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On H-antimagic decomposition of toroidal grids and triangulations

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ABSTRACT

Let $G=(V,E)$ be a finite simple graph with p vertices and q edges. A decomposition of a graph G into isomorphic copies of a graph H is called $(a,\ d)$-H-antimagic if there is a bijection $f: V\cup E \to \{1,2,\dots,p+q\}$ such that for all subgraphs H' isomorphic to H in the decomposition of G, the sum of the labels of all the edges and vertices belonging to H' constitutes an arithmetic progression with the initial term a and the common difference d. When $f(V)=\{1,2,\dots,p\}$, then G is said to be *super* $(a,\ d)$-H-antimagic and if $d=0$ then G is called H-supermagic. In the paper we examine the existence of such labelings for toroidal grids and toroidal triangulations.

KEYWORDS
Toroidal grid; toroidal triangulation; *H-decomposition*; *H-supermagic graph*; *H-antimagic graph*

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1. Introduction

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A *covering* of G is a family of subgraphs, say $\mathcal{B}=\{G_1,G_2,\dots,G_t\}$, having the property that each edge of G belongs to at least one of the subgraphs G_i, for some $i=1,2,\dots,t$. If all G_i, $i=1,2,\dots,t$, are isomorphic to a graph H, such a covering is called an *H-covering* of G. A family $\mathcal{B}=\{G_1,G_2,\dots,G_t\}$ of subgraphs of G is an *H-decomposition* of G if all subgraphs are isomorphic to graph H, $E(G_i)\cap E(G_j)=\emptyset$ for $i\neq j$, and $\cup_{i=1}^{t}E(G_i)=E(G)$. In such a case G is said to be *H-decomposable*. If G is an H-decomposable graph, then we write $H|G$. Hence an H-decomposition of the host graph G is a special case of its H-covering.

Suppose that a $(p,\ q)$-graph $G=(V,E)$ with p vertices and q edges admits H-covering, say $\mathcal{B}=\{G_1,G_2,\dots,G_t\}$. The graph G is called $(a,\ d)$-H-antimagic if there exists a bijection $f: V\cup E\to\{1,2,\dots,p+q\}$ such that for every subgraph $H'\in\mathcal{B}$, the H-weights,

$$wt_f(H')=\sum_{v\in V(H')}f(v)+\sum_{e\in E(H')}f(e)$$

form an arithmetic progression $a, a+d, a+2d,\dots,a+(t-1)d$, where $a>0$ and $d\geq 0$ are two integers, and t is the number of all subgraphs isomorphic to H in $\mathcal{B}$. Such a labeling is called *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) $(a,\ d)$-H-antimagic labeling is called *(super) $(a,\ d)$-H-antimagic*. An H-covering (or H-decomposition) of G is said to be a (super) $(a,\ d)$-H-antimagic covering (or decomposition) of G if G has a (super) $(a,\ d)$-H-antimagic labeling.

For $d=0$, a (super) $(a,\ d)$-H-antimagic graph is called *H-magic* and *H-supermagic*, respectively.

The H-(super)magic coverings were first studied by Gutiérrez and Lladó [11] as an extension of the edge-magic and super edge-magic labelings introduced by Kotzig and Rosa [15] and Enomoto, Lladó, Nakamigawa and Ringel [9], respectively. In [11] are considered star-(super)magic and path-(super)magic labelings of some connected graphs and it is proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó and Moragas [16] studied the cycle-(super)magic behavior of several classes of connected graphs. They proved that wheels, windmills, books and prisms are C_h-magic for some h. Maryati, Salman, Baskoro, Ryan and Miller [20] and also Salman, Ngurah and Izzati [22] proved that certain families of trees are path-supermagic. Ngurah, Salman and Susilowati [21] proved that chains, wheels, triangles, ladders and grids are cycle-supermagic. Maryati, Salman and Baskoro [19] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that the disjoint union of any paths is cP_h-supermagic for some c and h.

The H-magic decomposition and H-antimagic decomposition were first introduced by Inayah, Lladó and Moragas in [14]. They proved that if T is a graceful tree with m edges then the complete graph K_{2m+1} admits an $(a,\ d)$-T-antimagic decomposition for some a and all even differences $0\leq d\leq m+1$. Moreover they showed that if a tree T with m edges admits an α-labeling, then the complete bipartite graph $K_{m,m}$ admits an $(a,\ d)$-T-antimagic decomposition for some a and all $0\leq d\leq m$ with the same parity as m. Liang in [17] gave the conditions for the existence of C_{2k}-supermagic

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decomposition of the complete n-partite graph as well as of multiple copies of it. He used Sotteau's investigation of the complete bipartite graph decomposition into cycles of some fixed length given in [24] and results of Cavenagh and Billington [7] who listed certain necessary conditions for the existence of a $2k$-cycle decomposition of complete n-partite graphs.

The H-supermagic decomposition of antiprisms are described in [12] and the H-supermagic decompositions of the lexicographic product of graphs are discussed by Hendy, Sugeng and Salman in [13].

Let us introduce the following concept. Suppose that $\mathcal{B}$ is an H-decomposition of a $(p,\ q)$ graph G. Then $\mathcal{B}$ is said to be an *(a, d)-H-edge-antimagic* if there exists a bijection $g: E \to \{1, 2, ..., q\}$ such that for every subgraph $H' \in \mathcal{B}$, the H-weights,

$$w_f(H') = \sum_{e \in E(H')} g(e)$$

form an arithmetic progression $a, a+d, a+2d, ..., a+(t-1)d$, where $a > 0$ and $d \geq 0$ are two integers, and t is the number of all subgraphs isomorphic to H in $\mathcal{B}$. For $d = 0$, the $(a,\ d)$-H-edge-antimagic decomposition is called *H-edge-magic* and was introduced in [17].

For $H \cong K_2$, (super) $(a,\ d)$-H-antimagic labelings are also called (super) $(a,\ d)$-edge-antimagic total labelings and have been introduced in [23]. More results on $(a,\ d)$-edge-antimagic total labelings, can be found in [5, 18]. The vertex version of these labelings for generalized pyramid graphs is given in [3]. A detailed survey of graph labelings can be found in [10].

In this paper we mainly investigate the existence of super $(a,\ d)$-H-antimagic decompositions for toroidal grids and toroidal triangulations. We next summarize the contents of the forthcoming sections. In Section 2 we recall some useful facts on the partitions of a set of integers with given differences which will be used for constructing the requested labelings. In Section 3 we deal with the existence of super $(a,\ d)$-H-antimagic decompositions of a toroidal square grid, where H is a m-sun graph. In Section 4 we deal with the super $(a,\ d)$-H-antimagic decomposition of a toroidal triangulation with cycle m-sun graphs as H-subgraphs.

2. Partitions with given differences

For construction of the requested vertex and edge labelings of considered graphs we will use the partitions of a set of integers with given differences. This concept was introduced in [4] and we summarize below important relations for the convenience of the reader.

Let n, k, d and i be positive integers. We will consider the partition $\mathcal{P}^n_{k,d}$ of the set $\{1, 2, ..., kn\}$ into n, $n \geq 2$, k-tuples such that the difference between the sum of the numbers in the $(i+1)$ th k-tuple and the sum of the numbers in the ith k-tuple is always equal to the constant d, where $i = 1, 2, ..., n-1$. Thus they form an arithmetic sequence with the difference d. By the symbol $\mathcal{P}_{k,d}(i)$ we denote the ith k-tuple in the partition with the difference d, where $i = 1, 2, ..., n$.

Let $\sum \mathcal{P}^n_{k,d}(i)$ be the sum of the numbers in $\mathcal{P}^n_{k,d}(i)$. Evidently $\sum \mathcal{P}^n_{k,d}(i+1) - \sum \mathcal{P}^n_{k,d}(i) = d$. It is obvious that if there exists a partition of the set $\{1, 2, ..., kn\}$ with the difference d, there also exists a partition with the difference $-d$. By the notation $\mathcal{P}^n_{k,d}(i) \oplus c$ we mean that we add the constant c to every number in $\mathcal{P}^n_{k,d}(i)$.

If $k = 1$ then only the following partition of the set $\{1, 2, ..., n\}$ is possible

$$\mathcal{P}^n_{1,1}(i) = \{i\}, \quad \text{for } i = 1, 2, ..., n.$$

If $k = 2$ then we have several partitions of the set $\{1, 2, ..., 2n\}$. Let us define the partitions into 2-tuples in the following way:

$$\begin{aligned} &\mathcal{P}^n_{2,0}(i) = \{i, 2n+1-i\}, \\ &\sum \mathcal{P}^n_{2,0}(i) = 2n+1, && \text{for } i = 1, 2, ..., n. \\ &\mathcal{P}^n_{2,2}(i) = \{i, n+i\}, \\ &\sum \mathcal{P}^n_{2,2}(i) = n+2i, && \text{for } i = 1, 2, ..., n. \\ &\mathcal{P}^n_{2,4}(i) = \{2i-1, 2i\}, \\ &\sum \mathcal{P}^n_{2,4}(i) = 4i-1, && \text{for } i = 1, 2, ..., n. \end{aligned}$$

Moreover, for $3 \leq n \equiv 1 \pmod 2$

$$\mathcal{P}^n_{2,1}(i) = \begin{cases} \{\frac{n+1}{2} + \frac{i-1}{2}, n+1+\frac{i-1}{2}\} & \text{for } i \equiv 1 \pmod 2, \\ \{\frac{i}{2}, n+\frac{n+1}{2}+\frac{i}{2}\} & \text{for } i \equiv 0 \pmod 2, \end{cases}$$

$$\sum \mathcal{P}^n_{2,1}(i) = n + \frac{n+1}{2} + i, \qquad \text{for } i = 1, 2, ..., n.$$

Note that we are able to obtain the partitions into 2-tuples $\mathcal{P}^n_{2,0}(i)$ and $\mathcal{P}^n_{2,2}(i)$ as $\mathcal{P}^n_{1,s}(i) \cup (\mathcal{P}^n_{1,t}(i) \oplus n)$, where $s, t = \pm 1$. We can use this idea to construct the other partitions. More precisely,

$$\mathcal{P}^n_{k,d}(i) = \mathcal{P}^n_{l,s}(i) \cup (\mathcal{P}^n_{m,t}(i) \oplus ln),$$

where $k = l + m$ and $d = s + t$.

For example, we are able to obtain $\mathcal{P}^n_{3,d}(i)$ from the partitions $\mathcal{P}^n_{1,s}(i), s = \pm 1$ and $\mathcal{P}^n_{2,t}(i), t = 0, \pm 2, \pm 4$ and also $t = \pm 1$ for n odd. It means, $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5$ and if $n \equiv 1 \pmod 2$ also for $d = 0, \pm 2$. Moreover, we are able to construct $\mathcal{P}^n_{3,9}$ in the following way

$$\begin{aligned} &\mathcal{P}^n_{3,9}(i) = \{3(i-1)+1, 3(i-1)+2, 3(i-1)+3\}, \\ &\sum \mathcal{P}^n_{3,9}(i) = 9i - 3, \quad \text{for } i = 1, 2, ..., n. \end{aligned}$$

Thus $\mathcal{P}^n_{3,d}$ exists for $d = \pm 1, \pm 3, \pm 5, \pm 9$ and if $n \equiv 1 \pmod 2$ also for $d = 0, \pm 2$.

For the partition into 4-tuples we can use the following fact

$$\mathcal{P}^n_{4,d}(i) = \mathcal{P}^n_{l,s}(i) \cup (\mathcal{P}^n_{m,t}(i) \oplus ln),$$

where $l = 3, m = 1$ or $l = 2, m = 2$. Also

$$\begin{aligned} \mathcal{P}^n_{4,16}(i) = \{&4(i-1)+1, 4(i-1)+2, 4(i-1) \\ &+3, 4(i-1)+4\}, \end{aligned}$$

$$\sum \mathcal{P}^n_{4,16}(i) = 16i - 6, \quad \text{for } i = 1, 2, ..., n.$$

Thus $\mathcal{P}^n_{4,d}$ exists for $d = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$ and if $n \equiv 1 \pmod 2$ also for $d = \pm 1, \pm 3, \pm 5$.

Let us note that each of the defined partition $\mathcal{P}^n_{k,d}$ has the property that

$$\sum \mathcal{P}^n_{k,d}(i) = C^n_{k,d} + d(i-1), \tag{1}$$

where $C^n_{k,d} = \sum \mathcal{P}^n_{k,d}(1)$. For example $C^n_{2,0} = 2n+1, C^n_{2,2} = n+2, C^n_{2,4} = 3$ and $C^n_{2,1} = (3n+3)/2$ for n odd.

3. *H*-antimagic decompositions for toroidal grids

Let G_1 and G_2 be two graphs. The Cartesian product $G_1 \Box G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ where two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1y_2 \in E(G_2)$ or $y_1 = y_2$ and $x_1x_2 \in E(G_1)$.

If we consider graph G_1 as the cycle C_m with $V(C_m) = \{x_j : 1 \le j \le m\}, E(C_m) = \{x_jx_{j+1} : 1 \le j \le m-1\} \cup \{x_mx_1\}$ and graph G_2 as the cycle C_n with $V(C_n) = \{y_i : 1 \le i \le n\}, E(C_n) = \{y_iy_{i+1} : 1 \le i \le n-1\} \cup \{y_ny_1\}$ then $V(C_m \Box C_n) = \{(x_j, y_i) : 1 \le j \le m, 1 \le i \le n\}$ is the vertex set of $C_m \Box C_n$ and $E(C_m \Box C_n) = \{(x_j, y_i)(x_{j+1}, y_i) : 1 \le j \le m-1, \quad 1 \le i \le n\} \cup \{(x_m, y_i) \quad (x_1, y_i) : 1 \le i \le n\} \cup \{(x_j, y_i)(x_j, y_{i+1}) : 1 \le j \le m, \quad 1 \le i \le n-1\} \cup \{(x_j, y_n)(x_j, y_1) : 1 \le j \le m\}$ is the edge set of $C_m \Box C_n$. So $C_m \Box C_n$ is a *toroidal square grid* of order mn and size $2mn$ that can be embedded in a torus or in other words, the vertices of $C_m \Box C_n$ can be placed on a torus such that no edges cross. Let T^n_m be a toroidal square grid, $T^n_m \cong C_m \Box C_n$.

Anitha and Lekshmi [2] and also Marr and Wallis [18] use the term *m-sun graph*, say $\mathcal{S}_m$, to the graph on $2m$ vertices obtained by attaching m pendant edges to a cycle graph C_m where one pendant edge is attached at each vertex in C_n.

It is not difficult to see that the toroidal square grid T^n_m can be decomposed into m-sun graphs $\mathcal{S}_m$ or n-sun graphs $\mathcal{S}_n$. Due to symmetry it is enough to deal with an $\mathcal{S}_m$-decomposition of T^n_m.

Theorem 1. *Let $m, n \ge 3$ and $\mathcal{S}_m$ be the m-sun graph. Then the toroidal square grid T^n_m admits a $\mathcal{S}_m$-supermagic decomposition.*

Proof. Define the bijection $f_1 : V(T^n_m) \cup E(T^n_m) \to \{1, 2, \ldots, 3mn\}$ as follows:

$$\begin{aligned} f_1((x_j, y_i)) &= m(n-i) + j, \\ &\quad \text{for} \quad 1 \le j \le m, 1 \le i \le n, \\ f_1((x_j, y_i)(x_{j+1}, y_i)) &= m(n+i) + 1 - j, \\ &\quad \text{for} \quad 1 \le j \le m, 1 \le i \le n, \\ f_1((x_j, y_i)(x_j, y_{i+1})) &= m(2n+i+1) + 1 - j, \\ &\quad \text{for} \quad 1 \le j \le m, 1 \le i \le n-1, \\ f_1((x_j, y_n)(x_j, y_1)) &= m(2n+1) + 1 - j, \\ &\quad \text{for} \quad 1 \le j \le m. \end{aligned}$$

One can see that the vertices of the toroidal square grid are labeled by the smallest possible labels $1, 2, \ldots, mn$. Since $\mathcal{S}_m | T^n_m$ then the family $\mathcal{B} = \{\mathcal{S}^1_m, \mathcal{S}^2_m, \ldots, \mathcal{S}^n_m\}$ is a $\mathcal{S}_m$-decomposition of T^n_m and for every $i = 1, 2, \ldots, n-1$ the subgraph $\mathcal{S}^i_m$ in T^n_m has the vertex set

$$V(\mathcal{S}^i_m) = \{(x_j, y_i), (x_j, y_{i+1}) : 1 \le j \le m\},$$

the edge set

$$E(\mathcal{S}^i_m) = \{(x_j, y_i)(x_{j+1}, y_i), (x_j, y_i)(x_j, y_{i+1}) : 1 \le j \le m\}$$

and the corresponding vertex set and edge set of the subgraph $\mathcal{S}^n_m$ are as follows

$$\begin{aligned} V(\mathcal{S}^n_m) &= \{(x_j, y_n), (x_j, y_1) : 1 \le j \le m\}, \\ E(\mathcal{S}^n_m) &= \{(x_j, y_n)(x_{j+1}, y_n), (x_j, y_n)(x_j, y_1) : 1 \le j \le m\}. \end{aligned}$$

For the weight of the m-sun graphs $\mathcal{S}^n_m$ and $\mathcal{S}^i_m, i = 1, 2, \ldots, n-1$, under the labeling f_1 we get

$$\begin{aligned} wt_{f_1}(\mathcal{S}^n_m) &= \sum_{j=1}^m f_1((x_j, y_n)) + \sum_{j=1}^m f_1((x_j, y_1)) \\ &+ \sum_{j=1}^m f_1((x_j, y_n)(x_{j+1}, y_n)) \\ &+ \sum_{j=1}^m f_1((x_j, y_n)(x_j, y_1)) = \sum_{j=1}^m j \\ &+ \sum_{j=1}^m (m(n-1) + j) + \sum_{j=1}^m (2nm + 1 - j) \\ &+ \sum_{j=1}^m (m(2n+1) + 1 - j) = \sum_{j=1}^m j + m^2(n-1) \\ &+ \sum_{j=1}^m j + 2nm^2 + m - \sum_{j=1}^m j \\ &+ m^2(2n+1) + m - \sum_{j=1}^m j = 5m^2n + 2m. \end{aligned}$$

$$\begin{aligned} wt_{f_1}(\mathcal{S}^i_m) &= \sum_{j=1}^m f_1((x_j, y_i)) + \sum_{j=1}^m f_1((x_j, y_{i+1})) \\ &+ \sum_{j=1}^m f_1((x_j, y_i)(x_{j+1}, y_i)) \\ &+ \sum_{j=1}^m f_1((x_j, y_i)(x_j, y_{i+1})) = \sum_{j=1}^m (m(n-i) + j) \\ &+ \sum_{j=1}^m (m(n-i-1) + j) \\ &+ \sum_{j=1}^m (m(n+i) + 1 - j) + \sum_{j=1}^m (m(2n+i+1) \\ &+ 1 - j) = m^2(n-i) + \sum_{j=1}^m j + m^2(n-i-1) \\ &+ \sum_{j=1}^m j + m^2(n+i) + m - \sum_{j=1}^m j + m^2(2n+i+1) \\ &+ m - \sum_{j=1}^m j = 5m^2n + 2m. \end{aligned}$$

Thus the weight for each m-sun graph from the family $\mathcal{B}$ is the same. This implies that f_1 is an $\mathcal{S}_m$-supermagic labeling of the toroidal square grid T^n_m. □

Now we will deal with the super $\mathcal{S}_m$-antimagic decompositions of the toroidal square grid T_m^n by using the partitions of a set of integers with given differences.

Theorem 2. *Let $\mathcal{S}_m$ be the m-sun graph with m even, $m \geq 4$. Then the toroidal square grid $T_m^n, n \geq 3$, admits a super (a, d)-$\mathcal{S}_m$-antimagic decomposition for some a and all even d, $0 \leq d \leq 2m$. Moreover, if n is odd, then the same statement holds for every d, $1 \leq d \leq m/2$.*

Proof. Assume that m is an even integer, $m \geq 4$. Consider the following bijection $f_2 : V(T_m^n) \cup E(T_m^n) \to \{1, 2, \ldots, 3mn\}$, where for every $i = 1, 2, \ldots, n$

$$f_2((x_j, y_i)) = \begin{cases} m(n-i) + j & \text{for } j \text{ odd}, 1 \leq j \leq m-1, \\ m(i-1) + j & \text{for } j \text{ even}, 2 \leq j \leq m \end{cases}$$

and for every $i = 1, 2, \ldots, n-1$

$$f_2((x_j, y_i)(x_j, y_{i+1})) = \begin{cases} m(n+i+1) + 1 - j & \text{for } j \text{ odd}, 1 \leq j \leq m-1, \\ m(2n-i) + 1 - j & \text{for } j \text{ even}, 2 \leq j \leq m, \end{cases}$$

$$f_2((x_j, y_n)(x_j, y_1)) = \begin{cases} m(n+1) + 1 - j & \text{for } j \text{ odd}, 1 \leq j \leq m-1, \\ 2mn + 1 - j & \text{for } j \text{ even}, 2 \leq j \leq m \end{cases}$$

and for every $i = 1, 2, \ldots, n$ and $t = 0, 1, \ldots, m/2 - 1$ we put

$$\{f_2((x_{2t+1}, y_i)(x_{2t+2}, y_i)), f_2((x_{2t+2}, y_i)(x_{2t+3}, y_i))\} = \mathcal{P}_{2,s_t}^n(i) \oplus (2mn + 2tn).$$

It is not difficult to see that the vertices are labeled by numbers $1, 2, \ldots, mn$. Moreover, for the weight of the m-sun graph $\mathcal{S}_m^i, i = 1, 2, \ldots, n-1,$ we have

$$\begin{aligned}
wt_{f_2}(\mathcal{S}_m^i) &= \sum_{j=1}^{m} f_2((x_j, y_i)) + \sum_{j=1}^{m} f_2((x_j, y_{i+1})) + \sum_{j=1}^{m} f_2((x_j, y_i)(x_j, y_{i+1})) \\
&+ \sum_{j=1}^{m} f_2((x_j, y_i)(x_{j+1}, y_i)) = \sum_{j=1}^{m/2} f_2((x_{2j-1}, y_i)) + \sum_{j=1}^{m/2} f_2((x_{2j}, y_i)) + \sum_{j=1}^{m/2} f_2((x_{2j-1}, y_{i+1})) \\
&+ \sum_{j=1}^{m/2} f_2((x_{2j}, y_{i+1})) + \sum_{j=1}^{m/2} f_2((x_{2j-1}, y_i)(x_{2j-1}, y_{i+1})) + \sum_{j=1}^{m/2} f_2((x_{2j}, y_i)(x_{2j}, y_{i+1})) \\
&+ \sum_{j=1}^{m} f_2((x_j, y_i)(x_{j+1}, y_i)) = \sum_{j=1}^{m/2} (m(n-i) + 2j - 1) + \sum_{j=1}^{m/2} (m(i-1) + 2j) \\
&+ \sum_{j=1}^{m/2} (m(n-i-1) + 2j - 1) + \sum_{j=1}^{m/2} (mi + 2j) + \sum_{j=1}^{m/2} (m(n+i+1) + 1 - (2j-1)) \\
&+ \sum_{j=1}^{m/2} (m(2n-i) + 1 - 2j) + \sum_{t=0}^{m/2-1} \left[\mathcal{P}_{2,s_t}^n(i) \oplus (2mn + 2tn)\right] \\
&= \frac{m^2}{2}(n-i) + \sum_{j=1}^{m/2} (2j-1) + \frac{m^2}{2}(i-1) + \sum_{j=1}^{m/2} 2j + \frac{m^2}{2}(n-i-1) + \sum_{j=1}^{m/2} (2j-1) \\
&+ \frac{m^2}{2} i + \sum_{j=1}^{m/2} 2j + \frac{m^2}{2}(n+i+1) + \frac{m}{2} - \sum_{j=1}^{m/2} (2j-1) \\
&+ \frac{m^2}{2}(2n-i) + \frac{m}{2} - \sum_{j=1}^{m/2} 2j + \mathcal{P}_{2,s_0}^n(i) \oplus 2mn + \mathcal{P}_{2,s_1}^n(i) \oplus (2mn + 2n) + \ldots \\
&+ \mathcal{P}_{2,s_{m/2-1}}^n(i) \oplus (2mn + 2n(m/2-1)) = A + C_{2,s_0}^n + s_0(i-1) + 4mn + C_{2,s_1}^n \\
&+ s_1(i-1) + (4mn + 4n) + \ldots + C_{2,s_{m/2-1}}^n + s_{m/2-1}(i-1) + \left(4mn + 4n\left(\frac{m}{2} - 1\right)\right) \\
&= A + B + C_{2,s_0}^n + C_{2,s_1}^n + \ldots + C_{2,s_{m/2-1}}^n + (s_0 + s_1 + \ldots + s_{m/2-1})(i-1),
\end{aligned}$$

where $A = (5m^2n + 3m)/2$ and $B = (5m^2n)/2 - mn$.

For the weight of the m-sun graph $\mathcal{S}_m^n$ we have

$$\begin{aligned}
wt_{f_2}(\mathcal{S}_m^n) &= \sum_{j=1}^{m} f_2((x_j, y_n)) + \sum_{j=1}^{m} f_2((x_j, y_1)) + \sum_{j=1}^{m} f_2((x_j, y_n)(x_j, y_1)) \\
&+ \sum_{j=1}^{m} f_2((x_j, y_n)(x_{j+1}, y_n)) = \sum_{j=1}^{m/2} f_2((x_{2j-1}, y_n)) + \sum_{j=1}^{m/2} f_2((x_{2j}, y_n)) + \sum_{j=1}^{m/2} f_2((x_{2j-1}, y_1)) \\
&+ \sum_{j=1}^{m/2} f_2((x_{2j}, y_1)) + \sum_{j=1}^{m/2} f_2((x_{2j-1}, y_n)(x_{2j-1}, y_1)) + \sum_{j=1}^{m/2} f_2((x_{2j}, y_n)(x_{2j}, y_1)) \\
&+ \sum_{j=1}^{m} f_2((x_j, y_n)(x_{j+1}, y_n)) = \sum_{j=1}^{m/2} (2j-1) + \sum_{j=1}^{m/2} (m(n-1) + 2j) + \sum_{j=1}^{m/2} (m(n-1) + 2j - 1) \\
&+ \sum_{j=1}^{m/2} 2j + \sum_{j=1}^{m/2} (m(n+1) + 1 - (2j-1)) + \sum_{j=1}^{m/2} (2mn + 1 - 2j) \\
&+ \sum_{t=0}^{m/2-1} \left[\mathcal{P}_{2,s_t}^n(n) \oplus (2mn + 2tn)\right] = \sum_{j=1}^{m/2} (2j-1) + \frac{m^2}{2}(n-1) + \sum_{j=1}^{m/2} 2j \\
&+ \frac{m^2}{2}(n-1) + \sum_{j=1}^{m/2} (2j-1) + \sum_{j=1}^{m/2} 2j + \frac{m^2}{2}(n+1) + \frac{m}{2} - \sum_{j=1}^{m/2} (2j-1) \\
&+ m^2n + \frac{m}{2} - \sum_{j=1}^{m/2} 2j + \mathcal{P}_{2,s_0}^n(n) \oplus 2mn + \mathcal{P}_{2,s_1}^n(n) \oplus (2mn + 2n) + \ldots \\
&+ \mathcal{P}_{2,s_{m/2-1}}^n(n) \oplus (2mn + 2n(m/2 - 1)) = A + C_{2,s_0}^n + s_0(n-1) + 4mn + C_{2,s_1}^n \\
&+ s_1(n-1) + (4mn + 4n) + \ldots + C_{2,s_{m/2-1}}^n + s_{m/2-1}(n-1) + \left(4mn + 4n\left(\frac{m}{2} - 1\right)\right) \\
&= A + B + C_{2,s_0}^n + C_{2,s_1}^n + \ldots + C_{2,s_{m/2-1}}^n + (s_0 + s_1 + \ldots + s_{m/2-1})(n-1).
\end{aligned}$$

Under the labeling f_2 the weights of m-sun graphs $\mathcal{S}_m^i, i = 1, 2, \ldots, n,$ form an arithmetic sequence with the difference $s_0 + s_1 + \ldots + s_{m/2-1}$.

As $s_t = 0, \pm 2, \pm 4$ and also $s_t = \pm 1$ for n odd, $t = 0, 1, \ldots, m/2 - 1$, we obtain that f_2 is a super (a, d)-$\mathcal{S}_m$-antimagic labeling of the toroidal square grid T_m^n for $a = A + B + C_{2,s_0}^n + C_{2,s_1}^n + \ldots + C_{2,s_{m/2-1}}^n$ and $d = 0, 2, \ldots, 2m$, moreover for n odd also $d = 1, 2, \ldots, m/2$. □

Theorem 3. *Let $\mathcal{S}_m$ be the m-sun graph with m odd, $m \geq 3$. Then the toroidal square grid $T_m^n, n \geq 3$, admits a super (a, d)-$\mathcal{S}_m$-antimagic decomposition for some a and all even d, $0 \leq d \leq 2m - 2$. Moreover, if n is odd, then the same statement holds for every d, $1 \leq d \leq (m-1)/2$.*

Proof. Define the labeling $f_3 : V(T_m^n) \cup E(T_m^n) \to \{1, 2, \ldots, 3mn\}$ in the following way.

If $i = 1, 2, \ldots, n$ then

$$f_3((x_j, y_i)) = \begin{cases} (m-1)(n-i) + j & \text{for } j \text{ odd}, 1 \leq j \leq m-2, \\ (m-1)(i-1) + j & \text{for } j \text{ even}, 2 \leq j \leq m-1, \\ (m-1)n + i & \text{for } j = m, \end{cases}$$

$$f_3((x_m, y_i)(x_1, y_i)) = (m+1)n + 1 - i,$$

and if $i = 1, 2, \ldots, n-1$ then

$$f_3((x_j, y_i)(x_j, y_{i+1})) = \begin{cases} (m-1)(n+i+1) + 3n + 1 - j & \text{for } j \text{ odd}, 1 \leq j \leq m-2, \\ (m-1)(n+4-i) + 3n + 1 - j & \text{for } j \text{ even}, 2 \leq j \leq m-1, \\ n(m+2) - i & \text{for } j = m, \end{cases}$$

$$f_3((x_j, y_n)(x_j, y_1)) = \begin{cases} m(n+1) + 2n - j & \text{for } j \text{ odd}, 1 \leq j \leq m-2, \\ (m-1)(n+4) + 3n + 1 - j & \text{for } j \text{ even}, 2 \leq j \leq m-1, \\ n(m+2) & \text{for } j = m. \end{cases}$$

If $i = 1, 2, \ldots, n$ and $t = 0, 1, \ldots, m/2 - 1$ we put

$$\{f_3((x_{2t+1}, y_i)(x_{2t+2}, y_i)), f_3((x_{2t+2}, y_i)(x_{2t+3}, y_i))\} = \mathcal{P}_{2,s_t}^n(i) \oplus (2mn + 4 + 2tn).$$

There is no problem in seeing that under the labeling f_3 the vertices of T_m^n use integers $1, 2, \ldots, mn$ and this implies that the labeling f_3 is super. We observe that for every $i = 1, 2, \ldots, n-1$ the weight of the m-sun graph $\mathcal{S}_m^i$ is

$$
\begin{aligned}
wt_{f_3}(\mathcal{S}_m^i) &= \sum_{j=1}^{m} f_3((x_j,y_i)) + \sum_{j=1}^{m} f_3((x_j,y_{i+1})) + \sum_{j=1}^{m} f_3((x_j,y_i)(x_j,y_{i+1}))\\
&+ \sum_{j=1}^{m} f_3((x_j,y_i)(x_{j+1},y_i)) = \sum_{j=1}^{(m-1)/2} f_3((x_{2j-1},y_i)) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j},y_i)) + f_3((x_m,y_i))\\
&+ \sum_{j=1}^{(m-1)/2} f_3((x_{2j-1},y_{i+1})) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j},y_{i+1}))\\
&+ f_3((x_m,y_{i+1})) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j-1},y_i)(x_{2j-1},y_{i+1})) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j},y_i)(x_{2j},y_{i+1}))\\
&+ f_3((x_m,y_i)(x_m,y_{i+1})) + f_3((x_m,y_i)(x_1,y_i)) + \sum_{j=1}^{m-1} f_3((x_j,y_i)(x_{j+1},y_i))\\
&= \sum_{j=1}^{(m-1)/2} ((m-1)(n-i)+2j-1) + \sum_{j=1}^{(m-1)/2} ((m-1)(i-1)+2j) + (m-1)n + i\\
&+ \sum_{j=1}^{(m-1)/2} ((m-1)(n-i-1)+2j-1) + \sum_{j=1}^{(m-1)/2} ((m-1)i+2j) + (m-1)n+i+1\\
&+ \sum_{j=1}^{(m-1)/2} ((m-1)(n+i+1)+3n+1-(2j-1))\\
&+ \sum_{j=1}^{(m-1)/2} ((m-1)(n+4-i)+3n+1-2j) + n(m+2) - i + (m+1)n + 1 - i\\
&+ \sum_{t=0}^{(m-1)/2-1} \left[\mathcal{P}_{2,s_t}^n(i) \oplus (2mn+4+2tn)\right]\\
&= \frac{(m-1)^2}{2}(n-i) + \sum_{j=1}^{(m-1)/2} (2j-1) + \frac{(m-1)^2}{2}(i-1) + \sum_{j=1}^{(m-1)/2} 2j + (m-1)n + i\\
&+ \frac{(m-1)^2}{2}(n-i-1) + \sum_{j=1}^{(m-1)/2} (2j-1) + \frac{(m-1)^2}{2} i + \sum_{j=1}^{(m-1)/2} 2j + (m-1)n + i + 1\\
&+ \frac{(m-1)^2}{2}(n+i+1) + \frac{m-1}{2}(3n+1) - \sum_{j=1}^{(m-1)/2} (2j-1) + \frac{(m-1)^2}{2}(n+4-i)\\
&+ \frac{m-1}{2}(3n+1) - \sum_{j=1}^{(m-1)/2} 2j + n(m+2) - i + (m+1)n + 1 - i\\
&+ \mathcal{P}_{2,s_0}^n(i) \oplus (2mn+4) + \mathcal{P}_{2,s_1}^n(i) \oplus (2mn+4+2n) + \ldots\\
&+ \mathcal{P}_{2,s_{(m-1)/2-1}}^n(i) \oplus (2mn+4+2n((m-1)/2-1)) = D + C_{2,s_0}^n + s_0(i-1)\\
&+ (4mn+8) + C_{2,s_1}^n + s_1(i-1) + (4mn+8+4n) + \ldots + C_{2,s_{(m-1)/2-1}}^n + s_{(m-1)/2-1}(i-1)\\
&+ (4mn+8+4n((m-1)/2-1)) = D + F\\
&+ C_{2,s_0}^n + C_{2,s_1}^n + \ldots + C_{2,s_{(m-1)/2-1}}^n + (s_0+s_1+\ldots+s_{(m-1)/2-1})(i-1),
\end{aligned}
$$

where $D = (m-1)^2(4n+3)/2 + (m-1)(14n+m+2)/2 + 5n + 2$ and $F = (m-1)(5mn-3n+8)/2$.

The weight of the m-sun graph $\mathcal{S}_m^n$ is

$$
\begin{aligned}
wt_{f_3}(\mathcal{S}_m^n) &= \sum_{j=1}^{m} f_3((x_j,y_n)) + \sum_{j=1}^{m} f_3((x_j,y_1)) + \sum_{j=1}^{m} f_3((x_j,y_n)(x_j,y_1)) \\
&\quad + \sum_{j=1}^{m} f_3((x_j,y_n)(x_{j+1},y_n)) = \sum_{j=1}^{(m-1)/2} f_3((x_{2j-1},y_n)) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j},y_n)) \\
&\quad + f_3((x_m,y_n)) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j-1},y_1)) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j},y_1)) + f_3((x_m,y_1)) \\
&\quad + \sum_{j=1}^{(m-1)/2} f_3((x_{2j-1},y_n)(x_{2j-1},y_1)) + \sum_{j=1}^{(m-1)/2} f_3((x_{2j},y_n)(x_{2j},y_1)) \\
&\quad + f_3((x_m,y_n)(x_m,y_1)) + f_3((x_m,y_n)(x_1,y_n)) \\
&\quad + \sum_{j=1}^{m-1} f_3((x_j,y_n)(x_{j+1},y_n)) = \sum_{j=1}^{(m-1)/2} (2j-1) + \sum_{j=1}^{(m-1)/2} ((m-1)(n-1)+2j) \\
&\quad + (m-1)n + n + \sum_{j=1}^{(m-1)/2} ((m-1)(n-1)+2j-1) + \sum_{j=1}^{(m-1)/2} 2j + (m-1)n + 1 \\
&\quad + \sum_{j=1}^{(m-1)/2} (m(n+1)+2n-(2j-1)) + \sum_{j=1}^{(m-1)/2} ((m-1)(n+4)+3n+1-2j) \\
&\quad + n(m+2) + (m+1)n + 1 - n + \sum_{t=0}^{(m-1)/2-1} \left[\mathcal{P}_{2,s_t}^n(n) \oplus (2mn+4+2tn)\right] \\
&= \sum_{j=1}^{(m-1)/2} (2j-1) + \frac{(m-1)^2}{2}(n-1) + \sum_{j=1}^{(m-1)/2} 2j + (m-1)n + n \\
&\quad + \frac{(m-1)^2}{2}(n-1) + \sum_{j=1}^{(m-1)/2} (2j-1) + \sum_{j=1}^{(m-1)/2} 2j + (m-1)n + 1 + \frac{(m-1)^2}{2}(n+1) \\
&\quad + \frac{m-1}{2}(3n+1) - \sum_{j=1}^{(m-1)/2} (2j-1) + \frac{(m-1)^2}{2}(n+4) + \frac{m-1}{2}(3n+1) \\
&\quad - \sum_{j=1}^{(m-1)/2} 2j + mn + 2n + mn + n + 1 - n + \mathcal{P}_{2,s_0}^n(n) \oplus (2mn+4) \\
&\quad + \mathcal{P}_{2,s_1}^n(n) \oplus (2mn+4+2n) + \ldots + \mathcal{P}_{2,s_{(m-1)/2-1}}^n(n) \oplus (2mn+4+2n((m-1)/2-1)) \\
&= D + C_{2,s_0}^n + s_0(n-1) + (4mn+8) + C_{2,s_1}^n + s_1(n-1) + (4mn+8+4n) \\
&\quad + \ldots + C_{2,s_{(m-1)/2-1}}^n + s_{(m-1)/2-1}(n-1) + (4mn+8+4n((m-1)/2-1)) \\
&= D + F + C_{2,s_0}^n + C_{2,s_1}^n + \ldots + C_{2,s_{(m-1)/2-1}}^n + (s_0+s_1+\ldots+s_{(m-1)/2-1})(n-1).
\end{aligned}
$$

Again one can see that under the labeling f_3 the weights of m-sun graphs $\mathcal{S}_m^i, i=1,2,\ldots,n,$ constitute an arithmetic sequence with the difference $s_0+s_1+\ldots+s_{(m-1)/2-1}$.

Since for every $t=0,1,\ldots,(m-1)/2-1$ the value $s_t = 0, \pm 2, \pm 4$ and also $s_t = \pm 1$ for n odd, it follows that f_3 is a super $(a,\ d)$-$\mathcal{S}_m$-antimagic labeling of the toroidal square grid T_m^n for $a = D+F+C_{2,s_0}^n + C_{2,s_1}^n + \ldots + C_{2,s_{(m-1)/2-1}}^n$ and $d = 0,2,\ldots,2m-2$, moreover for n odd also $d=1,2,\ldots,(m-1)/2$. □

The next theorem shows the existence of C_4-edge-antimagic decomposition of the toroidal square grid T_m^n.

Theorem 4. *Let C_4 be the cycle on four vertices. If m and n are two even integers, $m,n \geq 4$, then the toroidal square grid T_m^n admits an $(a,\ d)$-C_4-edge-antimagic decomposition for some a and differences $d \in \{0,2,4,6,8,10,16\}$.*

Proof. For m, n even the existence of the C_4-decomposition of the toroidal square grid T_m^n is evident. Let

$\mathcal{B}=\left\{C_4^1, C_4^2, \ldots, C_4^{mn/2}\right\}$ be a C_4-decomposition of the toroidal square grid T_m^n. Since the edge sets $E(C_4^1)$, $E(C_4^2), \ldots, E(C_4^{mn/2})$ are pairwise disjoint and g_1 must be a bijection from $E(T_m^n)$ into the set $\{1,2,\ldots,2mn\}$, the family of sets $\{g_1(E(C_4^1)), g_1(E(C_4^2)), \ldots, g_1(E(C_4^{mn/2}))\}$ will form a partition of the set $\{1,2,\ldots,2mn\}$. From the existence of a partition $\mathcal{P}_{4,d}^{mn/2}$ of the set $\{1,2,\ldots,2mn\}$ into 4-tuples it follows that $d=0,\pm2,\pm4,\pm6,\pm8,\pm10,\ \pm16$. From Equation (1) it follows that $\sum\mathcal{P}_{4,d}^{mn/2}(i)=C_{4,d}^{mn/2}+d(i-1)$, where the parameter $a=C_{4,d}^{mn/2}=\sum\mathcal{P}_{4,d}^{mn/2}(1)$. This concludes the proof. □

4. *H*-antimagic decompositions for toroidal triangulations

The discovery of the fullerene molecules and related forms of carbon such as nanotubes has generated an explosion of activity in chemistry, physics, and materials science. Classical fullerene is an all-carbon molecule in which the atoms are arranged on a pseudospherical framework made up entirely of pentagons and hexagons. Deza et al. [8] considered fullerene's extension to other closed surfaces and showed that only four surfaces are possible, namely sphere, torus, Klein bottle and projective plane. Toroidal fullerenes contain only hexagons. The family of toroidal six-regular graphs are the duals of toroidal fullerenes, see [1]. One class of the family of toroidal six-regular graphs we can obtain from the toroidal square grid T_m^n by adding edges $(x_j,y_{i+1})(x_{j+1},y_i), 1\le j\le n, 1\le i\le n-1$, and $(x_j,y_1)(x_{j+1},y_n), 1\le j\le n$, with indices j taken modulo m. The resulting graph we call a toroidal triangulation and denote by $\mathbb{T}_m^n$.

A *complete m-sun graph* on $2m$ vertices (sometimes also known as a *trampoline graph*, see [6], page 112) consists of a central complete graph K_m with an outer ring of m vertices, each of which is joined to both endpoints of the closest outer edge of the central core. Note that when $x_1,x_2,\ldots,x_m$ are vertices of a cycle C_m in K_m, then for each $i=1,2,\ldots,m$ there is a vertex y_i adjacent to vertices x_i and $x_{(i+1)}$ (where $m+1$ is replaced by 1).If we replace the central complete graph K_m by the cycle C_m then the complete m-sun graph we call a *cycle m-sun graph* and denote it by $\mathcal{R}_m$.

It is not difficult to see that the toroidal triangulation $\mathbb{T}_m^n$ can be decomposed into cycle m-sun graphs $\mathcal{R}_m$ or cycle n-sun graphs $\mathcal{R}_n$. Due to symmetry it is enough to deal with $\mathcal{R}_m$-decomposition of $\mathbb{T}_m^n$.

Now we will deal with the super $\mathcal{R}_m$-antimagic decompositions of the toroidal triangulation $\mathbb{T}_m^n$.

Theorem 5. *Let $\mathcal{R}_m$ be the cycle m-sun graph with m even, $m\ge4$. Then the toroidal triangulation $\mathbb{T}_m^n$, $n\ge3$, admits a super $(a,\ d)$-$\mathcal{R}_m$-antimagic decomposition for some a and all even, d, $0\le d\le 5m$ and $d=5m+6,5m+12,\ldots,8m$. Moreover, if n is odd, then the same statement holds for all odd d, $1\le d\le 5m/2$.*

Proof. For a construction of a $\mathcal{R}_m$-antimagic decomposition of the toroidal triangulation $\mathbb{T}_m^n$ we will use the existence of $\mathcal{S}_m$-antimagic decompositions of the toroidal square grid T_m^n described in the proof of Theorem 2, only we complete the labels for added edges $(x_j,y_{i+1})(x_{j+1},y_i)$ and $(x_j,y_1)(x_{j+1},y_n)$ for corresponding indices j and i.

Let m be even, $m\ge4$. Define a bijection $f_4: V(\mathbb{T}_m^n)\cup E(\mathbb{T}_m^n)\to\{1,2,\ldots,4mn\}$ in the following way:

$$f_4((x_j,y_i))=f_2((x_j,y_i)),\qquad \text{for } i=1,2,\ldots,n,$$
$$f_4((x_j,y_i)(x_j,y_{i+1}))=f_2((x_j,y_i)(x_j,y_{i+1})),\quad \text{for } i=1,2,\ldots,n-1$$

and

$$f_4((x_j,y_n)(x_j,y_1))=f_2((x_j,y_n)(x_j,y_1)).$$

For every $i=1,2,\ldots,n$ and $t=0,1,\ldots,m/2-1$ we put

$$\begin{aligned}&\{f_4((x_{2t+1},y_i)(x_{2t+2},y_i)), f_4((x_{2t+1},y_{i+1})(x_{2t+2},y_i)),\\ &f_4((x_{2t+2},y_i)(x_{2t+3},y_i)), f_4((x_{2t+2},y_{i+1})(x_{2t+3},y_i))\}\\ &=\mathcal{P}_{4,s_t}^n(i)\oplus(2mn+4tn).\end{aligned}$$

Then for the weight of the cycle m-sun graph $\mathcal{R}_m^i$, $i=1,2,\ldots,n-1$ we get

$$\begin{aligned}wt_{f_4}(\mathcal{R}_m^i)&=\sum_{j=1}^m f_4((x_j,y_i))+\sum_{j=1}^m f_4((x_j,y_{i+1}))\\ &\quad+\sum_{j=1}^m f_4((x_j,y_i)(x_j,y_{i+1}))\\ &\quad+\sum_{j=1}^m\left[f_4((x_j,y_i)(x_{j+1},y_i))+f_4((x_j,y_{i+1})(x_{j+1},y_i))\right]\\ &=A+\sum_{t=0}^{m/2-1}\left[\mathcal{P}_{4,s_t}^n(i)\oplus(2mn+4tn)\right]\\ &=A+\mathcal{P}_{4,s_0}^n(i)\oplus 2mn+\mathcal{P}_{4,s_1}^n(i)\oplus(2mn+4n)+\ldots\\ &\quad+\mathcal{P}_{4,s_{m/2-1}}^n(i)\oplus(2mn+4n(m/2-1))\\ &=A+C_{4,s_0}^n+s_0(i-1)+8mn+C_{4,s_1}^n+s_1(i-1)\\ &\quad+(8mn+16n)+\ldots+C_{4,s_{m/2-1}}^n+s_{m/2-1}(i-1)\\ &\quad+(8mn+16n(m/2-1))\\ &=A+C+C_{4,s_0}^n+C_{4,s_1}^n+\ldots+C_{4,s_{m/2-1}}^n\\ &\quad+(s_0+s_1+\ldots+s_{m/2-1})(i-1),\end{aligned}$$

where $C=6m^2n-4mn$.

For the weight of the cycle m-sun graph $\mathcal{R}_m^n$ we get

$$\begin{aligned}wt_{f_4}(\mathcal{R}_m^n)&=\sum_{j=1}^m f_4((x_j,y_n))+\sum_{j=1}^m f_4((x_j,y_1))\\ &\quad+\sum_{j=1}^m f_4((x_j,y_n)(x_j,y_1))\\ &\quad+\sum_{j=1}^m\left[f_4((x_j,y_n)(x_{j+1},y_n))+f_4((x_j,y_1)(x_{j+1},y_n))\right]\\ &=A+\sum_{t=0}^{m/2-1}\left[\mathcal{P}_{4,s_t}^n(n)\oplus(2mn+4tn)\right]\\ &=A+C+C_{4,s_0}^n+C_{4,s_1}^n+\ldots\\ &\quad+C_{4,s_{m/2-1}}^n+(s_0+s_1+\ldots+s_{m/2-1})(n-1).\end{aligned}$$

The weights of the cycle m-sun graphs $\mathcal{R}_m^i, i = 1, 2, ..., n$, under the labeling f_4, form an arithmetic progression with the initial term $a = A + C + C_{4,s_0}^n + C_{4,s_1}^n + ... + C_{4,s_{m/2-1}}^n$ and difference $s_0 + s_1 + ... + s_{m/2-1}$, where $s_t = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$ and for n odd also $s_t = \pm 1, \pm 3, \pm 5$, for every $t = 0, 1, ..., m/2 - 1$.

It means that f_4 is a super (*a*, *d*)-$\mathcal{R}_m$-antimagic labeling for all even $d = 0, 2, 4, ..., 5m$ and $d = 5m + 6, 5m + 12, ..., 8m$ and if n is odd then also for all odd d, $1 \le d \le 5m/2$. □

Theorem 6. *Let $\mathcal{R}_m$ be the cycle m-sun graph with m odd, $m \ge 3$. Then the toroidal triangulation $\mathbb{T}_m^n, n \ge 3$, admits a super (a, d)-$\mathcal{R}_m$-antimagic decomposition for some a and all odd d, $1 \le d \le 5m - 4$ and $d = 5m + 2, 5m + 8, ..., 8m - 7$. Moreover, if n is odd, then the same statement holds for all even d, $0 \le d \le (5m - 3)/2$.*

Proof. If m is odd, $m \ge 3$, then we define a bijection $f_5 : V(\mathbb{T}_m^n) \cup E(\mathbb{T}_m^n) \to \{1, 2, ..., 4mn\}$ in the following way:

$$f_5((x_j, y_i)) = f_3((x_j, y_i)) \quad \text{for } i = 1, 2, ..., n,$$
$$f_5((x_m, y_i)(x_1, y_i)) = f_3((x_m, y_i)(x_1, y_i)) \quad \text{for } i = 1, 2, ..., n,$$
$$f_5((x_j, y_i)(x_j, y_{i+1})) = f_3((x_j, y_i)(x_j, y_{i+1})) \quad \text{for } i = 1, 2, ..., n - 1$$

and

$$f_5((x_j, y_n)(x_j, y_1)) = f_3((x_j, y_n)(x_j, y_1)).$$

For every $i = 1, 2, ..., n - 1$ and $t = 0, 1, ..., (m - 1)/2 - 1$ we give

$$\{f_5((x_{2t+1}, y_i)(x_{2t+2}, y_i)), f_5((x_{2t+1}, y_{i+1})(x_{2t+2}, y_i)), f_5((x_{2t+2}, y_i)(x_{2t+3}, y_i)), f_5((x_{2t+2}, y_{i+1})(x_{2t+3}, y_i))\} = \mathcal{P}_{4,s_t}^n(i) \oplus (2mn + 4 + 4tn)$$

and

$$\{f_5((x_m, y_{i+1})(x_1, y_i))\} = \mathcal{P}_{1,1}^n(i) \oplus (2mn - 2n + 4).$$

Then for the weight of the cycle m-sun graph $\mathcal{R}_m^i, i = 1, 2, ..., n - 1$ we obtain

$$\begin{aligned} wt_{f_5}(\mathcal{R}_m^i) &= \sum_{j=1}^m f_5((x_j, y_i)) + \sum_{j=1}^m f_5((x_j, y_{i+1})) \\ &+ \sum_{j=1}^m f_5((x_j, y_i)(x_j, y_{i+1})) \\ &+ \sum_{j=1}^m \left[f_5((x_j, y_i)(x_{j+1}, y_i)) + f_5((x_j, y_{i+1})(x_{j+1}, y_i)) \right] \\ &+ f_5((x_m, y_{i+1})(x_1, y_i)) \\ &= D + \sum_{t=0}^{(m-1)/2-1} \left[\mathcal{P}_{4,s_t}^n(i) \oplus (2mn + 4 + 4tn) \right] \\ &+ \mathcal{P}_{1,1}^n(i) \oplus (4mn - 2n + 4) \\ &= D + \mathcal{P}_{4,s_0}^n(i) \oplus (2mn + 4) \\ &+ \mathcal{P}_{4,s_1}^n(i) \oplus (2mn + 4 + 4n) + ... + \mathcal{P}_{4,s_{(m-1)/2-1}}^n(i) \\ &\oplus \left(2mn + 4 + 4n\left(\frac{m-1}{2} - 1 \right) \right) \\ &+ \mathcal{P}_{1,1}^n(i) \oplus (4mn - 2n + 4) \\ &= D + C_{4,s_0}^n + s_0(i - 1) + (8mn + 16) + C_{4,s_1}^n \\ &+ s_1(i - 1) + (8mn + 16 + 16n) + ... \\ &+ C_{4,s_{(m-1)/2-1}}^n + s_{(m-1)/2-1}(i - 1) \\ &+ (8mn + 16 + 16n(m - 3)/2) + C_{1,1}^n + (i - 1) \\ &+ (4mn - 2n + 4) = D + K + C_{4,s_0}^n + C_{4,s_1}^n \\ &+ ... + C_{4,s_{(m-1)/2-1}}^n + C_{1,1}^n \\ &+ (s_0 + s_1 + ... + s_{(m-1)/2-1} \pm 1)(i - 1), \end{aligned}$$

where $K = (m - 1)(12mn - 4n + 16/2 + 2n + 4$.

For the weight of the cycle m-sun graph $\mathcal{R}_m^n$ we obtain

$$\begin{aligned} wt_{f_5}(\mathcal{R}_m^n) &= \sum_{j=1}^m f_5((x_j, y_n)) + \sum_{j=1}^m f_5((x_j, y_1)) \\ &+ \sum_{j=1}^m f_5((x_j, y_n)(x_j, y_1)) \\ &+ \sum_{j=1}^m \left[f_5((x_j, y_n)(x_{j+1}, y_n)) + f_5((x_j, y_1)(x_{j+1}, y_n)) \right] \\ &+ f_5((x_m, y_1)(x_1, y_n)) \\ &= D + \sum_{t=0}^{(m-1)/2-1} \left[\mathcal{P}_{4,s_t}^n(n) \oplus (2mn + 4 + 4tn) \right] \\ &+ \mathcal{P}_{1,1}^n(n) \oplus (4mn - 2n + 4) \\ &= D + K + C_{4,s_0}^n + C_{4,s_1}^n + ... + C_{4,s_{(m-1)/2-1}}^n + C_{1,1}^n \\ &+ (s_0 + s_1 + ... + s_{(m-1)/2-1} \pm 1)(n - 1). \end{aligned}$$

Under the labeling f_5 the weights of cycle m-sun graphs $\mathcal{R}_m^i, i = 1, 2, ..., n$, constitute an arithmetic sequence with $a = D + K + C_{4,s_0}^n + C_{4,s_1}^n + ... + C_{4,s_{(m-1)/2-1}}^n + C_{1,1}^n$ and difference $s_0 + s_1 + ... + s_{(m-1)/2-1} \pm 1$.

Since for every $t = 0, 1, ..., m/2 - 1$ the value $s_t = 0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 16$ and for n odd also $s_t = \pm 1, \pm 3, \pm 5$, then there exists a super (*a*, *d*)-$\mathcal{R}_m$-antimagic labeling of the toroidal triangulation for all odd $d = 1, 3, ..., 5m - 4$ and $d = 5m + 2, 5m + 8, ..., 8m - 7$ and if n is odd then also for all even $0 \le d \le (5m - 3)/2$. □

The next theorem asserts the existence of C_3-edge-antimagic decomposition of the toroidal triangulation $\mathbb{T}_m^n$.

Theorem 7. *Let C_3 be the cycle on three vertices. Then the toroidal triangulation $\mathbb{T}_m^n, n, m \ge 3$, admits an (a, d)-$C_3$-edge-antimagic decomposition for some a and differences $d \in \{1, 3, 5, 9\}$. Moreover, if mn is odd, then the same statement holds for $d \in \{0, 2\}$.*

Proof. The existence of the C_3-decomposition of the toroidal triangulation $\mathbb{T}_m^n$ is obvious. Let $\mathcal{B} = \{C_3^1, C_3^2, ..., C_3^{mn}\}$ be a C_3-decomposition of the toroidal triangulation $\mathbb{T}_m^n$. Since the edge sets $E(C_3^1), E(C_3^2), ..., E(C_3^{mn})$ are pairwise disjoint and g_2 must be a bijection from $E(\mathbb{T}_m^n)$ into the set $\{1, 2, ..., 3mn\}$, the family of sets $\{g_2(E(C_3^1)), g_2(E(C_3^2)), ..., g_2(E(C_3^{mn}))\}$ will form a partition of the set $\{1, 2, \ldots, 3mn\}$. From the existence of a partition $\mathcal{P}^{mn}_{3,d}$ of the set $\{1, 2, \ldots, 3mn\}$ into mn 3-tuples it follows that $d = \pm 1, \pm 3, \pm 5, \pm 9$ and if $mn \equiv 1 \pmod 2$ also $d = 0, \pm 2$. Using the property (1) we have that $\sum \mathcal{P}^{mn}_{3,d}(i) = C^{mn}_{3,d} + d(i-1)$, where the parameter $a = C^{mn}_{3,d} = \sum \mathcal{P}^{mn}_{3,d}(1)$. This concludes the proof. □

5. Conclusion

In the foregoing sections we studied the existence of super $(a,\ d)$-H-antimagic decompositions for toroidal grids and toroidal triangulations. We have shown that the toroidal square grid $T^n_m, n \geq 3$, admits a super $(a,\ d)$-$\mathcal{S}_m$-antimagic decomposition for all even differences $0 \leq d \leq 4\lfloor m/2 \rfloor$ and if n is odd, then for every difference $1 \leq d \leq \lfloor m/2 \rfloor$. Moreover, we proved the existence of a super $(a,\ d)$-$\mathcal{R}_m$-antimagic decomposition of the toroidal triangulation $\mathbb{T}^n_m$ for certain differences.

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References

[1] Altshuler, A. (1973). Construction and enumeration of regular maps on the torus. *Discrete Math.* 4(3):201–217.

[2] Anitha, R, Lekshmi, R. S. (2008). *n*-sun decomposition of complete, complete bipartite and some Harary graphs. *Int. J. Math. Sci.* 2:33–38.

[3] Arumugam, S., Miller, M., Phanalasy, O., Ryan, J. (2014). Antimagic labeling of generalized pyramid graphs. *Acta Math. Sin-English. Ser.* 30(2):283–290.

[4] Baca, M., Brankovic, L., Lascsakova, M., Phanalasy, O., Semanicova–Fenovcikova, A. (2013). On *d*-antimagic labelings of plane graphs. *Electron. J. Graph Theory Appl.* 1(1):28–39.

[5] Bača, M., Miller, M. (2008). *Super Edge-Antimagic Graphs: A Wealth of Problems and Some Solutions*. Boca Raton, FL: Brown Walker Press.

[6] Brandstädt, A., Le, V. B., Spinrad, J. P. (1999). *Graph Classes: A Survey SIAM Monographs on Discrete Mathematics and Applications*, Vol. 3. Philadelphia, PA: SIAM.

[7] Cavenagh, N. J., Billington, E. J. (2000). Decompositions of complete multipartite graphs into cycles even length. *Graphs Comb.* 16(1):49–65.

[8] Deza, M., Fowler, P. W., Rassat, A., Rogers, K. M. (2000). Fullerenes as tilings of surfaces. *J. Chem. Inf. Comput. Sci.* 40(3):550–558.

[9] Enomoto, H., Lladó, A. S., Nakamigawa, T., Ringel, G. (1998). Super edge-magic graphs. *Sut J. Math.* 34:105–109.

[10] Gallian, J. (2017). A dynamic survey of graph labeling. *Electron. J. Comb.* #DS6.

[11] Gutiérrez, A., Lladó, A. (2005). Magic coverings. *J. Comb. Math. Comb. Comput.* 55:43–56.

[12] Hendy, (2016). The *H-super(anti)magic* decomposition of antiprism graphs. *AIP Conference Proceedings* 1707, 020007.

[13] Hendy, K. A., Sugeng, A. N. M. Salman, (2018). An *H-super* magic decompositions of the lexicographic product of graphs. *AIP Conference Proceedings* 2023, 020193

[14] Inayah, N., Lladó, A., Moragas, J. (2012). Magic and antimagic *H*-decompositions. *Discrete Math.* 312(7):1367–1371.

[15] Kotzig, A., Rosa, A. (1970). Magic valuations of finite graphs. *Can. Math. Bull.* 13(4):451–461.

[16] Lladó, A. S., Moragas, J. (2007). Cycle-magic graphs. *Discrete Math.* 307(23):2925–2933.

[17] Liang, Z. (2012). Cycle-supermagic decompositions of complete multipartite graphs. *Discrete Math.* 312(22):3342–3348.

[18] Marr, A. M., Wallis, W. D. (2013). *Magic Graphs*. New York: Birkhäuser.

[19] Maryati, T. K., Salman, A. N. M., Baskoro, E. T. (2013). Supermagic coverings of the disjoint union of graphs and amalgamations. *Discrete Math.* 313(4):397–405.

[20] Maryati, T. K., Salman, A. N. M., Baskoro, E. T., Ryan, J., Miller, M. (2010). On *H*-supermagic labelings for certain shackles and amalgamations of a connected graph. *Utilitas Math.* 83:333–342.

[21] Ngurah, A. A. G., Salman, A. N. M., Susilowati, L. (2010). *H*-supermagic labelings of graphs. *Discrete Math.* 310(8):1293–1300.

[22] Salman, A. N. M., Ngurah, A. A. G., Izzati, N. (2010). On (super)-edge-magic total labelings of subdivision of stars *Sn*. *Utilitas Math.* 81:275–284.

[23] Simanjuntak, R., Miller, M., Bertault, F. (2000). Two new *(a, d)-antimagic* graph labelings. *Proc. Eleventh Australas. Workshop Combin. Alg. (AWOCA)*:179–189.

[24] Sotteau, D. (1981). Decompositions of $K_{m,n}$ $(K*_{m,n})$ into cycles(circuits) of length $2k$. *J. Combin. Theory Ser. B* 30(1):75–81.