AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 788–795
https://doi.org/10.1016/j.akcej.2019.12.011

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ABSTRACT

Let T be an edge weighted tree and d_{min}, d_{max} be two non-negative real numbers where $d_{min} \leq d_{max}$. The pairwise compatibility graph (PCG) of T for d_{min}, d_{max} is a graph G such that each vertex of G corresponds to a distinct leaf of T and two vertices are adjacent in G if and only if the weighted distance between their corresponding leaves lies within the interval $[d_{min}, d_{max}]$. A graph G is a PCG if there exist an edge weighted tree T and suitable d_{min}, d_{max} such that G is a PCG of T. The class of pairwise compatibility graphs was introduced to model evolutionary relationships among a set of species. Since not all graphs are PCGs, researchers become interested in recognizing and characterizing PCGs. In this paper, we review the results regarding PCGs and some of its variants.

KEYWORDS

Pairwise compatibility graphs; phylogenetic trees; leaf power graphs; multi-interval pairwise compatibility graphs

1. Introduction

Let T be an edge weighted tree and let d_{min}, d_{max} be two non-negative real numbers where $d_{min} \leq d_{max}$. The *pairwise compatibility graph* (PCG) of T for d_{min} and d_{max}, denoted by $PCG(T, d_{min}, d_{max})$, is a graph $G = (V, E)$ where each vertex $u' \in V$ corresponds to a distinct leaf of T and there is an edge $(u', v') \in E$ if and only if the weighted distance between their corresponding leaves lies within the interval $[d_{min}, d_{max}]$. The tree T is called a *pairwise compatibility tree* (PCT) of G. A given graph is a PCG if there exist suitable T, d_{min}, d_{max} such that $G = PCG(T, d_{min}, d_{max})$. Construction of a PCG is trivial for a given weighted tree T and two real numbers d_{min}, d_{max}. However, the reverse problem of recognizing a given graph as a PCG is difficult. Figure 1(b) illustrates the pairwise compatibility graph G of the edge weighted tree T in Figure 1(a) for $d_{min} = 4$ and $d_{max} = 5$. For a pairwise compatibility graph G, pairwise compatibility tree T may not be unique. For example, Figure 1(c) shows another edge weighted tree T' such that the graph G in Figure 1(b) is a PCG of T' for $d_{min} = 4$ and $d_{max} = 7$.

PCGs have applications in modeling evolutionary relationship among set of organisms from biological data which is also called phylogeny. Phylogenetic relationships are normally represented as a tree called phylogenetic tree. The *phylogenetic tree reconstruction problem* asks to find a tree which provides the "best" explanation of the data of a set of taxa. A variety of biologically motivated heuristics are known for determining the "goodness" of a phylogenetic tree for a given data. Under most of the heuristics, phylogenetic tree reconstruction problem is known to be an NP-Complete problem [18, 21]. It is also difficult to find the heuristic for the optimal tree, hence experimental evaluations of trees generated from different algorithms are done to measure the relative performance of each algorithm. Since it is time expensive to find large trees consisting of many taxa, it is interesting to test a reconstruction algorithm on a subtree induced by a smaller sample of taxa. Thus it is important to efficiently find biologically significant samples of taxa from a large tree. While dealing with such sampling problem from large phylogenetic tree, Kearney et al. [22] introduced the concept of PCGs. They also showed that "the clique problem" can be solved in polynomial time for a PCG if a pairwise compatibility tree can be constructed in polynomial time.

Leaf Power Graphs (LPGs), a subclass of PCGs, are a well-studied class of graphs which is obtained by setting d_{min} to 0 [23, 26, 32]. A similar but relatively new subclass of PCGs named Min-Leaf Power Graphs (mLPGs) are obtained by setting d_{max} to ∞ [10]. Recently a superclass of PCGs named multi-interval PCGs are introduced where more than one intervals are allowed [1]. If the number of intervals is k then the graphs are called k-interval PCGs. In Table 1, we provide an overview of graph classes which are in PCG, LPG, mLPG and 2-interval PCG. In this paper, we review the current state of art regarding PCGs. In addition to that, we review some variants of PCGs.

The rest of the paper is organized as follows. In Section 2, we give some necessary definitions. In Section 3, we review the results on PCGs. In Section 4, we review some variants of PCGs. In Section 5, we review the results

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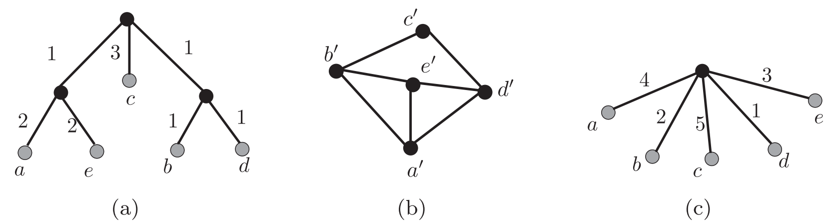


Figure 1. (a) An edge weighted tree T, (b) a pairwise compatibility graph G of T for $d_{min} = 4$ and $d_{max} = 5$, and (c) another edge weighted tree T' such that G is a PCG of T' for $d_{min} = 4$ and $d_{max} = 7$.

Table 1. Overview of graphs classes in PCG, LPG, mLPG, and 2-interval PCG.

PCGs	Every graph with at most seven vertices [7], Cycles, Single chord cycles [35]; Ladder graphs, Block-cycle graphs, Triangle-free outerplanar graphs [30]; Graphs with Dilworth number at most two [9]; A restricted subclass of split matrogenic graphs [11]; A restricted subclass of bipartite graphs, Tree power graphs [34]
LPGs	Trees [20], Interval graphs, Ptolemaic graphs [3]; Threshold graphs, Split matching graphs [8], A superclass of directed rooted graphs [4]
mLPGs	Threshold tolerance graphs [12], Split antimatching graphs [8]
2-interval PCGs	Wheel graphs, A restricted subclass of series-parallel graphs [1]

on PCGs of some specific tree topologies. Finally we conclude in Section 6.

2. Preliminaries

In this section, we define some terms which will be used throughout this paper.

Let, $G = (V,E)$ be a simple, undirected graph with vertex set V and edge set E. An edge between two vertices u and v is denoted by (u, v). If $(u,v) \in E$, then u and v are *adjacent* and the edge (u, v) is incident to u and v. The *degree* of a vertex is the number of edges incident to it. The complement graph $\overline{G}$ of G is the graph with vertex set V and edge set $\overline{E}$, where $\overline{E}$ consists of the edges that are determined by the non-adjacent pairs of vertices of G. For a vertex v of a graph G, $N(v) = \{u|(u,v) \in E\}$ denotes the *open neighborhood*, while $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood*. A *path* P_{uv} in G is a sequence of distinct vertices $w_1, w_2, w_3, \cdots, w_n$ in V such that $u = w_1$ and $v = w_n$ and $(w_i, w_{i+1}) \in E$ for $1 \leq i < n$. The vertices u and v are called *end-vertices* of path P_{uv}. If the end-vertices are same then the path is called a *cycle*. A *tree* T is a connected graph with no cycle. A vertex with degree one in a tree is called *leaf* of the tree. All the vertices other than leaves are called *internal nodes*. A *weighted tree* is a tree where each edge is assigned a number as the weight of the edge. The weight of an edge (u, v) is denoted as $w(u, v)$. The distance between two nodes u, v in T is the sum of the weights of the edges on path P_{uv} and denoted by $d_T(u,v)$. A *subtree induced by a set of leaves* of T is the minimal subtree of T which contains those leaves. We denote by T_{uvw} the subtree of a tree induced by three leaves u, v and w. A star graph S_n is a tree on n nodes with one node having degree $n - 1$ and all other nodes having degree 1. A *caterpillar* is a tree for which deletion of leaves together with their incident edges produces a path. The *spine* of a caterpillar is the longest path to which all other vertices of the caterpillar are adjacent. A *chord* of a cycle is an edge, not in the cycle, whose end-points lie on the cycle. An *odd chord* connects two vertices that have odd distance between them in the cycle. A graph is *chordal* if it does not contain any cycle of length greater than three as an induced subgraph. A chordal graph is *strongly chordal* if every even cycle of length at least six has an odd chord [5]. A vertex is called a *cut-vertex* if removing it disconnects the graph. A *block* of a graph is a maximal biconnected subgraph of the graph. A *block-cycle graph* is a graph with cycles as its maximum biconnected components. The blocks and cut-vertices of a graph can be represented by *block-cut-vertex tree* of the graph. A graph is *bipartite* if the vertices can be expressed as the union of 2 independent sets.

A graph $G' = (V',E')$ is a *k-root* of a graph $G = (V,E)$ if $V' = V$ and there is an edge $(u,v) \in E$ if and only if the shortest path connecting u and v in G' is at most k. Alternatively, G is called the *k-power* of G' [23]. A special case of graph power is tree power which requires G' to be a tree. The k-th power of G is represented as G^k. A graph G is a *tree compatible graph* if there exists a tree T such that all leaves and a subset of internal nodes of T correspond to the vertex set V of G, and for any two vertices $u, v \in V$; $(u,v) \in E$ if and only if $k_{min} \leq d_T(u,v) \leq k_{max}$ where k_{min} and k_{max} are real numbers. A *ladder graph* consists of two path graphs $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ where u_i and v_i are adjacent. A *dual graph* of a plane graph G is a graph which has a vertex for each face of G, and an edge for each edge in G joining two neighboring faces. A *weak dual* of a plane graph G is the subgraph of the dual graph of G whose vertices correspond to the bounded faces of G. A graph is *outerplanar* if it has a planar embedding where all vertices are on the outer face. *Subdividing an edge* (u, v) of a graph G is an operation to replace the edge (u, v) with a path $u, w_1, w_2, \cdots, w_k, v$. A graph G' is called a *subdivision of a graph* G if it can be obtained from G by subdividing some edges of G. The *Dilworth number* of a graph is defined as the size of the largest subset of vertices in which the closed neighborhood of no node contains the neighborhood of another [15]. The graphs with Dilworth number one are called *threshold graphs* [14]. A *split graph* is a graph whose vertex set can be partitioned as the disjoint union of an independent set and a clique. A split graph is called a *split matching* and *a split antimatching* if the subset of edges not

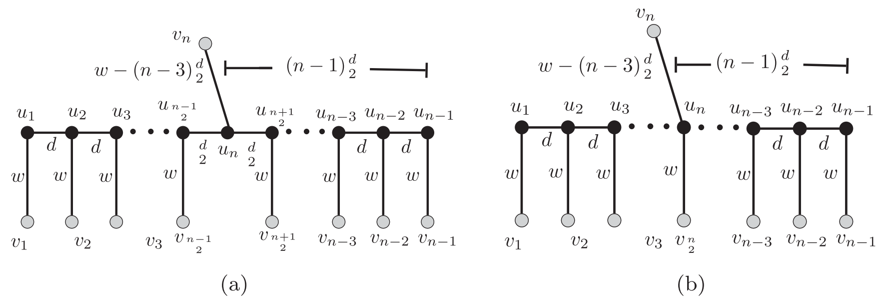


Figure 2. (a) A pairwise compatibility tree of a cycle with odd number of vertices and (b) a pairwise compatibility tree of a cycle with even number of vertices.

belonging to the clique forms a perfect matching and an anti-matching respectively [10]. A graph is an *intersection graph* if the vertices corresponds to a family of sets and there is an edge between two vertices if the corresponding sets have non-empty intersection. *Interval graphs*, *circular arc graphs*, *disk graphs*, *rectangle intersection graphs* and *rooted path graphs* are intersection graphs of intervals on real line, arcs of circle, disks in the plane, rectangles on the plane and directed subpaths of a rooted tree respectively. A graph is a *tolerance graph* if each vertex u corresponds to an interval I_u on real line and a tolerance t_u, and two vertices u, v are adjacent if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$ [19]. A graph G is *ptolemaic graph* if any four vertices u, v, w, x in the same connected component of G satisfy $d(u,v)d(w,x) \leq d(u,w)d(v,x) + d(u,x)d(v,w)$. A *wheel graph* with n vertices, denoted by W_n, is obtained from a cycle graph C_{n-1} with n – 1 vertices by adding a new vertex p and joining an edge from p to each vertex of C_{n-1}. A graph $G = (V, E)$ is called a *series-parallel* (*SP*) graph with source s and sink t if either G consists of a pair of vertices connected by a single edge or there exists two series-parallel graphs $G_i(V_i, E_i)$ with source s_i and sink t_i for $i = 1$, 2 such that $V = V_1 \cup V_2, E = E_1 \cup E_2$ and either $s = s_1$, $t_1 = s_2$ and $t = t_2$ or $s = s_1 = s_2$ and $t = t_1 = t_2$ [29].

3. Pairwise compatibility graphs

In this section, we review the results on PCGs.

3.1. Graphs that are PCGs

In 2003, Kearney et al. introduced PCGs while dealing with a sampling problem from large phylogenetic tree [22]. Since its inception many graph classes are proved to be PCGs. Initially, Phillips showed that every graph with at most five vertices is PCG [27] and later Calamoneri et al. showed that all graphs with at most seven vertices are PCGs [7]. It is easy to show that trees are PCGs [30]. The following theorem shows that every cycle is a PCG [35].

Theorem 1. *Every cycle is a PCG.*

Outline of the Proof: Let C_n be a cycle with n vertices $v'_1, v'_2, \cdots, v'_n$ where (v'_i, v'_{i+1}) are adjacent for $1 \leq i < n$ and (v'_1, v'_n) are also adjacent. We construct an edge weighted caterpillar T as follows. Let $v_1, v_2, \cdots, v_{n-1}$ be the leaves of T and $u_1, u_2, \cdots, u_{n-1}$ be the vertices on the spine of T such that u_i is adjacent to v_i. We assign weight d to edge (u_i, u_{i+1}) for $1 \leq i < n-1$ and weight w to the edges incident to a leaf where $w > (n+1)\frac{d}{2}$. If n is odd then we put a vertex u_n in the middle of the path $P_{u_1 u_{n-1}}$ as illustrated in Figure 2(a). If n is even then we use $u_{\frac{n}{2}}$ as u_n which is shown in Figure 2(b). We then place the last vertex v_n as a leaf adjacent to u_n. We assign weight $w_n = w - (n-3)\frac{d}{2}$ to the edge (u_n, v_n). The leaf v_i of T corresponds to the vertex v'_i of C_n. The tree T is a PCT of C_n for $d_{min} = 2w + d$ and $d_{max} = 2w + d$. Note that $d_T(v_i, v_j) > 2w + d$ for $j \neq i-1, i, i+1$ and $i, j \neq n$, and $d_T(v_n, v_j) < 2w + d$ for $j \neq 1, n-1$. □

Using Theorem 1 we get the following theorem [35].

Theorem 2. *Every cycle with a single chord is a PCG.*

Using Theorem 1 we can also prove that every block-cycle graph is a PCG [35].

Theorem 3. *Every block-cycle graph is a PCG.*

Outline of the Proof: Let G be a block-cycle graph. We give an outline of the construction of a PCT of G. For each connected component G_i of G we first construct block-cutvertex tree $\mathcal{T}_i$ of G_i. Since G_i is a connected block-cycle graph, each block of G_i is either a cycle or an edge. We start with an empty tree T_i. We perform e depth first search in $\mathcal{T}_i$ and for each node corresponding to a block B_{ij}, we construct a PCT T_{ij} and add T_{ij} with T_i to construct PCT of the nodes visited so far. Finally we construct PCT of G by merging PCT of each connected component of G by adding edge between trees. □

Although not every bipartite graphs are PCGs, some restricted class of biparite graphs are proved to be PCGs [34]. Yanhaona et al. showed that a class of graphs called *tree compatible graphs*, which contains tree power graphs, are PCGs [34]. Interval graphs are proved to be PCGs [3]. Calamoneri et al. showed that threshold graphs, split matching graphs are PCGs [8]. Every graph with Dilworth number at most two is proved to be PCG [9]. Calamoneri et al. proved that threshold tolerance graphs and a subclass of split matrogenic graphs are PCGs [11]. Salma et al. showed that ladder graphs are PCGs as in the following theorem [30].

Theorem 4. *Every ladder graph is a PCG.*

Outline of the Proof: Let G be a ladder graph with $2n$ vertices and $u'_1, u'_2, \cdots, u'_n, v'_1, v'_2, \cdots, v'_n$ be the vertices of G where there is edges between $(u'_i, u'_{i+1}), (v'_i, v'_{i+1})$ and (u'_i, v'_i). We construct an edge weighted tree T as shown in Figure 3. T is a PCT of G for $d_{min} = 2w + d + 1$ and $d_{max} = 2w + 2d$. Note that, $d_T(u_i, u_{i+1}) = d_T(v_i, v_{i+1}) = 2w + 2d$ and $d_T(u_i,$

$v_i) = 2w + d + 1$. On the other hand $d_T(u_i, u_j), d_T(v_i, v_j)$, $d_T(u_i, v_j) > 2w + 2d$ for $j > i+1$ and $d_T(u_{i-1}, v_i) < 2w + d + 1$. □

By providing a method to combine PCTs of two cycles of length at least four, Salma et al. gave the following theorem [30].

Theorem 5. *Outer subdivision of ladder graphs are PCGs.*

Salma et al. provided an algorithm to construct PCT of a triangle free outerplanar graph. The idea of the algorithm is to construct PCT of each biconnected component by performing DFS on the weak dual of the biconnected component and finally merging the PCTs of biconnected components. Thus we have following theorem [30].

Theorem 6. *Triangle-free outer planar 3-graphs are PCGs.*

Despite recent progress, we are far from a complete characterization of PCGs. Hence interesting open problems are remained open. It is not known whether grid graphs are PCGs or not [30].

Open Problem 3.1.1. Whether grid graphs are PCGs or not.

The following open problem asks about the largest class of outerplanar graphs which are PCGs [28].

Open Problem 3.1.2. What is the largest class of outerplanar graphs which are PCGs?

Although a subclass of split matrogenic graphs are PCGs, It is not known whether all split matrogenic graphs are included in PCGs [10].

Open Problem 3.1.3. Whether every split matrogenic graphs are PCGs or not.

3.2. Graphs that are not PCGs

Since there are exponentially increasing number of tree topologies, initially Kearney et al. conjectured that every graph

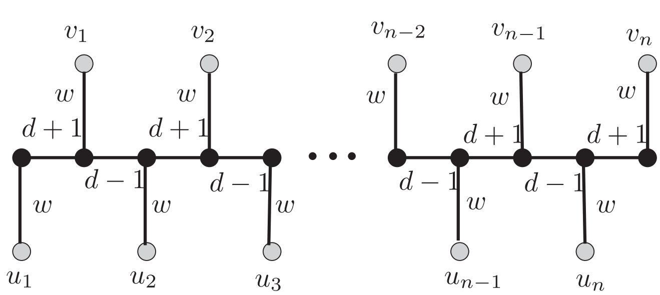


Figure 3. Pairwise compatibility tree of a ladder graph.

is a PCG [22]. However the conjecture is refuted by Yanhaona et al. as stated in the following theorem [34].

Theorem 7. *Not all graphs are PCGs.*

Yanhaona et al. provided an example of a bipartite graph of 15 vertices, as illustrated in Figure 4(a), which is not a PCG. The key idea behind the proof of Theorem 7 is the positions of leaves in a weighted tree put constraint on the distances between pair of leaves. The following lemma illustrates this idea [34].

Lemma 8. *Let T be an edge weighted tree and u, v, w be three leaves of T such that P_{uv} be the largest path in T_{uvw}. Let x be another leaf except u, v, w. Then $d_T(w,x) \le d_T(u,x)$ or $d_T(w,x) \le d_T(v,x)$.*

Outline of the Proof: Let o be the degree 3 vertex of T_{uvw}. Then each of the paths P_{uv}, P_{uw} and P_{wv} is composed of two of the three subpaths P_{uo}, P_{ow} and P_{ov}. Since $d_T(u,v)$ is the largest path in $T_{uvw}, d_T(u,v) \ge d_T(u,w)$. This implies that $d_T(u,o) + d_T(o,v) \ge d_T(u,o) + d_T(o,w)$. Hence $d_T(o,v) \ge d_T(o,w)$. Similarly, $d_T(u,o) \ge d_T(o,w)$ since $d_T(u,v) \ge d_T(w,v)$. Since T is a tree, there is a path from x to o. Let o_x be the first vertex in $V(T_{uvw}) \cap V(P_{xo})$ along the path P_{xo} from x. Then clearly o_x is on P_{uo}, P_{vo} or P_{wo}. We first assume that o_x is on P_{uo}. Then $d_T(v,x) \ge d_T(w,x)$ since $d_T(w,x) = d_T(x,o) + d_T(w,o), d_T(v,x) = d_T(x,o) + d_T(v,o)$ and $d_T(v,o) \ge d_T(w,o)$. We now assume that o_x is on P_{vo}. Then $d_T(u,x) \ge d_T(w,x)$ since $d_T(w,x) = d_T(x,o) + d_T(w, o)$, $d_T(u,x) = d_T(x,o) + d_T(o,u)$ and $d_T(u,o) \ge d_T(w,o)$. We finally assume that o_x is on P_{wo}. Then $d_T(u,x) = d_T(u,o) + d_T(o,o_x) + d_T(o_x,x)$ and $d_T(w,x) = d_T(w,o_x) + d_T(o_x,x)$. As $d_T(w,o_x) \le d_T(w,o)$ and $d_T(u,o) \ge d_T(w,o)$, $d_T(u,x) \ge d_T(w,x)$. Likewise, $d_T(v,x) \ge d_T(w,x)$. Thus, in each case, at least one of u and v is at a distance from x that is either larger than or equals to the distance between w and x. □

Mehnaz and Rahman generalized the counter-example provided by Yanhaona et al. and showed a subclass of bipartite graphs are not PCG [24].

Theorem 9. *Let $G = (V, E)$ be a bipartite graph with two partite sets V_1, V_2 such that $|V_1| = k$ and $|V_2| = \binom{k}{r}$ where $3 \le r \le n-2$. If each vertex in V_2 has exactly r neighbors in V_1 and no two vertices in V_2 has the same r neighbors in V_1, then G is not a PCG.*

Durocher et al. showed a graph of eight vertices, as illustrated in Figure 4(b), and a planar graph of sixteen vertices which are not a PCG [16]. The following lemma illustrates a

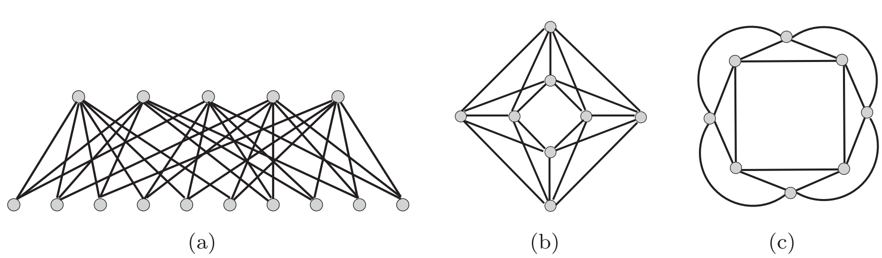
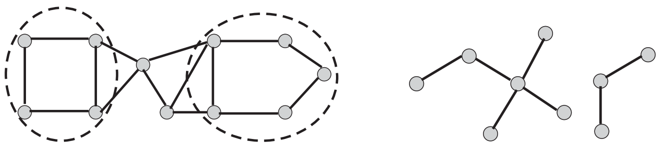


Figure 4. (a) A bipartite graph with 15 vertices which is not a PCG, (b) a graph with 8 vertices which is not a PCG, and (c) the smallest planar graph which is not a PCG.



(a) (b)

Figure 5. (a) A graph with two disjoint chordless cycle (highlighted by dotted ellipses) whose compliment is not a PCG and (b) a graph no cycle whose complement is a PCG.

constraint on the distances between pair of leaves based on which the graph with 8 vertices is proved to be not a PCG [16].

Lemma 10. *Let C be the cycle a', b', c', d' of four vertices. If $C = PCG(T, d_{min}, d_{max})$ for some tree T and values d_{min} and d_{max}, then $d_T(a, c)$ and $d_T(b, d)$ cannot be both greater than d_{max}.*

Calamoneri et al. showed that the graph with eight vertices, which is not a PCG, is a circular arc graph, disk graph and rectangle intersection graph. Thus the circular arc graphs, disk graphs and rectangle intersection graphs are not contained in PCGs [11]. Although split permutation graphs are PCGs [9], permutations graphs are not PCGs [11]. Calamoneri et al. also showed that tolerance graphs are not PCGs [11]. Recently, Baiocchi et al. gave a new proof technique for a graph to be not PCG, and proved that square of a cycle with at least 8 vertices are not PCGs [2]. The graph C_8^2, as shown in Figure 4(c), is proved to be a minimal not PCG and the smallest planer graph known which is not PCG [2]. Although wheel graphs with at most 8 vertices are proved to be PCGs, wheel graphs with at least 9 vertices are not PCGs [2]. Moreover, strong product of C_n and P_2 are not PCGs [2]. Since many graph classes are not yet proved to be PCGs or not PCGs, we have the following open problem [13].

Open Problem 3.2.1. Find other graph classes which are not PCGs.

3.3. Complexity of recognizing PCGs

Given a graph, the *PCG recognition problem* asks to decide whether the graph is a PCG or not. Durocher et al. considered a generalized PCG recognition problem and proved the hardness of the problem [16]. Given a graph $G = (V, E)$ and a set $S \subseteq \overline{E}$, the *generalized PCG recognition problem* asks to find a PCG $G' = PCG(T, d_{min}, d_{max})$ which contains G as a subgraph but does not contain any edge of S. If the maximum number of edges of S are required to have distance between their corresponding leaves greater than d_{max}, then the generalized PCG recognition problem is NP-hard. Formally the decision version of the problem is given below.

Problem: Max-Generalized PCG Recognition Problem.
Instance: A graph G, a subset S of the edges of the complement graph of G, and a positive integer k.
Question: Is there a PCG $G' = PCG(T, d_{min}, d_{max})$ such that G' contains G as a subgraph but does not contain any edges of S, and there are at least k edges of S such that the distance between their corresponding leaves in T is greater than d_{max}.

Durocher et al. proved that the Max-Generalized PCG Recognition Problem is NP-hard [16] by reduction from the Monotone-One-In-Three-3-Sat problem [31]. However the complexity of the PCG recognition problem is still unknown [16].

Open Problem 3.3.1. What is the complexity of PCG recognition problem?

3.4. Necessary and sufficient conditions of PCGs

Hossain et al. gave a necessary condition and a sufficient condition for a graph to be PCG [20]. The necessary condition is stated as follows [20].

Theorem 11. *Let G be a graph. Let H_1 and H_2 be two disjoint induced subgraphs of $\overline{G}$. If each of H_1 and H_2 is either a chordless cycle of at least four vertices or $\overline{C_n}$ for $n \geq 5$, then G is not a PCG.*

The graph shown in Figure 5(a) has two disjoint chordless cycles and hence by Theorem 11 the complement of the graph is not a PCG. The correctness of Theorem 11 is immediate from the following lemmas [20].

Lemma 12. *Let $G = (V, E)$ be a graph. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two induced subgraphs of G with no common vertices in G_1 and G_2. Assume that $V_2 \subset N(u')$ and $V_1 \subset N(v')$ in G for every $u' \in V_1$ and every $v' \in V_2$. Let T_1 be a PCT of G_1 such that $G_1 = PCG(T_1, d_{min1}, d_{max1})$ and T_2 be a PCT of G_2 such that $G_2 = PCG(T_2, d_{min2}, d_{max2})$. If there exist two leaves a, c in T_1 such that $d_{T_1}(a, c) > d_{max1}$, and two leaves b, d in T_2 such that $d_{T_2}(b, d) > d_{max2}$, then G is not a PCG.*

Lemma 13. *Let G be a graph. Let H_1 and H_2 be two disjoint subgraphs of G^c. For any $\overline{H_1} = PCG(T_1, d_{min1}, d_{max1})$ and $\overline{H_2} = PCG(T_2, d_{min2}, d_{max2})$, if T_1 has a pair of leaves whose weighted distance is greater than d_{max1} and T_2 has a pair of leaves whose weighted distance is greater than d_{max2}, then G is not a PCG.*

Lemma 14. *Let C_n be a cycle of length $n \geq 5$, then both C_n and $\overline{C}_n$ are in PCG but not in $\{mLPG, LPG\}$.*

We now state the sufficient condition given by Hossain et al. [20].

Theorem 15. *Let G be a graph. If $\overline{G}$ has no cycle then G is a PCG.*

By Theorem 15, the complement of the graph shown in Figure 5(b) is a PCG. Despite being the first necessary condition and first sufficient condition, there is a gap between the conditions. Hossain et al. identified four interesting graph classes between these two conditions as follows [20].

$\mathcal{G}^1$: containing graphs whose complement does not contain of any chordless cycle.

$\mathcal{G}^2$: containing graphs whose complement consisting of two induced chordless cycles which share some common vertices.

$\mathcal{G}^3$: containing graphs whose complement consisting of two induced chordless cycles where some edges are incident to both cycles.

$\mathcal{G}^4$: containing graphs whose complement contain only one chordless cycle.

Hossain et al. showed that there are some graphs that belong to $\mathcal{G}^1$ and $\mathcal{G}^2$ which are not PCGs. Hence following problems are remained open [20].

Open Problem 3.4.1. Whether every graph that belong to $\mathcal{G}^2$ are PCGs.

Open Problem 3.4.2. Whether every graph that belong to $\mathcal{G}^4$ are PCGs.

4. Variants of PCGs

In this section, we review some subclass and superclass of PCGs.

4.1. Leaf power graphs

A graph $G = (V, E)$ is called a *Leaf Power Graph* (LPG) is there is an edge weighted tree T and a nonnegative real number d_{max} such that each vertex of G corresponds to a leaf of T and there is an edge $(u', v') \in E$ if and only if $d_T(u, v) \leq d_{max}$ where u, v are the leaves of T corresponding to the vertices u', v' [26]. We denote the LPG of T, d_{max} by $LPG(T, d_{max})$. If $d_{max} = k$, then $LPG(T, d_{max})$ is called a k-leaf power graph. Thus LPGs are special case of PCGs when d_{min} is set to 0. Although every LPG is strongly chordal, not all strongly chordal graphs are LPGs [17]. Ptolemaic graphs, interval graphs and a superclass of directed rooted path graphs are PCGs [3, 4]. Nevries and Rosenke gave a list of seven graphs which are not LPGs [25], and one of the seven graphs is shown in Figure 6.

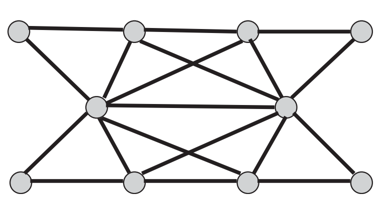


Figure 6. A graph which is not an LPG.

4.2. Min-leaf power graphs

A graph $G = (V, E)$ is called a *Min-Leaf Power Graph* (mLPG) is there is an edge weighted tree T and a nonnegative real number d_{min} such that each vertex of G corresponds to a leaf of T and there is an edge $(u', v') \in E$ if and only if $d_T(u, v) \geq d_{min}$ where u, v are the leaves of T corresponding to the vertices u', v' [10]. We denote the mLPG of T, d_{min} by $mLPG(T, d_{min})$. Thus mLPGs are special cases of PCGs when d_{max} is set to ∞. Threshold tolerance graphs are proved to be mLPGs [12]. Split matching graphs are not mLPGs [10].

4.3. Multi-interval PCGs

A graph G a *k-interval PCG* if there is an edge weighted tree T for mutually exclusive intervals $I_1, I_2, \cdots, I_k$ of nonnegative real numbers such that each vertex of G corresponds to a leaf of T and there is an edge between two vertices in G if the distance between their corresponding leaves lies in $I_1 \cup I_2 \cup \cdots I_k$ [1]. Figure 7(b) illustrates an edge weighted tree T and Figure 7(a) shows the corresponding 2-interval PCG where $I_1 = [1, 3]$ and $I_2 = [5, 6]$. Note that the graph shown in Figure 7(b) is not a PCG [16].

Since we are allowed to take as many intervals as needed, it is conceivable that every graph is a k-interval PCGs for some k. We have the following theorem [1].

Theorem 16. *Every graph is an $|E|$-interval PCG.*

Ahmed and Rahman showed that wheel graphs with at least 9 vertices, which are proved to be not PCGs, are 2-interval PCGs [1]. They also showed that a restricted class of series-parallel graphs, called the SQQ series-parallel graphs, are 2-interval PCGs. Being e relatively new graph class, many questions are yet open as listed below [1].

Open Problem 4.3.1. What is the smallest value of k for which every graph is a k-interval PCG?

Open Problem 4.3.2. Is every series-parallel graph a 2-interval PCG?

4.4. Relationship among PCGs, k-interval PCGs, LPGs, mLPGs

It is easy to observe that k-interval PCGs are superclass of PCGs and both LPGs and mLPGs are subsets of PCGs. Calamoneri et al. showed that co-LPG class coincides with mLPG and co-mLPG class coincides with LPG [8]. It is proved that LPG $\cap$ mLPG $\neq \emptyset$, and threshold graphs are both LPG and mLPG [10]. It is also proved that there are graphs such as cycles which are PCGs but neither LPGs nor

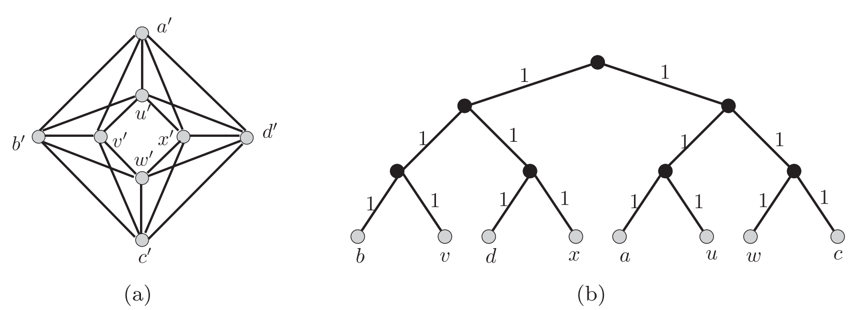


Figure 7. (a) A graph G which is not a PCG, (b) a 2-interval PCT G for $I_1 = [1, 3], I_2 = [5, 6]$.

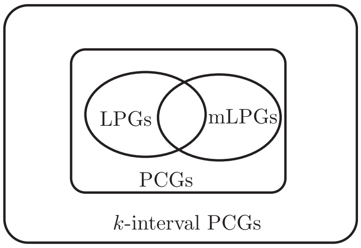


Figure 8. Relationship among LPG, mLPG, PCG, k-interval PCG.

mLPGs [10]. Both the set LPG $\setminus$ mLPG and mLPG $\setminus$ LPG are non-empty [10]. The following theorem [13] and Figure 8 illustrates the relationship among the four classes.

Theorem 17. *For the classes of LPG, mLPG, PCG and k-interval PCG, the following relationship holds: (a) LPG, mLPG $\subset$ PCG $\subset$ k-interval PCG, (b) LPG $\cap$ mLPG $\neq \emptyset$, (c) LPG $\setminus$ mLPG $\neq \emptyset$, and (d) mLPG $\setminus$ LPG $\neq \emptyset$.*

5. PCGs of specific tree topology

In this section, we review the results regarding PCGs of some specific tree topologies.

In most of the works, different graph classes are proved to be PCGs by showing a PCT which witnesses a graph of the graphs class as PCG. Mostly the structure of such PCTs are simple such as stars, caterpillars. Some researchers considered the reverse direction where the subclasses of PCGs of specific tree topologies are investigated.

5.1. PCGs of stars

PCGs of star topology are studied by some researchers. Threshold graphs are PCGs of stars [10]. Moreover, it is proved that PCGs of stars are a special superclass of threshold graphs [10]. Recently, Xiao and Nagamochi gave a complete characterization of PCGs of stars [33]. The authors showed that vertices of PCGs of stars admits a special ordering and gave an $O(n^6)$ time algorithm to recognize a graph as a PCG of stars [33].

5.2. PCGs of caterpillars

Different graph classes such as cycles, ladder graphs are proved to be PCGs of caterpillars [30, 35]. LPGs of unit weight caterpillars are unit interval graphs [3]. Calamoneri et al. gave some properties of PCGs of unit weight caterpillars [6]. Wheel graphs with more than 9 vertices are proved to be 2-interval PCGs of caterpillars [1]. However, complete characterization of PCGs of caterpillars are not known yet. Hence we have the following open problem [6].

Open Problem 5.2.1. Characterize PCGs of caterpillars.

It is also interesting to characterize the PCGs of other tree topologies [13].

Open Problem 5.2.2. Characterize PCGs of other tree topologies.

6. Conclusion

In this paper, we review the current state of art regarding the graph class PCG and some of its variants. Recent results show some important characteristics of these graph classes. However, the complete characterization is not known yet and some interesting problems are yet open.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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