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AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: B. S. Panda, Shaily Verma & Yash Keerti (2020) On the total and AVD-total coloring of graphs, AKCE International Journal of Graphs and Combinatorics, 17:3, 820-825, DOI: 10.1016/j.akcej.2019.10.004

To link to this article: https://doi.org/10.1016/j.akcej.2019.10.004

Published online: 22 Apr 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 820–825
https://doi.org/10.1016/j.akcej.2019.10.004

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On the total and AVD-total coloring of graphs

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ABSTRACT

A *total coloring* of a graph G is an assignment of colors to the vertices and the edges such that (*i*) no two adjacent vertices receive same color, (*ii*) no two adjacent edges receive same color, and (*iii*) if an edge e is incident on a vertex v, then v and e receive different colors. The least number of colors sufficient for a total coloring of graph G is called its *total chromatic number* and denoted by $\chi''(G)$. An *adjacent vertex distinguishing (AVD)-total coloring* of G is a total coloring with the additional property that for any adjacent vertices u and v, the set of colors used on the edges incident on u including the color of u is different from the set of colors used on the edges incident on v including the color of v. The *adjacent vertex distinguishing (AVD)-total chromatic number* of G, $\chi''_a(G)$ is the minimum number of colors required for a valid AVD-total coloring of G. It is conjectured that $\chi''(G) \leq \Delta(G)+2$, which is known as *total coloring conjecture* and is one of the famous open problems. A graph for which the total coloring conjecture holds is called totally colorable graph. The problem of deciding whether $\chi''(G)=\Delta(G)+1$ or $\chi''(G=\Delta(G)+2$ for a totally colorable graph G is called the classification problem for total coloring. However, this classification problem is known to be NP-hard even for bipartite graphs. In this paper, we give a sufficient condition for a bipartite biconvex graph G to have $\chi''(G)=\Delta(G)+1$. Also, we propose a linear time algorithm to compute the total chromatic number of chain graphs, a proper subclass of biconvex graphs. We prove that the total coloring conjecture holds for the central graph of any graph. Finally, we obtain the AVD-total chromatic number of central graphs for basic graphs such as paths, cycles, stars and complete graphs.

KEYWORDS

Total coloring; AVD-total coloring; total coloring conjecture; chain graphs; central graphs

1. Introduction

The graphs considered in this paper are simple and undirected. For a graph $G=(V,E)$, the sets $N(v)=\{u \in V(G) | uv \in E\}$ and $N[v]=N(v)\cup\{v\}$ denote the *open neighborhood* and *closed neighborhood* of a vertex v, respectively. The *degree* of a vertex v is $|N(v)|$ and is denoted by $d(v)$. If $d(v)=1$, then v is called a *pendant vertex*. For $S\subseteq V$, let $G[S]$ denote the subgraph induced by G on S. A central graph of a graph $G=(V,E)$, denoted by $C(G)$, is obtained by joining all the non-adjacent vertices in G and subdividing each edge of G exactly once. The set of vertices of $C(G)$ is then given by $V(C(G))=V_1\cup V_2$ where V_1 contains the vertices of graph G and V_2 contains the new added vertices. The edge set $E(C(G))=E(\bar{G})\cup\{uv, uv' | u$ subdivide the edge e where $e=vv'\in E(G)\}$.

A bipartite graph is a graph whose vertex set can be partitioned into two sets X and Y such that every edge has its endpoints in different sets. Given a bipartite graph $G=(X,Y,E)$, a *chain ordering* of X is an ordering $\sigma(X)=(x_1,x_2,...,x_n)$ such that the neighborhoods of the vertices of X form a chain, that is, $N(x_1)\subseteq N(x_2)\subseteq\cdots\subseteq N(x_n)$. A bipartite graph $G(X,Y,E)$ is said to be *chain graph* if there exists a chain ordering of X. If G is a chain graph, then the neighborhoods of the vertices of Y also form a chain, that is, there exists $\sigma(Y)=\{y_1,y_2,...,y_t\}$ such that $N(y_1)\supseteq N(y_2)\supseteq\cdots\supseteq N(y_t)$. Next, given a bipartite graph $G=(X,Y,E)$ a *convex ordering* of set X is a linear ordering $\sigma(X)=(x_1,x_2,...,x_n)$ such that for every vertex y in Y, neighbors of y are consecutive vertices in $\sigma(X)$. A bipartite graph is said to be a *biconvex graph*, if there exist convex orderings for both X and Y, simultaneously. The relationship between these two subclasses of bipartite graphs is as follows:

$$\text{Bipartite graphs} \supset \text{Biconvex graphs} \supset \text{Chain graphs}$$

A *total coloring* of a graph G is an assignment of colors to the vertices and the edges such that (*i*) no two adjacent vertices receive same color, (*ii*) no two adjacent edges receive same color, and (*iii*) if an edge e is incident on a vertex v, then v and e receive different colors. The least number of colors sufficient for a total coloring of graph G is called its *total chromatic number* and denoted by $\chi''(G)$. The concept was first introduced by Behzad in 1965 [1]. Clearly, $\chi''(G)\geq\Delta(G)+1$. The well known *total coloring conjecture* which has been posed independently by Behzad [1] and Vizing [14] states that, $\chi''(G)\leq\Delta(G)+2$. The graphs with total chromatic number equals to $(\Delta(G)+1)$ and $(\Delta(G)+2)$ are called *type 1 graph* and *type 2 graph*, respectively. The conjecture is still open for general graphs. However, it has

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been verified for many graph classes such as interval graphs [2], dually chordal graphs [6], unicord-free graphs [9], split graphs [3], bipartite graphs etc. The classification problem, that is, to classify the graphs as type 1 and type 2 graphs, for bipartite graphs is known to be NP-complete [10]. In this paper, we investigate the classification problem in subclasses of bipartite graphs.

Let $G = (V, E)$ be a graph with order at least 2, k be a positive integer and f be a total k-coloring of G. Define the set, $C(u) = \{f(u)\} \cup \{f(uv) | uv \in E(G)\}$ for each vertex $u \in V(G)$. The coloring f is called an *adjacent vertex distinguishing (AVD) total k-coloring* of graph G, if for every edge $uv \in E(G), C(u) \neq C(v)$. The least integer k such that there exists an AVD-total k-coloring of G, is called *adjacent vertex distinguishing (AVD) total chromatic number* of G and denoted by $\chi'_a\prime(G)$. This concept was introduced by Zhang et al. [18] in 2005. The authors also posed the *AVD-total coloring conjecture* which states that, for any graph G, $\chi'_a\prime(G) \leq \Delta(G) + 3$. This conjecture is open for general graphs. However, the conjecture has known to be true for many families of graphs such as complete graphs [18], hypercubes [4], indifference graphs [12], planar graphs with $\Delta(G) \geq 14$ [16], outerplanar graphs [17], 4-regular graphs [11], graphs with $\Delta(G) = 3$ [5, 8, 15].

The method of pullbacks for total colorings was first used in [6] to establish total coloring results for dually chordal graphs. We use this method in the next section to establish result for biconvex bipartite graphs. First, we explain the method of pullback here.

A pullback from G to G' is a function $f : V(G) \to V(G')$, such that:

1. f is a homomorphism, i.e., if $xy \in E(G)$ then $f(x)f(y) \in E(G')$.
2. f is injective when restricted to $N(x)$, for all $x \in V(G)$.

The main use of pullbacks is to transfer colorings.

Theorem 1.1. [6] *If f is a pullback from G to G' and τ' is a total coloring of G', then the color assignment τ defined by:*

$\tau(x) = \tau'(f(x))$

$\tau(xy) = \tau'(f(x)f(y))$ *is a total coloring of G.*

The organization of the paper is as follows: In Section 2, we study the total coloring in biconvex graphs and give an algorithm to compute total chromatic number of chain graphs. In Section 3, we validate the total coloring conjecture for central graph of any graph. In Section 4, we investigate AVD-total coloring in central graph of path, cycle, star and complete graph.

2. Total coloring in subclasses of bipartite graphs

In this section, we study the total coloring problem for chain graphs and biconvex graphs, which are subclasses of bipartite graphs. We give a sufficient condition for a biconvex bipartite graph to be type 1 graph.

Theorem 2.1. *Let $G = (X, Y, E)$ be a bipartite biconvex graph. If all the vertices of maximum degree Δ belong to the same part, then $\chi''(G) = \Delta(G) + 1$.*

Proof. Without loss of generality, assume that all the vertices of maximum degree Δ belong to X. Let $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_t\}$. To prove this result we are going to use the method of pullbacks from the given graph G to complete bipartite graph $K_{\Delta-1,\Delta}$. We know that $K_{\Delta-1,\Delta}$ has a total coloring using $\Delta + 1$ colors. So if we can show a valid pullback function, using Theorem 1.1 we are done. Assume that $K_{\Delta-1,\Delta} = (U, V, E)$ where $U = \{u_1, u_2, u_3, ..., u_{\Delta-1}\}$ and $V = \{v_1, v_2, v_3, ..., v_\Delta\}$. Consider the following pullback function f from G to $K_{\Delta-1,\Delta}$:

$f(x_i) = u_{\text{i mod } \Delta-1}$ for all $i, 1 \leq i \leq n$,
$f(y_j) = v_{\text{j mod } \Delta}$ for all $j, 1 \leq j \leq t$.

It is easy to see that f is a homomorphism as for any edge $uv \in G, f(u)f(v) \in K_{\Delta-1,\Delta}$. Now we have to show that, f is injective on $N(v)$ for $v \in G$. Note that since G is biconvex, the neighbors of any vertex v in X (or Y) are consecutive in Y (or X). From this, it follows that the indices of the neighbors of v are always going to give different residues modulo Δ, if $v \in X$ and modulo $\Delta - 1$ if $v \in Y$. It implies that f is injective for any $v \in G$. □

Next, we study the total coloring in chain graphs which is a subclass of biconvex graphs.

Theorem 2.2. *A connected chain graph $G = (X, Y, E)$ with $|X| \neq |Y|$, is a type 1 graph.*

Proof. Let $G = (X, Y, E)$ be a connected chain graph with $|X| \neq |Y|$. Let $|X| = n$ and $|Y| = m$, where $n > m$. It implies that, $\Delta(G) = n$. We take a pullback function f which is identity function from graph G to complete graph $K_{m,n}$. We know that, $\chi''(K_{m,n}) = n + 1$ as $n > m$. Assume that τ' be the $(n + 1)$-total coloring of $K_{m,n}$. Therefore, the total coloring obtained from τ', defined in Theorem 1.1 is a $(n + 1)$-total coloring of graph G. Hence, G is type 1 graph. □

Lemma 2.1. *Let $G = (X, Y, E)$ be a connected chain graph such that $|X| = |Y|$. If G has a pendant vertex, then G is type 1 graph.*

Proof. Let $G = (X, Y, E)$ be a connected chain graph with $|X| = |Y| = n$. Let v be a pendant vertex of G. Note that $G' = G \setminus \{v\}$ is a connected chain graph with partite sets of different sizes. Therefore G' can be total-colored using $(n + 1)$ colors, by Theorem 2.2. Let u be the only vertex adjacent to v in G. It follows that, u has at most $n - 1$ neighbors in G' and so at least one color is free on vertex u which can be used to color the edge uv. Now vertex v can be colored with any available color. Thus, we obtained a total coloring of G using $n + 1$ colors. Hence G is a type 1 graph. □

Lemma 2.2. *In any total coloring of a complete bipartite graph $K_{n,n-1} = (X, Y, E), |X| = n,$ with $(n + 1)$ colors, the vertices in X must all get distinct colors.*

Proof. Let f be a total coloring of $K_{n,n-1}$ with $(n + 1)$ colors. Assume that the vertices $u, v \in X$ such that $f(u) = f(v) = c$. Now, for any vertex $w \in Y$, $d(w) = n$ and so $f(w) \neq c$. Therefore, exactly one of the edges incident on w will be colored with color c. It follows that every vertex in Y must have an edge colored c incident on it because degree of each vertex in Y is n and number of colors used is $n+1$. Note that, none of the c colored edge is incident on u and v. Since there are $(n-2)$ vertices in $X-\{u,v\}$, two c colored edges must be adjacent which is a contradiction. Thus, all vertices in X get distinct colors. □

Next result we prove with the help of Lemma 2.2.

Theorem 2.3. *Let $G=(X,Y,E)$ be a connected chain graph with $|X|=|Y|=n$ and G has no pendant vertex. If the minimum degree in any partite set is $(n-1)$, then G is type 2 graph.*

Proof. Let the minimum degree in partite set X be $(n-1)$. Therefore, all the vertices in Y except one will have degree n. Let $v\in Y$ be such that $d(v)<n$. Note that the graph $G-\{v\}$ is complete bipartite graph $K_{n,n-1}$. It implies that, G be type 1 and there exists a $(n+1)$-total coloring of G. Using the previous Lemma 2.2, all the vertices in X (the larger part) get different colors. It implies that, all the vertices in $Y\setminus\{v\}$ get the same color, say color 1. Also, $d(v)>1$ as G does not have a pendant vertex. Suppose that v is adjacent to two vertices $y,z\in X$. Now, 1 must be the only color missing at both these vertices in the total coloring of $G\setminus\{v\}$. Thus, the edges vy and vz both must be colored with color 1 which is a contradiction. Thus, G must be type 2. □

Theorem 2.4. *Let $G=(X,Y,E)$ be a connected chain graph with $|X|=|Y|=n$. If k^{th} minimum degree in any partite set is at most k for any $k, 1\le k\le n-1$, then G is type 1 graph.*

Proof. Let G be a chain graph with degree sequence of X being $D(X)=(k,k,\ldots k(k \text{ times}),\ \ n,n,\ldots,n(n-k \text{ times}))$ for any $k, 1\le k\le n-1$. We show that the result holds for graph G. Any other graph G' satisfying the hypothesis of the theorem is a subgraph of graph G, for some value of k, with same maximum degree and so G' can be colored by restricting the total coloring of G to G'. Observe that, the degree sequence of Y is $D(Y)=(n,n,\ldots,n(k\ \textit{times}),n-k,n-k,\ldots,(n-k\ \textit{times}))$ as G is a chain graph. Therefore without loss of generality, $k\le n/2$. We partition the set X into two sets X_1 and X_2 where the former has the vertices with degree n, and the latter has the vertices with degree k. Now, we consider the subgraph of $G-G(X_1,Y)$ and totally color it using $n+1$ colors as follows:

The vertices in X_1 get color $n+1$, the ith vertex in chain ordering of Y gets color i and the edge x_iy_j gets color $(i+j)(\text{mod } n)$, for $1\le i\le n-k$ and $1\le j\le n$.

Now, we extend above coloring to a total coloring of G as follows:

Color the edge y_ix_j from Y to X_2 with color $(i+n-k+j)(\text{mod}(n+1))$ and color the vertex x_j in X_2 with color $n-k+j$, for $1\le j\le k$.
Since $k\le n/2$, the coloring is proper. Thus the graph G is Type 1. □

Unfortunately, none of the above conditions is necessary and thus the results presented above are only partial. The complete result, i.e., a necessary and sufficient condition for a chain graph to be type 1 turned out to be a little more involved and could only be derived using Hilton's [7] result for Nearly Complete Graphs. Thus, we must take a detour and briefly discuss his result before presenting the final result for Chain graphs. A subset $M\subset E$ of a graph $G=(V,E)$ is called a matching if no two edges in M are adjacent. The matching problem for a graph is to find a matching of maximum cardinality.

Theorem 2.5. [7] *Let J be a subgraph of $K_{n,n}$ with $n\ge 1$ and $e=|E(J)|$ and $j(J)$ be the maximum size (i.e., number of edges) of a matching in J. Then $\chi''(K_{n,n}\setminus E(J))=n+2$ if and only if $e+j\le n-1$.*

Hilton successfully linked the problem of total coloring in bipartite graphs to the maximum matching problem in bipartite graphs. Note that the maximum matching problem in bipartite graphs can be solved in polynomial time. It can only be used to determine the total chromatic number of subgraphs of $K_{n,n}$ with maximum degree n. But for general bipartite graphs which have arbitrary maximum vertex degree, this result is not useful. Fortunately for us, chain graphs are indeed such graphs with maximum degree n. As a corollary of the above result, we can easily get the following necessary and sufficient condition for a chain graph to be type 1.

Corollary 2.1. *A connected chain graph $G=(X,Y,E)$ with $|X|=|Y|=n$ is type 1 if and only if $n^2-m\ge n-j(K_{n,n}\setminus E(G))$ where m is number of edges in G and $j(K_{n,n}\setminus E(G))$ is the size of maximum matching in graph $K_{n,n}\setminus E(G)$.*

In our problem, we need to only find the size of the maximum matching and that too in $K_{n,n}\setminus E(G)$ which can shown to be a chain graph. Now the Hopcroft–Karp algorithm to find a maximum matching in bipartite graph, takes $O(\sqrt{nm})$ time, where $m=|E|$. For chain graphs, we can find the size of the maximum matching faster as the vertices can be ordered by inclusion. Therefore, we present a new approach for finding the size of maximum matching in a chain graph.

We are given degree sequence of vertices of X as $D(X)=(d_1,d_2,\ldots,d_n)$ of chain graph $G=(X,Y,E)$. We first convert degree sequence of vertices of X into a frequency map $M=\{(d_1,f_1),(d_2,f_2),\ldots,(d_k,f_k)\}$ where d_i's are strictly increasing and f_i's denote the number of vertices in X with degree d_i. This frequency map would be the input for our algorithm.

Algorithm 1: Algorithm for finding size of maximum matching in chain graph

function *MaxMatching* (M);
Input: frequency map $M=\{(d_1,f_1),(d_2,f_2),\ldots,(d_k,f_k)\}$
Output: j, size of maximum matching in G
if $|M|$=1 **then**
  return $\min\{f_1,d_1\}$;
else
  if $d_1\le f_1$ **then**
    return d_1+MaxMatching($\{(d_2-d_1,f_2),(d_3-d_1,f_3),\ldots,(d_k-d_1,f_k)\}$);
  else
    return f_1+ MaxMatching($\{(d_2-f_1,f_2),(d_3-f_1,f_3),\ldots,(d_k-f_1,f_k)\}$);

The Algorithm 1 can be implemented in $O(n)$ as the same value needs to be deleted from each degree and can be separately stored. Thus it is indeed faster than Hopcroft-Karp algorithm. Also, the algorithm is simple and easy to implement and requires no extra space. Next, we argue the correctness of our algorithm.

Theorem 2.6. *The Algorithm 1 generates maximum matching in chain graph $G = (X, Y, E)$ where $|X| = |Y| = n$.*

Proof. At each step in the above algorithm, we take the edges incident on minimum degree vertices of X, to the matching M and then remove these vertices from G along with their matched neighbors. We repeat this procedure on the remaining graph. The proof is by induction on the number of distinct degrees of vertices in X. For base case, $k=1$ and G is K_{f_1,d_1}. Thus, the maximum matching size in G is j, where $j = \min\{f_1, d_1\}$. For the induction hypothesis, assume that the algorithm gives the maximum matching of chain graph $G[\{x_1, x_2, ..., x_{d_i+f_i}\}]$ for all $i, i \leq k$.

Now, assume that number of distinct degrees of vertices in X in a chain graph G is $k+1$. Consider the subgraph of G induced by the minimum degree vertices of X and their neighbors in Y, say this graph G'. It is easy to see from the definition of chain graphs that G' is K_{f_1,d_1}. Note that, the number of maximum degree vertices in Y is equal to minimum degree in X. Thus, the maximum matching size in G' is j', where $j' = \min\{f_1, d_1\}$. Also, observe that after removing the matched vertices from graph G, the new graph G'' is again a chain graph and its frequency map is as given in the algorithm and number of distinct degrees of vertices in X is k. By Induction Hypothesis, the algorithm gives the correct maximum matching of G''. Since j' is the maximum number of minimum degree vertices that can be included in a matching, adding j' to the size of maximum matching in G'' gives the maximum matching in G. □

Henceforth, given a chain graph we can compute its total chromatic number from Theorem 2.2 and Corollary 2.1. Therefore, next corollary immediately follows from Algorithm 1:

Corollary 2.2. *Let $G = (X, Y, E)$ be a chain graph. Then $\chi''(G)$ can be computed in $O(n)$-time.*

3. Total coloring in central graph of a graph

Recently, in 2017 S. Sudha and K. Manikandan [13] studied the total coloring of central graph and give the total chromatic number for the central graph of some specific graph classes such as star, path and cycle. In this section, we prove that the total coloring conjecture holds for central graph of any graph. Observe that, the maximum degree of the central graph of a graph G is $n - 1$, where n is the order of the graph G. Therefore, we need to show that the central graph $C(G)$ of any graph G can be totally colored with $n+1$ colors.

Theorem 3.1. *For any graph $G = (V, E)$ of order n such that $n \geq 5$, $\chi''(C(G)) \leq \Delta(G) + 2$.*

Proof. Let $C(G)$ be the central graph of given graph G, where $V_1 = V(G)$ and $V_2 = \{u_i | u_i$ subdivide the edge e_i for every $e_i \in E(G)\}$. Note that, the structure of central graph $C(G)$ of any graph G can be seen as a complete bipartite graph K_n, where some of the edges of K_n are subdivided. We start with the total coloring f of K_n with optimal number of colors. Now restrict this total coloring of K_n to the central graph $C(G)$ such that for any subdivided edge e_i, the edges incident on vertex u_i get the same color as $f(e_i)$, where u_i is the vertex which subdivide the edge e_i. The set V_1 in the central graph gets the same color as of the vertices in K_n and the set of vertices in V_2 uncolored. Now we know that $\chi''(K_n) = n$ if n is odd, otherwise $n+1$ and $\Delta(C(G)) = n - 1$. Therefore we have two cases:

Case 1: n is even

In this case, $\chi''(K_n) = n+1$. Now observe that the set of vertices in V_1 have degree $n - 1$. Therefore the color set of any vertex in V_1 have n colors, that is, exactly one color would be missing from their color set. Now consider a subdivided edge, say $vv' \in E(G)$ and the vertex u subdivide the edge vv'. Note that, the color of edge vu and uv' is same, and so it is not a valid total coloring a u. So we recolor the edge vu with a color which is missing from the color set of v. Now after this step, we need to color the vertex u. Since we are assuming $n \geq 5$, there exists a color which is differ from color of vu, uv', v and v'. We color the given vertex u with that available color. We can repeat the same process on the graph till we recolor all the subdivided edges.

Case 2: n is odd

Therefore, $\chi''(K_n) = n$. Now in this case, each vertex in V_1 has all the n colors present in their color set. So, we take a new color $n+1$. Note that the color $n+1$ is missing from the color set of every vertex. Now consider a subdivided edge, say $vv' \in E(G)$ and the vertex u subdivide the edge vv'. Note that, the color of edge vu and uv' is same, and so it is not a valid total coloring a u. So we recolor the edge vu with color $n+1$. Since we are assuming $n \geq 5$, there exists a color which is differ from color of vu, uv', v and v'. We color the given vertex u with that available color. We can repeat the same process on the graph till we recolor all the subdivided edges.

Thus we can always totally color the graph $C(G)$ with $n+1$ colors and because $\Delta(C(G)) = n - 1$. Hence, $\chi''(C(G)) \leq \Delta + 2$. □

4. AVD-total coloring in central graphs

Now, we explore the problem of AVD-total coloring in central graphs for some basic graph classes. Assume that, for the central graph $C(G)$ of any given graph G, $V_1 = V(G) = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{u_1, u_2, ..., u_m\}$, where $V(C(G)) = V_1 \cup V_2, |V(G)| = n$ and $|E(G)| = m$.

Theorem 4.1. *For a path P_n, $\chi''_a(C(P_n)) = n+1$ if n is even and $\chi''_a(C(P_n)) \leq n+2$ otherwise.*

Proof. For this we first look at the total coloring of $C(P_n)$. Now it has been shown that $\chi''(C(P_n)) = n$ if n is odd and $n+1$ if n is even [13]. Suppose that, k is an odd integer. Therefore, $\chi''(C(P_k)) = k$. Let f be a total coloring of $C(P_k))$ using k colors. The coloring f is as follows:
$f(v_k) = k$ and $f(v_i) \equiv (2i+1) (\mathrm{mod}\ k)$ for $1 \leq i \leq k-1$.

$f(u_{k-1})=1$ and $f(u_i)\equiv 2(i+1)(\bmod k)$, for $1\leq i\leq (k-2)$.
For $1 \leq i,j, \leq k, f(v_iv_j) \equiv (i+j)(\bmod\ k)$, if $(i+j) \not\equiv 0 \ (\bmod\ k), j > i+1$ and $f(v_iv_j) = n$, otherwise.
For $1 \leq i,j, \leq k, f(v_iu_j) \equiv (i+j)(\bmod\ k)$, if $(i+j) \not\equiv 0 \ (\bmod\ k)$ and $f(v_iu_j) = n$, otherwise.

Note that the degree of each vertex v_i in V_1 is k – 1. It implies that each vertex v_i with respect to the coloring f have the same set of colors used on v_i and contain all the colors. Now remove the vertices v_1 and u_1. Observe that the edge u_1v_2 has a different color from the edges v_1v_j. Hence a unique and distinct color is removed from the color set of the vertices $v_2, \ldots, v_k$. Since their color sets were same earlier same, now their color sets would be different. Hence, we have obtained an AVD-total coloring for $C(P_{k-1})$ where k is odd. It follows that, $\chi''_a C((C_n)) = n+1$ where n is even as the vertices of maximum degree are adjacent in this graph.

Now again delete the vertex v_2 and u_2. Our claim is that it is still a valid AVD-total coloring of $C(P_{k-2})$ with $\Delta+3$ colors, where k is odd. Now let us prove our claim. Suppose that after the removal of the four vertices $v_1, u_1, v_2, u_2,$ it is not a valid AVD-total coloring. It implies that, there exists a pair of adjacent vertices having the same color set. Observe that, in the first operation a unique color was removed from the color set of each of the vertices in V_1, and similarly in the second operation as well. Assume that the conflicting vertices does not contain v_3. Therefore, we have a cycle of length 4 whose edges alternate in color. But we will prove this case is not possible. Let the two colors be a, b and indices of the vertices in cycle would be $1, i, 2, j$ in the same order.

If the edge v_1v_i is colored with a, then v_2v_j is also colored with a. From the definition of the total coloring which we have used,

$$\begin{gathered}(1+i)(\bmod\ k) \equiv a(\bmod\ k)\\ (2+j)(\bmod\ k) \equiv a(\bmod\ k)\\ \text{i.e}, (1+i+2+j)(\bmod\ k) \equiv 2a(\bmod\ k)\end{gathered}$$

Also if the edge v_iv_2 is colored with b then v_jv_1 is also colored with b. From the definition of the total coloring which we have used,

$$\begin{gathered}(i+2)(\bmod\ k) \equiv b(\bmod\ k)\\ (j+1)(\bmod\ k) \equiv b(\bmod\ k)\\ \text{i.e}, (1+i+2+j)(\bmod\ k) \equiv 2b(\bmod\ k)\end{gathered}$$

Hence, $2a(\bmod\ k) \equiv 2b(\bmod\ k)$. Also, we know that k is odd, which implies $a(\bmod\ k) \equiv b(\bmod\ k)$. Since a and b are both less than k, $a=b$ which is a contradiction. Also observe that the vertex v_3 cannot be a conflicting vertex because color 4 which is removed from its color set in the first iteration has not been removed from the color set of any other vertex. Therefore, it is a valid AVD-total coloring. Thus, we obtained an AVD-total coloring for $C(P_{k-2})$ with $\Delta+3$ colors, where k is odd. Hence, $\chi''_a(C(P_n)) \leq \Delta+3$ where n is odd. □

In the next theorem, we study the AVD-total chromatic number of central graph of a cycle.

Theorem 4.2. *For a cycle C_n, $\chi''_a(C(C_n)) = n+1$ if n is even and $\chi''_a(C(C_n)) \leq n+2$ otherwise.*

Proof. It is known that $\chi''(C(C_n)) = n$ if n is odd and $n+1$ if n is even (19). Assume that k is an odd integer. We look at the total coloring of $C(C_k)$. Define the total coloring f of $C(C_k)$ as follows:

$$f(v_i) \equiv (2i+1)(\bmod\ k) \text{ for } 1 \leq i \leq k-1 \text{ and } f(v_k) = k.$$
$$f(u_i) \equiv 2(i+1)(\bmod\ k) \text{ for } 1 \leq i \leq k-2 \text{ and } f(u_{k-1}) = k.$$

For $1 \leq i,j, \leq k, f(v_iv_j) \equiv (i+j)(\bmod\ k)$, if $(i+j) \not\equiv 0 \ (\bmod\ k), j > i+1$ and $f(v_iv_j) = n$, otherwise.
For $1 \leq i,j, \leq k, f(v_iu_j) \equiv (i+j)(\bmod\ k)$, if $(i+j) \not\equiv 0 \ (\bmod\ k)$ and $f(v_iu_j) = n$, otherwise.

The above coloring is a total coloring of the central graph of C_k using k colors. Since the degree of vertices v_i are k – 1, the color set of each vertex $v_i \in V_1$ would be same and contains all the colors. Now we remove the vertices v_1, u_1 and edge v_kv_2 and we add the edge u_kv_2 with the same color as the color of the edge u_1v_2. Now observe the color sets of v_2 and v_k are same, however the color set of all other vertices are different, mutually. Note that, v_2 and v_k are not adjacent and so the obtained coloring is an AVD-total coloring of $C(C_{k-1})$ with k colors where k is odd. It follows that, $\chi''_a(C(C_n)) = \Delta+2$ for n is even.

Now again we operate the same operation on the same $C(C_{k-1})$ as above. Our claim is that, it is still a valid AVD-total coloring of obtained graph $C(C_{k-2})$ where k is odd. Suppose that, the obtained coloring of graph $C_{(C_{k-2})}$ is not a valid AVD-total coloring. It implies that, there exists a pair of adjacent vertices which have the same color set. Suppose that, the conflicting vertices does not contain v_3. Therefore, there is a cycle of length four whose edges have alternate colors. Let the two colors be a, b and indices of the vertices in cycle would be $1, i, 2, j$ in the same order.

Let edge v_1v_i be colored with color a. Therefore, v_2v_j will also be colored a. Now, from the definition of the total coloring which we have used,

$$\begin{gathered}(1+i)(\bmod\ k) \equiv a(\bmod\ k)\\ (2+j)(\bmod\ k) \equiv a(\bmod\ k)\\ \text{i.e}, (1+i+2+j)(\bmod\ k) \equiv 2a(\bmod\ k)\end{gathered}$$

Also the edge v_iv_2 is colored with color b and the edge v_jv_1 is also colored with color b. Therefore, from the definition of the total coloring which we have used,

$$\begin{gathered}(i+2)(\bmod\ k) \equiv b(\bmod\ k)\\ (j+1)(\bmod\ k) \equiv b(\bmod\ k)\\ \text{i.e}, (1+i+2+j)(\bmod\ k) \equiv 2b(\bmod\ k)\end{gathered}$$

Hence, $2a(\bmod\ k) \equiv 2b(\bmod\ k)$. Also we know that k is odd, which implies that $a(\bmod\ k) \equiv b(\bmod\ k)$. Since a and b are both less than k, $a=b$ which is a contradiction. Also observe that the vertex v_3 and v_k cannot be a conflicting vertex because color 3 which is removed from the color sets of vertices v_3 and v_k in the second iteration but it has not been removed from the color set of any other vertex. Therefore, our claim is true. The obtained coloring is an AVD-total coloring of $C(C_{k-2})$ using k colors where k is odd. In other words, $\chi''_a(C(C_n)) \leq \Delta+3$ for n is odd. □

Theorem 4.3. *For a star $K_{1,n}$, $\chi''_a(C(K_{1,n}) = n+2$.*

Proof. It is known that $\chi''(C(K_{1,n})) = n+1$ [13]. We take a total coloring f of $C(K_{1,n})$ defined in [13] and construct an AVD-total coloring from this. The total coloring f is defined as follows:
$f(v_0) = n+1$ and $f(v_i) = i$ for $1 \leq i \leq n$.
$f(u_1) = 3$, $f(u_n) = 1$ and $f(u_i) = i-1$ for $2 \leq i \leq n-1$.

For $1 \leq i, \leq n, f(v_0u_i) = i, f(v_iu_i) \equiv 2i(\text{mod } (n+1))$ if $2i \not\equiv 0(\text{mod } (n+1))$ and $f(v_iu_i) = n+1$, otherwise.
For $1 \leq i,j, \leq n, f(v_iv_j) \equiv (i+j)(\text{mod } (n+1))$, if $2i \not\equiv 0(\text{mod } (n+1)), j \geq i$ and $f(v_iu_j) = n+1$, otherwise.

Now, if we remove the vertex v_1, then a unique color is get removed from the color sets of each of the vertices $v_2, \ldots, v_n$. Hence, the obtained coloring is an AVD-total coloring of the obtained graph $C(K_{1,n-1})$. Note that in $C(K_{1,n-1})$ there are vertices of maximum degree which are adjacent. Therefore, the obtained coloring is optimal also. Hence, $\chi''_a(K_{1,n}) = \Delta + 2$. □

Theorem 4.4. *For a complete graph K_n, $\chi''_a(C(K_n)) = n$.*

Proof. Observe that in graph $C(K_n)$, no two vertices of same degree are adjacent ($n \geq 5$). Hence, any total coloring would also be AVD-total coloring. In fact, $C(K_n)$ is a bipartite graph with partite sets V_1 and V_2 as there is no odd cycle and the maximum degree Δ of this graph is $n - 1$. Therefore it can be edge colored using n colors. So we take an edge coloring of $C(K_n)$. Since every vertex in V_1 has degree $n - 1$, there would be one color missing from the color set of each of the vertex in V_1. We color each vertex $v_i \in V_1$ with the missing color on the v_i. Since $n \geq 5$ and u_i has degree two, give it a color out of n colors which is different from the color of its neighbors and edges incident on it. Thus the obtained coloring is an AVD-total coloring of $C(K_n)$ using n colors. □

5. Conclusion and open problems

In this paper, we study the total coloring and AVD-total coloring problems. We obtained a sufficient condition for a biconvex graph G to be type 1. Also, we proposed a linear time algorithm to compute the total chromatic number of chain graphs, a proper subclass of biconvex graphs. We proved that the total coloring conjecture holds for the central graph of any graph. Finally, we obtained the AVD-total chromatic number of central graphs for basic graph classes such as paths, cycles, stars and complete graphs. The total coloring classification problem on central graph of any graph remains open. It would be interesting to prove the AVD-total coloring conjecture for central graph of any graph and then to solve the AVD-total coloring classification problem on central graph of any graph.

Disclosure statement

No conflicts of interest have been reported by the author(s).

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