AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 1045–1051
https://doi.org/10.1016/j.akcej.2019.12.022

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ABSTRACT

A vertex deleted unlabeled subgraph of a graph is a *card*. A *dacard* specifies the degree of the deleted vertex along with the card. The *adversary degree associated reconstruction number* of a graph G, denoted $adrn(G)$, is the minimum number k such that every collection of k dacards of G uniquely determines G. A *strong double broom* is the graph on at least 5 vertices obtained from a union of (at least two) internally vertex disjoint paths with same ends u and v by appending leaves at u and v. The strong double broom, obtained from a union of m internally vertex disjoint paths of order k with same ends u and v by appending n leaves at each u and v, is denoted by $B(n,n,mP_k)$. In this paper, we show that the *drn* of all strong double brooms is two and we determine $adrn(B(n,n,mP_k))$ for all n, m, k. For $n \geq 1$ and $m \geq 2$, usually $adrn(B(n,n,mP_3)) = n+2$ and $adrn(B(n,n,mP_4)) = n+m+2$, except $adrn(B(1,1,2P_3)) = 4$. For $n \geq 1, m \geq 2$ and $k > 4$, usually $adrn(B(n,n,mP_k)) = n+2$, except $adrn(B(n,n,mP_5)) = max\{m,n\}+2$ when $(n,m) \neq (1,2)$ and $adrn(B(1,1,2P_5)) = 5$.

KEYWORD
Reconstruction; reconstruction number; dacard

AMS SUBJECT CLASSIFICATION (2010)
Primary 05C60;
Secondary 05C05

1. Introduction

All graphs considered in this paper are finite, simple and undirected. We shall mostly follow the graph theoretic terminology of [4]. A vertex of degree m is called an *m-vertex* and a 1-vertex is called an *end vertex*. The neighbour of a 1-vertex is called a *base* and a base of degree m is called an *m-base*. A neighbour of v with degree k is called a *k-neighbour* of v. A *double broom* is a tree obtained from a path by appending leaves at both ends of the path. A *strong double broom*, denoted by B, is the graph on at least 5 vertices obtained from a union of (at least two) internally vertex disjoint (u, v)-paths by appending leaves at u and v. More precisely, $B(n_1, n_2, m_1P_{k_1}, m_2P_{k_2}, \ldots, m_tP_{k_t})$ denotes the strong double broom with n_1 leaves at one end u, n_2 leaves at the other end v and there are m_i internally vertex disjoint (u, v)-paths on k_i vertices for $1 \leq i \leq t$, $m_i \geq 0$ and $k_1 < k_2 < \ldots < k_t$ $(m_1 = 1$ when $k_1 = 2)$. The vertices u and v are called the *hub vertices* and the 2-vertices are called *middle vertices*.

We distinguish the dacards of B into three types: a leaf dacard L, a middle dacard M and a hub dacard K are obtained, respectively, by deleting a leaf vertex, a middle vertex and a hub vertex (Figure 1).

A *vertex deleted subgraph* or a *card* $G - v$ of a graph G is the unlabeled graph obtained from G by deleting a vertex v and all edges incident with v. The ordered pair $(d(v), G-v)$ is called a *degree associated card* (or *dacard*) of the graph G, where $d(v)$ is the degree of v in G. We denote m copies of the dacard $(d(v), G-v)$ by $m(d(v), G-v)$ or simply by $m(G-v)$. The *deck* (*dadeck*) of a graph G is the collection of all its cards (dacards). The *Ulam's Conjecture* [3], also called the *Reconstruction Conjecture* (RC) asserts that every graph on at least three vertices is determined uniquely (up to isomorphism) by its deck. Graphs that obey the RC are called *reconstructible*.

For a reconstructible graph G, Harary and Plantholt [5] have defined the *reconstruction number* $rn(G)$ to be the size of the smallest subcollection of the deck of G which is not contained in the deck of any other graph H, $H \ncong G$. Myrvold [11] referred to this number as *ally-reconstruction number* of G and also studied *adversary reconstruction number* of G, which is the smallest k such that no subcollection of the deck of G of size k is contained in the deck of any other graph H, $H \ncong G$.

An extension of the RC to digraphs - the *Digraph Reconstruction Conjecture* was disproved when Stockmeyer exhibited [14] several infinite families of counter-examples. In view of this, Ramachandran [12, 13] studied the degree (degree triple) associated reconstruction of graphs (digraphs) and their reconstruction number. For a reconstructible graph (digraph) G from its dadeck, the *degree (degree triple) associated reconstruction number* of G, denoted $drn(G)$, is the size of the smallest subcollection of the dadeck of G which is not contained in the dadeck of any other graph (digraph) H, $H \ncong G$. Monikandan and Sundar Raj [10] introduced the degree associated analogue of $arn(G)$ (attributing the notion to Ramachandran). When G is reconstructible from its dadeck, the adversary degree-associated reconstruction number, denoted $adrn(G)$, is the least k such

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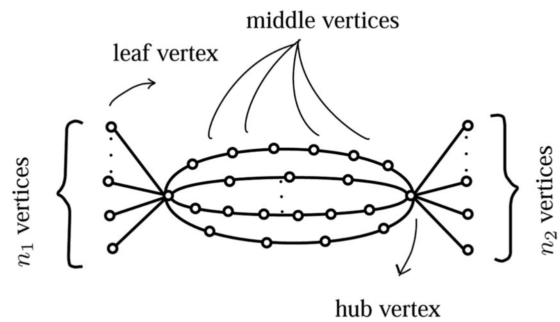

Figure 1. A strong double broom.

that every set of k dacards determines G. From the definition, $drn(G) \leq adrn(G)$. Equality holds when G is vertex-transitive, since then the dacards are pairwise isomorphic. The value of the *adrn* is known [6, 10] for complete graphs, complete bipartite graphs, cycles and wheels. In a subsequent paper, Monikandan and Sundar Raj [9] determined the *adrn* for double-stars, for subdivisions of stars, and for the disjoint union of t complete graphs of order n and s cycles of length m. If G is an r-regular of order n, then Barrus and West [1] have shown that $drn(G) \leq \min\{r+2, n-r+1\}$, which also implies that $adrn(G) \leq \min\{r+2, n-r+1\}$. (For elementary results on the edge version of adversary degree-associated reconstruction, see [8].)

Myrvold [11] showed $arn(G) = 3$ for almost every graph G. Since always $adrn(G) \leq arn(G)$, it is thus of some interest to find graphs G where $adrn(G)$ is large. Bowler et al. [2] constructed infinite families of pairs of graphs in which the pairs with n vertices have $2\lfloor\frac{n-4}{3}\rfloor$ common dacards, so $adrn(G)$ can be as large as $2\lfloor\frac{n-4}{3}\rfloor + 1$. They conjecture that this is the largest value for graphs of order n. Among vertex-transitive graphs, $G = 2K_{\frac{n}{4},\frac{n}{4}}$ in [1] achieves $adrn(G) = drn(G) = \frac{1}{4}|V(G)| + 2$. Recently Ma et al. [7] have proved that the *adrn* of most of the double brooms $B(m, n, P_k)$ is $\min\{m, n\} + 2$. In this paper, we show that the *drn* of all strong double brooms is two and we determine $adrn(B(n, n, mP_k))$ for all n, m, k. For $n \geq 1$ and $m \geq 2$, usually $adrn(B(n, n, mP_3)) = n + 2$ and $adrn(B(n, n, mP_4)) = n + m + 2$, except $adrn(B(1, 1, 2P_3)) = 4$. For $n \geq 1, m \geq 2$ and $k > 4$, usually $adrn(B(n, n, mP_k)) = n + 2$, except $adrn(B(n, n, mP_5)) = max\{m, n\} + 2$ when $(n, m) \neq (1, 2)$ and $adrn(B(1, 1, 2P_k)) = 5$.

2. *drn* and *adrn* of strong double brooms

We first determine the *drn*.

Theorem 1. *If G is a strong double broom, then* $drn(G) = 2$.

Proof. Let $G = B(n_1, n_2, m_1P_{k_1}, m_2P_{k_2}, \ldots, m_tP_{k_t})$. dacards $(1, L)$ and $\left(n_2 + \sum_{i=1}^{t} m_i, K\right)$ of G. Any leaf dacard forces G to be connected with $\binom{m_1}{2}$ cycles of length $2(k_1 - 1)$ (if $k_1 > 2$), $\binom{m_2}{2}$ cycles of length $2(k_2 - 1)$ (if $k_2 > 2$), $\ldots, \binom{m_t}{2}$ cycles of length $2(k_t - 1)$ (if $k_t > 2$) and to have m_1m_2 cycles of length $k_1 + k_2 - 2$, m_1m_3 cycles of length $k_1 + k_3 - 2, \ldots, m_1m_t$ cycles of length $k_1 + k_t - 2$, m_2m_3 cycles of length $k_2 + k_3 - 2, \ldots, m_2m_t$ cycles of length $k_2 + k_t - 2, \ldots, m_{t-1}m_t$ cycles of length $k_{t-1} + k_t - 2$. Hence to a hub dacard, add a new vertex v and join it to all isolated vertices and to $m_1, m_2, \ldots, m_t$ end vertices at distance $k_1 - 2, k_2 - 2, \ldots, k_t - 2$ respectively from a base of maximum degree if $k_1 > 2$, and joining v to m_j, $2 \leq j \leq t$ end vertices at distance $k_j - 2$ from a base of maximum degree and to this base if $k_1 = 2$. The resulting graph thus obtained is isomorphic to G. □

We denote the collection of m dacards $(d(v), G - v)$ by $m(d(v), G - v)$ or simply $m(G - v)$. An *extension* of a dacard $(d(v), G - v)$ of G is a graph obtained from the dacard by adding a new vertex x and joining it to $d(v)$ vertices of the dacard and it is denoted by $H(d(v), G - v)$ (or simply H). Throughout this paper, H and x are used in the sense of this definition.

Theorem 2. *If* $G = B(n, n, mP_3)$, *then* $adrn(G) = \begin{cases} 4 & \text{if } (n, m) = (1, 2) \\ n + 2 & \text{otherwise} \end{cases}$

Proof. The dacards of G are $2n(1, L)$, $m(2, M)$ and $2(n + m, K)$.

Lower bound: For $(n, m) = (1, 2)$, the graph G shares two middle dacards and a leaf dacard with a graph obtained from $(1, L)$ by annexing a new vertex x and joining it to the unique leaf vertex and therefore $adrn(G) \geq 4$. For $(n, m) \neq (1, 2)$, G shares $n + 1$ leaf dacards with $B(n + 1, n - 1, mP_3)$ and therefore $adrn(G) \geq n + 2$.

Upper bound: We proceed by three cases and prove that the collection of dacards considered under each case determines G uniquely.

Case 1. Two distinct dacards, except when one is L and the other is M for $(n, m) = (1, 2)$.

Case 1.1. K and M (or L).

Since the two dacards K and L determine G uniquely (Theorem 1), we consider the two dacards K and M. The dacard M forces G to be connected and G to have every base with at most n neighbours of degree 1. Hence G can be obtained uniquely from K by annexing a vertex and joining it to n isolated vertices and to m vertices of degree 1.

Case 1.2. L and M when $(n, m) \neq (1, 2)$.

The dacard L forces G to have $\binom{m}{2}$ cycles of length $2(k - 1)$ and to have maximum degree at least $n + m$. Hence in M, the newly added vertex x must be joined to the two $(n + m - 1)$-vertices. But then the extension is isomorphic to G.

Case 2. For $(n, m) = (1, 2)$, any collection S of four dacards including at least one L and M.

In view of Case 1, we take S to contain only the dacards L and M.

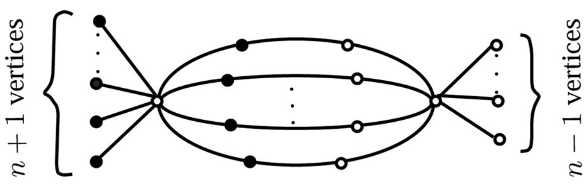

Figure 2. $B(n+1, n-1, mP_4)$.

The dacard L forces every extension to have C_4 and hence the only possibility to obtain an extension H ($\ncong G$) from M is to join the newly added vertex to the 1-vertex and to a 2-vertex at distance 2. The graph H has exactly one dacard L (obtained by removing the unique leaf) and exactly two dacards M (obtained by removing the two 2-neighbours of the unique 3-vertex). The removal of any other 2-vertex from K results in a dacard with a 3-vertex or an isolated vertex but M does not have these vertices. Hence S uniquely determines G.

Case 3. '$(n+1)L$ and D' or '$2M$ and D' for $(n, m) \neq (1, 2)$.

In view of Case 1, we assume that the dacards of S are isomorphic.

Case 3.1. $(n+2)L$ (this case does not arise when $n = 1$).

Now $H(1, L)$ is obtained by joining x to one vertex of L. If x is joined to a 1-vertex or a 2-vertex, then H has exactly one dacard L (obtained by removing x) and the removal of any other 1-vertex from H results in a dacard with a 2-base or two adjacent bases but L does not have these. If x is joined to the $(n+m)$-vertex, then H has exactly $n+1$ leaf dacards (obtained by removing each 1-neighbour of the $(n+m+1)$-vertex) and the removal of any other leaf from H results in a dacard with an $(n+m+1)$-vertex but L does not have an $(n+m+1)$-vertex.

Case 3.2. $3M$ (this case does not arise when $m = 2$).

The extension ($\ncong G$) obtained from M, by joining x to a 1-vertex and its base, has exactly two dacards isomorphic to M (obtained by removing x and its neighbour) and the removal of any other 2-vertex results in a dacard having C_3 but M does not have C_3. For all other extensions, the removal of any 2-vertex other than x, results in a dacard having at most $\binom{m}{2} - 1$ cycles of length $2(k-1)$ or two adjacent bases or a 2-base or a base with $n+1$ neighbours of degree 1 but L does not have these. Thus, from all preceding cases, we have $adrn(B(n, n, mP_3)) = 4$ if $(n, m) = (1, 2)$ (Cases 1.1 and 2) and $adrn(B(n, n, mP_3)) = n+2$ otherwise (Cases 1.2 and 3). □

Theorem 3. *If $G = B(n, n, mP_4)$, then $adrn(G) = n+m+2$.*

Proof. The dadeck of G consists of $2n(1, L)$, $2m(2, M)$ and $2(n+m, K)$. Let D denote any dacard of G.

Lower bound: The graph G shares $n+1$ leaf dacards and m middle dacards with $B(n+1, n-1, mP_4)$ (Figure 2). Hence $adrn(G) \geq n+m+2$.

Upper bound: We proceed by three cases and prove that the collection of dacards considered under each case determines G uniquely.

Case 1. M and K when $(n, m) \neq (1, 2)$.

The dacard M forces G to be connected and hence K forces G to have two vertices of degree at least $n+m$ and every base with at most n neighbours of degree 1. Hence G can be obtained from M by joining x to the 1-vertex whose base has $n+1$ neighbours of degree 1 and to the $(n+m-1)$-vertex.

Case 2. L and K.

Proof follows by Theorem 1.

Case 3. M, K and one more dacard D when $(n, m) = (1, 2)$.

We take D to be other than L as otherwise D and K together will determine G by Case 2.

Using M and K, we can say that every base in G has one 1-neighbour. Hence M forces every extension to have at most two 1-vertices. Therefore the only possible extension H non-isomorphic to G of K can be obtained by joining x to the isolated vertex and to the two 1-vertices at distance 3. The extension H then has exactly one dacard isomorphic to M (obtained by removing a 2-vertex adjacent to the 2-neighbour of the 3-base) as the removal of any other 2-vertex results in a dacard having every base with one 1-neighbour or an isolated vertex and exactly one dacard isomorphic to K (obtained by removing x) as the removal of any other 3-vertex results in a dacard having two non-trivial components.

Case 4. The collection S containing αL and βM where $\alpha, \beta \geq 1$ and $\alpha + \beta \leq n+m+1$, together with one more dacard.

Let us assume that S contains no K as otherwise, by Case 2, S uniquely determines G. The dacard L forces G to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in M, to get an extension non-isomorphic to G, join x to the unique $(n+m)$-vertex and to a 1-vertex at distance $2(k-2)$. The resulting extension then has exactly $n+1$ dacards isomorphic to L (obtained by removing the 1-neighbours of $(n+m+1)$-vertex), has no more leaf dacards (since there is no more leaf or the removal of any leaf results in a dacard having an $(n+m+1)$-vertex), exactly m dacards isomorphic to M (obtained by removing the 2-neighbours of the $(n+m+1)$-base), no more middle dacards (since the removal of any 2-vertex results in a dacard having an $(n+m+1)$-vertex) and no hub dacard (since there is no $(n+m)$-vertex).

Case 5. $(n+2)L$ (when $n \neq 1$) or $(m+1)M$.

Case 5.1. $(n+2)L$.

If, in L, vertex x is joined to a 1-vertex or an $(n+m)$-vertex, then the proof is similar to Case 3.1 of Theorem 2. If x is joined to a 2-vertex, then the resulting

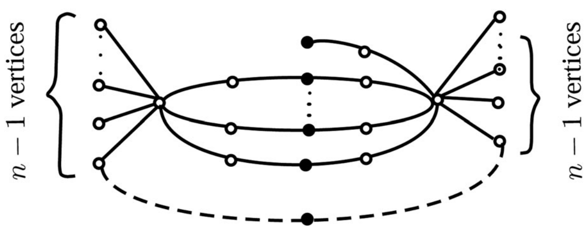

Figure 3. The extension H.

extension non-isomorphic to G has exactly one dacard isomorphic to L, since the removal of any other leaf results in a dacard having two bases at distance at most 2.

Case 5.2. $(m+1)M$.

If, in M, vertex x is joined to an $(n+m)$-vertex and a 1-neighbour of $(n+m-1)$- base, then the resulting extension has exactly m dacards isomorphic to M (obtained by removing the 2-neighbours of an $(n+m+1)$-base) and no more middle dacards, since the removal of any other 2-vertex results in a dacard with an $(n+m+1)$-base. If x is joined to a 1-vertex and its base, then the resulting extension has exactly two dacards isomorphic to M (obtained by removing x and its 2-neighbour) or if x is adjacent to a 1-vertex and a 2-vertex when $(n,m)=(1,2)$, the resulting extension has two dacards isomorphic to M (obtained by removing the 2-neighbours of 3-vertex lying on a cycle) and the removal of any other 2-vertex results in a disconnected dacard (when $(n,m)=(1,2)$) or in a dacard having a cycle C_3 or two bases at distance at most 2 (when $(n,m)=(1,2)$). All other extensions of M has at most two dacards isomorphic to M, since the removal of any 2-vertex other than x results in a dacard having two bases at distance at most 2 or at most $\binom{m}{2}-1$ cycles of length $2(k-1)$ or a 2-base or in a disconnected dacard. □

It is clear that all the $2m$ dacards of $B(n,n,mP_k)$ obtained by deleting middle vertices at equal distance i from the nearest hub vertex are mutually isomorphic; let M_i denote such a dacard. Then i can be 1, 2,..., or $\lceil \frac{k}{2} \rceil - 1$. These dacards will be used in proving the next main result.

Theorem 4. *For $k>4$,*

$$adrn(B(n,n,mP_k)) = \begin{cases} 5 & \text{if } (n,m)=(1,2) \\ max\{m,n\}+2 & \text{if } k=5 \text{ and } (n,m)\neq(1,2) \\ n+2 & \text{otherwise} \end{cases}$$

Proof. The dadeck of $G=B(n,n,mP_k)$ consists of $2n(1,L)$, $m(k-2)(2,M_i)$ and $2(n+m,K)$. Let D denote any dacard of G.

Lower bound: The graph G shares $n+1$ leaf dacards with the graph $B(n+1,n-1,mP_k)$. Hence $adrn(G)\geq n+2$. … … (Eq-1)

For $(n,m)=(1,2)$, the graph G shares two leaf dacards and two middle dacards with a graph obtained from L by joining x to a 2-vertex at distance $k-4$ from the base. Hence $adrn(G)\geq 5$.

For $k=5$ and $(n,m)\neq(1,2)$, G shares a leaf dacard and m middle dacards with a graph obtained from M_1 by joining x to two 1-vertices at distance $2(k-2)$. Hence from (Eq-1), $adrn(G)\geq max\{m,n\}+2$.

Upper bound: We proceed by eleven cases and we prove that the collection of dacards considered under each case determines G uniquely.

Case 1. L and any dacard D other than L and M_1.

The dacard L forces G to be connected and to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in K, join x to the n isolated vertices and to m vertices of degree 1 at distance $k-2$ from the unique $(n+m)$-vertex or in middle dacard (other than M_1) join x to two 1-vertices at distance $2(k-2)$ and the extension then obtained is G.

Case 2. L, M_1 and D when $(n,m)\neq(1,2)$ and $k\neq 5$.

The dacard L forces G to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in M_1, to get an extension non-isomorphic to G, join x with a 1-neighbour of the $(n+m-1)$-base and to a vertex at distance $2(k-2)$. The resulting graph has exactly one dacard isomorphic to M_1 (obtained by removing x), no middle dacards (since the removal of any other 2-vertex results in a dacard with two bases of degree at least 3 at distance less than $k-1$ or a vertex of degree at least 3 with $n-1$ neighbours of degree 1), exactly one dacard L (obtained by removing the 1-neighbour of the 3-base adjacent to x) as the removal of any other 1-vertex results in a dacard with two bases at distance less than $k-1$ and has no dacards H (since the removal of any $(n+m)$-vertex results in a dacard with no $(n+m)$-base).

Claim 3. αL, βM_1, $\alpha,\beta\geq 1$, $\alpha+\beta=m+1$ and D when $(n,m)\neq(1,2)$ and $k=5$.

The dacard L forces every extension to have $\binom{m}{2}$ cycles of length $2(k-1)$. Hence in the dacard M_1, to get an extension non-isomorphic to G, vertex x must be joined to a 1-neighbour of the $(n+m-1)$-base and to some vertex at distance 6. If x is joined to a 1-neighbour of an $(n+m)$-base, the the resulting extension H (Figure 3) has exactly m dacards isomorphic to M_1 (obtained by removing each 2-neighbour at distance 2 from the $(n+m)$-base lying on a cycle), has no more middle dacards (since the removal of any other 2-vertex results in a dacard with a base of degree at least 3 having $n-1$ neighbours of degree 1), exactly one leaf dacard (obtained by removing 1-neighbour of the 2-base) as the removal of any leaf results in a dacard having a 2-base and has no hub dacard (since the removal of the unique $(n+m)$-vertex results in a dacard with two nontrivial components). If x is joined to a 2-base, then the resulting extension has exactly one dacard isomorphic to each L (obtained by removing the 1-neighbour of the 3-base adjacent to x) and M_1 (obtained by removing x), has no

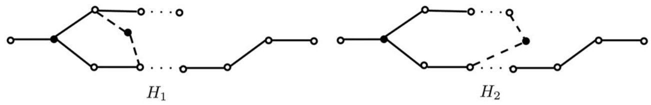

Figure 4. The extensions H_1 and H_2.

more leaf or middle dacards (since the removal of any leaf or 2-vertex results in a dacard with three vertices of degree at least 3 or two adjacent vertices each of degree at least 3) and has no dacards K (since the removal of an $(n+m)$-vertex results in a dacard having no $(n+m)$-vertex).

Case 4. $\alpha L,\ \beta M_1,\ \alpha,\beta \geq 1,\ \alpha+\beta = 4$ and one D when $(n,m) = (1,2)$.

The dacard L forces every extension to have a cycle of length $2(k-1)$. Hence the only two extensions non-isomorphic to G of M_1 are obtained by annexing a vertex and joining it to the 2-base at minimum distance from the 3-base or the 1-neighbour of the 3-base (when $k=5$) and to the 1-vertex at maximum distance from the 3-base. The first extension has exactly two dacards isomorphic to L (obtained by removing the two leaves) and has exactly two dacards isomorphic to M_1 (obtained by removing the 2-neighbours of the bases that are not common neighbours). The second extension has exactly one dacard isomorphic to L (obtained by removing the unique leaf) and has exactly two dacards isomorphic to M_1 (obtained by removing the 2-vertices at distance 2 from the unique 3-vertex). Both of these two extensions have no more dacards M_1 (since the removal of any other 2-vertex results in a dacard having two 3-bases or exactly two leaves or having two leaves at distance 2 or 3 from a 3-vertex or in a disconnected dacard or a base with two 1-neighbours), no more middle dacards M_i $\left(2 \leq i \leq \lceil \frac{k}{2} \rceil\right)$ (since the removal of 2-vertex results in a dacard with two 3-bases at distance either less than $k-1$ or greater than $k-1$ or with at most one 3-base) and no hub dacard (since the removal of any 3-vertex results in a dacard with no 3-base or having a 1-vertex at distance greater than $k-2$ from a 3-base).

Case 5. $M_{\lceil \frac{k}{2} \rceil - 1}$ and K when $k=5$.

As in Case 1.1 of Theorem 2, the dacards $M_{\lceil \frac{k}{2} \rceil - 1}$ and K force G to have every base with at most n neighbours of degree one. Hence in $M_{\lceil \frac{k}{2} \rceil - 1}$, the newly added vertex x must be adjacent to a 1-neighbour of each base with $n+1$ neighbours of degree 1 and the resulting extension is isomorphic to G.

Case 6. $\alpha M_1,\ \beta K,\ \alpha,\beta \geq 1,\ \alpha+\beta = 4$ and one D when $k=5$.

Case 6.1. $(n,m) = (1,2)$.

The dacard M_1 forces every extension to be connected and hence every extension of K is obtained by joining x to the isolated vertex in K. If the other neighbours of x are a 2-base and the unique 1-neighbour of the 3-base, then the resulting extension non-isomorphic to G has exactly two middle dacards M_1 (obtained by removing the two 2-neighbours adjacent to the 3-bases, exactly two hub dacards K (obtained by removing the two 3-bases) and no other hub dacards (since the removal of any other 3-vertex results in a dacard with no isolated vertex) and has no leaf dacards (since the removal of any leaf results in a dacard with a cycle of length less than $2(k-1)$)). If x is joined to the 1-neighbour of a 2-base and to the other 2-base or to the 1-neighbour of a 2-base and the 1-neighbour of the 3-base, then the resulting extension non-isomorphic to G has exactly one dacard K (since the removal of any 3-vertex other than x results in a dacard having a base with two 1-neighbours), exactly one dacard M_1 (obtained by removing a 2-vertex adjacent to two 3-bases or a 2-vertex lying on a cycle at distance 3 from the 3-base). The above extensions have no more middle dacards and the remaining extensions non-isomorphic to G have no middle dacards (since the removal of any other 2-vertex results in a disconnected dacard or in a dacard having a 1-vertex at distance $k-2$ from the nearest 3-base or 4-base or having no 3-base).

Case 6.2. $(n,m) \neq (1,2)$.

The dacards K and M_1 force G to be connected and to have two vertices of degree at least $n+m$. Hence in every extension non-isomorphic to G of M_1, one neighbour of the newly added vertex must be the unique $(n+m-1)$-vertex. If the other neighbour is the 2-base, then the resulting graph has exactly one dacard M_1 (obtained by removing x) and no more middle dacards (since the removal of any other 2-vertex results in a dacard having two adjacent bases of degree at least 3), exactly one dacard K (obtained by removing an $(n+m)$-vertex adjacent to the 3-base) as the removal of any other $(n+m)$-vertex results in a dacard having a 1-vertex at distance $k-3$ from the nearest $(n+m)$-base). The remaining extensions other than G have no dacard K (since the removal of any $(n+m)$-vertex results in a dacard having two nontrivial components having a 1-vertex at distance 2 from an $(n+m)$-base or an $(n+m+1)$-base or a cycle of length less than $2k-2$).

Case 7. $M_i,\ 1 \leq i \leq \lceil \frac{k}{2} \rceil - 1,\ K$ and D when $k \neq 5$.

Case 7.1. $M_1,\ K$ and D.

The dacards K and M_1 force G to be connected and to have two vertices of degree at least $n+m$. Hence in every extension non-isomorphic to G of M_1, one neighbour of the newly added vertex must be the unique $(n+m-1)$-vertex. The extensions H_1, H_2 (Figure 4), H_3 (Figure 5) when $(n,m) = (1,2)$ and H_4 when $(n,m) \neq (1,2)$ (Figure 5) have exactly one dacard M_1 (obtained by removing x), no more middle dacards (since the removal of any other 2-vertex results in a dacard having two 3-vertices at distance less than $k-1$ or no 3-base or two nontrivial components) and exactly one hub dacard (obtained by removing the unique 3-base in H_1 and H_2, removing the 3-base adjacent to a 3-vertex in H_3 and removing the $(n+m)$-base adjacent to the 3-vertex in H_4). The above extensions H_1, H_2, H_3 and H_4 have no more dacards K and the remaining extensions other than G have no dacards K (since the removal of any other $(n+m)$-vertex results in a dacard having two nontrivial

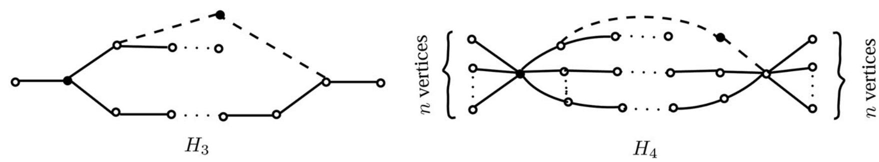

Figure 5. The extensions H_4 and H_5.

components or no $(n+m)$-base or an $(n+m)$-base with $n+1$ neighbours of degree 1 or an $(n+m)$-vertex or having a 1-vertex at distance 2 or less than $k-2$ (>1) or an $(n+m+1)$-vertex).

Case 7.2. M_i $\left(2 \le i \le \lceil \frac{k}{2} \rceil - 1\right)$, K and D.

Let the middle dacard have 1-vertices at distance $l_1 > 1$ and $l_2 \ge 1$ such that $l_1 \ge l_2$ from the nearest $(n+m)$-base. The extension non-isomorphic to G of M_i (obtained by joining x to a 1-vertex at distance l_2 (or l_1) from the nearest $(n+m)$-base and to the 2-neighbour of the other $(n+m)$-base which is at distance $l_1 - 1$ (or $l_2 - 1$ if $l_2 > 1$) from the other 1-vertex) has exactly one middle dacard (obtained by removing x) as the removal of any other 2-vertex results in a dacard having two bases of degree at least 3 at distance less than k – 1, no dacard L as the removal of any leaf results in a dacard having a cycle of length less than $2(k-1)$ and exactly one hub dacard (obtained by removing the $(n+m)$-neighbour of a 3-base adjacent to x). The above extensions have no more hub dacard and the remaining extensions non-isomorphic to G has no hub dacard (since the removal of any $(n+m)$-vertex results in a dacard having two nontrivial components or a 1-vertex at distance less than $k-1$ (>1) from the nearest $(n+m)$-base or at most $n+m-1$ isolated vertices or an $(n+m+1)$-vertex.

Case 8. $M_{\lceil \frac{k}{2} \rceil - 1}$ and $M_{\lceil \frac{k}{2} \rceil - 2}$ when k is even ($k \ne 6$ for $(n,m) = (1,2)$).

Every extension non-isomorphic to G of $M_{\lceil \frac{k}{2} \rceil - 1}$ does not have the dacard $M_{\lceil \frac{k}{2} \rceil - 2}$ (since the removal of any 2-vertex other than x results in a dacard having an $(n+m+1)$-vertex or a 1-vertex at distance $\frac{k}{2} - 1$ or $\frac{k}{2} - 2$ $(k \ge 8)$ from the nearest $(n+m)$-base or a cycle of length less than $2(k-1)$ or exactly one $(n+m)$-base (for $k \ge 8$) or results in a disconnected dacard).

Case 9. $M_{\lceil \frac{k}{2} \rceil - 1}$ and '$M_{\lceil \frac{k}{2} \rceil - 2}$ or $M_{\lceil \frac{k}{2} \rceil - 3}$' when k is odd ($k \ne 7$ for $(n,m) = (1,2)$) but only the first two dacards determine G when $k = 7$ and $(n,m) = (1,2)$.

Every extension non-isomorphic to G of $M_{\lceil \frac{k}{2} \rceil - 1}$ does not have the second dacard $M_{\lceil \frac{k}{2} \rceil - 2}$, since the removal of any 2-vertex (other than x) results in a dacard having a base with $n+1$ or $n+2$ neighbours of degree 1 (when $k = 5$) or a 1-vertex at distance $\lceil \frac{k}{2} \rceil - 2$ from the nearest $(n+m)$-vertex or having an $(n+m+1)$-vertex or a cycle of length less than $2(k-1)$ or exactly one $(n+m)$-base (for $k \ge 9$) or results in a disconnected dacard.

Case 10. M_i, M_j, $i \ne j$, $1 \le i < j \le \lceil \frac{k}{2} \rceil - 1$ and D.

We exclude Cases 8 and 9 that come under Case 10.

Case 10.1 M_i has exactly one vertex of maximum degree.

Let the other middle dacard, say M_j, have 1-neighbours at distance $l_1 > 1$ and $l_2 \ge 1$ such that $l_1 \ge l_2$, respectively from the nearest $(n+m)$-base.

Case 10.1.1. $m = 2$.

The extension $(\ncong G)$ of the dacard M_i (obtained by joining x to the 2-base at a maximum distance from the 3-base (when $n = 1$) or to the $(n+m-1)$-base (when $n > 1$) and to a 2-vertex at distance $l_1 - 2$ (if $l_1 > 3$) or $l_2 - 2$ (if $l_2 > 3$) from the 1-neighbour of another 2-base) has exactly one dacard M_i (obtained by removing x) and exactly one dacard M_j (obtained by removing a 2-vertex adjacent to a 3-neighbour of x which is at minimum distance from an $(n+m)$-vertex non adjacent to x) and the extension obtained by joining x to the 1-neighbour of the 2-base which is at minimum distance from the 3-base and to a 2-vertex at distance l_1 (if $l_1 > 1$) or l_2 (if $l_2 > 1$) from the 1-neighbour of another 2-base) have exactly one dacard M_i (obtained by removing x) and have exactly one dacard M_j (obtained by removing a central vertex of a path connecting two vertices of degree at least 3, disjoint from a path containing x where the central vertex is at minimum distance from x). The above extensions have no more dacards M_j (since the removal of any other 2-vertex results in a dacard having a cycle or two vertices of degree at least 3 at distance less than $k-1$, no more dacards M_i (since the removal of any other 2-vertex results in a dacard having two bases of degree at least 3 or a cycle or no $(n+m)$-base or having a 1-vertex at distance less than $k-3$ (>1) from a base of degree at least 3) and have no leaf dacards (since the removal of any leaf results in a dacard having a cycle of length less than $2(k-1)$) and have no hub dacards (since the removal of any $(n+m)$-vertex results in a dacard with no $(n+m)$-base). All other extensions non-isomorphic to G have either two isomorphic dacards M_i or exactly one dacard M_i (since the removal of any other 2-vertex results in a dacard having a cycle or at least $2n+2$ vertices of degree 1 or a base with $n+1$ neighbours of degree 1 or a 1-vertex at distance less than $k-3$ (>1) from a base of degree at least 3 or having no $(n+m)$-base.

Case 10.1.2. $m > 2$.

The extension non-isomorphic to G of the dacard M_i, obtained by joining x to a $(n+m)$-base to a 2-vertex at distance $l_1 - 2$ (if $l_1 > 2$) or $l_2 - 2$ (if $l_2 > 2$) from 1-neighbour of another 2-base, has exactly two middle dacards (obtained by removing x and by removing a 2-neighbour of a 3-base adjacent to x) and has no more dacards (because of the similar reasons described in Case 10.1).

Case 10.2. M_i has two vertices of maximum degree.

Proof is similar to Case 10.1.

Case 11. $(n+2)L$ (this case arise when $n>1$) or $3M_i$ $(1 \leq i \leq \lceil \frac{k}{2} \rceil - 1)$.

Proof is similar to Case 4 of Theorem 2.

From all the preceding cases, we conclude that

$$adrn(B(n,n,mP_k)) = \begin{cases} 5 & \text{if } (n,m)=(1,2) \text{ (Cases 1, 3, } 5-11) \\ max\{m,n\}+2 & \text{if } k=5 \text{ and } (n,m) \neq (1,2) \\ & \text{(Cases 1, } 4-6, 9-11) \\ n+2 & \text{otherwise (Cases 1, } 4-6, 9-11) \end{cases}$$

□

From the full deck of a graph G, it is easy to compute the degrees of the deleted vertices; hence the RC is equivalent to the reconstruction of graphs from their dadecks. From the partial deck of a graph, the degree of the deleted vertex is hard to compute, and the dacard (degree associated card) provides more information. Hence graphs may be reconstructible using fewer dacards than cards.

Disclosure statement

No conflicts of interest have been reported by the authors.

Funding

The second author's research is supported through a research project sponsored by DST-SERB, Govt. of India. Grant No. EMR/2016/000157.

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