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Folding trees gracefully

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ABSTRACT
When a graceful labeling of a bipartite graph assigns the smaller labels to the vertices of one of the stable sets of the graph, the assignment is called an α-labeling. Any graph that admits such a labeling is an α-graph. In this work we extend the concept of vertex amalgamation to generate a new class of α-graphs obtained by a sequence of k-vertex amalgamations of t copies of an α-tree. This procedure is also applied to any collection of α-trees such that any pair of trees in this collection have stable sets with the same cardinalities. We also use this idea on other types of α-graphs. In addition, we present a family of α-trees of even diameter formed with four caterpillars of the same size.

KEYWORDS
α-labeling; graceful labeling; k-vertex amalgamation

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1. Introduction

A *difference vertex labeling* of a graph G of size n is an injective mapping f from $V(G)$ into a set N of nonnegative integers, such that every edge uv of G has assigned a *weight* defined by $|f(u)-f(v)|$. The labeling f is called *graceful* when $N=\{0,1,...,n\}$ and the set of induced weights is $\{1,2,...,n\}$. In this case, G is called a *graceful graph*. Let G be a bipartite graph and $\{A, B\}$ be the natural bipartition of $V(G)$, we refer to A and B as the *stable sets* of G and assume that $|A|=a$ and $|B|=b$. A *bipartite labeling* of G is an injection $f: V(G) \to \{0,1,...,s\}$ for which there is an integer λ, named the *boundary value* of f, such that $f(u) \leq \lambda < f(v)$ for every $(u,v) \in A \times B$, that induces n different weights. This is an extension of the definition given by Rosa and Širáň [7]. From the definition we may conclude that $s \geq |E(G)|$; furthermore, the labels assigned by f on the vertices of A and B are in the integer interval $[0,\lambda]$ and $[\lambda+1,s]$, respectively. If $s=n$, the function f is an *α-labeling* and G is an *α-graph*. If f is an α-labeling of a tree and $f^{-1}(0) \in A$, then its boundary value is $\lambda=a-1$.

Suppose that $f: V(G) \to \{0,1,...,n\}$ is a graceful labeling of a graph G of size n:

- $g: V(G) \to \{c, c+1, \ldots, c+n\}$, defined for every $v \in V(G)$ and $c \in \mathbb{N}$ as $g(v)=c+f(v)$, is the *shifting* of f in c units. Note that this labeling preserves the weights induced by f.
- $\hat{f}: V(G) \to \{0,1,\ldots,n\}$, defined for every $v \in V(G)$ as $\hat{f}(v)=\lambda-f(v)$ if $f(v) \leq \lambda$, and $\hat{f}(v)=n+\lambda+1-f(v)$ if $f(v)>\lambda$, is the *reverse* labeling of f. Thus, $\hat{f}$ is also an α-labeling with boundary value λ.
- $g: V(G) \to \{0,1,\ldots,n+d-1\}$, defined for every $v \in V(G)$ and $d \in \mathbb{N}$ as $g(v)=f(v)$ if $f(v) \leq \lambda$ and $g(v)=f(v)+d-1$ if $f(v)>\lambda$, is the *d-graceful labeling* of G obtained from f. The labels assigned by g on the stable sets of $V(G)$ are in the intervals $[0,\lambda]$ and $[\lambda+d, n+d-1]$ and the set of induced weights is $\{d, d+1, \ldots, n+d-1\}$.

For example, let f be an α-labeling of a tree T of size n with boundary value λ. Suppose that f is transformed into a d-graceful labeling shifted c units. Then the elements of A are labeled with the integers in $[c, \lambda+c]$, the elements of B are labeled with the integers in $[c+\lambda+d, c+n++d-1]$, and the induced weights form the interval $[d, n+d-1]$.

In Section 2, we study α-labelings for graphs that result of k-vertex amalgamations of smaller α-graphs. Section 3 is devoted to α-labelings of trees, there we prove the existence of an α-labeling for trees obatined by amalgamating caterpillars. The reader interested in graph labelings is refered to Gallian' survey [4] for more information about the subject. In this paper, we follow the notation and terminology used in [3] and [4].

2. Folding α-trees

For $i=1, 2$, let G_i be a graph of order n_i and size m_i. A graph G, of order n_1+n_2-k and size m_1+m_2, is said to be a *k-vertex amalgamation* (or *strong vertex amalgamation*) of G_1 and G_2 if it is obtained identifying k independent vertices of G_1 with k independent vertices of G_2. We use here strong vertex amalgamation of α-graphs to construct new classes of α-graphs.

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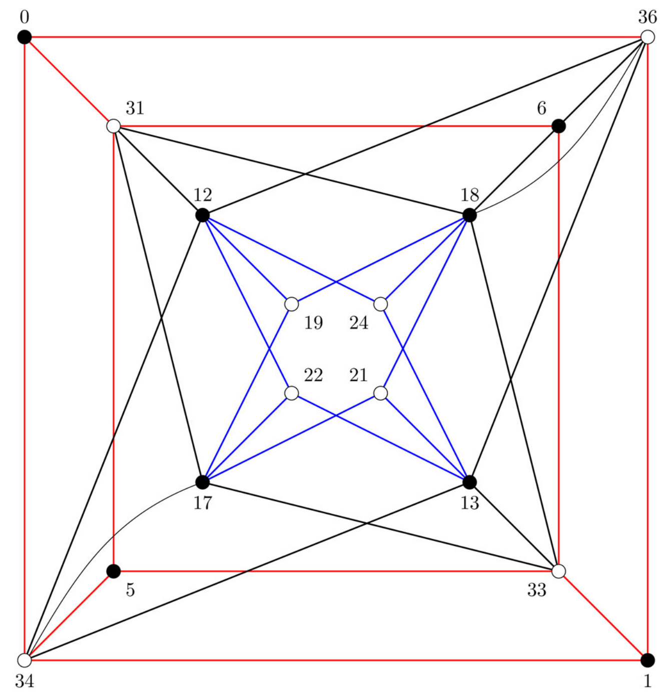

Figure 1. α-labeling of the 3-fold of Q_3.

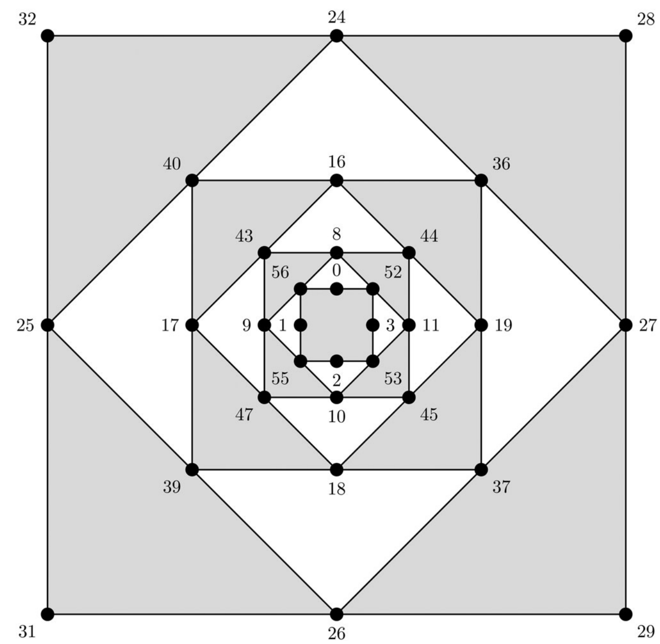

Figure 2. α-labeling of the 7-fold of C_8.

Suppose That G is a bipartite graph of order n and size m, with stable sets $A = \{u_1, u_2, \ldots, u_a\}$ and $B = \{v_1, v_2, \ldots, v_b\}$. Let $G_1, G_2, \ldots, G_t$ be disjoint copies of G with $A_i = \{u_1^i, u_2^i, \ldots, u_a^i\}$ and $B_i = \{v_1^i, v_2^i, \ldots, v_b^i\}$. If for every even value of $i \in \{1, 2, \ldots, t\}$, the vertices of B_i are identified with the vertices of B_{i-1}, in such a way that $v_j^i = v_j^{i-1}$ for every $1 \leq j \leq b$, and the vertices of A_i are identified with the vertices of A_{i+1}, in such a way that $u_j^i = u_j^{i+1}$ for every $1 \leq j \leq a$, then we obtain a bipartite graph of size tm and order $\frac{tn}{2} + a$ when t is even or $\frac{(t+1)n}{2}$ when t is odd. We call this graph the *t-fold* of G. We claim that the t-fold of G is an α-graph when G is an α-graph.

Notice that when t is odd, there is only one t-fold of G; that is, the t-fold of G is independent of the stable set of G chosen to be B. On the other hand, when t is even, there are, in general, two non-isomorphic t-folds of G, depending on the stable set chosen to be B. When G is vertex transitive, this selection is irrelevant and there is only one t-fold of G for any value of t.

Theorem 1. *If G is an α-graph, then any t-fold of G is an α-graph.*

Proof. Suppose that G is an α-graph of order n and size m with stable sets $A = \{u_1, u_2, \ldots, u_a\}$ and $B = \{v_1, v_2, \ldots, v_b\}$. Let f be an α-labeling of G with boundary value λ such that $f^{-1}(0) \in A$. Consider t copies of G, each labeled using f. For every value of $i \in \{1, 2, \ldots, t\}$, let f_i be the d_i-graceful labeling of G_i, obtained from f, shifted c_i units, where $d_i = 1 + m(t - i)$ and $c_i = m\lfloor \frac{i}{2} \rfloor$.

Thus, the set of weights induced by f_i on the edges of G_i is $[d_i, d_i + m - 1]$. This implies that the set of weights on the edges of the t-fold of G is

$$\bigcup_{i=1}^{t} [d_i, d_i + m - 1] = \bigcup_{i=1}^{t} [1 + m(t - i), 1 + m(t - i) + m - 1]$$
$$= \bigcup_{i=1}^{t} [1 + m(t - i), m(t + 1 - i)]$$
$$= [1, \ mt]$$

Suppose that $r_1 < r_2 < \ldots < r_a$ and $\rho_1 < \rho_2 < \ldots < \rho_b$ are the labels assigned by f on the vertices of A and B, respectively. Thus, f_i assigns the labels $r_1 + c_i, r_2 + c_i, \ldots, r_a + c_i$ to the elements of A_i and the labels $\rho_1 + d_i - 1 + c_i, \rho_2 + d_i - 1 + c_i, \ldots, \rho_b + d_i - 1 + c_i$ to the elements of B_i. Note that when i is even, $i = 2s$, the labels on the vertices of B_i are $\rho_j + m(t - s), 1 \leq j \leq b$, and the labels on the vertices of B_{i-1} are $\rho_j + m(t - s)$, as well. Similarly, the labels on the vertices of A_i and A_{i+1} are $r_1 + ms, r_2 + ms, \ldots, r_a + ms$. Therefore, the vertices of the t-fold of G are labeled with integers from $[0, mt]$ that induce the weights $1, 2, \ldots, mt$. Since the labelings of the G_i are bipartite, the final labeling is an α-labeling with boundary value $m + \lfloor \frac{t}{2} \rfloor + \lambda$. Hence, the t-fold of G is an α-graph. □

In Figure 1, we show an α-labeling of the 3-fold of the hypercube Q_3.

When the graph G in Theorem 1 is the path P_n, with $n \geq 4$, the t-fold of G results in a graph that contains as a subgraph a convex Eulerian polyomino, the induced labeling of this polyomino can be transformed into the α-labeling used by Acharya [1] to prove that all convex Eulerian polyominoes are arbitrarily graceful. Another nice family of α-graphs that can be obtained using Theorem 1 is the one containing the fractal type structure constructed by t-folding the cycle $C_{4m}, m \geq 2$. In Figure 2 we show an example of one of these structures constructed using C_8 folded 7 times.

Note that the α-labeling of the t-fold of an α-graph G can be also obtained using the labeling scheme of the weak tensor product of G and P_{t+1}, introduced by Snevily [8] and extended by López and Muntaner-Batle [5]. The labeling technique used in the proof of Theorem 1 is applied to

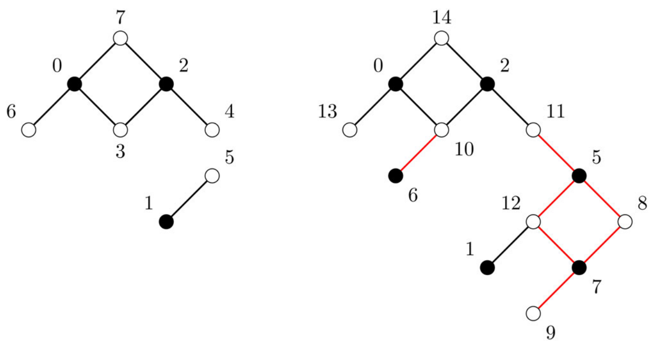

Figure 3. A 3-vertex amalgamation of an α-graph of size 7.

other types of folded graphs that cannot be explained using the weak tensor product.

In the next proposition we prove that for any $1 \le k \le b$, there is an α-graph G that results of a k-vertex amalgamation of two copies of an α-tree. Note that when $k = 1$, G is the well-known vertex amalgamation (or one-point union) of two copies of T. The gracefulness of this type of graph was proven in [2]. When $k = b$, an α-labeling of G can be obtained using Theorem 1 with $t = 2$.

Proposition 1. *Let T be an α-tree of size m with one stable set of cardinality b. If $k \le b$ is a positive integer, then there is a k-vertex amalgamation of two copies of T that is an α-graph.*

Proof. For $i = 1$, 2, let T_i be a copy of T with stable sets A_i and B_i, where $|B_i| = b$. Suppose that f is an α-labeling of T such that $f^{-1}(0) \in A$; so, its boundary value is $\lambda = |A| - 1 = a - 1$. Let f_i be the d_i-graceful labeling of T_i, obtained from f, shifted c_i units, where

$$(d_i, c_i) = \begin{cases} (m+1, 0) & \text{if } i = 1, \\ (1, \lambda + k) & \text{if } i = 2. \end{cases}$$

Thus, the vertices in A_1 are labeled with the integers in $\{0, 1, \ldots, \lambda\}$, the vertices in B_1 are labeled with the integers in $\{\lambda + m + 1, \lambda + m + 2, \ldots, 2m\}$, the vertices in A_2 are labeled with the integers in $\{\lambda + k, \lambda + k + 1, \ldots, 2\lambda + k\}$, and the vertices in B_2 are labeled with the integers in $\{2\lambda + k + 1, 2\lambda + k + 2, \ldots, m + \lambda + k\}$.

Identifying the vertices of B_1 and B_2 labeled $2\lambda + k + 1, 2\lambda + k + 2, \ldots, m + \lambda + k$, we obtain a graph G that is a k-vertex amalgamation of T_1 and T_2. Since the weights on T_1 are $m + 1, m + 2, \ldots, 2m$ and on T_2 are $1, 2, \ldots, m$, we have that all edges of G have different weights; in addition, the vertices of T_1 and T_2, with the same label, were amalgamated, hence there is no repetition of labels. Observe that the labels assigned to the elements of $A_1 \cup A_2$ are smaller that those assigned to the elements of $B_1 \cup B_2$, so the number $2\lambda + k$ is the boundary value of the α-labeling of G, therefore G is an α-graph (Figure 3). □

Suppose that T is an α-labeled tree of size m, where $A = \{0, 1, \ldots, \lambda\}$ and $B = \{\lambda + 1, \lambda + 2, \ldots, m\}$ are considered ordered sets. By a *generalized t-fold* of T we mean a graph G obtained using t α-labeled copies of T, where for every even value of i, the copy T_i is merged with the copies T_{i-1} and T_{i+1} in such a way that the last k_i vertices in B_i are amalgamated with the first k_i vertices in B_{i-1}, and the last k'_i vertices in A_i are amalgamated with the first k'_i vertices in A_{i+1}. These amalgamations must be done in ascending order; for example, suppose that x is amalgamated with y and x' is amalgamated with y', if $x < x'$, then $y < y'$.

Theorem 2. *If T is an α-tree, then any generalized t-fold of T is an α-graph.*

Proof. Suppose that T is an α-tree of size m with stable sets A and B. Let $T_1, T_2, \ldots, T_t$ be disjoint copies of T. Assume that f is an α-labeling of T with boundary value λ. Without loss of generality, suppose that $f^{-1}(0) \in A$. Thus, the labels assigned to the vertices of A and B form the intervals $L_A = [0, \lambda]$ and $L_B = [\lambda + 1, m]$, respectively. For each $i \in \{1, 2, \ldots, t\}$, suppose that f is the initial labeling of T_i. The final labeling of T_i, denoted by f_i, is obtained by transforming f into a d_i-graceful labeling shifted c_i units, where $d_i = (t - i)m + 1$ and $c_i = \sum_{j=1}^{i} \xi_j$ with

$$\xi_i \in \begin{cases} [\lambda + 1, m] & \text{if } n \text{ is even,} \\ [0, \lambda] & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the labels assigned by f_i to the vertices of A_i and B_i are $L_{A_i} = [c_i, c_i + \lambda]$ and $L_{B_i} = [\lambda + c_i + d_i, m + c_i + d_i - 1]$. In addition, the weights induced by f_i on the edges of T_i form the interval $[(t - i)m + 1, (t - i)m + m]$. Therefore, $\cup_{i=1}^{t} [(t - i)m + 1, (t - i)m + m] = [1, tm]$.

Since $L_{A_i} = [c_i, c_i + \lambda]$ where $c_i = \sum_{j=1}^{i} \xi_j$, we have that for every feasible even value of i, $L_{A_i} \cap L_{A_{i+1}} = [c_{i+1}, c_i + \lambda]$ and $|L_{A_i} \cap L_{A_{i+1}}| = c_i + \lambda - c_{i+1} + 1 = \lambda + 1 - \xi_{i+1} = k'_i$. Thus, the number k'_i, of vertices shared by A_i and A_{i+1} is bounded by $1 \le k'_i \le \lambda + 1 = a$. Similarly, $L_{B_{i-1}} \cap L_{B_i} = [\lambda + c_{i-1} + d_{i-1}, m + c_i + d_i - 1]$ and $|L_{B_{i-1}} \cap L_{B_i}| = \xi_i - \lambda = k_i$. Since $\xi_i \in [\lambda + 1, m] = [\lambda + 1, a + b - 1]$, the number k_i of vertices shared by B_{i-1} and B_i is bounded by $1 \le k_i \le b$.

Identifying the vertices with the same label, we form the graph G that is a generalized t-fold of T. Since the labelings used on the T_i are bipartite, the boundary value of the labeling of G is the largest number in L_{A_i}, that is, $\lambda + c_t$. Therefore, G is an α-graph. □

In Figure 4, we show an example of an α-labeling for a generalized 5-fold of an α-tree of size 10, where $k_2 = 4, k'_2 = 2, k_4 = 5, k'_4 = 5, \xi_1 = 0, \xi_2 = 8, \xi_3 = 3, x_4 = 9$, and $\xi_5 = 0$.

Let T_1 and T_2 be two α-trees of size m. We say that T_1 and T_2 are *analogous* if $|A_1| = |A_2|$ and $|B_1| = |B_2|$. We use this concept to extend the result of Theorem 2 by replacing any number of copies of T with analogous trees.

Theorem 3. *If G is a generalized t-fold of an α-tree T, then any of the copies of T, used to construct G, can be replaced by any tree T' analogous to T, and the resulting graph is an α-graph.*

Proof. Since T and T' are analogous, there exist α-labelings f and g, of T and T', respectively, such that $f^{-1}(0) \in A$ and $g^{-1}(0) \in A'$. Let G be a generalized t-fold of T, suppose that the α-labeling of G has been obtained using the procedure in Theorem 2 and T_i is a copy of T in G. By transforming g in the same way that the labeling of T_i was transformed before, we obtain a labeling of T' that assigns the same

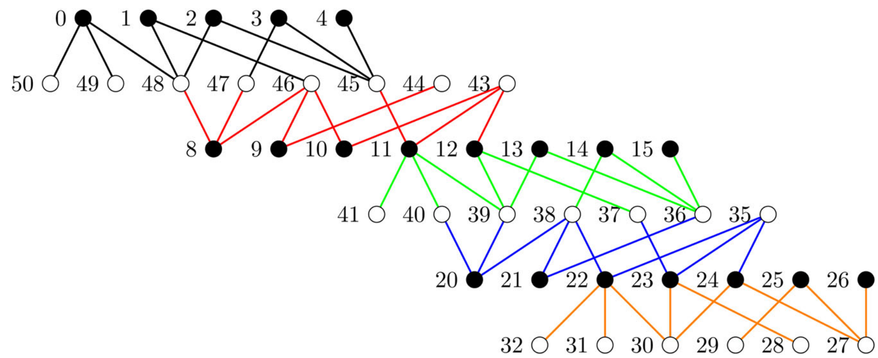

Figure 4. α-labeling of a generalized 5-fold of a tree.

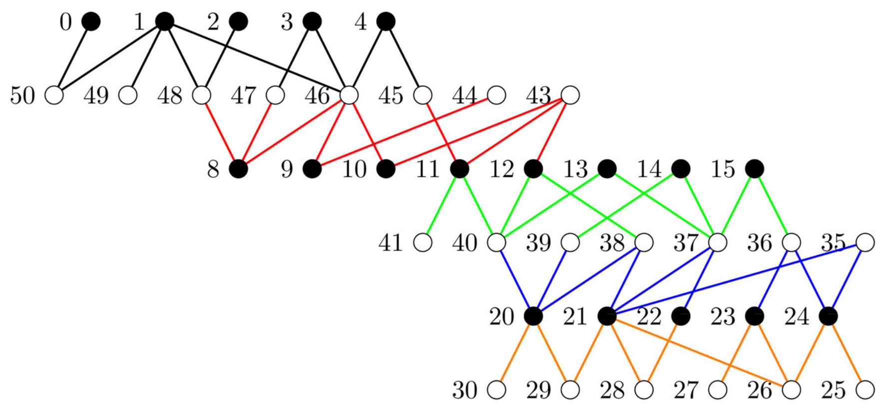

Figure 5. α-labeling of a modified 5-fold of a tree.

labels on the corresponding stable sets of T_i and T'. Thus, we can replace the edges of T_i in G with the edges of T' and the resulting graph is still an α-graph. This procedure can be applied as many times as necessary to obtain the desired α-labeling of the aimed graph. □

In Figure 5, we show this substitution of edges on the graph shown in Figure 4, where all the copies of T, except T_2, were replaced with analogous trees.

3. α-trees of even diameter

A *caterpillar* is a tree with a single path containing at least one endpoint of every edge. Suppose that T is a caterpillar of size $2m \geq 4$ such that $m = |A| = |B| - 1$. We say that $T \in \boldsymbol{F}_k$ if T has diameter $2k$ or $2k+1$, for some positive integer $k \geq 2$. For each $i \in \{1, 2, 3, 4\}$, let $T_i \in \boldsymbol{F}_k$ and T_5 be the tree consisting of one central vertex, denoted by w, which is attached to $t \geq 0$ pendant vertices, that is, $T_5 \cong K_1$ or $T_5 \cong K_{1,t}$. Recall that the *eccentricity* of a vertex in a graph is the maximum distance to other vertices. For each $i \in \{1, 2, 3, 4\}$, let $v_i \in V(T_i)$ such that its eccentricity equals $2k$. Thus, v_i is a leaf of T_i when diam $T_i = 2k$ or v_i is adjacent to a vertex of maximum eccentricity when diam $T_i = 2k+1$. Consider the tree T of size $4(2m+1)+t$ obtained by connecting, with an edge, all the vertices v_i to the vertex w of T_5. By $\boldsymbol{T}_{m,t}$ we understand the family of all trees of size $4(2m+1)+t$ obtained in the form described above. We claim that all the elements of $\boldsymbol{T}_{m,t}$ are α-trees.

Proposition 2. *If $T \in \boldsymbol{T}_{m,t}$, then T is an α-tree.*

Proof. Suppose that for every $i \in \{1, 2, 3, 4\}$, v_i is in the largest stable set of T_i; let f_i be an α-labeling of T_i such that $f_i(v_i) = 2m$. The existence of this labeling was proven by Rosa [6]. The labeling f_5 of T_5 is the α-labeling, also given by Rosa in the same work, that assigns the label 0 to the vertex w. For $i = 2$, 4, the initial labeling of T_i is the reverse of f_i, that is, $\hat{f}_i$.

For each $i \in \{1, 2, 3, 4, 5\}$, the initial α-labeling of T_i is transformed into a d_i-graceful labeling shifted c_i units, where

$$(d_i, c_i) = \begin{cases} (6m+t+5, 0) & \text{if } i = 1, \\ (4m+t+4, m) & \text{if } i = 2, \\ (2m+2, 2m+1) & \text{if } i = 3, \\ (1, 3m+1) & \text{if } i = 4, \\ (4m+3, 2m) & \text{if } i = 5. \end{cases}$$

Hence, the labels assigned to the vertices of T_i form the set $[0, m-1] \cup [7m+t+4, 8m+t+4]$ when $i = 1$, $[m, 2m-1] \cup [6m+t+3, 7m+t+3]$ when $i = 2$, $[2m+1, 3m] \cup [5m+2, 6m+2]$ when $i = 3$, $[3m+1, 4m] \cup [4m+1, 5m+1]$ when $i = 4$, and $\{2m\} \cup [6m+3, 6m+t+2]$ when $i = 5$. It follows that the labels assigned on the vertices of T form the interval $[0, 8m+t+4] = [0, 4(2m+1)+t]$.

The weights induced on the edges of T_i form the interval $[6m+t+5, 8m+t+4]$ when $i = 1$, $[4m+t+4, 6m+t+3]$ when $i = 2$, $[2m+2, 4m+1]$ when $i = 3$, $[1, 2m]$ when $i = 4$, and $[4m+3, 4m+t+2]$ when $i = 5$.

Notice that the labels assigned to v_i are $8m+t+4$ when $i = 1$, $6m+t+3$ when $i = 2$, $6m+2$ when $i = 3$, and $4m+1$ when $i = 4$. Since the label of w is $2m$, the edges $v_i w$ have weights $6m+t+4, 4m+t+3, 4m+2$, and $2m+1$, respectively. Thus, the weighst induced on the edges of T form the

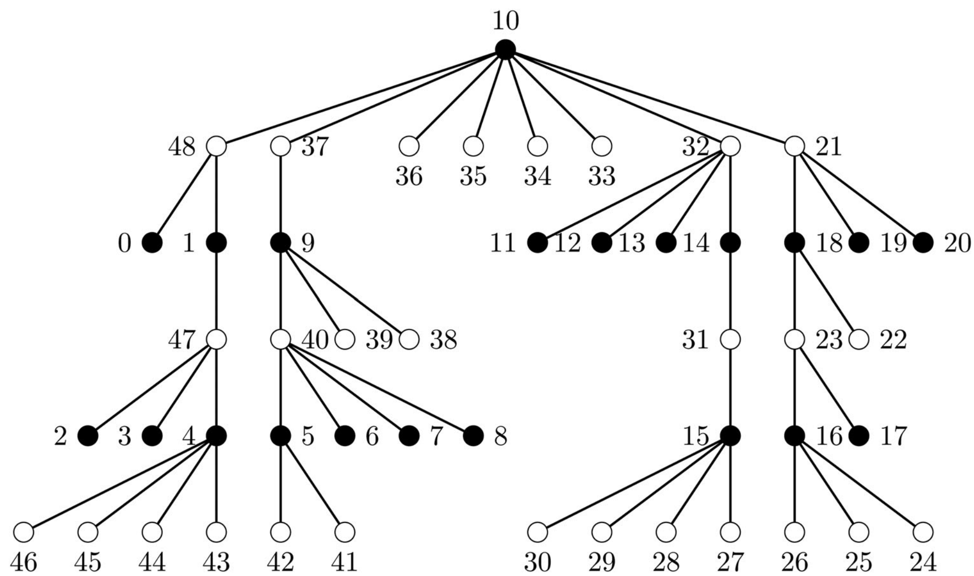

Figure 6. α-labeling of a tree in $\mathscr{T}_{5,4}$.

interval $[1, 8m + t + 4] = [1, 4(2m + 1) + t]$. The form in which the initial α-labelings are combined guarantees that the final labeling is also an α-labeling; its boundary value is $\lambda = 4m$. □

We must observe that the caterpillars used above can be replaced by α-trees for which there exist α-labelings that place the label 0 on vertices u_i satisfying the same conditions that the v_i. In Figure 6, we show an example of this labeling on a tree with four branches of length 5.

A *rooted tree* is a tree with a distinguished vertex r, called the *root*. The last proposition tells us that any rooted tree T is an α-tree if $T - r$ consists of four caterpillars of equal size and diameter $2k$ or $2k + 1$ for certain positive integer k. Is it possible to extend this result to trees T such that $T - r$ result in any number of caterpillars of equal size and similar diameters?

Disclosure statement

No potential conflict of interest was reported by the authors.

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