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Crossing number of Cartesian product of prism and path

Yip C. Yiew, Gek L. Chia, and Poh-Hwa Ong

Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, Kajang, Selangor, Malaysia

ABSTRACT
An *m-prism* is the Cartesian product of an m-cycle and a path with 2 vertices. We prove that the crossing number of the join of an m-prism ($m \geq 4$) and a graph with k isolated vertices is km for each $k \in \{1, 2\}$. We then use this result to prove that the crossing number of the Cartesian product of a 5-prism and a path with n vertices is $10(n-1)$. This answers partially the conjecture raised by Peng and Yiew (in 2006) in the affirmative.

KEYWORDS
Crossing number; Cartesian product; prism; path

2010 MATHEMATICS SUBJECT CLASSIFICATION
68R10; 05C10; 05C62

1. Introduction

By a *good drawing* of a graph G we mean an embedding of G on the plane such that (i) no edge intersect itself, (ii) adjacent edges do not intersect each other, (iii) any pair of edges do not touch each other and they intersect each other at most once, and (iv) no three edges intersect at the same point. If D is a good drawing of G, we let $cr_D(G)$ denote the number of pair-wise intersections of the edges of G in D. If $cr_D(G)$ achieves the minimum number, then D is called an *optimal drawing* of G and the minimum number of crossings is called the *crossing number* of G, denoted $cr(G)$.

The *Cartesian product* of two graphs G and H, denoted $G\Box H$ is the graph with vertex set $V(G) \times V(H)$ and having edges of the form $(u, u')(v, v')$ where either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$.

Let P_n and C_n denote the path and cycle on n vertices respectively. By an *m-prism*, denoted $P(m)$, we mean the Cartesian product $C_m\Box P_2$. The crossing numbers of the Cartesian product of some special graphs with the path (or other graph) have been the subject of investigation (see [2–13]). In particular, in [7], it was proved that $cr(P(3)\Box P_n) = 4(n-1)$ for all natural numbers $n \geq 1$. Further, it was conjectured that $cr(P(m)\Box P_n) = 2m(n-1)$ for all natural numbers $m \geq 4$. In this paper, we prove that the conjecture is true for $m = 5$ (see Theorem 3). The case $m = 4$ has earlier been established in [9].

Let $G + H$ denote the join of two graphs G and H. The proof of Theorem 3 rests on the result which states that $cr(P(m) + \overline{K_r}) = rm, 1 \leq r \leq 2$ (see Theorems 1 and 2). Here $\overline{K_n}$ denotes a graph with n isolated vertices. The proof of this is given in Section 2.

2. Join of graphs

Throughout, unless otherwise stated, we let the two m-cycles in $P(m)$ be denoted by $Z_1 = a_0a_1a_2...a_{m-1}a_0$ and $Z_2 = b_0b_1b_2...b_{m-1}b_0$. Also, we assume that a_ib_i is an edge in $P(m)$ for each $i \in \{0, 1, ..., m-1\}$. Each edge a_ib_i is called a *spoke* of $P(m)$.

Let D be a good drawing of a graph G and let H be a subgraph of G. The *responsibility* of H in D, denoted $r_D(H)$, is the total number of times edges in H are crossed. Note that if two edges of H cross each other, then a contribution of two is added to its responsibility.

Lemma 1. *Let D be a good drawing of $P(m) + \{w\}$ such that no triangle of the form wa_ib_iw has an edge crossed, $0 \leq i \leq m-1$. Then D has at least $2\lceil \frac{m}{2} \rceil$ crossings, $m \geq 3$.*

Proof. By the condition of the lemma, in D, no triangle of the form wa_ib_iw is enclosed by another triangle of the form wa_jb_jw. As such, we may assume that edges of the form a_ib_i are drawn on the x-axis in an arbitrary manner with the vertex w lying on the upper region of the x-axis and all edges of $Z_1 \cup Z_2$ are on the x -axis or are lying on the lower region of the x-axis.

Alternatively, we may assume that edges of the form a_ib_i are on the boundary of convex polygon P on $2m$ vertices with w lying on the "exterior" and the edges of $Z_1 \cup Z_2$ are on the boundary of P or lying in the "interior" of P. Figure 1 depicts an example of such drawing with $m = 5$.

We shall show that the number of crossings on the edges of Z_1 made by the edges of Z_2 is at least $2\lceil \frac{m}{2} \rceil$.

We have the following observations.

(i) It is easy to see that an edge of Z_1 which is on the boundary of P has no crossing.
(ii) Clearly, any vertex of Z_1 is incident to at most one edge of Z_1 which is a boundary edge of P.
(iii) We assert that every edge of Z_1 not lying on the boundary of P is crossed by at least 2 edges of Z_2.

CONTACT Gek L. Chia chiagl@utar.edu.my Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, Kajang, Selangor, Malaysia.

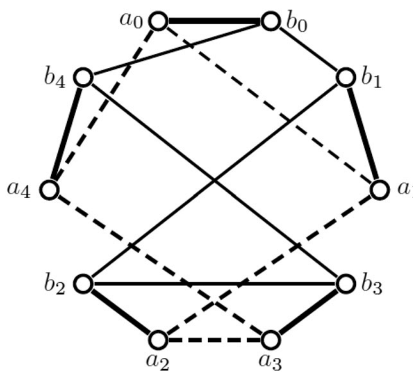

Figure 1. An illustration on the proof of Lemma 1 with $m = 5$.

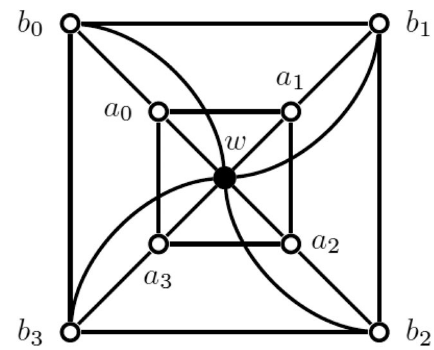

Figure 2. $P(4) + \{w\}$.

Evidently, the above observations imply the conclusion of the lemma.

To prove (iii), let a_ia_{i+1} be an edge of Z_1 not on the boundary of P and consider the subgraph H induced by the vertices of Z_2 which are enclosed in the "interior" of the triangle $\triangle_i = wa_ia_{i+1}w$. Here we just need to consider the case $|V(H)| \leq m/2$. Clearly H is a union of paths.

If H has an isolated vertex v, then v is adjacent to two vertices in the "exterior" region of $\triangle_i$ giving two crossings on the edge a_ia_{i+1} (see for example, $v = b_3$ for the case $i = 1$ in Figure 1). If H has no isolated vertices, then H has two vertices u_1, u_2 each of degree 1 in H (see for example, $u_1 = b_0, u_2 = b_1$ for the case $i = 0$ in Figure 1). Each of u_1, u_2 is adjacent to some vertex in the "exterior" region of $\triangle_i$ again giving two crossings on the edge a_ia_{i+1}. This completes the proof. □

Theorem 1. *$cr(P(m) + \{w\}) = m$ if $m \geq 4$.*

Proof. By induction on m. We first show that the result holds for $m = 4$.

The drawing of $P(4) + \{w\}$ in Figure 2 shows that its crossing number is at most 4. It remains to show the reverse inequality. We prove this by contradiction. Assume that there is an optimal drawing D of $P(4) + \{w\}$ with fewer than 4 crossings.

We claim that no two vertex-disjoint 4-cycles of $P(4)$ cross each other.

Suppose the contrary and let C_4' and C_4'' be two vertex-disjoint 4-cycles which cross each other.

Case (i) No edges in $C_4' \cup C_4''$ cross more than once.

Then C_4' encloses at least one vertex of C_4'' and vice versa. Since every vertex of $P(4)$ is adjacent to w, the number of crossings in D is at least 4, a contradiction.

Case (ii) Some edges in $C_4' \cup C_4''$ cross at least two times.

Without loss of generality, assume that an edge, say xy of C_4' crosses two edges of C_4''. Delete the edges xy, wx, wy resulting in a graph H with $d_H(x) = 2 = d_H(y)$. Note that H is a subdivision of $P(3) + \{w\}$ which has crossing number 2 (see [7, Lemma 1]). But this implies that the number of crossings in D is at least 4, a contradiction.

We now show that any 4-cycle in D has no self-crossing. Suppose C is a 4-cycle with edges e_1 and e_2 which intersect each other. Clearly e_1 and e_2 are non-adjacent. Hence there is a 4-cycle C' containing e_1 but not e_2 and there is a 4-cycle C'' containing e_2 but not e_1. It is not difficult to show that $V(C') \cap V(C'') = \emptyset$. But this is a contradiction by the preceding claim (since C' and C'' are two vertex-disjoint 4-cycles crossing each other).

It follows from the above observation that the sub-drawing of $P(4)$ induced by D yields a plane drawing of $P(4)$ which divides the plane into six 4-faces. Hence w is in one of the 4-faces F. Since w is adjacent to every vertex of $P(4)$, the edges joining w and the vertices in $V(P(4)) - V(F)$ must cross the boundaries of F which means that D has at least 4 crossings. This contradiction proves that $cr(P(4) + \{w\}) = 4$.

Note that the drawing of $P(4) + \{w\}$ in Figure 2 can easily be generalized to obtain a drawing of $P(m) + \{w\}$ with at most m crossings.

Assume that $cr(P(m) + \{w\}) = m$ where $m \geq 4$.

Let D be a good drawing of $P(m+1) + \{w\}$. Assume that D has at most m crossings. By Lemma 1, there is a triangle of the form wa_ib_iw in $P(m+1) + \{w\}$ with an edge crossed in D. Now delete the edges wa_j, wb_j and a_jb_j from $P(m+1) + \{w\}$. The resulting graph is a subdivision of $P(m) + \{w\}$ drawn with fewer than m crossings, a contradiction. This proves the theorem. □

Lemma 2. *$cr(P(m) + \{x, y\}) \leq 2m$ for any natural number $m \geq 4$.*

Proof. We shall describe a drawing of $P(m) + \{x, y\}$ with $2m$ crossings. Draw the cycle $Z_1 = a_0a_1a_2...a_{m-1}a_0$ in the form of a cycle on the plane. Draw $Z_2 = b_0b_1b_2...b_{m-1}b_0$ also in the form of a cycle with Z_2 enclosing Z_1 and join all the edges $a_ib_i, i = 0, 1, 2, ..., m-1$ to obtain a plane drawing of $P(m)$.

Now put x in the region enclosed by Z_1 and join x to all the vertices in Z_1 (with no crossing). Now join x and b_i with an edge so that xb_i crosses only the edge $a_ia_{i+1}, i = 0, 1, 2, ..., m-1$. Here $a_m = a_0$. See Figure 2 for the case $m = 4$ with $x = w$. Put y on the unbounded region of $P(m)$ and join y to all vertices in Z_2. Then join y and a_i with an edge so that ya_i crosses

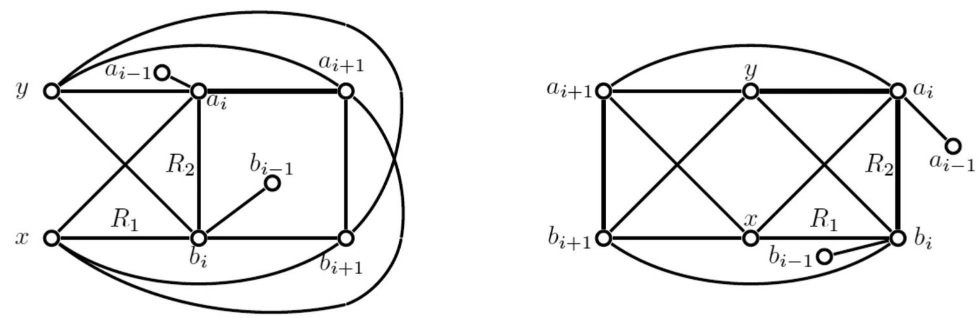

Figure 3. Some drawings of D'.

only the edge $b_ib_{i-1}, i = 0, 1, 2, ..., m-1$. Here $b_{-1} = b_{m-1}$. The resulting drawing has only $2m$ crossings. □

The following lemma has been proved in [9]. Hence we omit the proof.

Lemma 3. ([9], Lemma 4) $cr(P(4) + \{x, y\}) = 8$.

Lemma 4. *In any good drawing of $P(m) + \{x, y\}$ where $m \geq 4$, there is a 4-cycle of the form xa_iyb_ix having at least two crossings.*

Proof. Assume on the contrary that every 4-cycle of the type xa_iyb_ix has at most one crossing.

Consider the triangles $\triangle_x = xa_ib_ix$ and $\triangle_y = ya_ib_iy$.

We assert that x is not enclosed in $\triangle_y$. To see this, suppose the contrary. Let S_1, S_2 be the regions bounded by $\triangle_x, ya_ixb_iy$ respectively, and let S_3 denote the exterior region of $\triangle_y$.

Note that if there is a pair of vertices a_j, b_j $(j \neq i)$ that lie in S_1 or in S_3, then xa_jyb_jx is a 4-cycle with at least 2 crossings (which is impossible). Also, if a_j is in S_3 and b_j is in S_1, then xa_jyb_jx is a 4-cycle with at least 2 crossings.

Now, suppose a_j is in S_3 and b_j is in S_2. Let a_{j+1} be a neighbor of a_j and $a_{j+1} \neq a_i$. Note that this is possible because the degree of a_j is 5. Apply the same argument to the pair of vertices $\{a_{j+1}, b_{j+1}\}$ (which plays the role of $\{a_j, b_j\}$), it follows that a_{j+1} is in S_3 and b_{j+1} is in S_2. Since a_j is adjacent to b_j, and a_{j+1} is adjacent to b_{j+1}, there are at least 2 crossings on xa_iyb_ix.

Hence we assume no vertex (other than a_i, b_i, y) is in S_3 (so that T^x crosses no boundary of $\triangle_y$). If a_j is in S_1 and b_j is in S_2, we proceed as in the proceeding case to obtain a similar contradiction. Hence assume that all the vertices are in S_2. If the path $a_ia_{i-1}b_{i-1}b_i$ lies completely in S_2, then it separates x from y. As such, $xa_{i+1}yb_{i+1}x$ is a 4-cycle that intersects with the edges of this path at least two times. If the path $a_ia_{i-1}b_{i-1}b_i$ does not lie completely in S_2, then a_ia_{i-1} crosses the edge xb_i; in this case the path $a_ia_{i+1}b_{i+1}b_i$ must lie completely in S_2 (otherwise a_ia_{i+1} crosses the edge xb_i giving 2 crossings on xa_iyb_ix) and this path separates x from y yielding a 4-cycle of the form xa_jyb_jx with 2 crossings for some $j \in \{i-1, i+1\}$. This proves the assertion.

Likewise y is not enclosed by $\triangle_x$.

(i) First, we consider the case where no edge of Q is crossed by an edge of Q. Let vw be an edge of $P(m) - Q$ which crosses xa_i, and let C^* be a cycle containing vw. Then one end of vw, say w is enclosed either by $\triangle_x$ or by $\triangle_y$. If w is adjacent to neither a_i nor b_i, then C^* crosses Q at least twice, a contradiction. Hence we assume that w is adjacent to a_i. As such, $w \in \{a_{i-1}, a_{i+1}\}$, say $w = a_{i-1}$.

Since w is enclosed by either $\triangle_x$ or $\triangle_y, a_ib_i$ is crossed by either yw or by xw.

If b_{i-1} is enclosed by Q, then $xa_{i-1}yb_{i-1}x$ is a 4-cycle of the same type having 2 crossings, a contradiction. Hence we assume that b_{i-1} is not enclosed by Q. But this forces the 4-cycle xa_iyb_ix to have at least 2 crossings.

(ii) Next, suppose there is a crossing on the edges of the 4-cycle $Q = xa_iyb_ix$.

First we consider the case where there is an edge, say xa_i of Q which crosses an edge yb_i of Q.

Since y is not enclosed by $\triangle_x, yb_i$ partitions $\triangle_x$ into two regions R_1, R_2 with R_1 bounded by b_ix, a_ix, b_iy and R_2 bounded by b_ia_i, a_ix, b_iy. Note that no vertex in $Z_1 = a_0a_1 \cdots a_{m-1}a_0$ is enclosed by R_k for any $k = 1, 2$ otherwise there is a 4-cycle of the type xa_jyb_jx with 2 or more crossings. Likewise no vertex in $Z_2 = b_0b_1 \cdots b_{m-1}b_0$ is enclosed by R_k for any $k = 1, 2$.

With this restriction, we consider the sub-drawing D' on the subgraph induced by the edges of the 4-cycles $Q, Q_1 = xa_{i+1}yb_{i+1}x$ and $Q_{-1} = xa_{i-1}yb_{i-1}x$. In view of the observation in case (i), we may assume that any 4-cycle of the form xa_jyb_jx has a self-crossing. Note that among the various sub-drawings of D' on $Q \cup Q_1$, there is a vertex $z \in \{a_{i-1}, b_{i-1}\}$ such that z is separated from x (or from y) (see Figure 3 for some examples of drawings of D'). As such, the edge zx (or zy) must be crossed (in addition to the self-crossing in Q_{-1}), yielding 2 crossings on the 4-cycle $xa_{i-1}yb_{i-1}x$.

We now assume that the edges of any 4-cycle of the type xa_iyb_ix have no crossing. Since $P(m) + \{x, y\}$ is non-planar, for some i, the edge a_ib_i must be crossed by some edge, say vw.

We can assume without loss of generality that $\triangle_x$ contains one end, say w of vw. Then the edge wy (which belongs to the 4-cycle $xwyvx$) crosses a_ib_i. By the assumption in the preceding paragraph, $\{v, w\} \neq \{a_j, b_j\}$ for any j. If $\{v, w\} = \{a_j, a_{j+1}\}$ or $\{v, w\} = \{b_j, b_{j+1}\}$, then it is easy to see that there is a 4-cycle xa_jyb_jx with at least two crossings, a contradiction. □

Theorem 2. $cr(P(m) + \{x, y\}) = 2m$ *for any natural number* $m \geq 4$.

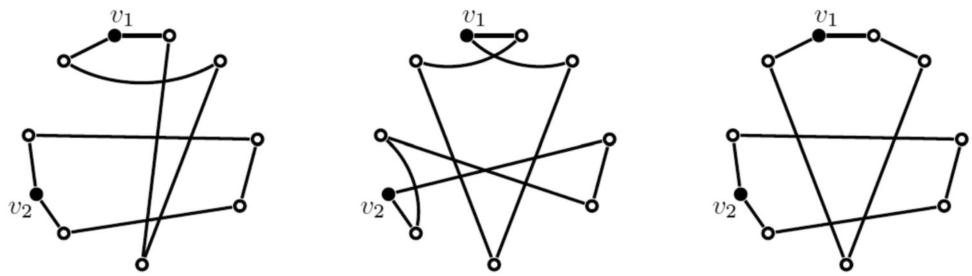

Figure 4. Some drawings of $Z_1 \cup Z_2$.

Proof. By Lemma 2, we have $cr(P(m) + \{x, y\}) \le 2m$.

We prove the reverse inequality by induction on m. By Lemma 3, the result is true for $m = 4$. Assume that the result is true for $m = k$ where $k \ge 4$. Suppose there is a good drawing of $P(k+1) + \{x, y\}$ with fewer than $2(k+1)$ crossings.

By Lemma 4, there exists a 4-cycle xa_iyb_ix with at least two crossings. Now delete all the edges of these 4-cycles together with the edge a_ib_i, the resulting graph is a subdivision of the graph $P(k) + \{x, y\}$ drawn with fewer than $2k$ crossings. This contradiction proves the result. □

3. Cartesian product of graphs

In [1], Beineke and Ringeisen proved that $cr(C_4 \square C_n) = 2n$. Since $P(m) \square P_2$ is isomorphic to $C_m \square C_4$, we have $cr(P(m) \square P_2) = 2m$ where $m \ge 4$.

For each $i = 1, 2, \ldots, n$, let $P_i(m)$ denote the i-th copy of $P(m)$ in $P(m) \square P_n$ and $E_i = E(P_i(m))$. Also, let L_j denote the set of all edges joining $P_j(m)$ and $P_{j+1}(m), j = 1, 2, \ldots, n-1$.

Suppose D is a good drawing of $P(m) \square P_n$ and suppose $H, J \subseteq E(P(m) \square P_n)$. Let $cr_D(H, J)$ denote the number of crossings in D which are made on H by J. In particular, $cr_D(E_i, E_i)$ stands for the number of self-crossings among the edges in E_i. On the other hand, let $cr_D(E_i)$ denote the total number of crossings in D on the edges in E_i.

Lemma 5. *Let D be a good drawing of $P(5) \square P_n$. If $cr_D(E_i, L_{i-1}) = 0$ or $cr_D(E_i, L_i) = 0$, then $cr_D(E_i, E_i) \ge 6$.*

Proof. It suffices to prove the following statement.

Let D be a good drawing of $P(5)$ with all its vertices lying in the same region. Then D has at least 6 crossings.

Since all the vertices are in the same region, we may assume without loss of generality, as in the proof of Lemma 1 that the vertices of $P(5)$ are all on the boundary of a convex 10-gon P and that any edge of $P(5)$ is either on the boundary or in the "interior" region of P. Suppose the vertices of P are $x_0, x_1, \ldots, x_9$ (in cyclic order).

Let e be an edge of Z_i not on the boundary of $P, i \in \{1, 2\}$. Call e a *separating diagonal edge* (*s.d.edge*) on Z_{3-i} if the vertices of Z_{3-i} are being separated by e into two different segments of P.

If Z_i has at least three *s.d.* edges, then, as in the proof of Lemma 1, D has at least 6 crossings since each *s.d.* edge is crossed by at least two edges of Z_{3-i}. Hence we assume that Z_i has at most two *s.d.* edges. It is easy to see that neither Z_1 nor Z_2 can have precisely one *s.d.* edge.

Case (1): Z_i has no *s.d.* edge.

In this case, we may assume that the vertices of Z_1 lay on the segment $x_0, x_1, \ldots, x_4$ (which also denote the vertices of Z_1) while those of Z_2 lay on the segment $x_5, x_6, \ldots, x_9$. We shall show that the number of crossings on the edges of Z_i is at least 3 for each $i = 1, 2$.

Suppose x_0x_j is an edge of Z_1.

If $j = 4$, then each spoke of $P(5)$ incident to a vertex in $\{x_1, x_2, x_3\}$ must cross the edge x_0x_4.

If $j = 3$, then each spoke of $P(5)$ incident to a vertex in $\{x_1, x_2\}$ must cross the edge x_0x_3. Moreover the edge joining x_4 to a vertex in $\{x_1, x_2\}$ must cross the edge x_0x_3.

So assume that $j = 2$. If x_4 is adjacent to x_1, the situation is similar to the case $j = 3$. Hence x_4 is adjacent to x_2 and x_3. But this also implies that x_1x_3 is an edge of Z_1 which is crossed by x_0x_2, x_4x_2. Also, the spoke incident to x_2 must cross the edge x_1x_3.

By applying the same arguments to Z_2 we obtain the required conclusion.

Case (2): Z_i has only two *s.d.* edges.

Since the two *s.d.* edges e_1, e_2 of Z_1 give raise to at least 4 crossings on $\{e_1, e_2\}$, it remains to show that some edge of Z_i is crossed by some spoke of $P(5)$ for each $i = 1, 2$.

For this purpose, we observe that in any drawing of Z_i with only two *s.d.* edges (there are in fact only 12 such drawings as is depicted in Figure 5), there is a vertex v_i of Z_i whose spoke incident to it crosses some edge of $Z_i, i = 1, 2$. Figure 4 illustrates some of these drawings. We conclude that D has at least 6 crossings in this case.

This completes the proof. □

Lemma 6. *Suppose D is an optimal drawing of $P(5) \square P_n$ where $n \ge 3$. If $P_i(5)$ and $P_j(5)$ cross each other, $i \ne j$, then $cr_D(E_i, E_j) \ge 4$.*

Proof. Clearly every edge in $P(5)$ is an edge of some cycle in $P(5)$. Let Z be a cycle in $P_i(5)$ that crosses some edge of $P_j(5)$.

If Z encloses two or more vertices of $P_j(5)$, then clearly there are at least 4 crossings on the boundary of Z.

Suppose Z encloses only one vertex, say v of $P_j(5)$. Let Z' be a cycle in $P_j(5)$ containing an edge incidents to v. If Z' encloses a vertex of Z, then clearly $cr_D(E_i, E_j) \ge 4$. Hence we assume that Z' encloses no vertex of Z. We can further assume that any cycle in $P_j(5)$ which contains an edge incident to the vertex v encloses no vertex of Z. But this means that the three edges e_1, e_2, e_3 of $P_j(5)$ incident to v are all crossed by an edge e of Z (see Figure 6). Since the degree of

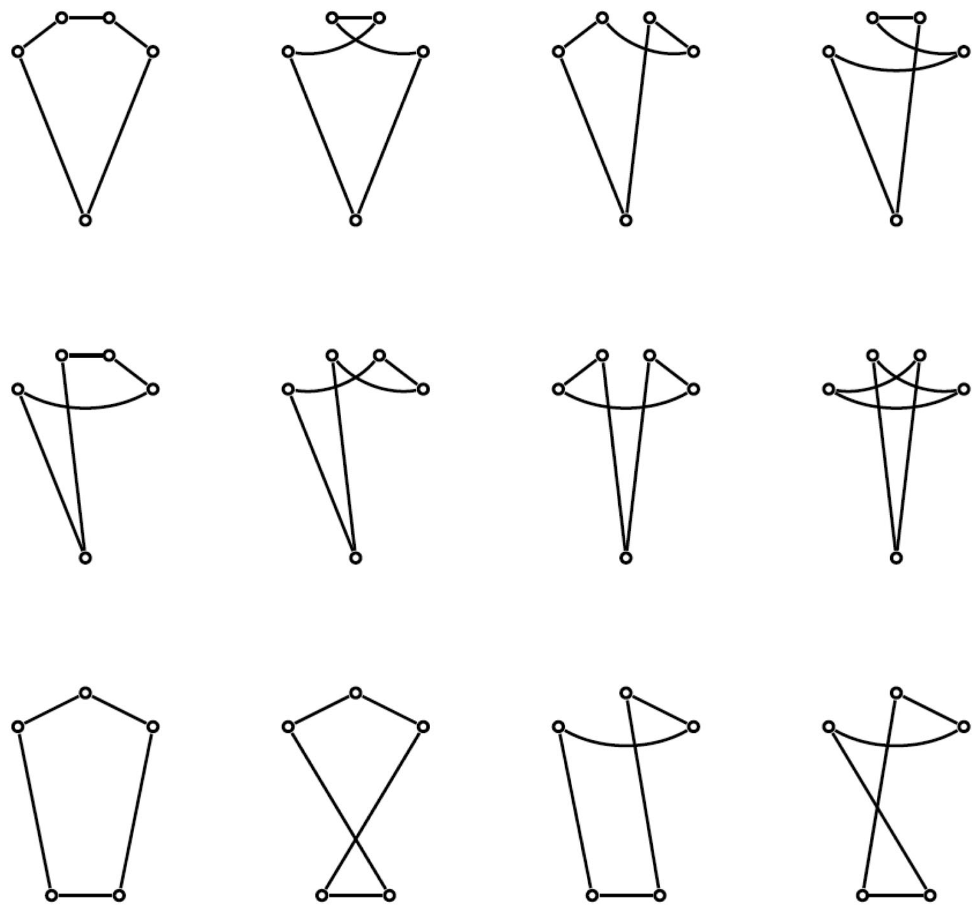

Figure 5. A list of all 5 -cycles Z_i with only 2 s.d. edges.

v in $P(5)\Box P_n$ is either 4 or 5, by detouring the edge e as shown in Figure 6, we obtain a drawing of $P(5)\Box P_n$ with fewer crossings. This contradicts the optimality of D. □

Lemma 7. *Let D be an optimal drawing of $P(5)\Box P_n$ where $n \geq 3$. Then either $cr_D(E_i) \geq 10$ for some $1 \leq i \leq n$ or else $cr_D(P(5)\Box P_n) \geq 10(n-1)$.*

Proof. Suppose each copy of $P(5)$ in $P(5)\Box P_n$ has at most 9 crossings.

To establish $cr_D(P(5)\Box P_n) \geq 10(n-1)$, we shall first show that

(1) any two distinct copies of $P(5)$ in $P(5)\Box P_n$ do not cross each other.

Suppose on the contrary that $P_i(5)$ crosses $P_j(5), i \neq j$.

Case (1): Suppose $2 \leq i \leq n-1$ and $1 \leq j \leq n$.

By Lemma 6, $cr_D(E_i, E_j) \geq 4$.

(i) Suppose $j \in \{i-1, i+1\}$.

Without loss of generality assume that $j = i-1$. If $P_{i+1}(5)$ does not cross $P_i(5)$, then we contract $P_{i+1}(5)$ to a single vertex w (to get a subgraph isomorphic to $P_i(5) + \{w\}$). Since $cr(P_i(5) + \{w\}) = 5$ by Theorem 1, we have $cr_D(E_i, E_i \cup L_i) \geq 5$.

If $cr_D(E_i, L_{i-1}) \geq 1$, we have $cr_D(E_i) \geq cr_D(E_i, E_{i-1}) + cr_D(E_i, E_i \cup L_i) + cr_D(E_i, L_{i-1}) \geq 4 + 5 + 1$ a contradiction.

If $cr_D(E_i, L_{i-1}) = 0$, we have $cr_D(E_i) \geq cr_D(E_i, E_i) + cr_D(E_i, E_{i-1}) \geq 6 + 4$ (by Lemma 5), a contradiction.

On the other hand, if $P_{i+1}(5)$ crosses $P_i(5)$, then $cr_D(E_i, E_{i+1}) \geq 4$ by Lemma 6. Further, if $cr_D(E_i, L_{i-1} \cup L_i) \geq 2$, then

$$\begin{aligned} cr_D(E_i) &\geq cr_D(E_i, L_{i-1} \cup L_i) + cr_D(E_i, E_{i-1}) + cr_D(E_i, E_{i+1}) \\ &\geq 2 + 4 + 4 \end{aligned}$$

a contradiction. If $cr_D(E_i, L_{i-1} \cup L_i) \leq 1$, then either $cr_D(E_i, L_{i-1}) = 0$ or $cr_D(E_i, L_i) = 0$. In either case, $cr_D(E_i, E_i) \geq 6$ by Lemma 5 and this leads to

$$\begin{aligned} cr_D(E_i) &\geq cr_D(E_i, E_i) + cr_D(E_i, E_{i-1}) + cr_D(E_i, E_{i+1}) \\ &\geq 6 + 4 + 4 \end{aligned}$$

again a contradiction.

(ii) Suppose $j \notin \{i-1, i+1\}$.

Note that $P_i(5)$ is crossed by at most one of its neighboring copies. This is true otherwise $cr_D(E_i, E_k) \geq 4$ for each $k \in \{i-1, i+1, j\}$ and we have $cr_D(E_i) \geq 12$.

Contract a neighboring copy of $P_i(5)$ which does not cross with $P_i(5)$ to a single vertex.

If the other neighboring copy of $P_i(5)$ crosses $P_i(5)$, then we have $cr_D(E_i) \geq 4 + 4 + 5$ (by Lemma 6 and Theorem 1).

If the other neighboring copy of $P_i(5)$ does not cross $P_i(5)$, we contract $P_{i-1}(5)$ (and $P_{i+1}(5)$) into a single vertex x (respectively y) to get a subgraph isomorphic to $P(5) + \{x, y\}$ which has crossing number 10 by Theorem 2. As such, we have a contradiction because

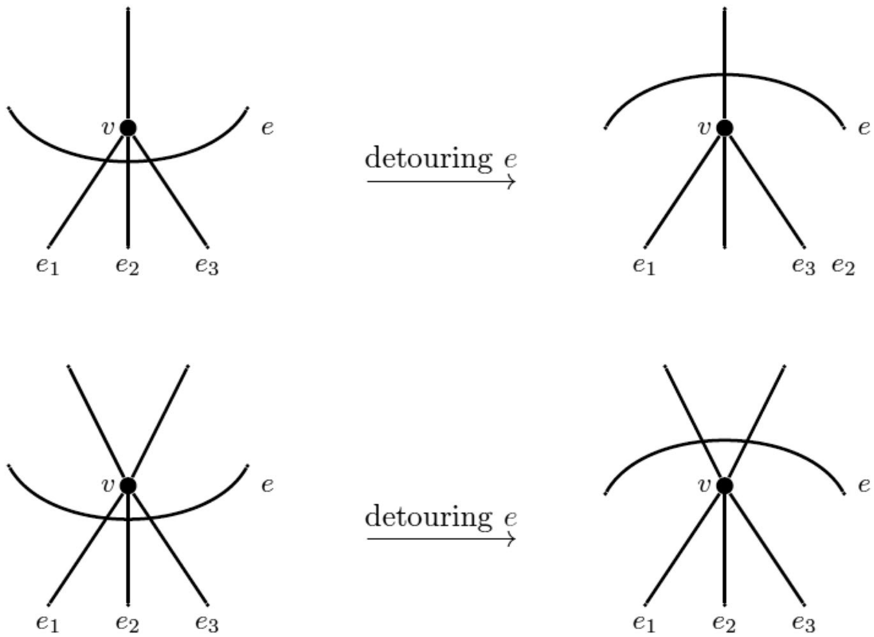

Figure 6. Detouring an edge.

$$cr_D(E_i) \geq cr_D(E_i, E_i) + cr_D(E_i, L_{i-1}) + cr_D(E_i, L_i) \geq cr(P(5) + \{x, y\}) = 10.$$

Case (2): Suppose $i = 1$ and $j = n$.

By Lemma 6, $cr_D(E_1, E_n) \geq 4$.

Let Z' be a cycle in $P_1(5)$ that intersects with a cycle Z'' of $P_n(5)$. Since D is a good drawing, we may assume without loss of generality that Z' encloses a vertex v of $P_n(5)$. As such, the edge in L_{n-1} incident to v must cross the boundary of Z'.

By the result in Case (1), $P_2(5)$ does not cross with $P_1(5)$. Contract $P_2(5)$ to a single vertex w. Since $cr(P_1(5) + \{w\}) \geq 5$, we have $cr_D(E_1) \geq 4 + 5 + 1$, a contradiction.

This shows that $P_i(5)$ does not cross $P_j(5)$ for $i \neq j$.

Next we show that

(2) $cr_D(E_i, L_j) = 0$ if $j \notin \{i-1, i\}$.

Suppose the contrary and let xy be an edge in L_j that crosses $P_i(5)$. Suppose $x \in V(P_j(5))$ and $y \in V(P_{j+1}(5))$. Consider a sub-drawing D' of D induced by the vertices of $P_i(5)$.

(a) Suppose x and y are in different regions of D'.

This means that all the vertices of $P_j(5)$ are in the same region as x while all the vertices of $P_{j+1}(5)$ are in the same region as y (because $P_i(5)$ does not cross other copies of $P(5)$). But this means that the edges in L_j cross the boundary of the region at least 10 times (yielding $cr_D(E_i) \geq 10$), a contradiction.

(b) Suppose x and y are in the same region of D'.

Let Z be a cycle containing xy which crosses E_i. Since D is a good drawing, Z must enclose some vertex of $P_i(5)$ (otherwise xy crosses the same edge of $P_i(5)$ at least twice).

Suppose Z encloses only one vertex v of $P_i(5)$. This means that the edge xy crosses three edges of $P_i(5)$ incident to v. By detouring the edge xy as shown in Figure 6, we obtain a drawing of $P(5)\square P_n$ with fewer crossings, contradicting the optimality of D. Hence assume that Z encloses at least two vertices of $P_i(5)$. As such, we have $cr_D(Z, E_i) \geq 4$. Since no neighboring copies of $P(5)$ cross each other (by the result in case (1)), we contract a neighboring copy of $P_i(5)$ to a single vertex w (to get $cr(P_i(5) + \{w\}) \geq 5$) so that $cr_D(E_i, L_{i-1}) \geq 5$ or $cr_D(E_i, L_i) \geq 5$. By using an argument similar to Case (1)(i), we obtain $cr_D(E_i) \geq 10$, a contradiction.

We shall now show that $cr_D(P(5)\square P_n) \geq 10(n-1)$.

For this purpose, let Q_i denote the subgraph of $P(5)\square P_n$ induced by the set of vertices in $P_{i-1}(5) \cup P_i(5) \cup P_{i+1}(5)$ for each $i = 2, \ldots, n-1$. Define

$$f(Q_i) = cr_D(E_i, L_{i-1} \cup L_i) + cr_D(L_{i-1}, L_i) + cr_D(E_i, E_i).$$

It is easy to see that a crossing in D contributes at most one to the sum $F = \sum_{i=2}^{n-1} f(Q_i)$. Let D_i denote the subdrawing of D that corresponds to Q_i. By observation (1), we have $cr_{D_i}(E_r, E_s) = 0$ for any distinct $r, s \in \{i-1, i, i+1\}$. Also, by observation (2), we have $cr_{D_i}(L_{i-1}, E_{i+1}) = 0$ and $cr_{D_i}(L_i, E_{i-1}) = 0$. As such, by contracting $P_{i-1}(5)$ and $P_{i+1}(5)$ into two vertices x_i and y_i respectively, the resulting graph is isomorphic to $P_i(5) + \{x_i, y_i\}$ which has crossing number 10 by Theorem 2. This means that $f(Q_i) \geq cr_D(P_i(5) + \{x_i, y_i\}) \geq 10$ and that $F = \sum_{i=2}^{n-1} f(Q_i) \geq 10(n-2)$.

Finally consider the subgraph H of $P(5)\square P_n$ induced by the set of vertices in $P_1(5) \cup P_2(5)$. By observation (1), we can contract $P_2(5)$ (of H) into a single vertex w to obtain a graph isomorphic to $P(5) + \{w\}$ which has crossing number 5 by Theorem 1. This means that, in D, there are at least 5 crossings on E_1. By symmetry (since we can just relabel the subscripts on $P_i(5)$ in reverse order), there are at least 5 crossings on E_n. None of these crossings are counted in F. Consequently, the number of crossings in D is at least $F + 10 = 10(n-1)$ and the proof is complete. □

Theorem 3. *$cr(P(5)\square P_n) = 10(n-1)$ for all natural number $n \geq 1$.*

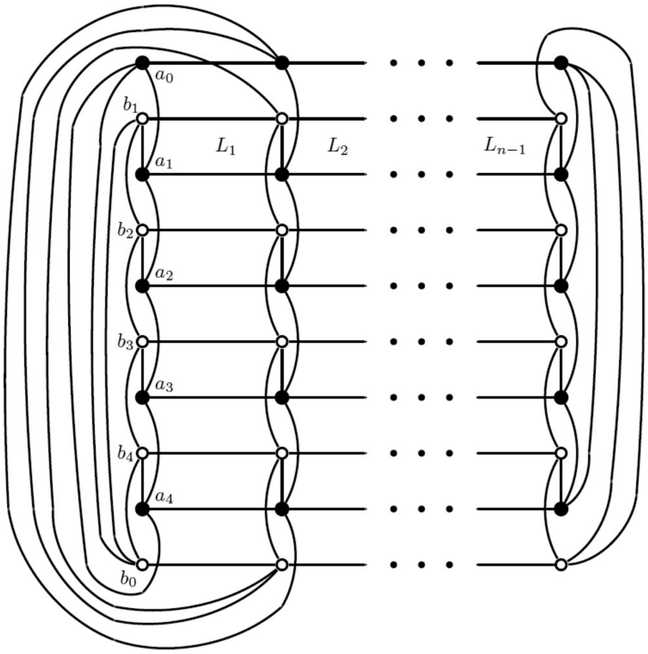

Figure 7. $P(5)\square P_n$ with $10(n-1)$ crossings.

Proof. Figure 7 depicts a drawing of $P(5)\square P_n$ with $10(n-1)$ crossings where $n \geq 3$. This proves the upper bound. It remains to show that $cr(P(5)\square P_n) \geq 10(n-1)$.

Since $P(5)$ is a planar graph and that $P(5)\square P_2$ is isomorphic to $C_5\square C_4$ as remarked earlier, the result is true for $n \leq 2$.

Assume that $cr(P(5)\square P_k) \geq 10(k-1)$ where $k \geq 3$ and that there is a drawing D of $P(5)\square P_{k+1}$ with fewer than $10k$ crossings. By Lemma 7, $P(5)\square P_{k+1}$ contains a copy $P_i(5)$ with at least 10 crossings. By deleting all edges of $P_i(5)$ in $P(5)\square P_{k+1}$, the resulting graph is either a subdivision of $P(5)\square P_k$ or else contains the subgraph $P(5)\square P_k$ each with fewer than $10(k-1)$ crossings, a contradiction. □

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