such defense matchings, say M_1 from B to A and M_2 from B to C.

Claim 3.5. *In either G_1 or G_3, if some vertex $b \in B$ is a member of an edge ba in defense matching M, then for any other defense matching M', either ba is in M' or b is not a incident to any edge in M'.*

Proof. Suppose without loss of generality that $b \in B$ in G_3 belongs to a defense matching M where ba is an edge of the matching. This means that an attack on a may be defended by a guard on b (although it may also be defended by another guard). However, if the guard from b moves to a, then by the definition of A and B, in the new configurations of guards, a becomes a vertex of B and b becomes a vertex of A. Since A is an independent set, b must not be adjacent to a' for any $a' \in A$ from the original configuration, other than a. This means that b cannot be adjacent to any other vertex of A (or symmetrically C). □

We now consider cases based on whether or not M_1 and M_2 have vertices in common.

Case 1. *Suppose that some vertex $b \in B$ belongs to M_1 but not to M_2.*

This means that we can force a guard to move away from $b \in B$ (in G_3) to defend an attack on a vertex $a \in A$ in G_3. We now claim that $\gamma^\infty(G) < k$, for a contradiction. In G_1, the copy of vertex a is occupied by a guard. However, if we remove this guard from a, so that G_1 now has $k - 1$ guards, the remaining $k - 1$ guards can defend G_1. The vertex a is defended by its neighbor in B (on the vertex b) from the perfect matching described above and that guard is only allowed to move between a and b. Vertices in C are defended using the perfect matching described above (and those guards just move back and forth along the edges of the matching) and all remaining vertices have guards that are stationary. This means that G can be dominated by fewer than k vertices, a contradiction.

Case 2. *Case 1 does not apply.*

Then every vertex of B in M_2 is also in every defense matching of G_3, including M_1 in particular. Suppose bc is an edge of M_2 within G_1, and that ba is an edge of M_1 within G_3.

Subcase (i) Suppose that when the first guard moves from b to c in G_1, a guard on the corresponding c in G_3 is used to defend an attack on the vertex a in G_3. In this case the vertices a, b, and c form a triangle (all the edges exist). Then we claim that $\gamma^\infty(G) < k$. To see this, we may remove the guards from (v_1, a) and (v_1, b) and place a guard on (v_1, c). This guard now does the job of defending all the vertices that the other two did previously.

Subcase (ii) Lastly, suppose that when the first guard moves from b to c in G_1, the guard on the corresponding c in G_3 cannot move to defend an attack on the vertex a in G_3. This means that a guard on some other vertex $b' \in B$ in G_3 must defend the attack on a.

This means that there exists a defense matching M in G_3 different from M_1 in which $b'a$ is an edge. Since every vertex of B in M must also be a vertex of M_2, there exists some vertex $a' \in A$ in G_3 so that ba' is a edge of M. However, this contradicts Claim 3.5. □

Corollary 3.6. *For any graph G with $n > 1$ vertices and no isolated vertices and any odd integer $k \geq 3$,*

$$\gamma^\infty(G \square P_k) > \gamma^\infty(G)\gamma^\infty(P_k).$$

Proof. We induct on k. The base case for $k = 3$ is covered by Theorem 3.4. Consider k so that the statement holds for all odd values less than k. Suppose, to obtain a contradiction, that $\gamma^\infty(G \square P_k) > \gamma^\infty(G)\gamma^\infty(P_k)$. Let $P_k = \{v_1, \ldots, v_k\}$. Attack the fibers with odd indices, $G \times \{v_1\}, G \times \{v_3\}, \ldots, G \times \{v_k\}$ sufficiently so that all guards are on those fibers.

We now repeat the argument in the proof of Theorem 3.4 for the P_3 subgraph of P_k on the vertices v_1, v_2, v_3. □

Generalizing to all graphs G and H seems challenging. The special case where $H = K_{1,n}$ seems to be particularly important in this direction. A partial result for this special case is given by Theorem 3.3, which includes the special case when both G and H are $K_{1,n}$.

4. Eternal eviction of product graphs

We now consider the eviction model of eternal domination in strong products.

Theorem 4.1. *For any graphs G and H,*

$$e^\infty(G \boxtimes H) \leq \theta_2(G)\gamma(H) + \theta_1(G)e^\infty(H).$$

Proof. Let Θ be a minimum clique partition of G where $\Theta_2 \subseteq \Theta$ are the cliques of size at least 2 and $\Theta_1 \subseteq \Theta$ are the cliques of size 1. Furthermore, suppose $|\Theta_2| = \theta_2(G)$. From each clique in Θ_2 choose a vertex, to produce the set $\{v_1, \ldots, v_{\theta_2(G)}\}$. Let $\{u_1, \ldots, u_{\theta_1(G)}\}$ be the set of vertices in Θ_1.

For $1 \leq i \leq \theta_2(G)$, choose a $\gamma(H)$-set of vertices in $v_i \times H$ and place guards on those vertices. Notice that this collection of guards eternally dominates $\Theta_2 \boxtimes H$. This follows since v_i belongs to a clique of Θ_2 of size at least 2, and thus there is another vertex v in its clique. Therefore, every guard in $v_i \times H$ may be evicted to $v \times H$ and continue to dominate its former neighbors in $v_i \times H$.

For $1 \leq j \leq \theta_1(G)$, choose a $e^\infty(H)$-set of vertices in $u_j \times H$ and place guards on those vertices. Notice that this set of guards eternally dominates $u_j \times H$ by definition.

Thus, the selected set of guards of the required size eternally dominates $G \boxtimes H$ in the eviction model. □

This upper bound is attained by many graphs, for example $P_2 \boxtimes P_3$. For the lower bound, we believe the following inequality is true:

Conjecture 4.2. *For any graphs G and H,*

$$e^\infty(G \boxtimes H) \geq \min\{\rho(G)e^\infty(H), \rho(H)e^\infty(G)\}.$$

We note that this inequality is attained, for example, when $G = P_2$ and $H = P_3$ as noted above.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Brešar, B., Dorbec, P., Goddard, W., Hartnell, B., Henning, M., Klavžar, S, Rall, D. (2012). Vizing's conjecture: a survey and recent results. *J. Graph Theory* 69(1):46–76.

[2] Burger, A. P., Cockayne, E. J., Gründlingh, W. R., Mynhardt, C. M., van Vuuren, J. H, Winterbach, W. (2004). Infinite order domination in graphs. *J. Comb. Math. Comb. Comput.* 50: 179–194.

[3] Chudnovsky, M., Robertson, N., Seymour, P, Thomas, R. (2006). The strong perfect graph theorem. *Ann. Math.* 164(1): 51–229.

[4] Goddard, W., Hedetniemi, S. M., Hedetniemi, S. T. (2005). Eternal security in graphs, *J. Comb. Math. Comb. Comput*. 52: 169–180.

[5] Goldwasser, J, Klostermeyer, W. (2008). Tight bounds for eternal dominating sets in graphs. *Discrete Math* 308(12): 2589–2593.

[6] Goldwasser, J., Klostermeyer, W. F, Mynhardt, C. M. (2013). Eternal protection in grid graphs. *Utilitas Math.* 91:47–64.

[7] Klostermeyer, W., Lawrence, M, MacGillivray, G. (2016). Dynamic dominating sets: the eviction model for eternal domination. *J. Comb. Math. Comb. Comput.* 97:247–269.

[8] Klostermeyer, W. F, MacGillivray, G. (2009). Eternal dominating sets in graphs. *J. Comb. Math. Comb. Comput.* 68:97–111.

[9] Klostermeyer, W. F, Mynhardt, C. M. (2015). Domination, eternal domination, and clique covering. *Discuss. Math. Graph Theory* 35(2):283–300.

[10] Klostermeyer, W. F, Mynhardt, C. M. (2016). Protecting a graph with mobile guards. *Appl. Anal. Discrete Math.* 10(1): 1–29.

[11] McInerney, F., Nisse, N, Perennes, S. (2018). Eternal domination in grids, manuscript.

[12] Vizing, V. G. (1968). Some unsolved problems in graph theory. *Uspehi Mat. Nauk* 23 (6):117–134.