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## Decomposition of product graphs into paths and stars on five vertices

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### ABSTRACT

Let  $P_k$ ,  $S_k$  and  $K_k$  respectively denote a path, a star and a complete graph on  $k$  vertices. By a  $(k; r, s)$ -decomposition of a graph  $G$ , we mean a decomposition of  $G$  into  $r$  copies of  $P_{k+1}$  and  $s$  copies of  $S_{k+1}$ . In this paper, it is shown that the graph  $K_m \times K_n$  admits a  $(4; r, s)$ -decomposition if and only if  $mn(m-1)(n-1) \equiv 0 \pmod{8}$ , where  $K_m \times K_n$  denotes a tensor product of complete graphs. Also we extend the existence of such a decomposition in complete  $m$ -partite graphs.

### KEYWORDS

Path; star; tensor product; graph decomposition

### 2010 MSC

05B30; 05C38

### 1. Introduction

All graphs considered here are finite. By a *decomposition* of  $G$ , we mean a list of edge-disjoint subgraphs of  $G$  whose union is  $G$ . For the graph  $G$ , if  $E(G)$  can be partitioned into  $E_1, \dots, E_k$  such that the subgraph induced by  $E_i$  is  $H_i$ , for all  $i$ ,  $1 \leq i \leq k$ , then we say that  $H_1, \dots, H_k$  decompose  $G$  and we write  $G = H_1 \oplus \dots \oplus H_k$ . For  $1 \leq i \leq k$ , if  $H_i \cong H$ , we say that  $G$  has a *H-decomposition*. If  $G$  can be decomposed into  $r$  copies of  $H_1$  and  $s$  copies of  $H_2$ , then we say that  $G$  has a  $\{rH_1, sH_2\}$ -decomposition or  $(H_1, H_2)$ -multidecomposition. If such a decomposition exists for all possible  $r$  and  $s$  satisfying trivial necessary conditions, then we say that  $G$  has a  $\{H_1, H_2\}_{\{r,s\}}$ -decomposition or *complete*  $\{H_1, H_2\}$ -decomposition.

Let  $K_n$  be a complete graph on  $n$  vertices and  $K_{k,k}$  be the *complete bipartite* graph with bipartition  $(X, Y)$ , where  $X = \{1_i\}$  and  $Y = \{2_i\}$ ,  $1 \leq i \leq k$ . For two graphs  $G$  and  $H$ , their *tensor product*  $G \times H$  and *wreath product*  $G \otimes H$  have the same vertex set  $V(G) \times V(H)$  and their edge sets are defined as follows:  $E(G \times H) = \{(g, h)(g', h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}$ ,  $E(G \otimes H) = \{(g, h)(g', h') \mid gg' \in E(G) \text{ or } g = g', hh' \in E(H)\}$ .

The above products are associative and distributive over edge-disjoint union of graphs and the tensor product is commutative. A graph  $G$  having partite sets  $X_1, \dots, X_m$  with  $|X_i| = n$ ,  $1 \leq i \leq m$  and  $E(G) = \{xy \mid x \in X_i \text{ and } y \in X_j, \forall i \neq j\}$  is called *complete  $m$ -partite* graph and is denoted by  $K_m \otimes \overline{K}_n$ . Let  $P_{k+1}$ ,  $C_k$  and  $S_{k+1}$  respectively denote a path, cycle and star each with having  $k$  edges. Also  $[x_1 \dots x_k x_{k+1}]$  and  $(y_1; x_1, \dots, x_k)$  respectively denotes a path  $P_{k+1}$  and a star  $S_{k+1} (\cong K_{1,k})$ . If there are  $t$  stars with same end vertices  $x_1, x_2, \dots, x_k$  and different centers  $y_1, y_2, \dots, y_t$ , we denote it by  $(y_1, y_2, \dots, y_t; x_1, x_2, \dots, x_k)$ .

The study of  $(G, H)$ -multidecomposition has been introduced by Atif Abueida and M. Daven [1]. Moreover, Atif Abueida and Theresa O'Neil [2] have settled the existence of

$(G, H)$ -multidecomposition of  $K_m(\lambda)$ , when  $(G, H) = (K_{1,n-1}, C_n)$  for  $n = 3, 4, 5$ . Priyadharsini and Muthusamy [11] gave necessary and sufficient condition for the existence of  $(G_n, H_n)$ -multidecomposition of  $\lambda K_{n,n}$ , where  $G_n, H_n \in \{C_n, P_n, S_n\}$ . H.C. Lee [8] established necessary and sufficient condition for the multidecomposition of  $K_{m,n}$  into at least one copy of  $C_k$  and  $S_{k+1}$ . H.C. Lee and J.J. Lin [10] have obtained necessary and sufficient condition for the decomposition of  $K_{m,m}$  graph minus a one factor into cycles and stars. Shyu [12, 13] respectively, considered the existence of a decomposition of  $K_{m,n}$ ,  $K_n$  into paths and stars, cycles and stars with  $k$  edges. Jeevadoss and Muthusamy [5–7] have proved that the necessary and sufficient condition for the existence of a decomposition of  $K_{m,n}$ ,  $\lambda K_{m,n}$  into paths and cycles each having  $k$  edges, also product graphs into paths and cycles of length four. H.C. Lee [9] established necessary and sufficient conditions for the existence of a decomposition of complete bipartite multigraph into cycles and stars with at least one copy of each. Recently, Shyu [13] has been proved that the necessary and sufficient conditions for the existence of decomposition of  $K_{m,n}$  and  $K_n$  into paths, stars and cycles with four edges each. M. Ilayaraja and A. Muthusamy [4] have obtained necessary and sufficient conditions for the existence of decomposition of  $K_{m,n}$  into cycles and stars with four edges. By a  $(k; r, s)$ -decomposition of a graph  $G$ , we mean a decomposition of  $G$  into  $r$  copies of  $P_{k+1}$  and  $s$  copies of  $S_{k+1}$ . In this paper, we prove that there exists a  $(4; r, s)$ -decomposition of  $K_m \times K_n$  if and only if  $mn(m-1)(n-1) \equiv 0 \pmod{8}$ , where  $K_m \times K_n$  denotes a tensor product of complete graphs. Also we extend the existence of such a decomposition in complete regular  $m$ -partite graphs.

### Remarks.

- (1) Let  $A + B = \{(x_1 + y_1, x_2 + y_2) \mid (x_1, x_2) \in A, (y_1, y_2) \in B\}$  and  $rA$  is the sum of  $r$  copies of  $A$ .

- (2) If  $G_1$  and  $G_2$  have a  $(4; r, s)$ -decomposition, then  $G_1 \oplus G_2$  has a such decomposition.

To prove our main results we require the following:

**Theorem 1.1.** [12] *Let  $r$  and  $s$  be nonnegative integers, and let  $n$  be a positive integer with  $n \geq 16$ . There exists a  $(4; r, s)$ -decomposition of  $K_n$  if and only if  $4(r + s) = e(K_n)$ .*

**Theorem 1.2.** [14] *Let  $r$  and  $s$  be nonnegative integers, and let  $n$  be a positive integer with  $n \geq 2$ . There exists a  $(4; r, s)$ -decomposition of  $K_{2,2n}$  if and only if  $4(r + s) = e(K_{2,2n})$  and  $s$  is even.*

**Theorem 1.3.** [12] *Let  $r$  and  $s$  be nonnegative integers, and let  $k$  and  $m$  be positive integers such that  $m \geq k$ . There exists a  $(k; r, s)$ -decomposition of  $K_{k,m}$  if and only if the following conditions are fulfilled:*

- (1)  $k(r + s) = e(K_{k,m})$ ;
- (2)  $r \leq \lfloor \frac{k}{2} \rfloor - 1 \Rightarrow (r \equiv 0 \pmod{2} \wedge m \geq k + r)$ ;
- (3)  $(\lfloor \frac{k}{2} \rfloor \leq r \leq k - 1 \wedge k \equiv 1 \pmod{2} \wedge r \equiv 1 \pmod{2}) \Rightarrow m \geq k + 1$ .

**Theorem 1.4.** [3] *A nontrivial connected graph  $G$  has a  $P_3$ -decomposition if and only if  $G$  has even size.*

**Theorem 1.5.** [15] *Let  $k$ ,  $m$  and  $n$  be positive integers with  $m \leq n$ . There exists an  $S_{k+1}$ -decomposition of  $K_{m,n}$  if and only if one of the following holds:*

- (i)  $k \leq m$  and  $mn \equiv 0 \pmod{k}$ ;
- (ii)  $m < k \leq n$  and  $n \equiv 0 \pmod{k}$ .

## 2. Constructions

In this section we present some basic constructions which are required to prove our main result.

**Lemma 2.1.** *There exists a  $(4; r, s)$ -decomposition of  $K_m$  when  $n = 8, 9$ .*

*Proof.* **Case 1** Let  $V(K_9) = \{\{i\} \mid 1 \leq i \leq 8\}$ . The required  $(4; r, s)$ -decompositions are given below:

- (1).  $r = 7$  and  $s = 0$ . The required paths are [71862], [27843], [42165], [32851], [25763], [13546], [14738].
- (2).  $r = 6$  and  $s = 1$ . The required paths and stars are [71862], [27843], [56124], [76328], [13546], [14738], (5;1,2,7,8).
- (3).  $r = 5$  and  $s = 2$ . The required paths and stars are [71862], [27843], [42165], [32851], [25763], (3;1,5,7,8), (4;1,5,6,7).
- (4).  $r = 4$  and  $s = 3$ . The required paths and stars are [71862], [27843], [56124], [76328], (3;1,5,7,8), (4;1,5,6,7).
- (5).  $r = 3$  and  $s = 4$ . The required paths and stars are [71862], [56124], [76328], (3;1,4,5,8), (4;1,5,6,8), (5;1,2,7,8), (7;2,3,4,8).
- (6).  $r = 2$  and  $s = 5$ . The required paths and stars are [71862], [27843], (2;1,3,4,8), (3;1,5,7,8), (4;1,5,6,7), (5;1,2,7,8).
- (7).  $r = 1$  and  $s = 6$ . The required paths and stars are

[71862], (2;1,3,4,8), (3;1,4,5,8), (4;1,5,6,8), (5;1,2,7,8), (6;1,3,5,7), (7;2,3,4,8).

- (8).  $r = 0$  and  $s = 7$ . The required stars are (1;5,6,7,8), (2;1,3,4,8), (3;1,4,5,8), (4;1,5,6,8), (5;2,6,7,8), (6;2,3,7,8), (7;2,3,4,8).

**Case 2.** Let  $V(K_9) = \{\{i\} \mid 1 \leq i \leq 9\}$ . The required  $(4; r, s)$ -decompositions are given below:

- (1).  $r = 9$  and  $s = 0$ . The required paths are [51326], [52436], [35846], [37456], [67689], [16879], [21496], [19275], [28395].
- (2).  $r = 8$  and  $s = 1$ . The required paths and stars are [51326], [52436], [35846], [37456], [19283], [41275], [67189], [16879], (9;3456).
- (3).  $r = 7$  and  $s = 2$ . The required paths and stars are [63715], [57216], [43569], [46259], [39485], [38679], [54789], (1;3489), (2;3489).
- (4).  $r = 6$  and  $s = 3$ . The required paths and stars are [51326], [52436], [35846], [37456], [17695], [18275], (1;2469), (8;3679), (9;2347).
- (5).  $r = 5$  and  $s = 4$ . The required paths and stars are [51326], [52436], [35967], [37546], [65847], (1;4678), (2;1789), (8;3679), (9;1347).
- (6).  $r = 4$  and  $s = 5$ . The required paths and stars are [51326], [52436], [35846], [37456], (1;4689), (2;1789), (7;1569), (8;3679), (9;3456).
- (7).  $r = 3$  and  $s = 6$ . The required paths and stars are [41239], [58349], [48765], (1;3579), (2;4679), (5;2349), (6;1349), (7;3459), (8;1269).
- (8).  $r = 2$  and  $s = 7$ . The required paths and stars are [12349], [14839], (1;3789), (2;5679), (4;2567), (5;1389), (6;1359), (7;3569), (8;2679).
- (9).  $r = 1$  and  $s = 8$ . The required paths and stars are [41237], (3;1456), (4;2567), (5;1279), (6;1259), (7;1269), (8;1234), (8;5679), (9;1279).
- (10).  $r = 0$  and  $s = 9$ . The required stars are (1;2345), (2;3456), (3;4567), (4;5678), (5;6789), (6;1789), (7;1289), (8;1239), (9;1234).  $\square$

**Lemma 2.2.** *There exists a  $(4; r, s)$ -decomposition of  $K_{5,5} - I$ , where  $I$  is a 1-factor of distance zero in  $K_{5,5}$ , and  $r \neq 1$ .*

*Proof.* Let  $V(K_{5,5} - I) = \{\{\{1_i\}, \{2_i\}\} \mid 1 \leq i \leq 5\}$ . The required  $(4; r, s)$ -decompositions are given below:

- (1)  $r = 5$  and  $s = 0$ . The required paths are  $[2_4 1_5 2_1 1_2 5]$ ,  $[2_2 1_4 2_5 1_3 2_4]$ ,  $[2_4 1_1 2_3 1_2 2_5]$ ,  $[2_2 1_3 2_1 1_4 2_3]$ ,  $[2_4 1_2 2_1 1_5 2_3]$ .
- (2)  $r = 4$  and  $s = 1$ . The required paths and stars are  $[2_1 1_2 2_3 1_1 2_4]$ ,  $[2_2 1_1 2_5 1_2 2_4]$ ,  $[2_1 1_3 2_2 1_4 2_3]$ ,  $[2_1 1_4 2_5 1_3 2_4]$ ,  $(1_5; 2_1, 2_2, 2_3, 2_4)$ .
- (3)  $r = 3$  and  $s = 2$ . The required paths and stars are  $[2_1 1_2 2_3 1_1 2_4]$ ,  $[2_2 1_3 2_4 1_2 2_5]$ ,  $[2_1 1_3 2_5 1_1 2_2]$ ,  $(1_4; 2_1, 2_2, 2_3, 2_5)$ ,  $(1_5; 2_1, 2_2, 2_3, 2_4)$ .
- (4)  $r = 2$  and  $s = 3$ . The required paths and stars are  $[2_1 1_2 2_3 1_1 2_4]$ ,  $[2_2 1_1 2_5 1_2 2_4]$ ,  $(1_3; 2_1, 2_2, 2_4, 2_5)$ ,  $(1_4; 2_1, 2_2, 2_3, 2_5)$ ,  $(1_5; 2_1, 2_2, 2_3, 2_4)$ .
- (5)  $r = 1$  and  $s = 4$ . It is easy to see that  $K_{5,5} - I$ , can not be decomposed into  $1P_5$  and  $4S_5$ .

- (6)  $r=0$  and  $s=5$ . The required stars are  $(1_1; 2_2, 2_3, 2_4, 2_5)$ ,  $(1_2; 2_1, 2_3, 2_4, 2_5)$ ,  $(1_3; 2_1, 2_2, 2_4, 2_5)$ ,  $(1_4; 2_1, 2_2, 2_3, 2_5)$ ,  $(1_5; 2_1, 2_2, 2_3, 2_4)$ .  $\square$

**Lemma 2.3.** *There exists a  $(4; r, s)$ -decomposition of  $K_{9,9} - I$ , where  $I$  is a 1-factor of distance zero in  $K_{9,9}$ .*

*Proof.* We can write,  $K_{9,9} - I = 2(K_{5,5} - I) \oplus 2K_{4,4}$ . By Lemma 2.2 and Theorem 1.3, the graphs  $K_{5,5} - I$  and  $K_{4,4}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_{9,9} - I$  has the desired decomposition except  $\{1P_5, 17S_5\}$ .

Finally,  $(4; r, s)$ -decomposition for the case  $\{1P_5, 17S_5\}$  is given as follows:  $[1_3 2_5 1_4 2_6 1_5]$ ,  $(1_1; 2_2, 2_3, 2_6, 2_8)$ ,  $(1_3; 2_2, 2_4, 2_6, 2_7)$ ,  $(1_5; 2_3, 2_4, 2_7, 2_8)$ ,  $(1_6; 2_2, 2_4, 2_5, 2_9)$ ,  $(1_7; 2_4, 2_5, 2_8, 2_9)$ ,  $(1_8; 2_2, 2_3, 2_4, 2_5)$ ,  $(1_9; 2_3, 2_4, 2_5, 2_8)$ ,  $(2_1; 1_2, 1_3, 1_6, 1_8)$ ,  $(2_1; 1_4, 1_5, 1_7, 1_9)$ ,  $(2_2; 1_4, 1_5, 1_7, 1_9)$ ,  $(2_3; 1_2, 1_4, 1_6, 1_7)$ ,  $(2_6; 1_2, 1_7, 1_8, 1_9)$ ,  $(2_7; 1_4, 1_6, 1_8, 1_9)$ ,  $(2_8; 1_2, 1_3, 1_4, 1_6)$ ,  $(2_9; 1_3, 1_4, 1_5, 1_8)$ ,  $(1_1, 1_2; 2_4, 2_5, 2_7, 2_9)$ .  $\square$

**Lemma 2.4.** *There exists a  $(4; r, s)$ -decomposition of  $K_{4x+1, 4x+1} - I$ , where  $I$  is a 1-factor of distance zero in  $K_{4x+1, 4x+1}$ ,  $x \geq 1$ , and  $r \neq 1$  for  $x = 1$ .*

*Proof.* When  $x = 1$  and 2, the graphs  $K_{5,5} - I$  and  $K_{9,9} - I$  have a  $(4; r, s)$ -decomposition, by Lemmas 2.2 and 2.3. For  $x \geq 3$ , we can write,  $K_{4x+1, 4x+1} - I = K_{9,9} - I \oplus (x - 2)(K_{5,5} - I) \oplus 2(x - 2)K_{4,8} \oplus (x - 2)(x - 3)K_{4,4}$ . By Lemmas 2.2 and 2.3 and Theorem 1.3, the graphs  $K_{5,5} - I, K_{9,9} - I, K_{4,4}$  and  $K_{4,8}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_{4x+1, 4x+1} - I$  has the desired decomposition.  $\square$

**Lemma 2.5.** *There exists a  $(4; r, s)$ -decomposition of  $P_3 \times K_3$ , where  $r \neq 1$ .*

*Proof.* Let  $V(P_3 \times K_3) = \cup_{i=1}^3 X_i$ , where  $X_i = \{i_j \mid 1 \leq j \leq 3\}$ . The required  $(4; r, s)$ -decompositions are given below:

- (1)  $r=3$  and  $s=0$ . The required paths are  $[1_1 2_2 1_3 2_1 1_2]$ ,  $[1_1 2_3 3_1 2_2 3_3]$ ,  $[1_2 2_3 3_2 2_1 3_3]$ .
- (2)  $r=2$  and  $s=1$ . The required paths and stars are  $[1_1 2_3 1_2 2_1 1_3]$ ,  $[3_1 2_3 3_2 2_1 3_3]$ ,  $(2_2; 1_1, 1_3, 3_1, 3_3)$ .
- (3)  $r=1$  and  $s=2$ . It is easy to see that  $P_3 \times K_3$  can not be decomposed into  $1P_5$  and  $2S_5$ .
- (4)  $r=0$  and  $s=3$ . The required stars are  $(2_1; 1_2, 1_3, 3_2, 3_3)$ ,  $(2_2; 1_1, 1_3, 3_1, 3_3)$ ,  $(2_3; 1_1, 1_2, 3_1, 3_3)$ .  $\square$

**Lemma 2.6.** *There exists a  $(4; r, s)$ -decomposition of  $P_3 \times K_6$ .*

*Proof.* Let  $V(P_3 \times K_6) = \cup_{i=1}^3 X_i$ , where  $X_i = \{i_j \mid 1 \leq j \leq 6\}$ . The required  $(4; r, s)$ -decompositions are given below: Now, we decompose the graph  $P_3 \times K_6$  into  $15S_5$ 's as follows:

- $\{(1_2, 3_2; 2_3, 2_4, 2_5, 2_6), (1_4, 3_4; 2_1, 2_2, 2_3, 2_6), (1_5, 3_5; 2_1, 2_2, 2_4, 2_6), (1_6, 3_6; 2_1, 2_2, 2_3, 2_4), (2_5, 2_6; 1_1, 1_3, 3_1, 3_3), (2_2, 2_4; 1_1, 1_3, 3_1, 3_3), (2_1; 1_2, 1_3, 3_2, 3_3), (2_3; 1_1, 1_5, 3_1, 3_5), (2_5; 1_4, 1_6, 3_4, 3_6)\}$ .

First, we decompose the given  $2S_5$ 's into  $\{2P_5\}$  as follows:

- (1)  $(1_2, 3_2; 2_3, 2_4, 2_5, 2_6) \Rightarrow \{[2_3, 1_2, 2_4, 3_2, 2_5], [2_5, 1_2, 2_6, 3_2, 2_3]\}$ .
- (2)  $(1_4, 3_4; 2_1, 2_2, 2_3, 2_6) \Rightarrow \{[2_1, 1_4, 2_2, 3_4, 2_3], [2_3, 1_4, 2_6, 3_4, 2_1]\}$ .
- (3)  $(1_5, 3_5; 2_1, 2_2, 2_4, 2_6) \Rightarrow \{[2_2, 1_5, 2_1, 3_5, 2_6], [2_2, 3_5, 2_4, 1_5, 2_6]\}$ .
- (4)  $(1_6, 3_6; 2_1, 2_2, 2_3, 2_4) \Rightarrow \{[2_1, 1_6, 2_2, 3_6, 2_3], [2_3, 1_6, 2_4, 3_6, 2_1]\}$ .
- (5)  $(2_5, 2_6; 1_1, 1_3, 3_1, 3_3) \Rightarrow \{[1_3, 2_6, 1_1, 2_5, 3_3], [1_3, 2_5, 3_1, 2_6, 3_3]\}$ .

Now, we consider the  $5S_5$ 's  $\{(2_1; 1_2, 1_3, 3_2, 3_3), (2_3; 1_1, 1_5, 3_1, 3_5), (2_5; 1_4, 1_6, 3_4, 3_6), (2_2, 2_4; 1_1, 1_3, 3_1, 3_3)\}$  which can be decomposed into either  $\{4P_5, 1S_5\}$  or  $\{3P_5, 2S_5\}$  or  $\{2P_5, 3S_5\}$  as follows:  $\{[1_2 2_1 1_3 2_2 3_1], [1_1 2_2 3_3 2_1 3_2], [1_3 2_4 3_1 2_3 1_5], [3_3 2_4 1_1 2_3 3_5], (2_5; 1_4, 1_6, 3_4, 3_6)\}$  or  $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 1_5], (2_1; 1_2, 1_3, 3_2, 3_3), (2_5; 1_4, 1_6, 3_4, 3_6)\}$  or  $\{[1_1 2_2 3_3 2_1 3_2], [1_2 2_1 1_3 2_2 3_1], (2_3; 1_1, 1_5, 3_1, 3_5), (2_4; 1_1, 1_3, 3_1, 3_3), (2_5; 1_4, 1_6, 3_4, 3_6)\}$ .

Further, we consider the  $7S_5$ 's  $\{(2_5, 2_6; 1_1, 1_3, 3_1, 3_3), (2_2, 2_4; 1_1, 1_3, 3_1, 3_3), (2_1; 1_2, 1_3, 3_2, 3_3), (2_3; 1_1, 1_5, 3_1, 3_5), (2_5; 1_4, 1_6, 3_4, 3_6)\}$  which can be decomposed into either  $\{7P_5\}$  or  $\{5P_5, 2S_5\}$  or  $\{4P_5, 3S_5\}$  or  $\{3P_5, 4S_5\}$  as follows:  $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 3_5], [1_2 2_1 1_3 2_5 1_4], [3_2 2_1 3_3 2_5 3_4], [1_3 2_6 1_1 2_5 1_6], [3_3 2_6 3_1 2_5 3_6]\}$  or  $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 3_5], [1_2 2_1 1_3 2_5 1_4], [3_2 2_1 3_3 2_5 3_4], (2_5; 1_1, 1_6, 3_1, 3_6), (2_6; 1_1, 1_3, 3_1, 3_3)\}$  or  $\{[1_2 2_1 1_3 2_5 1_4], [3_2 2_1 3_3 2_5 3_4], [1_3 2_6 1_1 2_5 1_6], [3_3 2_6 3_1 2_5 3_6], (2_2; 1_1, 1_3, 3_1, 3_3), (2_3; 1_1, 1_5, 3_1, 3_5), (2_4; 1_1, 1_3, 3_1, 3_3)\}$  or  $\{[1_1 2_2 1_3 2_4 3_3], [1_5 2_3 1_1 2_4 3_1], [3_3 2_2 3_1 2_3 3_5], (2_1; 1_2, 1_3, 3_2, 3_3), (2_5; 1_3, 1_4, 3_3, 3_4), (2_5; 1_1, 1_3, 3_1, 3_3), (2_6; 1_1, 1_3, 3_1, 3_3)\}$ .

Finally,  $(4; r, s)$ -decomposition for the case  $\{1P_5, 14S_5\}$  is given as follows:

- $[1_2 2_5 3_2 2_3 1_4], (2_1; 1_2, 1_3, 3_2, 3_3), (2_1; 1_4, 1_6, 3_4, 3_6), (2_2; 1_1, 1_3, 1_4, 3_4), (2_2; 1_6, 3_1, 3_3, 3_6), (2_3; 1_1, 1_5, 3_1, 3_5), (2_3; 1_2, 1_6, 3_4, 3_6), (2_4; 1_1, 1_2, 1_3, 3_2), (2_4; 1_6, 3_1, 3_3, 3_6), (2_5; 1_4, 1_6, 3_4, 3_6), (2_6; 1_2, 1_4, 3_2, 3_4), (2_5, 2_6; 1_1, 1_3, 3_1, 3_3), (1_5, 3_5; 2_1, 2_2, 2_4, 2_6)$ .

**Lemma 2.7.** *There exists a  $(4; r, s)$ -decomposition of  $P_3 \times K_7$ .*

*Proof.* Let  $V(P_3 \times K_7) = \cup_{i=1}^3 X_i$ , where  $X_i = \{i_j \mid 1 \leq j \leq 7\}$ . The required  $(4; r, s)$ -decompositions are given below: Now, we decompose the graph  $P_3 \times K_7$  into  $21S_5$ 's as follows:

- $\{(2_3, 2_4; 1_1, 1_7, 3_1, 3_7), (2_4, 2_7; 1_2, 1_3, 3_2, 3_3), (2_5, 2_6; 1_1, 1_2, 3_1, 3_2), (2_1; 1_3, 1_5, 3_3, 3_5), (2_2; 1_3, 1_5, 3_3, 3_5), (2_6; 1_3, 1_4, 3_3, 3_4), (2_1; 1_4, 1_6, 3_4, 3_6), (2_4; 1_5, 1_6, 3_5, 3_6), (2_7; 1_4, 1_5, 3_4, 3_5), (2_2; 1_6, 1_7, 3_6, 3_7), (2_3; 1_4, 1_6, 3_4, 3_6), (2_5; 1_4, 1_7, 3_4, 3_7), (2_2; 1_1, 1_3, 3_1, 3_3), (2_5; 1_3, 1_6, 3_3, 3_6), (2_7; 1_1, 1_6, 3_1, 3_6), (2_1; 1_2, 1_7, 3_2, 3_7), (2_3; 1_2, 1_5, 3_2, 3_5), (2_6; 1_5, 1_7, 3_5, 3_7)\}$ .

First, we decompose the given  $2S_5$ 's into  $\{2P_5\}$  as follows:

- (1)  $(2_3, 2_4; 1_1, 1_7, 3_1, 3_7) \Rightarrow \{[1_1 2_3 3_1 2_4 1_7], [1_1 2_4 3_7 2_3 1_7]\}$ .
- (2)  $(2_4, 2_7; 1_2, 1_3, 3_2, 3_3) \Rightarrow \{[1_2 2_4 3_3 2_7 1_3], [1_2 2_7 3_2 2_4 1_3]\}$ .

(3)  $(2_5, 2_6; 1_1, 1_2, 3_1, 3_2) \Rightarrow \{[1_1 2_5 3_2 2_6 1_2], [1_1 2_6 3_1 2_5 1_2]\}$ .  
 Now the remaining  $3S_5$ 's can be decomposed into  $\{3P_5\}$  as follows:

- (4)  $\{(2_1; 1_3, 1_5, 3_3, 3_5), (2_2; 1_3, 1_5, 3_3, 3_5), (2_6; 1_3, 1_4, 3_3, 3_4)\} \Rightarrow \{[1_3 2_1 3_3 2_6 1_4], [1_4 2_2 3_5 2_1 1_5], [1_3 2_6 3_4 2_2 1_5]\}$ .
- (5)  $\{(2_1; 1_4, 1_6, 3_4, 3_6), (2_4; 1_5, 1_6, 3_5, 3_6), (2_7; 1_4, 1_5, 3_4, 3_5)\} \Rightarrow \{[1_4 2_1 3_4 2_7 1_5], [1_5 2_4 3_6 2_1 1_6], [1_4 2_7 3_5 2_4 1_6]\}$ .
- (6)  $\{(2_2; 1_6, 1_7, 3_6, 3_7), (2_3; 1_4, 1_6, 3_4, 3_6), (2_5; 1_4, 1_7, 3_4, 3_7)\} \Rightarrow \{[1_4 2_3 3_6 2_2 1_6], [1_4 2_5 3_7 2_2 1_7], [1_6 2_3 3_4 2_5 1_7]\}$ .
- (7)  $\{(2_2; 1_1, 1_3, 3_1, 3_3), (2_5; 1_3, 1_6, 3_3, 3_6), (2_7; 1_1, 1_6, 3_1, 3_6)\} \Rightarrow \{[1_1 2_2 3_3 2_5 1_6], [1_1 2_7 3_6 2_5 1_3], [1_3 2_2 3_1 2_7 1_6]\}$ .
- (8)  $\{(2_1; 1_2, 1_7, 3_2, 3_7), (2_3; 1_2, 1_5, 3_2, 3_5), (2_6; 1_5, 1_7, 3_5, 3_7)\} \Rightarrow \{[1_2 2_1 3_7 2_6 1_5], [1_2 2_3 3_5 2_6 1_7], [1_5 2_3 3_2 2_1 1_7]\}$ .

Further, the decomposition of the case  $\{20P_5, 1S_5\}$  is given as follows: From (1) and (3), we obtain  $\{3P_5, 1S_5\}$  as  $\{[1_1 2_4 1_2 2_5 3_1], [1_7 2_3 1_1 2_6 3_2], [3_2 2_5 3_1 2_3 3_7], (2_4; 1_1, 1_7, 3_1, 3_7)\}$  and together  $\{17P_5\}$  given in (2) and (4) to (8) gives the required paths and stars except the case  $\{1P_5, 20S_5\}$ . Finally,  $(4; r, s)$ -decomposition for the case  $\{1P_5, 20S_5\}$  is given as follows:

- $[1_3 2_2 3_7 2_1 1_7], (2_1; 1_3, 1_4, 3_3, 3_4), (2_2; 1_4, 1_7, 3_3, 3_4),$
- $(1_1, 3_1; 2_2, 2_3, 2_4, 2_5), (1_2, 3_2; 2_1, 2_3, 2_4, 2_5), (1_3, 3_3; 2_4, 2_5, 2_6, 2_7),$
- $(1_4, 3_4; 2_3, 2_5, 2_6, 2_7), (1_5, 3_5; 2_3, 2_4, 2_6, 2_7), (1_6, 3_6; 2_3, 2_4, 2_5, 2_7),$
- $(1_7, 3_7; 2_3, 2_4, 2_5, 2_6), (2_1, 2_2; 1_5, 1_6, 3_5, 3_6), (2_6, 2_7; 1_1, 1_2, 3_1, 3_2).$  □

**Lemma 2.8.** *There exists a  $(4; r, s)$ -decomposition of  $K_{m,6}$  for  $m = 2, 4, 6$ .*

*Proof.* The cases  $m = 2, 4$  follows by Theorems 1.2 and 1.3. For  $m = 6$ , let  $V(K_{6,6}) = \{(\{1_i\}, \{2_i\}) \mid 1 \leq i \leq 6\}$ . We can write,  $K_{6,6} = K_{2,6} \oplus K_{4,6}$ . By Theorems 1.2 and 1.3, we obtain the required decomposition for the cases  $(r, s) \in \{(3, 0), (1, 2)\} + \{(6, 0), (5, 1), \dots, (2, 4), (0, 6)\} = \{(9, 0), (8, 1), \dots, (3, 6), (1, 8)\}$ . Further, the required decomposition for the case  $\{2P_5, 7S_5\}$  is given as follows:  $[2_1 1_1 2_2 1_2 2_3], [2_1 1_2 2_5 1_1 2_3], (1_3, 1_4; 2_1, 2_2, 2_3, 2_6), (1_5, 1_6; 2_1, 2_2, 2_3, 2_4), (2_4; 1_1, 1_2, 1_3, 1_4), (2_5; 1_3, 1_4, 1_5, 1_6), (2_6; 1_1, 1_2, 1_5, 1_6)$ . Finally, by Theorem 1.5, we get  $(r, s) = (0, 9)$ . □

**Theorem 2.1.** *Let  $r$  and  $s$  be nonnegative integers, and let  $m$  be a positive even integer with  $m \geq 4$ . Then there exists a  $(4; r, s)$ -decomposition of  $K_{m,m}$  if and only if  $4(r + s) = e(K_{m,m})$ , and  $r \neq 1$  for  $m = 4$ .*

*Proof.* Necessity. The condition  $4(r + s) = e(K_{m,m})$  is trivial by using counting arguments. Sufficiency. The cases  $m = 4, 6$  follows by Theorem 1.3 and Lemma 2.8. For  $m = 8$ , we can write,  $K_{8,8} = K_{6,6} \oplus K_{2,6} \oplus 2K_{2,4}$ . By Lemma 2.8 and Theorem 1.2, we obtain the required decomposition for the cases  $(r, s) \in \{(9, 0), (8, 1), \dots, (1, 8), (0, 9)\} + \{(3, 0), (1, 2)\} + \{(4, 0), (2, 2), (0, 4)\} = \{(16, 0), (15, 1), \dots, (2, 14), (1, 15)\}$ . Further, by Theorem 1.5, we get  $(r, s) = (0, 16)$ . For  $m \geq 10$ , we deal the proof is two cases as follows:

**Case 1.**  $m \equiv 2 \pmod{4} \geq 10$ . We can write,  $K_{m,m} = K_{6,6} \oplus (\frac{m-6}{4}K_{4,4} \oplus \{\oplus_i K_{4,i}\} \oplus \{\oplus_j K_{j,4}\})$ , where  $6 \leq i, j \equiv 2 \pmod{4} \leq m - 4$ . Note that  $K_{j,4} \cong K_{4,j}$ . By Theorem 1.3 and Lemma 2.8, the graphs  $K_{4,4}, K_{4,i}, K_{j,4}$  and  $K_{6,6}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_{m,m}$  has the desired decomposition.

**Case 2.**  $m \equiv 0 \pmod{4} \geq 12$ . We can write,  $K_{m,m} = K_{8,8} \oplus (\frac{m-12}{4}K_{4,4} \oplus \{\oplus_i K_{i,4}\} \oplus \{\oplus_j K_{4,j}\})$ , where  $4 \leq i \equiv 0 \pmod{4} \leq m - 4$  and  $8 \leq j \equiv 0 \pmod{4} \leq m - 4$ . Note that  $K_{i,4} \cong K_{4,i}$ . By Theorem 1.3 and the first paragraph of the proof, the graphs  $K_{4,4}, K_{i,4}, K_{4,j}$  and  $K_{8,8}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_{m,m}$  has the desired decomposition. □

### 3. $(4; r, s)$ -decomposition of $K_m \times K_n$

**Lemma 3.1.** *There exists a  $(4; r, s)$ -decomposition of  $K_4 \times K_4$ .*

*Proof.* Let  $V(K_4 \times K_4) = \cup_{i=1}^4 X_i$  and  $X_i = \{i_j \mid 1 \leq j \leq 4\}$ . Then the required  $(4; r, s)$ -decompositions are as given below: First, we decompose the given  $2S_5$ 's into  $\{2P_5\}$  as follows:

- (1)  $\{(1_1; 2_2, 2_3, 2_4, 3_2), (1_2; 2_1, 2_3, 2_4, 3_3)\} \Rightarrow \{[3_2 1_1 2_3 1_2 3_3], [2_1 1_2 2_4 1_1 2_2]\}$ .
- (2)  $\{(1_3; 2_1, 2_2, 2_4, 3_4), (1_4; 2_1, 2_2, 2_3, 3_1)\} \Rightarrow \{[3_1 1_4 2_2 1_3 3_4], [2_4 1_3 2_1 1_4 2_3]\}$ .
- (3)  $\{(2_1; 3_2, 3_3, 3_4, 4_4), (2_2; 3_1, 3_3, 3_4, 4_1)\} \Rightarrow \{[3_1 2_2 3_4 2_1 3_2], [4_1 2_2 3_3 2_1 4_4]\}$ .
- (4)  $\{(3_1; 4_2, 4_3, 4_4, 1_3), (3_2; 4_1, 4_3, 4_4, 1_3)\} \Rightarrow \{[4_1 3_2 1_3 3_1 4_3], [4_2 3_1 4_4 3_2 4_3]\}$ .
- (5)  $\{(3_3; 4_1, 4_2, 4_4, 1_1), (3_4; 4_1, 4_2, 4_3, 1_1)\} \Rightarrow \{[4_2 3_3 1_1 3_4 4_3], [4_2 3_4 4_1 3_3 4_4]\}$ .
- (6)  $\{(4_1; 1_2, 1_3, 1_4, 2_3), (4_4; 1_1, 1_3, 2_2, 2_3)\} \Rightarrow \{[1_1 4_4 1_3 4_1 1_2], [1_4 4_1 2_3 4_4 2_2]\}$ .
- (7)  $\{(4_2; 1_1, 1_3, 2_1, 2_4), (4_3; 1_1, 2_1, 2_2, 2_4)\} \Rightarrow \{[1_1 4_2 2_4 4_3 2_2], [1_1 4_3 2_1 4_2 1_3]\}$ .
- (8)  $\{(1_2; 3_1, 3_4, 4_3, 4_4), (2_3; 3_1, 3_2, 3_4, 4_2)\} \Rightarrow \{[3_2 2_3 3_1 1_2 4_3], [4_2 2_3 3_4 1_2 4_4]\}$ .
- (9)  $\{(1_4; 3_2, 3_3, 4_2, 4_3), (2_4; 3_1, 3_2, 3_3, 4_1)\} \Rightarrow \{[3_1 2_4 3_3 1_4 4_3], [4_1 2_4 3_2 1_4 4_2]\}$ .

Now, from (1) and (2), we have  $4S_5$ 's which can be decomposed into either  $\{4P_5\}$  or  $\{3P_5, S_5\}$  or  $\{2P_5, 2S_5\}$  as follows:

- $\{[3_2 1_1 2_3 1_2 3_3], [2_1 1_2 2_4 1_1 2_2], [3_1 1_4 2_2 1_3 3_4], [2_4 1_3 2_1 1_4 2_3]\}$  or
- $\{[2_2 1_3 2_4 1_2 3_3], [2_1 1_2 2_3 1_4 3_1], [2_2 1_4 2_1 1_3 3_4], (1_1; 2_2, 2_3, 2_4, 3_2)\}$
- or  $\{[3_2 1_1 2_3 1_2 3_3], [2_1 1_2 2_4 1_1 2_2], (1_3; 2_1, 2_2, 2_4, 3_4), (1_4; 2_1, 2_2, 2_3, 3_1)\}$ .

Finally,  $(4; r, s)$ -decomposition for the case  $\{1P_5, 17S_5\}$  is given as follows:

- $[1_1 3_2 2_4 3_3 2_2], (1_2; 2_1, 2_3, 2_4, 3_3), (1_2; 3_1, 3_4, 4_3, 4_4),$
- $(1_3; 2_1, 2_2, 2_4, 3_4), (1_4; 2_1, 2_2, 2_3, 3_1), (1_4; 3_2, 3_3, 4_2, 4_3),$
- $(2_1; 3_2, 3_3, 3_4, 4_4), (2_2; 1_1, 3_1, 3_4, 4_1), (2_3; 1_1, 3_1, 3_2, 3_4),$
- $(2_4; 1_1, 3_1, 4_1, 4_2), (3_1; 1_3, 4_2, 4_3, 4_4), (3_2; 1_3, 4_1, 4_3, 4_4),$
- $(3_3; 1_1, 4_1, 4_2, 4_4), (3_4; 1_1, 4_1, 4_2, 4_3), (4_1; 1_2, 1_3, 1_4, 2_3),$
- $(4_2; 1_1, 1_3, 2_1, 2_3), (4_3; 1_1, 2_1, 2_2, 2_4), (4_4; 1_1, 1_3, 2_2, 2_3).$  □



**Lemma 3.2.** *There exists a  $(4; r, s)$ -decomposition of  $K_4 \times K_5$ .*

*Proof.* We can write,  $K_4 \times K_5 = 6(K_{5,5} - I)$ . By Lemma 2.2, the graph  $K_{5,5} - I$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_4 \times K_5$  has the desired decomposition except the case  $\{1P_5, 29S_5\}$ . We can write,  $K_4 \times K_5 = K_4 \times K_4 \oplus 12K_{1,4}$ , then by Lemma 3.1 and Theorem 1.5, we have  $(r, s) \in (1, 17) + (0, 12) = (1, 29)$ .  $\square$

**Lemma 3.3.** *There exists a  $(4; r, s)$ -decomposition of  $K_4 \times K_6$ .*

*Proof.* We can write,  $K_4 \times K_6 = 3(P_3 \times K_6)$ . By Lemma 2.6, the graph  $P_3 \times K_6$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_4 \times K_6$  has the desired decomposition.  $\square$

**Lemma 3.4.** *There exists a  $(4; r, s)$ -decomposition of  $K_4 \times K_7$ .*

*Proof.* We can write,  $K_4 \times K_7 = 3(P_3 \times K_7)$ . By Lemma 2.7, the graph  $P_3 \times K_7$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_4 \times K_7$  has the desired decomposition.  $\square$

**Lemma 3.5.** *There exists a  $(4; r, s)$ -decomposition of  $K_5 \times K_3$ .*

*Proof.* We can write,  $K_5 \times K_3 = 3(K_{5,5} - I)$ . By Lemma 2.2, the graph  $K_{5,5} - I$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_5 \times K_3$  has the desired decomposition except the case  $\{1P_5, 14S_5\}$ . Further,  $(4; r, s)$ -decomposition for the case  $\{1P_5, 14S_5\}$  is given as follows:

$$\begin{aligned} & [1_3 2_2 1_1 3_2 1_4], (1_1; 2_3, 2_4, 2_5, 3_3), (1_2; 2_3, 2_4, 2_5, 3_4), \\ & (1_3; 2_1, 2_4, 2_5, 3_2), (1_4; 2_1, 2_2, 2_3, 2_5), (1_5; 2_2, 2_3, 2_4, 3_2), \\ & (2_1; 1_2, 1_5, 3_2, 3_5), (2_2; 3_1, 3_3, 3_4, 3_5), (2_3; 3_1, 3_2, 3_4, 3_5), \\ & (2_4; 3_1, 3_2, 3_3, 3_5), (2_5; 3_1, 3_2, 3_3, 3_4), (3_1; 1_2, 1_3, 1_4, 1_5), \\ & (3_3; 1_2, 1_4, 1_5, 2_1), (3_4; 1_1, 1_3, 1_5, 2_1), (3_5; 1_1, 1_2, 1_3, 1_4). \end{aligned}$$

**Lemma 3.6.** *There exists a  $(4; r, s)$ -decomposition of  $K_5 \times K_5$ .*

*Proof.* We can write,  $K_5 \times K_5 = 10(K_{5,5} - I)$ . By Lemma 2.2, the graph  $K_{5,5} - I$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_5 \times K_5$  has the desired decomposition except the case  $\{1P_5, 49S_5\}$ . Let  $K_5 \times K_5 = K_4 \times K_4 \oplus 3(4)K_{1,4} \oplus 4(5)K_{1,4}$ , then by Lemma 3.1 and Theorem 1.5, we have  $(r, s) \in (1, 17) + (0, 32) = (1, 49)$ .  $\square$

**Lemma 3.7.** *There exists a  $(4; r, s)$ -decomposition of  $K_5 \times K_6$ .*

*Proof.* We can write,  $K_5 \times K_6 = 5(P_3 \times K_6)$ . By Lemma 2.6, the graph  $P_3 \times K_6$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_5 \times K_6$  has the desired decomposition.  $\square$

**Lemma 3.8.** *If  $m, n \equiv 0 \pmod{4}$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \times K_n$ .*

*Proof.* Let  $m = 4x$  and  $n = 4y$ , where  $x, y \geq 1$ . Then we can write,

$$\begin{aligned} K_{4x} \times K_{4y} &= xy(K_4 \times K_4) \oplus \frac{xy(y-1)}{2}(K_4 \times K_{4,4}) \\ &\oplus \frac{xy(x-1)}{2}(K_{4,4} \times K_4) \\ &\oplus \frac{xy(x-1)(y-1)}{4}(K_{4,4} \times K_{4,4}) \\ &= xy(K_4 \times K_4) \oplus 2xy(x+y-2)(K_{4,12}) \\ &\oplus 2xy(x-1)(y-1)(K_{4,16}). \end{aligned}$$

By Lemma 3.1 and Theorem 1.3, the graphs  $K_4 \times K_4$ ,  $K_{4,12}$  and  $K_{4,16}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \times K_n$  has the desired decomposition.  $\square$

**Lemma 3.9.** *If  $m \equiv 0 \pmod{4}$  and  $n \equiv 2 \pmod{4}$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \times K_n$ .*

*Proof.* Let  $m = 4x$  and  $n = 4y + 2$ , where  $x, y \geq 1$ . Then we can write,

$$\begin{aligned} K_{4x} \times K_{4y+2} &= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_6) \\ &\oplus x(y-1)(K_4 \times K_{4,6}) \\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_4) \\ &\oplus \frac{x(x-1)}{2}(K_{4,4} \times K_6) \\ &\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_{4,6}) \\ &= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_6) \\ &\oplus 4x(y-1)(K_{4,18}) \\ &\oplus 2x(x-1)(y-1)(K_{4,12}) \oplus 3x(x-1)K_{4,20} \\ &\oplus 4x(x-1)(y-1)(K_{4,24}). \end{aligned}$$

By Lemmas 3.1 and 3.3 and Theorem 1.3, the graphs  $K_4 \times K_4, K_4 \times K_6, K_{4,12}, K_{4,18}, K_{4,20}$  and  $K_{4,24}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \times K_n$  has the desired decomposition.  $\square$

**Lemma 3.10.** *If  $m \equiv 0 \pmod{2}$  and  $n \equiv 1 \pmod{4}$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \times K_n$ .*

*Proof.* We deal the proof in two cases.

**Case 1.**  $m \equiv 0 \pmod{4}$  and  $n \equiv 1 \pmod{4}$ .

Let  $m = 4x$  and  $n = 4y + 1$ , where  $x, y \geq 1$ . Then we can write,

$$\begin{aligned}
K_{4x} \times K_{4y+1} &= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_5) \\
&\oplus x(y-1)(K_4 \times K_{4,5}) \\
&\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_4) \\
&\oplus \frac{x(x-1)}{2}(K_{4,4} \times K_5) \\
&\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_{4,5}) \\
&= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_5) \\
&\oplus 4x(y-1)(K_{4,15}) \\
&\oplus 2x(x-1)(y-1)(K_{4,18}) \oplus \frac{x(x-1)}{2}(5K_{4,16}) \\
&\oplus 4x(x-1)(y-1)(K_{4,20}).
\end{aligned}$$

By Lemmas 3.1 and 3.2 and Theorem 1.3, the graphs  $K_4 \times K_4, K_4 \times K_5, K_{4,15}, K_{4,16}, K_{4,18}$  and  $K_{4,20}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \times K_n$  has the desired decomposition.

**Case 2.**  $m \equiv 2 \pmod{4}$  and  $n \equiv 1 \pmod{4}$ .

Let  $m = 4x + 2$  and  $n = 4y + 1$ , where  $x, y \geq 1$ . Then we can write,

$$\begin{aligned}
K_{4x+2} \times K_{4y+1} &= (x-1)(y-1)(K_4 \times K_4) \oplus (x-1)(K_4 \times K_5) \\
&\oplus (x-1)(y-1)(K_4 \times K_{4,5}) \\
&\oplus (y-1)(K_6 \times K_4) \oplus (K_6 \times K_5) \\
&\oplus (y-1)(K_6 \times K_{4,5}) \\
&\oplus (x-1)(y-1)(K_{4,6} \times K_4) \\
&\oplus (x-1)(K_{4,6} \times K_5) \\
&\oplus (x-1)(y-1)(K_{4,6} \times K_{4,5}) \\
&= (x-1)(y-1)(K_4 \times K_4) \oplus (x-1)(K_4 \times K_5) \\
&\oplus 4(x-1)(y-1)(K_{4,15}) \oplus (y-1)(K_6 \times K_4) \\
&\oplus (K_6 \times K_5) \oplus 6(y-1)(K_{4,25}) \\
&\oplus 4(x-1)(y-1)(K_{4,18}) \oplus 5(x-1)(K_{4,24}) \\
&\oplus (x-1)(y-1)(5K_{4,24} \oplus 4K_{4,30}).
\end{aligned}$$

By Lemmas 3.1 to 3.3 and 3.7 and Theorem 1.3, the graphs  $K_4 \times K_4, K_4 \times K_5, K_4 \times K_6, K_5 \times K_6, K_{4,15}, K_{4,18}, K_{4,24}, K_{4,25}$  and  $K_{4,30}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \times K_n$  has the desired decomposition.  $\square$

**Lemma 3.11.** *If  $m \equiv 0 \pmod{4}$  and  $n \equiv 3 \pmod{4}$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \times K_n$ .*

*Proof.* Let  $m = 4x$  and  $n = 4y + 3$ , where  $x \geq 1$  and  $y \geq 0$ . When  $y = 0$ , we can write,  $K_{4x} \times K_3 = x(4x-1)(P_3 \times K_3)$ , by Theorem 1.4. Then the graph  $P_3 \times K_3$  has a  $(4; r, s)$ -decomposition, by Lemma 2.5. Hence, the graph  $K_{4x} \times K_3$  has the desired decomposition. When  $y \geq 1$ , we can write,

$$\begin{aligned}
K_{4x} \times K_{4y+3} &= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_7) \\
&\oplus x(y-1)(K_4 \times K_{4,7}) \\
&\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_4) \\
&\oplus \frac{x(x-1)}{2}(K_{4,4} \times K_7) \\
&\oplus \frac{x(x-1)(y-1)}{2}(K_{4,4} \times K_{4,7}) \\
&= x(y-1)(K_4 \times K_4) \oplus x(K_4 \times K_7) \\
&\oplus 4x(y-1)(K_{4,21}) \\
&\oplus 2x(x-1)(y-1)(K_{4,12}) \oplus 7\frac{x(x-1)}{2}(K_{4,24}) \\
&\oplus 4x(x-1)(y-1)(K_{4,28}).
\end{aligned}$$

By Lemmas 3.1 and 3.4 and Theorem 1.3, the graphs  $K_4 \times K_4, K_4 \times K_7, K_{4,12}, K_{4,21}, K_{4,24}$  and  $K_{4,28}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \times K_n$  has the desired decomposition.  $\square$

**Lemma 3.12.** *If  $m \equiv 1 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \times K_n$ .*

*Proof.* We deal the proof in two cases.

**Case 1.**  $m = 5$  and  $n \equiv 1 \pmod{2}$ .

Let  $m = 5$  and  $n = 2x + 1$ , where  $x \geq 1$ . For  $x = 1$ , then the graph  $K_5 \times K_3$  has a  $(4; r, s)$ -decomposition, by Lemma 3.5. For  $x > 1$ . Then we can write,

$$\begin{aligned}
K_5 \times K_{2x+1} &= (x-2)(K_5 \times K_2) \oplus K_5 \times K_5 \\
&\oplus (x-2)(K_5 \times K_{2,5}) \\
&= (x-2)(K_{5,5} - I) \oplus K_5 \times K_5 \oplus 5(x-2)(K_{2,20}).
\end{aligned}$$

By Lemmas 2.2 and 3.6 and Theorem 1.2, the graphs  $K_{5,5} - I, K_5 \times K_5$  and  $K_{2,20}$  have a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \times K_n$  has the desired decomposition.

**Case 2.**  $m \equiv 1 \pmod{4} > 5$  and  $n \equiv 1 \pmod{2}$ .

Let  $m = 4x + 1$  and  $n = 2y + 1$ , where  $x \geq 2$  and  $y \geq 1$ . Then  $K_{4x+1} \times K_{2y+1} = (2y+1)y(K_{4x+1, 4x+1} - I)$ , where  $I$  is a 1-factor of distance zero in  $K_{4x+1, 4x+1}$ . By Lemma 2.4, the graph  $K_{4x+1, 4x+1} - I$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_{4x+1} \times K_{2y+1}$  has the desired decomposition.  $\square$

Now, we prove our main result as follows:

**Theorem 3.1.** *Let  $r$  and  $s$  be nonnegative integers, and let  $m$  and  $n$  be positive integers. There exists a  $(4; r, s)$ -decomposition of  $K_m \times K_n$  if and only if  $mn(m-1)(n-1) \equiv 0 \pmod{8}$ , where  $K_m \times K_n$  denotes a tensor product of complete graphs.*

*Proof.* Necessity is trivial by counting the number of edges of the graph  $K_m \times K_n$ . Sufficiency follows from Lemmas 3.8 to 3.12.  $\square$

#### 4. $(4; r, s)$ -decomposition of $K_m \otimes \overline{K_n}$

In this section we obtain necessary and sufficient conditions for the existence of a  $(4; r, s)$ -decomposition of  $K_m \otimes \overline{K_n}$ .

**Lemma 4.1.** *If  $m \equiv 0$  (or)  $1 \pmod{2}$  and  $n \equiv 0 \pmod{2} \geq 4$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \otimes \overline{K_n}$ , and  $r \neq 1$  when  $m = 2$  and  $n = 4$ .*

*Proof.* We can write  $K_m \otimes \overline{K_n} = \frac{m(m-1)}{2} K_{n,n}$ . By Theorem 2.1, the graph  $K_{n,n}$  has a  $(4; r, s)$ -decomposition except  $r = 1$  when  $n = 4$ . Hence, by the remark, the graph  $K_m \otimes \overline{K_n}$  has the desired decomposition except when  $(n, r) = (4, 1)$ . Further,  $(4; 1, s)$ -decomposition of  $K_m \otimes \overline{K_4}$  is given as follows: For  $m = 2$ , then  $K_2 \otimes \overline{K_4} = K_{4,4}$  can not be decomposed into  $\{1P_5, 3S_5\}$ , by Theorem 2.1.

**Case 1.** For  $m = 3$ , then  $K_3 \otimes \overline{K_4}$  can be decomposed into  $\{1P_5, 11S_5\}$  is given as follows:  $[1_2 2_1 1_1 2_3 1_3]$ ,  $(1_2; 2_2, 2_3, 2_4, 3_1)$ ,  $(1_3; 2_1, 2_2, 2_4, 3_1)$ ,  $(1_4; 2_1, 2_2, 2_3, 2_4)$ ,  $(3_1; 1_1, 1_4, 2_2, 2_4)$ ,  $(2_1, 2_3; 3_1, 3_2, 3_3, 3_4)$ ,  $(2_2, 2_4; 1_1, 3_2, 3_3, 3_4)$ ,  $(3_2, 3_3, 3_4; 1_1, 1_2, 1_3, 1_4)$ .

**Case 2.** For  $m = 4$ , then we can write  $K_4 \otimes \overline{K_4} = K_3 \otimes \overline{K_4} \oplus K_{4,12}$ . By case 1 and Theorem 1.5, we obtain the required decomposition for the case  $(r, s) \in (1, 11) + (0, 12) = (1, 23)$ .

**Case 3.** For  $m > 4$ , then we can write  $K_m \otimes \overline{K_4} = K_4 \otimes \overline{K_4} \oplus \frac{(m-4)(m-5)}{2} K_{4,4} \oplus (m-4)K_{4,16}$ . By Case 2, and Theorem 1.5, we obtain the required decomposition for the case  $(r, s) \in (1, 23) + (0, 2(m-4)(m-5)) + (0, 16(m-4)) = (1, 23 + 2(m-4)\{8 + (m-5)\})$ .  $\square$

**Lemma 4.2.** *If  $m \equiv 1 \pmod{8}$  and  $n \equiv 1 \pmod{2}$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \otimes \overline{K_n}$ .*

*Proof.* We can write,  $K_m \otimes \overline{K_n} = nK_m \oplus (K_m \times K_n)$ . By Theorem 1.1, the graph  $K_m$  (If  $m = 9$ , the graph  $K_9$  has a  $(4; r, s)$ -decomposition, by Lemma 2.1) has a  $(4; r, s)$ -decomposition and by Lemma 3.12, the graph  $K_m \times K_n$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \otimes \overline{K_n}$  has the desired decomposition.  $\square$

**Lemma 4.3.** *If  $m \equiv 0 \pmod{8}$  and  $n \equiv 1 \pmod{2}$ , then there exists a  $(4; r, s)$ -decomposition of  $K_m \otimes \overline{K_n}$ .*

*Proof.* We can write,  $K_m \otimes \overline{K_n} = nK_m \oplus (K_m \times K_n)$ . By Theorem 1.1, the graph  $K_m$  (If  $m = 8$ , the graph  $K_8$  has a  $(4; r, s)$ -decomposition, by Lemma 2.1) has a  $(4; r, s)$ -decomposition and by Lemmas 3.10 and 3.11, the graph  $K_m \times K_n$  has a  $(4; r, s)$ -decomposition. Hence, by the remark, the graph  $K_m \otimes \overline{K_n}$  has the desired decomposition.  $\square$

**Theorem 4.1.** *Let  $r$  and  $s$  be nonnegative integers, and let  $m$  and  $n$  be positive integers. Then there exists a  $(4; r, s)$ -decom-*

*position of  $K_m \otimes \overline{K_n}$  if and only if  $mn^2(m-1) \equiv 0 \pmod{8}$ .*

*Proof.* Necessity is trivial by counting the number of edges of the graph  $K_m \otimes \overline{K_n}$ . Sufficiency follows from Lemmas 4.1 to 4.3.  $\square$

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