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AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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To cite this article: Zehui Shao, Zepeng Li, Bo Wang, Shaohui Wang & Xiujun Zhang (2020) Interval edge-coloring: A model of curriculum scheduling, AKCE International Journal of Graphs and Combinatorics, 17:3, 725-729, DOI: 10.1016/j.akcej.2019.09.003

To link to this article: https://doi.org/10.1016/j.akcej.2019.09.003

Published online: 22 Apr 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 725–729
https://doi.org/10.1016/j.akcej.2019.09.003

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Interval edge-coloring: A model of curriculum scheduling

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ABSTRACT
Considering the appointments that teachers plan to teach some courses for specific classes, the problem is to schedule the curriculum such that the time for each teacher is consecutive. In this work, we propose an integer linear programming model to solve consecutive interval edge-coloring of a graph. By using the proposed method, we give the interval edge colorability of some small complete multipartite graphs. Moreover, we find some new classes of complete multipartite graphs that have interval edge-colorings and disprove a conjecture proposed by Grzesik and Khachatrian (2014).

KEYWORDS
Interval coloring; complete tripartite graphs; complete multipartite graphs

1. Introduction

This paper considers graphs without multiple edges or loops. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. For a vertex $v \in V(G)$, the degree of v is denoted by $d(v)$, i.e., $d(v) = |\{u : uv \in E(G)\}|$ and we denote by $N_e(v)$ the set of edges incident to v, i.e., $N_e(v) = \{uv : uv \in E(G)\}$. If two edges e_1 and e_2 have a common vertex, we say they are adjacent and we write $e_1 \sim e_2$. A complete graph of order n is denoted by K_n. A hypercube of dimension n is denoted by Q_n. A complete k-partite ($k \geq 2$) graph is a graph whose vertices can be partitioned into k independent sets $V_1, \ldots, V_k$ with $|V_i| = n_i (1 \leq i \leq k)$ such that each vertex in V_i is adjacent to all the other vertices in V_j for $i \neq j$, and such a graph is denoted by $K_{n_1,\ldots,n_k}$. For graphs G and H, we write $G \cong H$ if G is isomorphic to H. The terms and concepts that we do not define can be found in [12].

For two positive integers a and b with $a \leq b$, the set $\{a, \ldots, b\}$ is denoted by $[a, b]$ and called an interval. A *proper edge-coloring* of a graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. For a proper coloring f of G and $v \in V(G)$, we denote by $S(v, f)$ the set of colors of edges incident to v under f. A proper edge-coloring f of a graph G with colors $1, \ldots, t$ is an *interval t-coloring* if all colors are used, and for any vertex v of G, the set $S(v, f)$ is an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the smallest and greatest value of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively. Not only because of the practical applications of interval coloring, but also for theirs interesting theoretical properties, the interval coloring and its variations have received a lot of attention [1–4, 5–8, 10].

Petrosyan presented the following conjecture in the "Cycles and Colorings 2012" workshop:

Conjecture 1. *For any $m, n \in \mathbb{N}, K_{1,m,n}$ is interval colorable if and only if $\gcd(m+1, n+1) = 1$.*

In [6], Grzesik and Khachatrian confirmed the above conjecture, and proposed the following conjectures:

Conjecture 2. [6] *The graph $K_{1,m,n}$ has an interval t-coloring if and only if $t = m + n$ and $\gcd(m+1, n+1) = 1$.*

Conjecture 3. [6] *The graph $K_{\ell,m,n}$, where $\ell \leq m \leq n$ and $n > \ell + m$ is interval colorable if and only if the graph $K_{\ell,m,n-\ell-m}$ is interval colorable.*

Conjecture 4. ([6], Conjecture 4) *The graph $K_{\ell,m,n}$, where $\ell \leq m \leq n$ and $n \leq \ell + m$ is interval colorable if and only if the sum $\ell + m + n$ is even.*

2. Mathematical model

Let G be a graph and the edges of G be labeled with $1, 2, \ldots, |E(G)|$. For a given positive integer k, we will determine whether G admits an interval coloring of G.

IEC ILP:
For an edge $e \in E(G)$, Let $x_{e,i} = 1$ if e is labeled with i and $x_{e,i} = 0$ otherwise. We will construct some constraints such that they are satisfied if and only if G admits an interval coloring.

In order to ensure each edge is assigned with one color, we have

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$$\sum_{i=1}^{k} x_{e,i} = 1, \qquad \forall e \in E(G) \tag{1}$$

Next, we must ensure two adjacent edges are colored with different color, so

$$x_{e_1,i} + x_{e_2,i} \leq 1, \quad \forall 1 \leq i \leq k, e_1 \sim e_2, 1 \leq e_1, e_2 \leq |E(G)| \tag{2}$$

Based on formula (1), it can be seen that $\sum_{i=1}^{k} i x_{e,i}$ is just the color assigned to an edge e. For a vertex v in G, we introduce variables $y_{v,max}$ and $y_{v,min}$ to obtain the constraints on interval coloring as in formulas (3) and (4).

$$y_{v,min} \leq \sum_{i=1}^{k} i x_{e,i} \leq y_{v,max}, \qquad \forall v \in V, e \in N_e(v) \tag{3}$$

$$y_{v,max} - y_{v,min} = d(v) - 1, \qquad \forall v \in V(G) \tag{4}$$

Binary requirements on the variables $x_{e,i}$ are given by formula (5).

$$x_{e,i} \in \{0,1\}, \qquad 1 \leq e \leq |E(G)|, 1 \leq i \leq k \tag{5}$$

If we require each color must be used, we have

$$\sum_{e \in E(G)} x_{e,i} = 1, \qquad \forall i \in \{1,2,\ldots,k\} \tag{6}$$

Now we have the following result:

Proposition 1. *The formulas (1)–(6) are satisfied if and only of G admits an interval coloring using each color from $\{1,2,\ldots,k\}$.*

3. On some conjectures on interval coloring of complete tripartite graphs

For convenience, we denote by (X, Y, Z) the tripartition of $K_{m,n,s}$, where $X = \{x_0, x_1, \ldots, x_{m-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n-1}\}$, and $Z = \{z_0, z_1, \ldots, z_{s-1}\}$.

The following result gives several counterexamples to Conjecture 4.

Theorem 1. *The complete bipartite graphs $K_{2,3,4}, K_{2,4,5}, K_{2,5,6}, K_{2,6,7}, K_{3,4,6}$ are interval edge colorable.*

Proof. Define $f: E(K_{2,3,4}) \to \{1,2,\ldots,10\}$ as follows. $f(x_0y_0) = 3$, $f(x_0y_1) = 5, f(x_0y_2) = 7, f(x_0z_0) = 2, f(x_0z_1) = 4, f(x_0z_2) = 6, f(x_0z_3) = 8$, $f(x_1y_0) = 4, f(x_1y_1) = 6, f(x_1y_2) = 8, f(x_1z_0) = 3, f(x_1z_1) = 5, f(x_1z_2) = 7, f(x_1z_3) = 9$, $f(y_0z_0) = 1, f(y_0z_1) = 2, f(y_0z_2) = 5, f(y_0z_3) = 6$, $f(y_1z_0) = 4, f(y_1z_1) = 3, f(y_1z_2) = 8, f(y_1z_3) = 7$, $f(y_2z_0) = 5$, $f(y_2z_1) = 6$, $f(y_2z_2) = 9$, and $f(y_2z_3) = 10$. Clearly f is an interval 10-coloring. Also the patterns P_1, P_2, P_3 and P_4 interval edge colorings of $K_{2,4,5}, K_{2,5,6}, K_{2,6,7}$ and $K_{3,4,6}$, respectively.

$$P_1 = \begin{bmatrix} 0 & 0 & 7 & 8 & 9 & 4 & 3 & 5 & 6 & 11 & 10 \\ 0 & 0 & 5 & 10 & 11 & 6 & 4 & 8 & 7 & 12 & 9 \\ 7 & 5 & 0 & 0 & 0 & 0 & 2 & 4 & 3 & 8 & 6 \\ 8 & 10 & 0 & 0 & 0 & 0 & 5 & 6 & 4 & 9 & 7 \\ 9 & 11 & 0 & 0 & 0 & 0 & 6 & 7 & 5 & 10 & 8 \\ 4 & 6 & 0 & 0 & 0 & 0 & 1 & 3 & 2 & 7 & 5 \\ 3 & 4 & 2 & 5 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & 4 & 6 & 7 & 3 & 0 & 0 & 0 & 0 & 0 \\ 6 & 7 & 3 & 4 & 5 & 2 & 0 & 0 & 0 & 0 & 0 \\ 11 & 12 & 8 & 9 & 10 & 7 & 0 & 0 & 0 & 0 & 0 \\ 10 & 9 & 6 & 7 & 8 & 5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 12 & 6 & 9 & 4 & 10 & 11 & 7 & 5 & 8 & 3 & 13 \\ 0 & 0 & 11 & 3 & 5 & 7 & 8 & 9 & 4 & 6 & 10 & 2 & 12 \\ 12 & 11 & 0 & 0 & 0 & 0 & 0 & 13 & 8 & 10 & 9 & 7 & 14 \\ 6 & 3 & 0 & 0 & 0 & 0 & 0 & 7 & 2 & 4 & 5 & 1 & 8 \\ 9 & 5 & 0 & 0 & 0 & 0 & 0 & 8 & 6 & 7 & 11 & 4 & 10 \\ 4 & 7 & 0 & 0 & 0 & 0 & 0 & 10 & 3 & 8 & 6 & 5 & 9 \\ 10 & 8 & 0 & 0 & 0 & 0 & 0 & 12 & 5 & 9 & 7 & 6 & 11 \\ 11 & 9 & 13 & 7 & 8 & 10 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 4 & 8 & 2 & 6 & 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 6 & 10 & 4 & 7 & 8 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 10 & 9 & 5 & 11 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 7 & 1 & 4 & 5 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 13 & 12 & 14 & 8 & 10 & 9 & 11 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 13 & 5 & 11 & 6 & 4 & 7 & 8 & 9 & 2 & 14 & 3 & 12 & 10 \\ 0 & 0 & 10 & 12 & 14 & 9 & 3 & 8 & 11 & 6 & 4 & 15 & 5 & 13 & 7 \\ 13 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 12 & 6 & 11 & 8 & 14 & 9 \\ 5 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 10 & 7 & 13 & 6 & 11 & 8 \\ 11 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 13 & 8 & 16 & 9 & 15 & 12 \\ 6 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 8 & 5 & 12 & 7 & 10 & 11 \\ 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 7 & 1 & 9 & 2 & 8 & 5 \\ 7 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 11 & 3 & 10 & 4 & 9 & 6 \\ 8 & 11 & 7 & 9 & 10 & 4 & 6 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 6 & 12 & 10 & 13 & 8 & 7 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & 7 & 8 & 5 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 15 & 11 & 13 & 16 & 12 & 9 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5 & 8 & 6 & 9 & 7 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 13 & 14 & 11 & 15 & 10 & 8 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 7 & 9 & 8 & 12 & 11 & 5 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P_4 = \begin{bmatrix} 0 & 0 & 0 & 2 & 9 & 7 & 6 & 4 & 11 & 3 & 10 & 5 & 8 \\ 0 & 0 & 0 & 8 & 11 & 5 & 13 & 10 & 9 & 4 & 12 & 6 & 7 \\ 0 & 0 & 0 & 3 & 10 & 4 & 12 & 5 & 7 & 6 & 11 & 8 & 9 \\ 2 & 8 & 3 & 0 & 0 & 0 & 0 & 7 & 5 & 1 & 9 & 4 & 6 \\ 9 & 11 & 10 & 0 & 0 & 0 & 0 & 6 & 8 & 5 & 13 & 7 & 12 \\ 7 & 5 & 4 & 0 & 0 & 0 & 0 & 9 & 6 & 2 & 8 & 3 & 10 \\ 6 & 13 & 12 & 0 & 0 & 0 & 0 & 8 & 10 & 7 & 14 & 9 & 11 \\ 4 & 10 & 5 & 7 & 6 & 9 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 9 & 7 & 5 & 8 & 6 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 6 & 1 & 5 & 2 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 12 & 11 & 9 & 13 & 8 & 14 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 6 & 8 & 4 & 7 & 3 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 7 & 9 & 6 & 12 & 10 & 11 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We denote by $K_{n\times k}$ the complete multipartite graph with k parts and n vertices in each part. Recently, Petrosyan [10] obtained the following result on interval edge-colorings of complete balanced multipartite graphs:

Theorem 2. (Petrosyan [10]) *$K_{n\times k} \in \mathfrak{N}$ if and only if nk is even. Moreover, if nk is even, then $w(K_{n\times k}) = n(k-1)$ and $W(K_{n\times k}) \geq \left(\frac{3}{2}k-1\right)n-1$.*

Note that the result that a balanced complete k-partite graph with n vertices in each part is interval colorable if and only if nk is even follows from a well-known (and trivial) fact that a regular graph is interval colorable if and only if it is Class 1, and the fact that such a balanced complete k-partite graph is Class 1 if and only if nk is even. The latter result was obtained by Laskar and Hare [9].

Moreover, Grzesik and Khachatrian [6] obtained the following result:

Theorem 3. [6] *If ℓ, m and n are odd, then $K_{\ell,m,n} \notin \mathfrak{N}$.*

By Theorem 2, we obtain the following result.

Corollary 1. *If n is even, then $K_{n\times 3}$ has an interval $2n$-coloring.*

Theorem 4 confirms Conjecture 3 in the case when $\ell = m$, n is multiple of m, and n or m is even.

Theorem 4. *For any $n, t \in \mathbb{N}$, if tn is even, then $K_{n,n,tn}$ has an interval $(tn+n)$-coloring.*

Proof. If t is even, let $G_0 = K_{n,n,tn}[X \cup Y]$ and $G = K_{n,n,tn} - E(G_0)$. Then $G_0 \cong K_{n,n}$. Let $Z_i = \{z_{2(i-1)n}, z_{2(i-1)n+1}, \ldots, z_{2in-1}\}$ and $G_i = G[X \cup Y \cup Z_i]$, where $i = 1, 2, \ldots, \frac{t}{2}$. Then $G_i \cong K_{2n,2n}$ for any $i \in \{1, 2, \ldots, \frac{t}{2}\}$. By the König's edge-coloring theorem, G_0 has an n-edge-coloring, say $f_0 : E(G_0) \to \{1, 2, \ldots, n\}$, and G_i has a $2n$-edge-coloring, say $f_i : E(G_i) \to \{(2i-1)n+1, (2i-1)n+2, \ldots, (2i+1)n\}$, where $i = 1, 2, \ldots, \frac{t}{2}$. Define $f(e) = f_\ell(e)$ if $e \in E(G_\ell)$, where $\ell = 0, 1, \ldots, \frac{t}{2}$. Then $S(x_j, f) = S(y_j, f) = [1, (t+1)n]$, $j = 0, 1, \ldots, n-1$; $S(z, f) = [(2i-1)n+1, (2i+1)n]$ for any $z \in Z_i$, $i = 1, 2, \ldots, \frac{t}{2}$. So f is an interval $(tn+n)$-coloring of $K_{n,n,tn}$.

If t is odd, then n is even. If $t=1$, then the result is true by Corollary 1. Now we consider the case of $t \geq 3$. Let $Z_0 = \{z_0, z_1, \ldots, z_{n-1}\}$ and $Z_i = \{z_{(2i-1)n}, z_{(2i-1)n+1}, \ldots, z_{(2i+1)n-1}\}$, where $i = 1, 2, \ldots, \frac{t-1}{2}$. Let $G_0 = G[X \cup Y \cup Z_0]$ and $G = K_{n,n,tn} - E(G_0)$. Then $G_0 \cong K_{n,n,n}$. By Corollary 1, G_0 has an interval $2n$-coloring $f_0 : E(G_0) \to \{1, 2, \ldots, 2n\}$. Let $G_i = G[X \cup Y \cup Z_i]$, where $i = 1, 2, \ldots, \frac{t-1}{2}$. Then $G_i \cong K_{2n,2n}$ for any $i \in \{1, 2, \ldots, \frac{t-1}{2}\}$. By the König's edge-coloring theorem, G_i has a $2n$-edge-coloring, say $f_i : E(G_i) \to \{2in+1, 2in+2, \ldots, (2i+2)n\}$, where $i = 1, 2, \ldots, \frac{t-1}{2}$. Define $f(e) = f_\ell(e)$ if $e \in E(G_\ell)$, where $\ell = 0, 1, \ldots, \frac{t-1}{2}$. Then $S(x_j, f) = S(y_j, f) = [1, (t+1)n]$, $j = 0, 1, \ldots, n-1$; $S(z, f) = [2in+1, (2i+2)n]$ for any $z \in Z_i$, $i = 0, 1, \ldots, \frac{t-1}{2}$. So f is an interval $(tn+n)$-coloring of $K_{n,n,tn}$. □

By Theorems 3 and 4, we obtain:

Corollary 2. *$K_{n,n,tn} \in \mathfrak{N}$ if and only if tn is even.*

Table 1. Values $W(G)$ of some complete balanced k-partite graphs $G = K_{n,n,\ldots,n}$.

k	n	$W'(G)$	$W(G)$	k	n	$W'(G)$	$W(G)$	k	n	$W'(G)$	$W(G)$
3	2	6	6	3	4	13	**14**	4	2	9	**10**
4	3	14	**16**	4	4	19	≥ 22	5	2	12	**13**
6	2	15	≥ 17	7	2	18	≥ 20				

4. Complete *k*-partite graphs

Let $G = K_{n,n,\ldots,n}$ be a k-partite graph and $W'(G) = \frac{3nk}{2} - n - 1$. In [11], it was shown that

Theorem 5. [11] If G is a complete balanced k-partite graph with n vertices in each part and nk is even, then $W(G) \geq W'(G)$.

In [11], the following three problems were put forward:

Problem 1. [11] Characterize all interval colorable complete multipartite graphs.

Problem 2. [11] Find the exact values of $w(G)$ and $W(G)$ for interval colorable complete multipartite graphs G.

Table 2. Interval edge colorability of complete k-partite graphs.

Instance	t	Result	Instance	t	Result	Instance	t	Result	Instance	t	Result
$K_{1,2,3}$	6	N	$K_{1,2,4}$	7	N	$K_{1,2,6}$	9	N	$K_{1,2,7}$	10	N
$K_{1,3,4}$	8	N	$K_{1,3,6}$	10	N	$K_{2,2,3}$	7,8,9	N	$K_{2,3,4}$	9,11	N
$K_{2,3,4}$	10	Y	$K_{2,4,5}$	11,13	N	$K_{2,4,5}$	12	Y	$K_{2,5,6}$	13,15	N
$K_{2,5,6}$	14	Y	$K_{2,6,7}$	15,17	N	$K_{2,6,7}$	16	Y	$K_{3,4,4}$	1,11,213	N
$K_{3,4,6}$	13,15	N	$K_{3,4,6}$	14	Y	$K_{3,6,6}$	18	N	$K_{1,1,1,2}$	3-9	N
$K_{1,1,1,3}$	4-11	N	$K_{1,1,1,4}$	5,8	N	$K_{1,1,1,4}$	6,7	Y	$K_{1,1,1,5}$	6-13	N
$K_{1,1,1,6}$	7-15	N	$K_{1,1,1,7}$	8,11	N	$K_{1,1,1,7}$	9,10	Y	$K_{1,1,1,10}$	12,13	Y
$K_{1,1,1,13}$	15,16	Y	$K_{1,1,2,2}$	4,8	N	$K_{1,1,2,2}$	5-7	Y	$K_{1,1,2,3}$	7,9	N
$K_{1,1,2,3}$	8	Y	$K_{1,1,2,4}$	6,10	N	$K_{1,1,2,4}$	7-9	Y	$K_{1,1,2,5}$	8-15	N
$K_{1,1,2,6}$	8,12	N	$K_{1,1,2,6}$	9-11	Y	$K_{1,1,2,7}$	13	N	$K_{1,1,2,7}$	12	Y
$K_{1,1,2,8}$	10,14	N	$K_{1,1,2,8}$	11-13	Y	$K_{1,1,3,3}$	6,10	Y	$K_{1,1,3,3}$	7-9	Y
$K_{1,1,3,4}$	9,11	N	$K_{1,1,3,4}$	10	Y	$K_{1,1,3,5}$	10,12	N	$K_{1,1,3,5}$	11	Y
$K_{1,1,4,4}$	9-11	Y	$K_{1,2,2,2}$	6-11	N	$K_{1,2,2,3}$	10	N	$K_{1,2,2,3}$	7-9	Y
$K_{1,2,2,4}$	9,11	N	$K_{1,2,2,4}$	10	Y	$K_{1,2,2,5}$	10	N	$K_{1,2,2,5}$	11	Y
$K_{1,3,3,3}$	11	N	$K_{2,2,2,3}$	7-10	N	$K_{2,2,2,4}$	13	N	$K_{2,2,2,4}$	8-12	Y
$K_{2,2,2,6}$	15	N	$K_{2,2,2,6}$	10-14	Y	$K_{2,2,2,8}$	17	N	$K_{2,2,2,8}$	12-16	Y
$K_{1,1,1,1,2}$	7	N	$K_{1,1,1,1,2}$	5,6	Y	$K_{1,1,1,1,3}$	7,9	N	$K_{1,1,1,1,3}$	8	Y
$K_{1,1,1,1,4}$	10	N	$K_{1,1,1,1,4}$	7-9	Y	$K_{1,1,1,1,5}$	8-17	N	$K_{1,1,1,1,6}$	9-11	Y
$K_{1,1,1,2,2}$	6-13	N	$K_{1,1,1,2,3}$	10	N	$K_{1,1,1,2,3}$	7-9	Y	$K_{1,1,2,2,2}$	11	N
$K_{1,1,2,2,2}$	7-10	Y	$K_{1,1,2,2,3}$	8	N	$K_{1,2,2,2,2}$	8-15	N	$K_{1,2,2,2,3}$	13	N
$K_{1,2,2,2,3}$	9-12	Y	$K_{2,2,2,2,3}$	9-19	N	$K_{2,2,2,2,4}$	17	N	$K_{2,2,2,2,4}$	10-16	Y

Problem 3. [11] Find the exact value of $W(K_{n,n,\dots,n})$ for interval colorable complete balanced k-partite graphs $K_{n,n,\dots,n}$.

By solving the instance of IEC ILP on some complete balanced k-partite graphs, we obtain some lower bounds for $W(G)$ greater than $W'(G)$ which are presented in Table 1.

Let $P^{c}_{k,n}$ be a pattern of the edge-coloring of the complete balanced k-partite graph $K_{n,n,\dots,n}$ with c colors. Then the following patterns $P^{14}_{3,4}$, $P^{16}_{4,3}$, $P^{10}_{4,2}$, $P^{17}_{6,2}$, $P^{20}_{7,2}$ give the corresponding interval edge-coloring which provide new lower bounds for $W(G)$ in Table 1.

$$P^{14}_{3,4}=\begin{bmatrix}
0&0&0&0&9&10&6&12&13&7&8&11\\
0&0&0&0&7&8&5&11&10&4&9&6\\
0&0&0&0&4&6&1&8&7&2&3&5\\
0&0&0&0&10&11&7&13&12&8&6&9\\
9&7&4&10&0&0&0&0&11&6&5&8\\
10&8&6&11&0&0&0&0&9&5&4&7\\
6&5&1&7&0&0&0&0&8&3&2&4\\
12&11&8&13&0&0&0&0&14&9&7&10\\
13&10&7&12&11&9&8&14&0&0&0&0\\
7&4&2&8&6&5&3&9&0&0&0&0\\
8&9&3&6&5&4&2&7&0&0&0&0\\
11&6&5&9&8&7&4&10&0&0&0&0
\end{bmatrix},
P^{16}_{4,3}=\begin{bmatrix}
0&0&0&11&13&7&10&15&8&12&14&9\\
0&0&0&6&7&1&4&9&3&5&8&2\\
0&0&0&10&11&6&9&12&5&8&13&7\\
11&6&10&0&0&0&8&13&7&9&12&5\\
13&7&11&0&0&0&12&14&9&10&15&8\\
7&1&6&0&0&0&5&8&2&4&9&3\\
10&4&9&8&12&5&0&0&0&7&11&6\\
15&9&12&13&14&8&0&0&0&11&16&10\\
8&3&5&7&9&2&0&0&0&6&10&4\\
12&5&8&9&10&4&7&11&6&0&0&0\\
14&8&13&12&15&9&11&16&10&0&0&0\\
9&2&7&5&8&3&6&10&4&0&0&0
\end{bmatrix},$$

$$P^{10}_{4,2}=\begin{bmatrix}
0&0&8&5&9&7&6&10\\
0&0&6&2&4&3&1&5\\
8&6&0&0&7&5&4&9\\
5&2&0&0&6&4&3&7\\
9&4&7&6&0&0&5&8\\
7&3&5&4&0&0&2&6\\
6&1&4&3&5&2&0&0\\
10&5&9&7&8&6&0&0
\end{bmatrix},
P^{22}_{4,4}=\begin{bmatrix}
0&0&0&0&21&15&18&11&17&22&16&13&12&19&20&14\\
0&0&0&0&18&13&14&9&15&19&11&10&8&16&17&12\\
0&0&0&0&11&6&8&2&7&12&5&3&1&9&10&4\\
0&0&0&0&15&9&10&6&12&14&8&5&4&11&13&7\\
21&18&11&15&0&0&0&0&16&20&14&12&10&17&19&13\\
15&13&6&9&0&0&0&0&11&16&10&7&5&12&14&8\\
18&14&8&10&0&0&0&0&13&17&12&9&7&15&16&11\\
11&9&2&6&0&0&0&0&8&13&7&4&3&10&12&5\\
17&15&7&12&16&11&13&8&0&0&0&0&9&14&18&10\\
22&19&12&14&20&16&17&13&0&0&0&0&11&18&21&15\\
16&11&5&8&14&10&12&7&0&0&0&0&6&13&15&9\\
13&10&3&5&12&7&9&4&0&0&0&0&2&8&11&6\\
12&8&1&4&10&5&7&3&9&11&6&2&0&0&0&0\\
19&16&9&11&17&12&15&10&14&18&13&8&0&0&0&0\\
20&17&10&13&19&14&16&12&18&21&15&11&0&0&0&0\\
14&12&4&7&13&8&11&5&10&15&9&6&0&0&0&0
\end{bmatrix},$$

$$P^{17}_{6,2}=\begin{bmatrix}
0&0&3&9&2&8&7&10&1&4&6&5\\
0&0&8&17&10&15&14&16&9&13&12&11\\
3&8&0&0&4&9&10&11&2&5&7&6\\
9&17&0&0&11&16&12&15&10&8&14&13\\
2&10&4&11&0&0&5&9&3&6&8&7\\
8&15&9&16&0&0&13&14&7&12&11&10\\
7&14&10&12&5&13&0&0&6&11&9&8\\
10&16&11&15&9&14&0&0&8&7&13&12\\
1&9&2&10&3&7&6&8&0&0&5&4\\
4&13&5&8&6&12&11&7&0&0&10&9\\
6&12&7&14&8&11&9&13&5&10&0&0\\
5&11&6&13&7&10&8&12&4&9&0&0
\end{bmatrix},
P^{20}_{7,2}=\begin{bmatrix}
0&0&4&13&5&9&8&7&14&3&10&12&11&6\\
0&0&5&12&1&7&11&3&9&4&6&8&10&2\\
4&5&0&0&6&12&13&9&10&8&7&14&15&11\\
13&12&0&0&11&16&17&14&19&10&15&18&20&9\\
5&1&6&11&0&0&7&4&8&2&12&10&9&3\\
9&7&12&16&0&0&14&10&15&11&13&17&18&8\\
8&11&13&17&7&14&0&0&18&12&9&15&16&10\\
7&3&9&14&4&10&0&0&13&6&8&11&12&5\\
14&9&10&19&8&15&18&13&0&0&11&16&17&12\\
3&4&8&10&2&11&12&6&0&0&5&9&13&7\\
10&6&7&15&12&13&9&8&11&5&0&0&14&4\\
12&8&14&18&10&17&15&11&16&9&0&0&19&13\\
11&10&15&20&9&18&16&12&17&13&14&19&0&0\\
6&2&11&9&3&8&10&5&12&7&4&13&0&0
\end{bmatrix}.$$

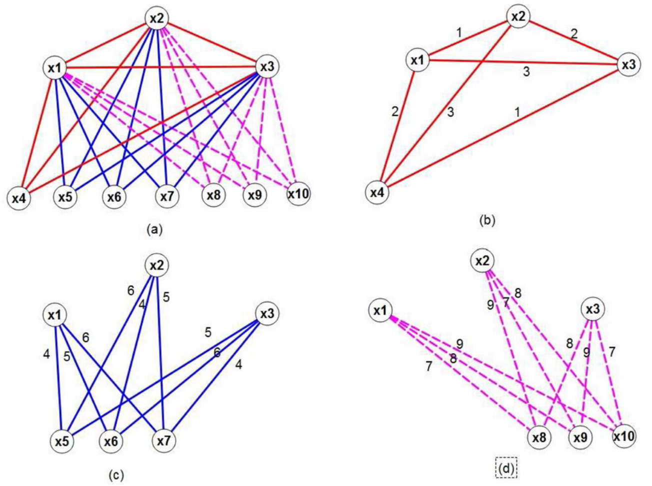

Figure 1. A interval 9-coloring of $K_{1,1,1,7}$.

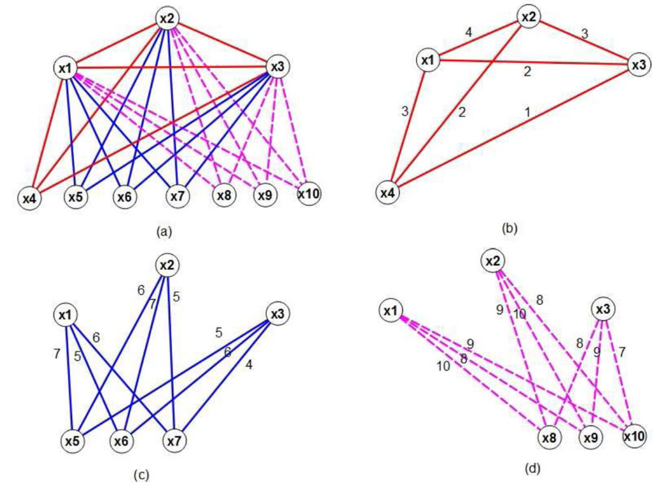

Figure 2. A interval 10-coloring of $K_{1,1,1,7}$.

Remark: We find five exact values of $K_{n,n,\dots,n}$ concerning Problem 3.

By solving the instance of IEC ILP, we are able to obtain exact values of $w(G)$ and $W(G)$ for a graph. Many exact values of $w(G)$ and $W(G)$ for interval colorable complete multipartite graphs are obtained in are shown in Table 2. The value N (resp. Y) in column *result* stand for the corresponding instance *admits no* (resp. *admits a)* interval t-coloring. For instance, we have $K_{1,2,3}$ admits no interval 6-coloring.

Theorem 6. *If $n \equiv 1 \pmod 3$ and $t \in \{n+2, n+3\}$, then the graph $K_{1,1,1,n}$ has an interval t-coloring.*

Proof. Let X_i be four partite parts of $K_{1,1,1,n}$ with $|X_i| = 1$ for $i = 1, 2, 3$ and $|X_4| = n$. Assume $X_1 = \{x_1\}, X_2 = \{x_2\}, X_3 = \{x_3\}$, and $X_4 = \{x_4, x_5, \dots, x_{n+3}\}$. It can be seen that the edge set of $K_{1,1,1,n}$ can be decomposed into a $K_{1,3}$ and $\frac{n-1}{3}$ disjoint $K_{3,3}$. That is, $E(K_{1,1,1,n}) = E(X, x_4) \cup E(X, Y_i)$ for $i = 1, 2, \dots, \frac{n-1}{3}$, where $X = \{x_1, x_2, x_3\}$ and $Y_i = \{x_{3i+2}, x_{3i+3}, x_{3i+4}\}$.

Case 1: $t = n+2$.
Consider an edge function f with $f(x_1x_2) = 1, f(x_2x_3) = 2, f(x_1x_3) = 3, f(x_1x_4) = 2, f(x_2x_4) = 3,$ and $f(x_3x_4) = 1$. We color the edges of $E(X, Y_i)$ with color set $\{3i+1, 3i+2, 3i+3\}$ for each $i \in \{1, 2, \dots, \frac{n-1}{3}\}$ such that $f(x_1x_{3i+k}) = 3i+k-1$ for $k = 2, 3, 4$, $f(x_2x_{3i+2}) = 3i+3, f(x_2x_{3i+3}) = 3i+1, f(x_2x_{3i+4}) = 3i+2$, $f(x_2x_{3i+2}) = 3i+2, f(x_2x_{3i+3}) = 3i+3$ and $f(x_2x_{3i+4}) = 3i+1$. Then f is an interval $(n+2)$-coloring (see e.g., Figure 1) in the case $n = 7$.

Case 2: $t = n+3$.
Consider an edge function f with $f(x_1x_2) = 4, f(x_2x_3) = 3, f(x_1x_3) = 2, f(x_1x_4) = 3, f(x_2x_4) = 2,$ and $f(x_3x_4) = 1$. We color the edges of $E(X, Y_i)$ with $f(x_1x_{3i+2}) = 3i+4, f(x_2x_{3i+3}) = 3i+2, f(x_2x_{3i+4}) = 3i+3$, $f(x_2x_{3i+2}) = 3i+3, f(x_2x_{3i+3}) = 3i+4, f(x_2x_{3i+4}) = 3i+2$, $f(x_2x_{3i+2}) = 3i+2, f(x_2x_{3i+3}) = 3i+3$ and $f(x_2x_{3i+4}) = 3i+1$. Then f is an interval $(n+3)$-coloring (see e.g., Figure 2) in the case $n = 7$. □

Inspired by the results we obtain, we propose the following conjecture:

Conjecture 5. *The graph $K_{1,1,1,n}$ has an interval t-colorable if and only if $n \equiv 1 \pmod 3$ and $t \in \{n+2, n+3\}$.*

Disclosure statement

The authors declare that there is no conflict of interest regarding the publication of this paper.

Funding

This work was supported by the National Key Research and Development Program under grants 2017YFB0802300 and 2017YFB0802303, the National Natural Science Foundation of China under grants 61309015, Applied Basic Research (Key Project) of Sichuan Province under grant 2017JY0095.

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