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## Gallai-Ramsey number of an 8-cycle

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### ABSTRACT

Given graphs  $G$  and  $H$  and a positive integer  $k$ , the Gallai-Ramsey number  $gr_k(G : H)$  is the minimum integer  $N$  such that for any integer  $n \geq N$ , every  $k$ -edge-coloring of  $K_n$  contains either a rainbow copy of  $G$  or a monochromatic copy of  $H$ . These numbers have recently been studied for the case when  $G = K_3$ , where still only a few precise numbers are known for all  $k$ . In this paper, we extend the known precise Gallai-Ramsey numbers to include  $H = C_8$  for all  $k$ .

### KEYWORDS

Gallai-Ramsey; rainbow triangle; 8-cycle

## 1. Introduction

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called *rainbow* if no two edges have the same color.

Colorings of complete graphs which contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai [7] first examined this structure under the guise of transitive orientations. This result was restated in [9] in the terminology of graphs and can also be traced back to [2]. For the following statement, a *trivial partition* is a partition into only one part.

**Theorem 1** ([7, 9]). *In any coloring of a complete graph with at least 2 vertices containing no rainbow triangle, there exists a non-trivial partition of the vertices (called a Gallai partition) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.*

In honor of this result, rainbow triangle-free colorings have been called *Gallai colorings*. The partition given by **Theorem 1** is called a *Gallai partition* or *G-partition* for short. Given a Gallai coloring of a complete graph and its associated G-partition, define the *reduced graph* of this partition to be the induced subgraph consisting of exactly one vertex from each part of the partition. Note that the reduced graph is a 2-colored complete graph.

When considering 2-colored complete graphs, a very natural problem to consider is the Ramsey problem of finding a monochromatic (one-colored) copy of some desired subgraph. Given a graph  $G$ , let  $R_k(G)$  denote the *k-color Ramsey number* of  $G$ , namely the minimum integer  $M$  such that for any  $m \geq M$ , any coloring of  $K_m$  using at most  $k$

colors contains a monochromatic copy of  $G$ . We refer to the dynamic survey [11] for results about Ramsey numbers.

Combining the concepts of Ramsey numbers and rainbow triangle free colorings, we arrive at the following definition of Gallai-Ramsey numbers.

**Definition 1.** Given two graphs  $G$  and  $H$ , the  $k$ -colored Gallai-Ramsey number  $gr_k(G : H)$  is defined to be the minimum integer  $n$  such that every coloring of the complete graph on  $n$  vertices using at most  $k$  colors contains either a rainbow copy of  $G$  or a monochromatic copy of  $H$ .

The general behavior of Gallai-Ramsey numbers when  $G$  is a triangle depends on the chromatic number of  $H$  in the following sense.

**Theorem 2** ([8]). *Let  $H$  be a fixed graph with no isolated vertices. Let  $k$  be an integer with  $k \geq 1$ . If  $H$  is not bipartite, then  $gr_k(K_3 : H)$  is exponential in  $k$ . If  $H$  is bipartite, then  $gr_k(K_3 : H)$  is linear in  $k$ .*

With this result in mind, the orders of magnitude in the following general bounds for cycles should not be surprising. For the sake of notation, let  $C_n$  be the cycle of order  $n$  and let  $P_n$  be the path of order  $n$ .

**Theorem 3** ([4, 10]). *Given integers  $n \geq 2$  and  $k \geq 1$ ,*  
$$(n - 1)k + n + 1 \leq gr_k(K_3 : C_{2n}) \leq (n - 1)k + 3n.$$

**Theorem 4** ([4, 10]). *Given integers  $n \geq 2$  and  $k \geq 1$ ,*  
$$n2^k + 1 \leq gr_k(K_3 : C_{2n+1}) \leq (2^{k+3} - 3)n \log n.$$

It is commonly believed that the lower bounds in these results are sharp. For  $gr_k(K_3 : C_n)$  with  $3 \leq n \leq 6$ , the exact numbers are shown below.

**Theorem 5** ([1, 3, 8]).

$$gr_k(K_3 : K_3) = \begin{cases} 5^{k/2} + 1 & \text{if } k \text{ is even,} \\ 2 \cdot 5^{(k-1)/2} + 1 & \text{otherwise.} \end{cases}$$

**Theorem 6** ([4]). For any positive integer  $k \geq 2$ ,  $gr_k(K_3 : C_4) = k + 4$ .

**Theorem 7** ([5]). For any positive integer  $k \geq 2$ ,  $gr_k(K_3 : C_5) = 2^{k+1} + 1$  and  $gr_k(K_3 : C_6) = 2k + 4$ .

These and other related results in the area are collected in the dynamic survey [6]. Our main result is the following which extends the known Gallai-Ramsey numbers for even cycles to include the next open case.

**Theorem 8.** For  $k \geq 1$ ,  $gr_k(K_3 : C_8) = 3k + 5$ .

The lower bound on the Gallai-Ramsey number in **Theorem 8** follows from **Theorem 3**. Our proof of **Theorem 8**, particularly the use of **Lemma 1** below, suggests that if the Gallai-Ramsey numbers were completely understood for all linear forests, then we may be able to establish the numbers for all cycles. This is somewhat complementary to the results of [10] where the bounds for even cycles were used to establish bounds for paths.

For more general notation, define  $gr_k(G : H_1, H_2, \dots, H_k)$  to be the minimum integer  $N$  such that every coloring of  $K_n$  for  $n \geq N$  using at most  $k$  colors contains either a rainbow copy of  $G$  or a monochromatic copy of  $H_i$  in color  $i$  for some  $i$ . For a shorthand version of this, we will also abuse notation and let  $gr_k(G : tH, (k-t)K) = gr_k(G : H, H, \dots, H, K, K, \dots, K)$  where  $H$  appears  $t$  times and  $K$  appears the remaining  $k-t$  times for some integer  $t$  with  $0 \leq t \leq k$ .

**Lemma 1.** For integers  $k$  and  $t$  with  $k \geq 2$  and  $0 \leq t \leq k$

$$gr_k(K_3 : tP_5, (k-t)P_3) \leq t + 4.$$

*Proof.* The proof is by induction on  $t$ . If  $t=0$ , the result is trivial since we are looking for a  $P_3$  in each color and it is easy to see that  $gr_k(K_3 : P_3) = 3$  for all  $k \geq 3$ . So suppose  $t \geq 1$ .

Let  $G$  be a Gallai colored  $K_n$  where  $n = t + 4$ , consider a  $G$ -partition of  $G$ , and let  $H$  be a largest part of this partition. If  $3 \leq |H| \leq n - 3$ , then there are three vertices in  $G \setminus H$  such that at least two of them have the same color on all edges to  $H$ . Since this graph contains a monochromatic  $K_{2,3}$ , this produces a monochromatic  $P_5$ , so we may assume that either  $|H| \leq 2$  or  $|H| \geq n - 2$ . Our first goal is to show that  $|H| \geq n - 2$ .

If  $|H| = 1$ , then  $G$  is simply a 2-coloring of  $K_n$  for  $n = t + 4$ . This contains the desired monochromatic  $P_5$  or  $P_3$  since  $R(P_3, P_5) = 5$  and  $R(P_5, P_5) = 6$ . So suppose  $|H| = 2$ . If  $t = 1$ , then to avoid creating a monochromatic  $P_5$ , there can be at most two vertices in  $G \setminus H$  with all color 1 on edges to  $H$  but since  $|G \setminus H| = 3$ , there must be exactly two such vertices. The remaining vertex has both edges in another color, making the desired monochromatic  $P_3$ . Next suppose  $t = 2$ , so  $n = t + 4 = 6$ . With  $|G \setminus H| = 4$ , there

must be precisely two pairs of vertices with each of the (first) two colors on edges to  $H$ , say red and blue. Each edge between these two pairs of vertices must be either red or blue, but any such edge would create a monochromatic  $P_5$ . Thus, we assume  $t \geq 3$ , so  $n = t + 4 \geq 7$ . Since  $|H| = 2$ , there are at least 5 vertices in  $G \setminus H$  so at least three of these vertices have the same color on edges to  $H$ . This contains a monochromatic  $K_{2,3}$ , which contains the desired monochromatic  $P_5$ . Together, these observations mean that we may assume that  $|H| \geq n - 2$ .

Since each vertex in  $G \setminus H$  has all one color on edges to  $H$ , the vertices of  $G \setminus H$  must have distinct colors on edges to  $H$  to avoid a monochromatic  $P_5$ . Note that these colors must be within the first  $t$  colors, say  $t$  (and  $t - 1$  if there are two such vertices), since otherwise this is already a monochromatic  $P_3$ . Also, if  $H$  contains a  $P_3$  in one of these colors, then using the vertex of  $G \setminus H$  with edges in the same color to  $H$ , we find a monochromatic  $P_5$ . By induction on  $t$  applied within  $H$ ,  $H$  contains either a monochromatic  $P_5$  in one of the first  $t - 1$  colors (or  $t - 2$  if  $|G \setminus H| = 2$ ) or a monochromatic  $P_3$  in one of the remaining  $k - (t - 1)$  colors (respectively  $k - (t - 2)$ ). This monochromatic path is either the desired path or can be used to construct the desired path as observed above, completing the proof of **Lemma 1**.  $\square$

In our arguments, we occasionally use classical Ramsey numbers. The following case will be helpful.

**Theorem 9** ([11]).  $R_2(C_8) = 11$ .

For the sake of our next lemma, we need an extra definition. Given sets of graphs  $\mathcal{G}$  and  $\mathcal{H}$ , define  $R(\mathcal{G}, \mathcal{H})$  to be the minimum integer  $N$  such that any 2-coloring of  $K_n$  (say using red and blue) for  $n \geq N$  contains either a copy of a graph in  $\mathcal{G}$  in red or a copy of a graph in  $\mathcal{H}$  in blue.

**Lemma 2.**  $R(\{C_4, P_5\}, \{C_4, P_5\}) = 5$ .

*Proof.* If we consider the unique 2-coloring of a  $K_5$  with no monochromatic triangles, then there is a  $C_5$  in each color. Thus, we also have the desired  $P_5$  in both colors. We may therefore assume that all other 2-colorings of  $K_5$  have a monochromatic triangle. Let  $a_1, a_2, a_3 \in A$  be a monochromatic  $K_3$ , say in red, and  $b_1, b_2 \in B$  be the two remaining vertices of the  $K_5$ . If all the edges from  $A$  to  $B$  are in one color, then there exists a monochromatic  $C_4$  in that color. Without loss of generality, let  $e$  be a red edge  $a_1b_1$ . To avoid a  $C_4$  in red, we get that the edges  $a_2b_1$  and  $a_3b_1$  are blue. To avoid getting a  $P_5$  in red we get that the edges  $a_2b_2$  and  $a_3b_2$  are also blue. Now we can clearly see that these blue edges make a  $C_4$  on  $b_1 - a_2 - b_2 - a_3 - b_1$ .  $\square$

## 2. Proof of Theorem 8

In order to prove **Theorem 8**, we actually prove the following slightly stronger result. For the precise statement, let  $G_3 = C_8$ ,  $G_2 = P_7$ ,  $G_1 = P_5$ , and  $G_0 = P_3$ . Note that all of these graphs are subgraphs of  $C_8$  and represent the results of

removing vertices from  $C_8$ . **Theorem 8** follows from **Theorem 10** by setting  $i_j = 3$  for all  $j$ .

**Theorem 10.** For  $k \geq 1$ , and for  $0 \leq i_j \leq 3$  for all  $1 \leq j \leq k$ ,

$$gr_k(K_3 : G_{i_1}, G_{i_2}, \dots, G_{i_k}) \leq \sum_{j=1}^k i_j + 5.$$

*Proof.* Let  $\Sigma = \sum i_j$ . The proof is by induction on  $\Sigma$ . If  $\Sigma = 0$ , the result is trivial since in each color we are only looking for  $P_3$  and it is easy to see that  $gr_k(K_3 : P_3) = 3$ . Thus, suppose  $\Sigma \geq 1$  so  $n \geq \Sigma + 5 \geq 6$ . Let  $G$  be a  $k$ -coloring of  $K_n$  with no rainbow triangle and no monochromatic  $G_{i_j}$  for any  $j$ . Let  $T$  be a largest set of vertices in  $G$  with the properties that

- each vertex in  $T$  has one color on all its edges to  $G \setminus T$ , and
- $|G \setminus T| \geq 4$ .

Note that  $T = \emptyset$  is possible. Let  $T_1, T_2, \dots, T_k$  denote the sets of vertices in  $T$  such that each vertex in  $T_j$  has all edges in color  $j$  to the vertices in  $G \setminus T$ . If  $|T_j| > i_j$ , then  $T_j \cup (G \setminus T)$  contains the desired monochromatic copy of a graph  $G_{i_j}$  in color  $j$ . Thus,  $|T_j| \leq i_j$  for all  $j$ . More generally, suppose  $T \neq \emptyset$ . Then, by induction on  $\Sigma$  applied within  $G \setminus T$ , there exists a copy, say  $H$ , of  $G_{i_{j-a}}$  in color  $j$  for some  $j$  where  $a = |T_j|$  with  $1 \leq a \leq 3$ . Then the graph consisting of edges of color  $j$  induced on  $H \cup T_j$  along with  $a - 1$  other vertices of  $G \setminus (T_j \cup H)$  contains a copy of  $G_{i_j}$  in color  $j$ , the desired subgraph. Thus, we may assume that  $T = \emptyset$ .

Consider a  $G$ -partition of  $G$  and let  $A$  be a largest part of this partition. Note that if  $|A| \geq 4$ , we can let  $T = G \setminus A$  and apply induction as above so we may assume  $|A| \leq 3$ . By the choice of  $A$ , the following fact becomes immediate.

**Fact 1.** Every part of the  $G$ -partition has order at most 3.

By **Lemma 2**, if there are at least five parts of order at least 2, then there is a monochromatic  $C_8$  since the 2-blow-up of a  $C_4$  or a  $P_5$ , replacing each vertex by a 2-set of vertices, each containing a  $C_8$ . We now prove several helpful claims, most of which provide a monochromatic  $C_8$  under certain restrictions.

**Claim 1.** If there are two parts of order 3 and at least five more vertices, then there exists a monochromatic  $C_8$ .

*Proof.* Let  $A$  and  $B$  be the two parts of order 3, say with all red edges between them. Let  $C = \{v_1, v_2, v_3, v_4, v_5\}$  be a set of 5 of the remaining vertices in  $G \setminus (A \cup B)$ . If all edges between  $C$  and  $A \cup B$  were blue, there is clearly a blue  $C_8$ , so suppose there are some red edges, say from  $v_1$  to  $A$ . To avoid creating a red  $C_8$ , all other vertices in  $C$  must have blue edges to  $B$ . To avoid creating a blue  $C_8$ , all of  $C$  must have red edges to  $A$  and so, by symmetry,  $v_1$  (and so all of  $C$ ) must also have blue edges to  $B$ . Any two red edges within  $C$  would produce a red  $C_8$  and any two blue edges within  $C$  would produce a blue  $C_8$  so there can be at most one red and at most one blue edge within  $C$ . Since, by Fact

1, all parts of the  $G$ -partition have order at most 3, this is clearly a contradiction, completing the proof of Claim 1.  $\square$

**Claim 2.** If there is one part of order 3, one part of order at least 2 and at least six additional vertices, then there exists a monochromatic  $C_8$ .

*Proof.* Let  $A$  be the set of order 3 and let  $B$  be the set of order at least 2 and assume all edges between  $A$  and  $B$  are red. By Claim 1, we may assume that  $|B| = 2$  and none of the additional vertices form a part of the  $G$ -partition of order 3. Label the additional vertices as  $v_i$  where  $1 \leq i \leq 6$ . If there is a vertex, say  $v_1$ , with red edges to  $B$  and two other vertices, say  $v_2$  and  $v_3$ , with red edges to  $A$ , then we have a red  $C_8$  using  $B - v_1 - B - A - v_2 - A - v_3 - A - B$ .

Suppose first that no vertex  $v_i$  has red edges to  $B$ , which means that all vertices  $v_i$  have all blue edges to  $B$ . No three vertices  $v_i$  can have blue edges to  $A$  since otherwise we could find a blue  $C_8$ , so this means that at least four vertices  $v_i$  must have red edges to  $A$ . Without loss of generality, let  $C = \{v_1, \dots, v_4\}$  be this set of four vertices. Any two red edges within  $C$  would allow for the construction of a red  $C_8$ . Also since no three of the vertices in  $C$  form a part of our  $G$ -partition, there can be at most two edges of colors other than red or blue within  $C$  and these must induce a matching. This means that at least three edges within  $C$  are blue and they must contain a blue  $P_4$ , say  $v_1 v_2 v_3 v_4$ . If both  $v_5$  and  $v_6$  have blue edges to  $A$ , then  $v_1 - v_2 - v_3 - B - v_5 - A - v_6 - B - v_1$  is the desired blue  $C_8$ . Thus, we may assume, without loss of generality, that  $v_5$  also has red edges to  $A$ . By the same argument, the blue graph induced on  $C \cup \{v_5\}$  contains a blue  $P_5$ , say  $P$  from  $v_1$  to  $v_5$ . Then  $v_1 - P - v_5 - B - v_6 - B - v_1$  produces a blue  $C_8$ .

The previous argument means that we may assume there is a vertex, say  $v_1$ , with red edges to  $B$ . As noted, this means that at most one other vertex, say  $v_2$ , can have red edges to  $A$ , so all other vertices in  $\{v_3, \dots, v_6\}$  have blue edges to  $A$ . Certainly no two of these vertices may have blue edges to  $B$ , meaning that at least three of them, say  $v_3, v_4, v_5$  have red edges to  $B$ . By the same argument as above with  $v_3$  in place of  $v_1$ , there can actually be at most one vertex  $v_i$  with red edges to  $A$ , meaning that there are five vertices with blue edges to  $A$ . Let  $C$  be this set of vertices and note that at least four of the vertices in  $C$  also have red edges to  $B$ . If there are two blue edges within  $C$ , we may construct a blue  $C_8$ , so suppose there is at most one. Since the vertices of  $C$  do not form parts of order 3 in our  $G$ -partition, there are at most two edges of colors other than red or blue within  $C$  and these must induce a matching. This leaves at least 7 red edges within  $C$ . Trivially  $C$  contains a red  $P_5$ , say  $P$ , starting and ending at vertices with red edges to  $B$ , say  $v_4$  and  $v_5$ . Then  $v_4 - P - v_5 - B - A - B - v_4$  is the desired red  $C_8$ , completing the proof of Claim 2.  $\square$

**Claim 3.** If there is one set of order at least 3 and at least nine more vertices, then there exist a monochromatic  $C_8$ .

*Proof.* Let  $A$  be the part of order 3. We define  $B$  to be the set of vertices with red edges to  $A$ , and  $C$  to be the set of



vertices with blue edges to  $A$ . By the pigeon hole principle at least five edges will have the same color edges to  $A$ , say  $|B| \geq 5$ .

If  $|B| = 5$ , then  $A \cup B$  induces a red  $K_{3,5}$  and  $|C| = 4$  so  $A \cup C$  induces a blue  $K_{3,4}$ . To avoid a rainbow triangle, each edge between  $B$  and  $C$  must be red or blue. Within this 2-colored  $K_{4,5}$ , the graph induced on the edges between  $B$  and  $C$ , there must be a monochromatic  $P_3$ . Regardless of the color or placement, this easily creates a monochromatic  $C_8$ .

Now suppose  $|B| \geq 6$ . Within  $B$ , there is at most one red edge since otherwise we could easily construct a red  $C_8$ . Also the edges that are neither red nor blue induce a matching since there is no part of the  $G$ -partition of order at least 3 within  $B \cup C$  (by Claim 1). In particular, this means that the minimum degree of the graph induced on the blue edges within  $B$  is at least  $|B| - 3$ , so there is a blue Hamiltonian cycle within  $B$ . If  $|B| = 8$ , then this is the desired blue  $C_8$  and if  $|B| = 9$ , we actually have even more blue edges so the blue graph is pancyclic and we again find a blue  $C_8$ . Otherwise, each vertex in  $C$  has at most one red edge to  $B$  because otherwise we could construct a red  $C_8$ . To avoid a rainbow triangle, this means that all but one edge from each vertex in  $C$  to  $B$  must be blue, meaning that each vertex in  $C$  can be absorbed into a blue Hamiltonian cycle of  $B$  to again create a blue  $C_8$ , completing the proof of Claim 3.  $\square$

**Claim 4.** *If there are three sets of order at least 2 and at least five more vertices, then there exists a monochromatic  $C_8$ .*

*Proof.* Let  $A$ ,  $B$  and  $C$  be the sets of order 2 and label the remaining vertices as  $v_i$  where  $1 \leq i \leq 5$ . First suppose  $A$ ,  $B$  and  $C$  have all red edges between them. If two of the other vertices, say  $v_1$  and  $v_2$ , have red edges to at least two of the sets, say  $A$  and  $B$ , we can find a red  $C_8$ ,  $v_1 - A - C - B - v_2 - B - C - A - v_1$ . Therefore, we can have either at most one vertex  $v_i$  with red edges to the sets or at most one set with red edges to the vertices  $v_i$ . This means that at least 4 vertices outside have blue edges to at least two of the sets. This induces a blue  $K_{4,4}$  which contains a blue  $C_8$ .

Thus, we may assume that the edges between  $A$  and  $C$  are blue while all edges from  $B$  to  $A \cup C$  are red. If none of the vertices  $v_i$  have red edges to  $A$  or  $C$ , then this induces a blue  $K_{4,5}$  which contains the desired blue  $C_8$ . Thus we may assume that at least one vertex, say  $v_1$ , has red edges to either  $A$  or  $C$ , say  $A$ . To avoid a red  $C_8$ , all other vertices  $v_i$  for  $i \geq 2$  must have blue edges to  $C$ . To avoid a blue  $C_8$ , no three of these vertices can have blue edges to  $A$ , so that means at least two of them, say  $v_2$  and  $v_3$ , have red edges to  $A$ . By symmetry, this means that  $v_1$  also has blue edges to  $C$ .

To avoid a red  $C_8$ , there can be at most one red edge within  $\{v_1, v_2, v_3\}$ . Since there is no part of our  $G$ -partition of order 3 among the vertices  $v_i$  and a part of order 2 within  $\{v_1, v_2, v_3\}$  would mean that the remaining vertex has two edges of the same color to the part, this means that there are at least 2 blue edges within these three vertices, say  $v_1v_2$  and  $v_2v_3$ . If both  $v_4$  and  $v_5$  have blue edges to  $A$ , then  $C - v_4 - A - v_5 - A - C - v_1 - v_2 - v_3 - C$  is a blue  $C_8$ . This means that one of  $v_4$  or  $v_5$ , say  $v_4$ , must have red edges to  $A$ .

To avoid a red  $C_8$ , there can be at most one red edge within  $\{v_1, v_2, v_3, v_4\}$  and since there is no part of order 3 and a 2-part would imply two edges of the same color, we must have a blue  $P_4$  within these vertices, say  $v_1v_2v_3v_4$ . Now if  $v_5$  has blue edges to  $A$ , then  $C - v_5 - A - C - v_1 - v_2 - v_3 - v_4 - C$  is a blue  $C_8$ . This means that  $v_5$  must also have red edges to  $A$ . By the same logic as above, there are at most 3 non-blue edges within  $\{v_1, \dots, v_5\}$ , so there is a blue  $P_5$ , say  $v_1v_2\dots v_5$ . Then  $C - A - C - v_1 - v_2 - v_3 - v_4 - v_5 - C$  is a blue  $C_8$ , completing the proof.  $\square$

By Theorem 9, there are at most 10 parts in our  $G$ -partition. By Fact 1, no part has order larger than 3 and by Lemma 2, there are at most 4 parts of order at least 2. By Claim 1, if there are 2 parts of order 3, then  $n \leq 10$ . By Claim 2, if there is one part of order 3 and at least one part of order 2, then  $n \leq 10$  again. By Claim 3, if there is any part of order 3, then  $n \leq 11$ . Thus, we may assume that either  $n \leq 11$  or all parts have order at most 2. By Claim 4, if there are 3 parts of order 2, then  $n \leq 10$  so we may assume there are at most 2 parts of order 2. With at most 10 parts total, this means that  $n \leq 12$ .

To complete the proof of Theorem 10, we consider cases based on small values of  $n$ , and therefore small values of  $\Sigma = n - 5$ .

**Case 1.**  $\Sigma = 1$ .

With loss of generality, suppose  $G_1 = P_5$  and  $G_i = P_3$  for  $i \geq 2$ . Therefore, we have  $G = K_6$  we want to show  $gr_k(K_3 : P_5, P_3, P_3, \dots, P_3) = 6$ . Since red is the only color allowed to contain adjacent edges, each other color induces only a matching. In fact, to avoid a rainbow triangle, the edges induced on all colors other than red together must induce a matching. The complement of this matching contains a  $P_5$  in red to easily complete the proof in this case.

**Case 2.**  $\Sigma = 2$ .

**Subcase 2.1.**  $gr_k(K_3 : P_7, P_3, \dots, P_3) = 7$

In this case, all colors other than red together induce a matching  $M$ . In  $K_7 \setminus M$ , it is easy to find a  $P_7$ .

**Subcase 2.2.**  $gr_k(K_3 : P_5, P_5, P_3, \dots, P_3) \leq 7$ .

This result follows from Lemma 1.

**Case 3.**  $\Sigma = 3$ .

**Subcase 3.1.**  $gr_k(K_3 : C_8, P_3, P_3, \dots, P_3) = 8$ .

In this case, all colors other than red together induce a matching  $M$ . In  $K_8 \setminus M$ , it is easy to find a  $C_8$ .

**Subcase 3.2.**  $gr_k(K_3 : P_7, P_5, P_3, \dots, P_3) = 8$ .

Since  $R_2(P_7, P_5) = 8$ , we may assume that there are at most 7 parts in the partition. Thus, there must exist a part of the partition of order at least 2. Other than the first two colors red and blue, all other colors together induce a matching so if we choose our  $G$ -partition to have the most possible parts, we may assume all parts have order at most 2.

First suppose there exists exactly one part of order 2, call it  $A$ . To avoid creating a blue  $P_5$ , there can be at most 2

vertices in  $G \setminus A$  with blue edges to  $A$ . Call these vertices  $A_{blue}$  and let  $A_{red}$  denote the remaining vertices of  $G \setminus A$ , those with all red edges to  $A$ . Note that  $|A_{red}| \geq 4$ . To avoid creating a red  $P_7$ , each vertex of  $A_{blue}$  has at most 2 red edges to  $A_{red}$ , so all other edges from  $A_{blue}$  to  $A_{red}$  must be blue. If  $|A_{blue}| = 2$ , then we have a blue  $P_5$  immediately using these blue edges to  $A_{red}$  so suppose  $|A_{blue}| \leq 1$ , meaning that  $|A_{red}| \geq 5$ . To avoid creating a red  $P_7$ , there can be at most 1 red edge within  $A_{red}$ , so there must be the claimed blue  $P_5$  within  $A_{red}$ .

Next suppose 2 sets have size 2, call them  $A$  and  $B$ . If blue appears between  $A$  and  $B$  then all other edges will be red to the 2 sets. This gives us a  $K_{4,4}$  which contains a  $P_7$ . Therefore the edges between  $A$  and  $B$  must be red. If there are at least 2 vertices outside with red to  $A$  and one vertex to  $B$  then there is a  $P_7$  in red. On the other hand if there are 2 vertices outside with blue to  $A$ , then we might as well have blue in between the 2 sets. Therefore we have found our desired  $P_7$  in one color and  $P_5$  in the other color.

**Subcase 3.3.**  $gr_k(K_3 : P_5, P_5, P_5, P_3, \dots, P_3) \leq 8$ .

This subcase follows from [Lemma 1](#).

The remaining cases, when  $\Sigma \in \{4, 5, 6, 7\}$ , follow from similar (albeit tedious) case analysis or by straightforward computer search.  $\square$

## Disclosure statement

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