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On the complexity of some quorum colorings problems of graphs

Rafik Sahbi

Department of Preparatory Training, Algiers Higher School of Applied Sciences, Algiers, Algeria

ABSTRACT

A partition $\pi = \{V_1, V_2, ..., V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $i \in \{1, ..., k\}$ is called a quorum coloring if for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N[v]$ of v have the same color as v. The maximum order of a quorum coloring of G is called the quorum coloring number of G and is denoted $\psi_q(G)$. In this paper, we give answers to two open problems stated in 2013 by Hedetniemi, Hedetniemi, Laskar and Mulder. In fact, we prove that the decision problem associated with $\psi_q(G)$ is NP-complete when the input graph is a 4-regular graph. We also show that the decision problem asks whether a given graph G has a quorum coloring of order at least 2 is NP-complete too.

KEYWORDS

Defensive alliance; quorum colorings; satisfactory partition; cost-effective partition; complexity

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05C15; 05C69

1. Introduction

Let $G = (V, E)$ be a simple graph with order $n = |V|$. The complement graph of G is denoted by $\bar{G}$. The graph induced in G by a subset S of V is denoted by $G[S]$. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $d_G(v) = |N_G(v)|$, while the degree of v in the complement graph $\bar{G}$ is denoted by $\bar{d}_G(v)$. More generally, the degree of a vertex v in $G[S]$ is denoted by $d_S(v)$, while the degree of v in $\bar{G}[S]$ is denoted by $\bar{d}_S(v)$.

A partition $\pi = \{V_1, V_2, ..., V_k\}$ of the vertex set V of a graph G into k color classes V_i, with $i \in \{1, ..., k\}$ is called a *quorum coloring* if for every vertex $v \in V$, at least half of the vertices in the closed neighborhood $N_G[v]$ have the same color as v. The color classes V_i are called *quorum classes*. The maximum order of a quorum coloring of G is called the *quorum coloring number* of G and is denoted by $\psi_q(G)$. A quorum coloring of order $\psi_q(G)$ is called a ψ_q*-coloring*. Quorum colorings were introduced by Hedetniemi, Hedetniemi, Laskar and Mulder [5]. The concept of Quorum colorings is closely related to the concept of defensive alliances in graphs introduced by Kristiansen, Hedetniemi and Hedetniemi [4]. Indeed, a *defensive alliance* in G is a subset S of V such that for every vertex $v \in S$, $|N_G[v] \cap S| \geq |N_G(v) \cap (V \setminus S)|$. Hence every color class of a ψ_q-coloring is a defensive alliance. Note that Haynes and Lachniet in [3] were the first to introduce the problem of partitioning the vertex set V into defensive alliances. This problem was also studied later by Eroh and Gera [2]. However, we will adopt in this paper the definitions and notations given in [5].

A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ having the property that no two edges in M have a vertex in common. The *matching number* $\beta_1(G)$ equals the maximum cardinality of a matching in G.

In [5], Hedetniemi et al. raised the following problems.

1. It is easy to see that for 1-regular graphs G of order n, $\psi_q(G) = n$. It is also easy to determine the value of $\psi_q(G)$ for any 2-regular graph G. In addition, since $\psi_q(G) = \beta_1(G)$ for 3-regular graphs G, it is easy to determine, in polynomial time, the value of $\psi_q(G)$ for 3-regular graphs. This leads us to the following decision problem:

   4-**REGULAR QUORUM**

   Instance: A $4-$regular graph $G = (V, E)$, positive integer $K \leq |V|$.

   Question: Does G have a quorum coloring of order at least K?

2. What is the complexity of the following decision problem:

   QUORUM-ONE

   Instance: Graph $G = (V, E)$.

   Question: Is $\psi_q(G) > 1$?

In 2018, Sahbi and Chellali [6] showed that the problem associated with $\psi_q(G)$, to which we refer as QUORUM-K, is NP-complete in general graphs, thus answering to an open problem stated in [5]. However, Questions 1 and 2 remain open. In this paper, we first show that the problem 4-REGULAR QUORUM is NP-complete by reducing the PARTITION INTO TRIANGLES problem in 4-regular graphs, to which we refer as 4-REGULAR PARTITION INTO TRIANGLES, to the problem 4-REGULAR QUORUM. The NP-hardness of the problem 4-REGULAR

CONTACT Rafik Sahbi r.sahbi@g.essa-alger.dz Department of Preparatory Training, Algiers Higher School of Applied Sciences, B.P. 474, Martyrs Square, Algiers 16001, Algeria.
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PARTITION INTO TRIANGLES was proven in 2011 by van Rooij et al. [7].

QUORUM-K
Instance: Graph $G=(V,E)$, positive integer $K \leq |V|$.
Question: Does G have a quorum coloring of order at least K?

4-**REGULAR PARTITION INTO TRIANGLES**
Instance: A 4-regular graph $G=(V,E)$.
Question: Can V be partitioned into 3-element sets $S_1, S_2, ..., S_{|V|/3}$ such that each S_i forms a triangle in G?

Then, we prove that the problem QUORUM-ONE is *NP*-complete by firstly showing that its balanced version, to which we refer as BALANCED QUORUM, is *NP*-complete and secondly by reducing the problem BALANCED QUORUM to the problem QUORUM-ONE. The problem BALANCED QUORUM was inspired from the BALANCED SATISFACTORY PARTITION problem introduced by Bazgan et al. [1] and the polynomial reduction from the problem BALANCED QUORUM to the problem QUORUM-ONE is an adaptation of the proof of Proposition 1 in [1].

BALANCED QUORUM
Instance: Graph $G=(V,E)$ of even order.
Question: Is there a partition $\{V_1, V_2\}$ of V such that $|V_1|=|V_2|$ and for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) +1 \geq d_{V_{3-i}}(v)$?

For a graph $G=(V,E)$ of even order, a bipartition $\{V_1, V_2\}$ of the vertex set V is a *balanced quorum coloring* if $|V_1|=|V_2|$ and each V_i, with $i \in \{1,2\}$ is a quorum class, that is, for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v)+1 \geq d_{3-i}(v)$.

BALANCED SATISFACTORY PARTITION
Instance: Graph $G=(V,E)$ of even order.
Question: Is there a partition $\{V_1, V_2\}$ of V such that $|V_1|=|V_2|$ and for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq d_{V_{3-i}}(v)$?

In fact, we prove the *NP*-completeness of the problem BALANCED QUORUM by reducing it to the BALANCED CO-SATISFACTORY PARTITION problem, to which we refer in this paper as BALANCED COST EFFECTIVE BIPARITITON. The *NP*-completeness of the problem BALANCED COST EFFECTIVE BIPARTITION was proven in 2005 by Bazgan et al. [1].

BALANCED COST EFFECTIVE PARTITION
Instance: Graph $G=(V,E)$ of even order.
Question: Is there a partition $\{V_1, V_2\}$ of V such that $|V_1|=|V_2|$ and for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \leq d_{V_{3-i}}(v)$?

2. Answer to Question 1

Theorem 1. *Problem 4-REGULAR QUORUM is NP-complete.*

Proof. 4-REGULAR QUORUM is a member of $\mathcal{NP}$, since we can check in polynomial time that any partition of the vertices of a 4-regular graph G into at least K color classes is a quorum coloring. Let G be a 4-regular graph of order $3q$, instance of both problems 4-REGULAR PARTITION INTO TRIANGLES and 4-REGULAR QUORUM, and set $K=q$. We will show that the instance G has a solution with respect to the first problem if and only if the same instance has a solution with respect to the second problem, that is, we will show that G has a partition into triangles if and only if $\psi_q(G) \geq K$.

Suppose that G has a partition into triangles and let $\pi = \{V_1, ..., V_q\}$ be such a partition. Then, one can easily see that each V_i is a quorum class. Hence, π is a quorum coloring of order $q=K$.

Conversely, suppose that G has a quorum coloring of order at least $K=q$ and let $\pi' = \{V'_1, ..., V'_q\}$ be such a coloring. Clearly, $|V'_i| \geq \lceil \frac{4+1}{2} \rceil = 3$, for every $i \in \{1, ..., q\}$. Therefore, $|V'_i| = 3$, for every $i \in \{1, ..., q\}$ for otherwise, $|V(G)| > 3q$. Since each class of π' is a quorum class, all vertices of each V'_i are necessarily pairwise adjacent, that is, each V'_i is a triangle. Consequently, π' is a partition of G into triangles. □

The NP-completeness of the problem 4-REGULAR QUORUM being now established, an interesting open problem is to determine more generally the complexity of the decision problem associated with the quorum coloring number in r-regular graphs, for $r > 4$.

3. Answer to Question 2

In [1], Bazgan et al. proved that problem BALANCED SATISFACTORY PARTITION is polynomial-time reducible to problem SATISFACTORY PARTITION, where SATISFACTORY PARTITION was defined as follows:

SATISFACTORY PARTITION
Instance: Graph $G=(V,E)$.
Question: Is there a partition $\{V_1, V_2\}$ of V such that for every $v \in V$, if $v \in V_i$ then $d_{V_i}(v) \geq d_{V_{3-i}}(v)$?

Proposition 2. [1] *BALANCED SATISFACTORY PARTITION is polynomial-time reducible to SATISFACTORY PARTITION.*

In the following proposition, we prove similarly that problem BALANCED QUORUM is polynomial time reducible to problem QUORUM-ONE by using the reduction of Proposition 2 and adapting its proof.

Proposition 3. *Problem BALANCED QUORUM is polynomial time reducible to problem QUORUM-ONE.*

Proof. Let $G=(V,E)$ be a graph, instance of BALANCED QUORUM on n vertices. The graph $G'=(V',E')$, instance of QUORUM-ONE, is obtained from G by adding two cliques of size $\frac{n}{2}, A=\{a_1, ..., a_{\frac{n}{2}}\}$ and $B=\{b_1, ..., \frac{n}{-2}\}$. In G', in addition to the edges of G, all vertices of V are adjacent to all vertices of A and B. Also, each vertex $a_i \in A$ is linked to all vertices of B except b_i, $i \in \{1, ..., n\}$.

Let $\{V_1, V_2\}$ be a balanced quorum coloring of G. Then, $\{V'_1, V'_2\} = \{V_1 \cup A, V_2 \cup B\}$ is a quorum coloring of G'. Indeed, we have

$$\begin{aligned}\text{for every } v \in A,\ d_{V_1'}(v)+1 &= d_A(v)+d_{V_1}(v)+1\\ &= |A|-1+|V_1|+1\\ &= |A|+|V_1| = |B|+\\ |V_2| &= d_{V_2'}(v)+1 > d_{V_2'}(v), \text{ and}\end{aligned}$$

$$\begin{aligned}\text{for every } v \in V_1,\ d_{V_1'}(v)+1 &= d_A(v)+d_{V_1}(v)+1\\ &= |A|+d_{V_1}(v)+1\\ &= |B|+d_{V_1}(v)+1 \geq\\ |B|+d_{V_2}(v) &= d_{V_2'}(v).\end{aligned}$$

By symmetry, we obtain analogously:

$$\text{for every } v \in B,\ d_{V_2'}(v)+1 > d_{V_1'}(v), \text{ and}$$

$$\text{for every } v \in V_2,\ d_{V_2'}(v)+1 \geq d_{V_1'}(v).$$

Conversely, let $\{V_1', V_2'\}$ be a quorum coloring of G'. Set $V_1' = V_1 \cup A_1 \cup B_1$ and $V_2' = V_2 \cup A_2 \cup B_2$ where $V_i \subseteq V, A_i \subseteq A$ and $B_i \subseteq B$, with $i \in \{1,2\}$. We claim that $\{V_1, V_2\}$ is a balanced quorum coloring of G. We consider the following two cases.

Case 1. Either $A_1 \cup B_1 = \emptyset$ or $A_2 \cup B_2 = \emptyset$.
Suppose without lost of generality that $A_2 \cup B_2 = \emptyset$. Therefore, $A_1 \cup B_1 = A \cup B$ (i.e.: $A_1 = A$ and $B_1 = B$). Since $\{V_1', V_2'\}$ is a quorum coloring of G', then we have by definition for every $v \in V_2, d_{V_2'}(v)+1 \geq d_{V_1'}(v)$. Hence, $d_{V_2}(v)+1 \geq d_{V_1}(v)+n$, or equivalently $d_{V_2}(v) \geq d_{V_1}(v)+n-1$. Thus, $G = K_n$ (n even) and any clique of even order has a balanced quorum coloring.

Case 2. Now assume that neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ is empty. We consider the following two subcases.

Subcase 2.1. $A_1 \cup B_1 = A$ and $A_2 \cup B_2 = B$ (i.e.: $B_1 = \emptyset$ and $A_2 = \emptyset$).
Since $\{V_1', V_2'\}$ is a quorum coloring of G', we have

$$\begin{aligned}\text{for every } v \in A,\ d_{V_1'}(v)+1 &= d_A(v)+d_{V_1}(v)+1\\ &= |A|-1+|V_1|+1\\ &= |A|+|V_1| \geq d_{V_2'}(v)\\ &= d_B(v)+d_{V_2}(v)\\ &= |B|-1+|V_2|\\ &= |A|-1+|V_2| \ (\text{since } |A|=|B|).\end{aligned}$$

Consequently,

$$|V_1| \geq |V_2|-1. \tag{1}$$

By symmetry, we obtain

$$\text{for every } v \in B,\ |V_2| \geq |V_1|-1. \tag{2}$$

Inequalities (1) and (2) imply that $|V_1| \in \{|V_2|-1, |V_2|, |V_2|+1\}$. Since n is even, we get $|V_1| = |V_2|$.
Then, by removing the vertices of $A \cup B$, we remove for each vertex of V_1 and each vertex of V_2, $\frac{n}{2}$ neighbors in its class and $\frac{n}{2}$ neighbors in the other class. Thus, $\{V_1, V_2\}$ is a balanced quorum coloring of G.

Subcase 2.2. Either A or B is cut by the partition, with one part in V_1' and the second part in V_2'.
We now show that if $a_i \in A_1$ for some i, then also $b_i \in B_2$ for the same i. Assume by contradiction that $b_i \in B_1$. We have,

$$\begin{aligned}d_{V_1'}(a_i)+1 \geq d_{V_2'}(a_i) &\iff d_{A_1}(a_i)+d_{B_1}(a_i)+d_{V_1}(a_i)+1\\ &\geq d_{A_2}(a_i)+d_{B_2}(a_i)+d_{V_2}(a_i)\\ &\iff |A_1|-1+|B_1|-1+|V_1|+1\\ &\geq |A_2|+|B_2|+|V_2| \iff |V_1'| \geq |V_2'|+1\end{aligned} \tag{3}$$

Now, let $a_j \in A_2$. Then, either $b_j \in B_1$ or $b_j \in B_2$.
If $b_j \in B_2$, by analogy with the previous case we obtain,

$$|V_2'| \geq |V_1'|+1 \tag{4}$$

However, inequality (4) contradicts inequality (3). Hence, $b_j \in B_1$.
Since $\{V_1', V_2'\}$ is a quorum coloring of G', then

$$\begin{aligned}d_{V_2'}(a_j)+1 \geq d_{V_1'}(a_j) &\iff d_{A_2}(a_j)+d_{B_2}(a_j)+d_{V_2}(a_j)+1\\ &\geq d_{A_1}(a_j)+d_{B_1}(a_j)+d_{V_1}(a_j)\\ &\iff |A_2|-1+|B_2|+|V_2|+1\\ &\geq |A_1|+|B_1|-1+|V_1| \iff |V_1'| \leq |V_2'|+1\end{aligned} \tag{5}$$

(3) and (5) imply that $|V_1'| = |V_2'|+1$, which is impossible since $|V(G')| = 2n$.
In conclusion, $|A_1| = |B_2|$ and $|A_2| = |B_1|$. So, $|A_1 \cup B_1| = |A_2 \cup B_2| = \frac{n}{2}$.
Moreover, we have

$$\begin{aligned}&\text{for every } v \in A_1:\ d_{V_1'}(v)+1 \geq d_{V_2'}(v)\\ &\iff d_{A_1}(v)+d_{B_1}(v)+d_{V_1}(v) \geq d_{A_2}(v)+d_{B_2}(v)+d_{V_2}(v)\\ &\iff |A_1|-1+|B_1|+|V_1|+1 \geq |A_2|+|B_2|-1\\ &\quad +|V_2| \iff |V_1| \geq |V_2|-1,\end{aligned} \tag{6}$$

and by symmetry for every $v \in A_2$,

$$|V_1| \leq |V_2|+1\ . \tag{7}$$

Inequalities (6) and (7) imply that $|V_1| \in \{|V_2|-1, |V_2|, |V_2|+1\}$. However, $n = |V_1|+|V_2|$ is even and hence, $|V_1| = |V_2|$. Thus, $\{V_1, V_2\}$ is a balanced quorum coloring of G. □

We state now our *NP*-completeness result.

Theorem 4. *Problem QUORUM-ONE is NP-complete.*

Proof. Clearly, QUORUM-ONE is in $\mathcal{NP}$. We reduce BALANCED COST EFFECTIVE PARITION to BALANCED QUORUM which shows the *NP*-completeness of BALANCED QUORUM. Proposition 3 implies the *NP*-completeness of QUORUM-ONE. The reduction is as follows.

Let G be a graph of even order, instance of BALANCED COST EFFECTIVE PARTITION, and consider $\bar{G}$ as instance of BALANCED QUORUM. We will show that the instance G has a solution if and only if the instance $\bar{G}$ has a solution.

Suppose that the instance G of BALANCED COST EFFECTIVE PARTITION has a solution $\{V_1, V_2\}$. This is equivalent to,

$$\begin{aligned}&\text{for every } i \in \{1,2\},\ d_{V_i}(v) \leq d_{V_{3-i}}(v)\\ &\iff |V_i|-1-d_{V_i}(v) \geq |V_i|-1-d_{V_{3-i}}(v)\\ &\iff \bar{d}_{V_i}(v)+1 \geq |V_{3-i}|-d_{V_{3-i}}(v)\\ &\iff \bar{d}_{V_i}(v)+1 \geq \bar{d}_{V_{3-i}}(v)\end{aligned} \tag{8}$$

Inequality (8) means that $\{V_1, V_2\}$ is a balanced quorum coloring of $\bar{G}$. □

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ORCID

Rafik Sahbi http://orcid.org/0000-0003-1175-5137

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