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AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

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AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 3, 1028–1038
https://doi.org/10.1016/j.akcej.2019.12.021

New resolvability parameters of graphs

Stephen T. Hedetniemi[a], Renu C. Laskar[b], and Henry Martyn Mulder[c]

[a]School of Computing, Clemson University, Clemson, SC, USA; [b]Department of Mathematical Sciences, Clemson University, Clemson, SC, USA; [c]Econometrisch Insituut, Erasmus Universiteit, Rotterdam, Netherlands

ABSTRACT

In this paper we introduce two concepts related to resolvability and the metric dimension of graphs. The *k*th dimension of a graph *G* is the maximum cardinality of a subset of vertices of *G* that is resolved by a set *S* of order *k*. Some first results are obtained. A pair of vertices *u*, *v* is totally resolved by a third vertex *x* if $d(u,x) \neq d(v,x)$. A total resolving set in G is a set *S* such that each pair of vertices of *G* is totally resolved be a vertex in S. The total metric dimension of a graph is the minimum cardinality of a total resolving set. We determine the total metric dimension of paths, cycles, and grids, and of the 3-cube, and the Petersen graph.

1. Introduction

In 1975 Slater [8] introduced the notion of the location number of a graph. In 1976 Harary and Melter [5] independently introduced the same notion, which they called the metric dimension of a graph. The latter terminology has become the standard in the literature, see e.g., [1–4]. Many papers have been written on the metric dimension of a connected graph since 1976. Let G be a connected graph. A vertex x is said to *resolve* a pair u, v of two vertices if the distances from x to u and v satisfy $d(x,u) \neq d(x,v)$. A set S is said to *resolve* a set T in G if each pair of vertices in T is resolved by some vertex x in S. If T is the vertex set V of G, then S resolves G. The *metric dimension* of G is the minimum cardinality of a resolving set of G.

The aim of this paper is to present two new parameters that are related to the metric dimension of a graph. The first one is the kth dimension of a connected graph. In this case we fix the order $k = |S|$ of a set S, and we seek maximal sets that are resolved by S. The *lower kth dimension* equals the minimum cardinality of a maximal set that is resolved by a set S of order k, and the *upper kth dimension* equals the maximum cardinality of set T that can be resolved by a set S of order k.

So far, in the definition of resolving a pair of vertices, a vertex u always resolves the pair u, v, since $d(u,u) = 0 < d(u,v)$. Hence, a set S always resolves the set S. In our second new parameter, we require that the resolving vertex x is always distinct from the vertices in the pair u, v. In this case we say that x *totally resolves* the pair u, v. The *total metric dimension* of a graph G equals the minimum cardinality of a set S that totally resolves the vertex set V of G. This new concept of resolvability is similar to the notion of total domination versus domination. A big difference between the metric dimension is that the total metric dimension is not always defined, in the same way that the total domination number is not always defined.

We determine the total metric dimension of paths and cycles. We determine the total metric dimension of the 3-cube Q_3, and pose the determination of the total metric dimension of Q_n, with $n \geq 4$ as an open problem. We also determine the total metric dimension of the Petersen graph, and of the grid graphs. In all these instances the differences between the usual metric dimension and this new total metric dimension become clear.

2. Preliminaries

Let $G = (V,E)$ be a graph of *order* $n = |V|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u | uv \in E\}$ of vertices adjacent to v. Each vertex $u \in N(v)$ is called a *neighbor* of v. The *closed neighborhood* of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is $N(S) = \cup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set $N[S] = \cup_{v \in S} N[v]$.

Let $d(u, v)$ denote the minimum length of a path (the *distance*), between vertices u and v. The *eccentricity* of a vertex $x \in V$ is defined as $ecc(x) = max\{d(x,v) | v \in V\}$. The *diameter* of G equals $diam(G) = max\{ecc(v) | v \in V\}$, and the *radius* of G equals $rad(G) = min\{ecc(v) | v \in V\}$.

A set $S \subseteq V$ is a *dominating set* in G if $N[S] = V$, that is, every vertex $w \in V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in G is called the *domination number* of G, denoted $\gamma(G)$. A set $S \subseteq V$ is a *total dominating set* if $N(S) = V$, that is, every vertex $w \in V$ is adjacent to a vertex $v \in S$ with $v \neq w$. The minimum cardinality of a total dominating set in G is the *total domination number*, denoted $\gamma_t(G)$.

CONTACT Henry Martyn Mulder hmmulder@few.eur.nl Econometrisch Insituut, Erasmus Universiteit, Rotterdam, Netherlands.

A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent. The *vertex independence number* $\alpha(G)$ equals the maximum cardinality of an independent set in G.

Let $S = \{v_1, v_2, ..., v_k\}$ be an ordered set of k vertices in a connected graph $G = (V, E)$, and let $w \in V$ be an arbitrary vertex. The *metric representation* of w with respect to the set S is the (ordered) k-vector

$$r(w|S) = (d(w, v_1), d(w, v_2), ..., d(w, v_k)).$$

A set S is said to be a *resolving set for G* if no two vertices in V have the same metric representation, with respect to S. The minimum cardinality of a resolving set for G is called its *metric dimension* and is denoted $dim(G)$.

It is worth pointing out that for any set $S \subset V$, any two vertices $v_i, v_j \in S$ have distinct metric representations with respect to S, that is, $r(v_i|S) \neq r(v_j|S)$ because $d(v_i, v_i) = 0 \neq d(v_j, v_i) > 0$. And for the same reason, the metric representation of any vertex in S is different than the metric representation of any vertex in $V - S$. Thus, in order to determine if a set S is a resolving set, one only has to check that every pair of distinct vertices in $V - S$ have distinct metric representations with respect to S.

An equivalent definition of the metric dimension of a graph G can be given as follows. A vertex $x \in V$ in a connected graph G is said to *resolve* two vertices $u, v \in V$ if $d(u, x) \neq d(v, x)$; we denote this $x \preceq u, v$. It follows from this definition that for any vertex $x \in S$, and any vertex $v \in V - \{x\}, x \preceq x, v$.

A set $S \subset V$ *resolves* a set $T \subset V$, denoted $S \preceq T$, if for every two vertices $u, v \in T$ there exists a vertex $x \in S$ such that $x \preceq u, v$. Note that according to this definition, every set S resolves itself and all subsets $S' \subseteq S$.

Proposition 1. *For any connected graph $G = (V, E)$ and any set $S \subseteq V, S \preceq S$.*

A set $S \subset V$ is a *resolving set* for G, if $S \preceq V$, that is, if for every two distinct vertices $u, v \in V$, there exists a vertex $x \in S$, such that $x \preceq u, v$. The *metric dimension* $dim(G)$ equals the minimum cardinality of a resolving set S for G. Such a minimum cardinality resolving set is called a *metric basis* for G. This concept was first introduced by Slater [8] in 1975, who called a resolving set a *locating set*, and the metric dimension of G the *location number* $loc(G)$. This concept was independently introduced by Harary and Melter [5] in 1976, who used the term "metric dimension". By now there exists a considerable literature on resolvability and the metric dimension of a graph, see e.g., [2–4].

A resolving set S for a connected graph G is said to be *minimal* if no proper subset of S is a resolving set for G. Note that the property of being a resolving set is *superhereditary*, which means that every superset of a resolving set is also a resolving set. In addition, the property of not being a resolving set is hereditary, that is, each subset of a non-resolving set is also a non-resolving set. Hence, in order to determine if a resolving set is minimal, all you have to do is to check that $S - \{v\}$ is a not resolving set, for every $v \in S$.

The maximum cardinality of a minimal resolving set for a connected graph G is called the *upper metric dimension* of G and is denoted $dim^+(G)$. The upper metric dimension was introduced by Chartrand, Poisson, and Zhang [2] in 2000. It follows from the definitions that for any connected graph G, $dim(G) \leq dim^+(G)$. In [2] it is shown that for every positive integer k, there exists a connected graph G for which $dim^+(G) - dim(G) \geq k$. The authors were not able to prove the following:

Conjecture [Chartrand, Poisson, Zhang]. *For every pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G for which $dim(G) = a$ and $dim^+(G) = b$.*

In the following sections we introduce several new parameters, all stemming from the concept of a resolving set and the definition of the metric dimension of a graph.

3. The upper and lower *k*th dimension of a graph

A third, and still equivalent, definition of metric dimension can be given as follows. The *(metric) upper dimension Dim(x) of a vertex $x \in V$* equals the maximum cardinality of a set $S \subseteq V$ such that $x \preceq u, v$. for any pair of distinct vertices $u, v \in S$. Thus, $Dim(x) = max\{|S||S \subseteq V \text{ and } x \preceq S\}$. Similarly, we can define the *lower dimension* $dim(x)$ of a vertex v to equal the minimum cardinality of a maximal set S such that $x \preceq u, v$, for any pair of distinct vertices $u, v \in S$.

Proposition 2. *For any vertex x in a connected graph G, $dim(x) = Dim(x) = 1 + ecc(x)$.*

Proof. Let x be a vertex in G, and let S be a set of maximum cardinality such that $x \preceq S$. It follows that no two vertices $u, v \in S$ can be at the same distance from x, else $d(x, u) = d(x, v)$, and therefore x does not resolve S. By definition, there can be at most $1 + ecc(x)$ distinct distances from vertices in V to x. Thus, $Dim(x) \leq 1 + ecc(x)$. Let u be any vertex for which $d(x, u) = ecc(x)$, let $x, u_1, u_2, ...u_{ecc(x)} = u$ be a shortest path from x to u, and let S be the set of vertices on this path. Then clearly $x \preceq S$, and $|S| = 1 + ecc(x)$. Thus, $Dim(x) \geq 1 + ecc(x)$.

Similarly, let S be a maximal set of minimum cardinality such that $x \preceq S$, that is, $dim(x) = |S|$. Define for x, the $1 + ecc(x)$ sets X_i, where for every vertex $w \in X_i, d(x, w) = i$, for $0 \leq i \leq ecc(x)$. By definition $|S \cap X_i| \leq 1$, and if there exists a j such that $S \cap X_j = \emptyset$, then S is not a maximal set. Thus, $|S| = 1 + ecc(x)$. □

Definition 1. For any connected graph G, the upper k-th dimension $Dim^k(G)$ of G equals the maximum cardinality of a set $T \subset V$ such that there exists a set S of cardinality k and $S \preceq T$. The lower k-th dimension $dim_k(G)$ of G equals the minimum cardinality of a maximal set $T \subset V$ such that there exists a set S of cardinality k and $S \preceq T$. Such a set T is called a dim_k-set of G, while the set S is called a dim_k-resolving set for T.

Notice from the definitions that if a set T is a dim_k-set of G and the set S is a dim_k-resolving set for T, then it must follow that $S \subseteq T$, since if $S \preceq T$ then by definition $S \preceq T \cup S$.

The following inequalities are easily established.

Proposition 3. *For any connected graph G of order n,*

$$k \leq Dim^k(G) \leq Dim^{k+1}(G) \leq n.$$

3.1. The first dimension of a graph

We determine the first dimension of a connected graph in terms of its diameter and radius.

Proposition 4. *For any connected graph G,* $Dim^1(G) = 1 + diam(G)$.

Proof. Let x be a vertex for which $ecc(x) = diam(G)$, let u_d be a vertex for which $d(x, u_d) = diam(G)$, let $x, u_1, u_2, \ldots u_d$ be a shortest path of length $d = diam(G)$ from x to u_d, and let S be the set of vertices on this path. Then, clearly $x \preceq S$, and therefore, $Dim^1(G) \geq 1 + diam(G)$.

To show that $Dim^1(G) \leq 1 + diam(G)$, let x be a vertex for which $Dim(x) = Dim^1(G)$, and let $S(x)$ be a largest set, such that $x \preceq S(x)$. Clearly no two vertices, say u and v in $S(x)$ can be at the same distance from x, else x cannot resolve the pair u, v. There can be at most $1 + diam(G)$ distinct distances from any vertex in V to all vertices in V. Thus, $|S(x)| \leq 1 + diam(G)$. □

In like manner, by selecting a vertex x such that $ecc(x) = rad(x)$, one can prove the following; we omit the proof.

Proposition 5. *For any connected graph G,* $dim_1(G) = 1 + rad(G)$.

3.2. The second dimension of a graph

Proposition 4 asserts that the upper first dimension of a connected graph is 1 plus its diameter. This raises the following question:

Question 1. *Can you characterize the upper second dimension* $Dim^2(G)$ *of a connected graph G?*

A partial answer to this question follows from the definitions.

Proposition 6. *For any graph G of order* $n \geq 2$, $dim(G) = 2$, *if and only if* $Dim^2(G) = n$.

Corollary 7. *For any cycle* C_n, $Dim^2(C_n) = n$.

Proof. It is well known that for any cycle C_n, $dim(C_n) = 2$, since any pair of adjacent vertices in a cycle is a resolving set for C_n. □

It should be pointed out that in general, $dim_2(C_n) < Dim^2(C_n)$ is possible. For example, we have just pointed out that $Dim^2(C_6) = 6$, since any pair of adjacent vertices of C_6 is a resolving set for $S = V(C_6)$. However, any set of two antipodal vertices of C_6, that is, any two vertices u, v, for which $d(u, v) = 3$ in C_6, is a set that can only resolve four vertices, and any such set of four vertices is a maximal resolving set for such a pair u and v. This shows that $dim_2(C_6) = 4 < Dim^2(C_6) = 6$.

Corollary 7 leads us to ask, what is $dim_2(C_n)$? We answer this question in the next two results. Recall that an *internal vertex* on a path is a vertex that is not one of the ends of the path.

Proposition 8. *For any cycle* C_{2k+1} *of odd order,*

$$dim_2(C_{2k+1}) = Dim^2(C_{2k+1}) = 2k + 1.$$

Proof. Let $S = \{u, v\}$ be a set of two vertices in C_{2k+1}. From the proof of Corollary 7 we know that if u is adjacent to v then $S \preceq V(C_{2k+1})$. Next let u and v not be adjacent. Let P_1 and P_2 be the two edge-disjoint paths between u and v on the cycle C_{2k+1}. It is easy to see that $S \preceq V(P_1)$ and $S \preceq V(P_2)$. Therefore, let x be an internal vertex of P_1, and let y be an internal vertex of P_2. We will show that $S \preceq \{x, y\}$.

We can assume, without loss of generality, that P_1 is shorter than P_2. This implies that the distances $d(u, x)$ and $d(v, x)$ are the lengths of the subpaths of P_1 between the respective vertices. Suppose that $d(x, u) = d(y, u)$. Then it follows that $d(x, y) = d(x, u) + d(u, y) = 2d(x, u)$. Now we compare $d(v, x)$ and $d(v, y)$. The value of $d(v, y)$ depends on the length of the two paths between v and y in C_{2k+1}. One of these contains both x and u. If this one is the shorter of the two, then, obviously, we have $d(v, x) < d(v, y)$, so $d(v, x) \neq d(v, y)$. If the other (v,y)-path is the shorter one, then $d(v, x) + d(v, y) = 2k + 1 - d(x, u) - d(u, y) = 2k + 1 - 2d(x, u)$, which is odd. So again we get $d(v, x) \neq d(v, y)$. In both cases it follows that $S \preceq \{x, y\}$.

Otherwise, we have $d(u, x) \neq d(u, y)$, in which case $S \preceq \{x, y\}$ trivially. □

Proposition 9. *For any cycle* C_{2k} *of even order,* $dim_2(C_{2k}) = k + 1$.

Proof. Let $S = \{u, v\}$ be any two vertices on the cycle C_{2k} such that $d(u, v) = k$, and let P_1 and P_2 be the two edge-disjoint paths between u and v on the cycle C_{2k}. By definition, $|V(P_1)| = |V(P_2)| = k + 1$. It follows that $S \preceq T = V(P_1)$. It also follows that the set T is a maximal set resolved by S of order $k + 1$.

Next let $S = \{u, v\}$ consist of two vertices of C_{2k} with $d(u, v) = m < k$. Let Q_1 and Q_2 be the two edge-disjoint paths between u and v on the cycle, with Q_1 of length m and Q_2 of length $2k - m \geq k + 1$. It is easy to see that $S \preceq V(Q_2)$. So S resolves at least $k + 2$ vertices.

Thus we have shown that the smallest maximal set that is resolved by two vertices is of order $k + 1$. □

Question 2. *What are good upper and lower bounds for the second dimensions* $Dim^2(G)$ *and* $dim_2(G)$ *of a graph?*

We know of course, for any graph G, that

$$1 + diam(G) = Dim^1(G) \leq Dim^2(G) \leq n.$$

But, are there better bounds? For the case of trees see [7].

3.3. The computational complexity of the second dimension of a graph

Question 3. *Can the second dimension of a connected graph be computed in polynomial time?*

We present some observations that might be a first step in answering this question.

For any pair u, v of vertices, define $Dim(u, v) = max\{|T| | T \subseteq V \text{ and } \{u, v\} \preceq T\}$. The *upper second dimension* of a connected graph G is therefore $Dim^2(G) = max\,\{Dim(u, v) | u, v \in V\}$.

Let $D(G)$ be the n-by-n *distance matrix* of a connected graph $G = (V, E)$, where $D(i,j) = D(j,i) = d(v_i, v_j)$, and consider any two vertices $v_i, v_j \in V$. We seek to determine $Dim(v_i, v_j)$, that is, the maximum cardinality of a set $T \subseteq V$ such that $\{v_i, v_j\} \preceq T$.

Consider the ith and jth rows of the matrix D and construct the following graph $G(i,j) = (V, E(i,j))$. Two vertices v_x and v_y, for $1 \le x < y \le n$, are adjacent in $G(i, j)$ if and only if (i) $d(v_i, v_x) = d(v_i, v_y)$ and (ii) $d(v_j, v_x) = d(v_j, v_y)$. Thus, the edges $v_x v_y$ of the graph $G(i, j)$ correspond to the pairs of vertices v_x, v_y that are not resolved by either v_i or v_j. Thus, if $\{v_i, v_j\} \preceq T$, then v_x and v_y cannot both be in T.

Proposition 10. *For any connected graph G, and any two vertices v_i, v_j in V,*

$$Dim(v_i, v_j) = \alpha(G(i,j)).$$

Proof. Let $T \subseteq V$ be an independent set of maximum cardinality $\alpha(G(i,j))$ in $G(i, j)$. Since no two vertices in T are joined by an edge in $G(i, j)$, it follows that for every pair $v_x, v_y \in T$, either $d(v_i, v_x) \ne d(v_i, v_y)$, or $d(v_j, v_x) \ne d(v_j, v_y)$, whence $\{v_i, v_j\} \preceq T$. Therefore, $Dim(v_i, v_j) \ge |T| = \alpha(G(i,j))$.

Conversely, let $T \subseteq V$ be a maximum cardinality set such that $\{v_i, v_j\} \preceq T$. Thus, $Dim(v_i, v_j) = |T|$. It follows that for any two vertices $v_x, v_y \in T$, either $d(v_i, v_x) \ne d(v_i, v_y)$, or $d(v_j, v_x) \ne d(v_j, v_y)$. But in this case there is no edge between v_x and v_y in $G(i, j)$. Thus, the set T must be an independent set in $G(i, j)$. Therefore, by definition, $Dim(v_i, v_j) = |T| \le \alpha(G(i,j))$. □

Corollary 11. *For any connected graph G,*

$$Dim^2(G) = max\{\alpha(G(i,j)) | v_i, v_j \in V\}.$$

It is well known that the decision problem corresponding to the independence number $\alpha(G)$ of a graph is NP-complete. However, we do not know whether this decision problems remain NP-complete, when it is restricted to such graphs $G(i, j)$. We leave this as an open problem.

We conclude this section with a simple observation on the relation between the kth upper dimensions of a graph $(k \ge 1)$ and its metric dimension.

Proposition 12. *For any connected graph G of order n,*

$$dim(G) = min\{k | Dim^k(G) = n\}.$$

Proof. Let k be a smallest integer for which there exists a set S with $|S| = k$ and $S \preceq V$. Then, by definition, $dim(G) \le k = min\{k | Dim^k(G) = n\}$.

Conversely, let S be a smallest set in G such that $S \preceq V$, that is, by definition, $dim(G) = |S|$. Then $min\{k | Dim^k(G) = n\} \le |S| = dim(G)$. □

4. Total metric dimension

A fourth, and still equivalent, definition of the metric dimension can be given as follows. A set $S \subset V$ is an *external resolving set* if $S \preceq V - S$, that is, for every two distinct vertices $u, v \in V - S$, there exists a vertex $x \in S$, such that $x \preceq u, v$.

The *external metric dimension* $xdim(G)$ equals the minimum cardinality of an external resolving set S in G; such a minimum cardinality external resolving set is called an *external metric basis* for G. Since every resolving set S for G is by definition an external resolving set, it follows that for any graph G,

$$xdim(G) \le dim(G).$$

However, the following is true.

Proposition 13. *For any graph G, $xdim(G) = dim(G)$.*

Proof. It follows from the definitions that every resolving set S for G is automatically a resolving set for $V - S$. It only remains to show that the converse is true: every resolving set S for $V - S$ is a resolving set for V, and therefore, for G. This follows from the previous observation above that any two vertices $u, v \in S$ have distinct metric representations with respect to S, since $d(u, u) = 0 < d(v, u)$. Furthermore, the same statement applies to any vertex $u \in S$ and any vertex $v \in V - S$. □

The property of being a resolving set for $V - S$ is, therefore, the same as being a resolving set for V. However, this $V - S$ versus S difference becomes clearer if we make a minor, but significant change in the definition.

Consider the following reworded definitions of a dominating set and a total dominating set of a graph G. A set $S \subseteq V$ is a *dominating set* of a graph G if for every vertex $v \in V$ there exists a vertex $u \in S$ such that u *dominates* v, or u is adjacent to v. Notice that we assume that a vertex v dominates itself, and therefore if $v \in V$ is in fact a vertex $v \in S \subseteq V$, then we may choose vertex $v \in S$ so that v dominates v. But in total domination we cannot choose a vertex to dominate itself.

A set $S \subseteq V$ is a *total dominating set* of a graph G if the set S *totally dominates* the set V, that is, for every vertex $v \in V$ there exists a distinct vertex $u \in S$ such that u *dominates* v, that is, v is a neighbor of u. This implies that $u \ne v$.

We can make the same distinction with regard to a vertex u resolving a pair of vertices v, w. Recall the definition, a set S is a *resolving set* of a graph G if, for every pair of distinct vertices $v, w \in V$, there exists a vertex $u \in S$, such that u resolves v, w, or $u \preceq v, w$. Notice that we assume that a vertex resolves itself and any other vertex, thus, $u \preceq u, v$. A total dominating version of resolving is *total resolving*.

Definition 2. A vertex $u \in S$ in a connected graph G totally resolves two distinct vertices $v, w \in V$, denoted $u \prec v, w$, if $d(u, v) \ne d(u, w)$ and $v \ne u \ne w$.

It no longer follows from this definition that for any vertex $u \in S$, and any distinct vertex $v \in V - \{u\}, u \prec u, v$, since all three vertices must be distinct.

Definition 3. A set $S \subseteq V$ totally resolves a set $T \subseteq V$, denoted $S \prec T$, if, for every two distinct vertices $v, w \in T$, there exists a distinct third vertex $u \in S$ such that $u \prec v, w$.

Note that according to this definition, it is not necessarily true that every set S totally resolves itself and all subsets $S' \subset S$.

Definition 4. A set $S \subseteq V$ is a total resolving set or a TRS for G, if $S \prec V$, that is, if for every two distinct vertices $v, w \in V$, there exists a distinct third vertex $u \in S$, such that $u \prec v, w$.

The *total metric dimension* $dim_t(G)$ equals the minimum cardinality of a total resolving set S for G; a minimum cardinality total resolving set is called a *total metric basis* for G. It follows from these definitions that for any graph G of order n,

$$dim(G) \leq dim_t(G).$$

This inequality raises the following question.

Question 4. *For which connected graphs G do we have $dim_t(G) = dim(G)$?*

Notice, first of all, that by definition, if S is a *TRS* of a graph G, then $|S| \geq 3$. Thus, if a graph G of order n has a *TRS*, then

$$3 \leq dim_t(G) \leq n.$$

The next lemma follows immediately from the definition, but is quite helpful in the sequel.

Lemma 14. *If a connected graph G has a total resolving set S of order 3, then the distances between the three pairs in S are all different.*

Proof. Let $S = \{u, v, w\}$. If, say, $d(u, w) = d(v, w)$, then S does not totally resolve the pair u, v. □

A few examples will serve to illustrate the properties of total resolving sets. Notice next that not all graphs have a *TRS*. These include complete graphs K_n with $n \geq 3$, and all graphs having a vertex that is adjacent to two or more leaves. This observation immediately raises the following two questions.

Question 5. *Can you characterize the family of graphs having a TRS?*

Question 6. *What is the complexity of the following decision problem?*

TOTAL RESOLVING SET
Instance: a graph $G = (V, E)$
Question: does G have a TRS?

4.1. Paths and cycles

For convenience, let us assume that the vertices of the path P_n of order n are simply labeled $1, 2, 3, \ldots, n$. Clearly, the three pairs of vertices in the path P_3 of order 3 do not have different distances. So, by Lemma 14, it does not have a *TRS*.

Proposition 15. *For any $n \geq 4, dim_t(P_n) = 3$.*

Proof. We show that the set $S = \{1, 2, 4\}$ is a *TRS*. It is easy to see that for every $2 \leq u < v \leq n$, we have $1 \prec u, v$, since $d(1, u) < d(1, v)$. This only leaves the following vertex pairs to be totally resolved, as follows: $4 \prec 1, 2$; $4 \prec 1, 3$; and $2 \prec 1, v$, for all $v \geq 4$. □

Vertex 1 resolves each pair in the path P_n, so $dim(P_n) = 1$, for all n.

Again, for convenience, let us assume that the vertices of the cycle C_n of order n are labeled in circular order $1, 2, 3, \ldots, n, 1$. Again by Lemma 14, it is obvious that the cycle C_3 of order 3 does not have a *TRS*.

Proposition 16. *The cycle C_4 does not have a TRS.*

Proof. No pair of non-adjacent vertices can be totally resolved. Hence there is no *TRS* in C_4. □

Proposition 17. $dim_t(C_n) = 4$, for $n = 5$ or 6.

Proof. It is easy to see that the 5-cycle does not contain three vertices with pairwise different distances. So $dim_t(C_5) > 3$.

The 6-cycle C_6 does have three vertices with pairwise different distances. Without loss of generality, this set is {1, 2, 4}. In this case we have $d(1, 2) = d(1, 6) = 1$ and $d(4, 6) = d(4, 2) = 2$. So the pair 2, 6 is not totally resolved. Thus we have $dim_t(C_6) > 3$.

It is straightforward to check that the path P on the four vertices is a *TRS* in C_5 as well as C_6. So, indeed, we have $dim_t(C_5) = dim_t(C_6) = 4$. □

Theorem 18. *For any $n \geq 7, dim_t(C_n) = 3$.*

Proof. Let $S = \{1, 2, 4\}$. We will show that S is a *TRS* of C_n, that is, for every $u, v \in \{1, 2, 3, \ldots, n\}$, there exists a vertex $w \in \{1, 2, 4\}$, such that $w \prec u, v$.

Consider first all six vertex pairs u, v for $1 \leq u < v \leq 4$. It is easy to see the following: first 4 totally resolves the pair 1, 2, the pair 1, 3, and the pair 2, 3, next 2 totally resolves the pair 1, 4, and the pair 3, 4, and finally 1 totally resolves the pair 2, 4. This deals with all pairs in the set $\{1, 2, 3, 4\}$.

Since 2 is adjacent to 1, but not to any vertex v with $v \geq 5$, it follows that $2 \prec 1, v$, for $v \geq 5$. Similarly, 2 is adjacent to 3, but not to any vertex v with $v \geq 5$, so $2 \prec 3, v$, for $v \geq 5$. Since 1 is adjacent to 2, but not to any vertex v with $5 \leq v \leq n-1$, it follows that $1 \prec 2, v$, for $5 \leq v \leq n-1$. Since $n \geq 7$, we have $d(4, n) \geq 3$. So $4 \prec 2, n$. Next either $1 \prec 4, v$ or $2 \prec 4, v$, for $v \geq 5$. Notice in this case that if $d(1, 4) = d(1, v) = 3$, for some $v \geq 5$, then $2 = d(2, 4) \neq 3 \leq d(2, v)$.

This leaves the vertex pairs u, v for $5 \leq u < v \leq n$. But it is easy to see that either $1 \prec u, v$ or $2 \prec u, v$, since, if $d(1, u) = d(1, v)$, for $5 \leq u < v$, then $d(2, u) \neq d(2, v)$. □

The set {1, 2} resolves each pair of the cycle C_n, so $dim(C_n) = 2$, for all $n \geq 3$.

4.2. The 3-cube Q_3

Observe that the property of being a total resolving set is superhereditary, similar to the case of resolving set, that is, every superset of a total resolving set is also a total resolving set. This implies that a total resolving set S is *minimal* if and only if for every vertex $x \in S$, the set $S - \{x\}$ is not a total resolving set. It follows therefore, that every vertex x in a minimal (total) resolving set must (totally) resolve some

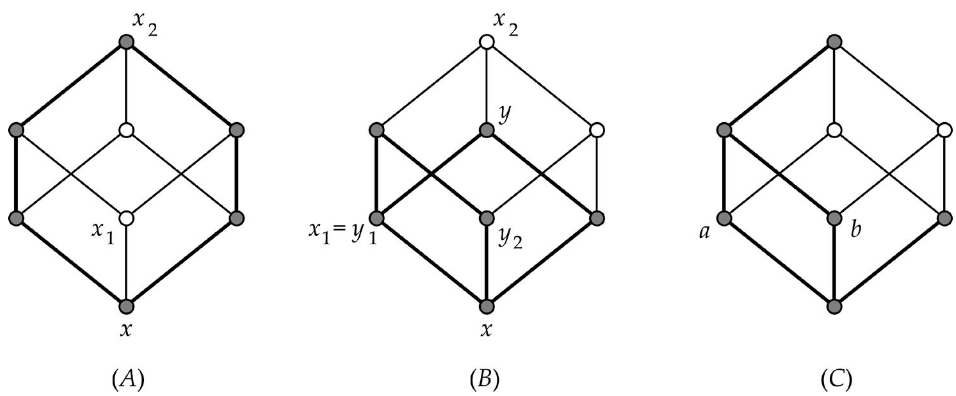


Figure 1. The sets of order 6 in Q_3.

pair of vertices u, v that is not (totally) resolved by any other vertex in S, that is $x \preceq u, v$. We speak of such a pair u, v as a *private* (*total*) *resolving pair* for vertex x, or a *private pair*, for short.

The *n-cube* Q_n is the graph whose vertices correspond 1-to-1 with all possible n-tuples of 0 s and 1 s, such that two vertices are adjacent if and only if the corresponding n-tuples differ in exactly one position. It has many nice properties, amongst which are abundant symmetries and automorphisms. We will use this without mention. The 3-cube Q_3 is the well-known 3-dimensional cube having order 8. Note that the minimal resolving sets in Q_3 are the 3-sets that induce either a path of three vertices or three independent vertices. So $dim(Q_3) = 3$.

Proposition 19. *For the 3-cube,* $dim_t(Q_3) = 6$.

Proof. In Figure 1 we see the three types of sets of order 6 in Q_3 depicted as the gray vertices. In Case (A), the two vertices not in the set have distance 3. In Case (B), they have distance 1, and in Case (C), they have distance 2. In Case (C), the pair a, b is not resolved by any other gray vertex. So this set of six vertices is not a *TRS*.

We will show that the sets in Cases (A) and (B) are minimal total resolving sets. Since any set of 7 or 8 vertices contain a set of type (A), such sets are not minimal total resolving sets. Moreover any set of 5 vertices is contained in a set of type (A) or type (B). Thus it will follow that the sets of type (A) and (B) are precisely the minimal total resolving sets of Q_3.

First we show that the sets are total resolving sets. Note that, since Q_3 is bipartite, any pair of vertices at odd distance is resolved by any other vertex. So one only needs to check vertices at distance 2. Note that, for any vertex u of Q_3, there is a unique vertex u' at distance 3, its *antipodal* vertex. Clearly, u' totally resolves any pair containing u.

In Case (A), the antipodal vertex of a gray vertex is again a gray vertex. Any pair u, v at distance 2 contains a gray vertex, so the antipodal vertex of this gray vertex totally resolves this pair.

In Case (B), there are two different types of gray vertices: vertices of degree 2, of which the antipodal vertex is also gray, and vertices of degree 3, of which the antipodal vertex is white. Any pair u, v at distance 2 containing a gray vertex u of degree 2 is totally resolved by u'. So we only need to check pairs u, v at distance 2 such that u is a gray vertex of degree 3, and v is a white vertex. Now the antipodal vertex v' of v is a gray vertex, and totally resolves u, v.

To show that each gray vertex has a private resolving pair, we have indicated for each type of vertex u such a pair u_1, u_2. In Case (A), there is only one type of vertex indicated as x. In Case (B), there are two types of vertices indicated as x and y.

Thus we have shown that the sets in Cases (A) and (B) are minimal total resolving sets, which concludes the proof. □

Since $Q_2 = C_4$, the 2-cube does not have a *TRS*.

Question 7. *Can you determine* $dim_t(Q_n)$, *for* $n > 3$?

4.3. The Petersen graph

Next consider the Petersen graph $\mathcal{P}$. We take the Petersen graph in its usual drawing, with an outer 5-cycle and an inner 5-cycle, and a connecting matching, see Figure 2. The Petersen graph has many symmetries and automorphisms. For instance, we may take any 5-cycle as the outer cycle, whereby the remaining five vertices form the inner cycle. We make use of this fact when we visualize the arguments in the proof below. Since the distances between two distinct vertices in the Petersen graph are either 1 or 2, a vertex x totally resolves a pair x_1, x_2 if and only if x is adjacent to one of the two and not adjacent to the other. We use these facts below in the proof without mention. We denote the path of order n by P_n.

First we observe that $dim(\mathcal{P}) = 3$. Take a P_3 on the outer cycle, say, with ends u and v, and middle vertex z. Now let $S = \{u, v, w\}$, where w is a vertex on the inner cycle that is not adjacent to u, v, or z. It is easy to see that S is a resolving set. Moreover, no 2-set in $\mathcal{P}$ is a resolving set.

Proposition 20. *For the Petersen graph* $\mathcal{P}, dim_t(\mathcal{P}) = 5$.

Proof. Let S be a set of at least three vertices in $\mathcal{P}$, and let $\mathcal{P}[S]$ be the subgraph of $\mathcal{P}$ induced by S. A pair of independent vertices in $\mathcal{P}[S]$ is not resolved by any other vertex in S. If P_2 is a component in $\mathcal{P}[S]$, then the pair of ends of this P_2 is not resolved by any other vertex in S. If P_3 is a component in $\mathcal{P}[S]$, then again the pair of ends of this P_3 is not resolved by any other vertex in S. If $K_{1,3}$ is a component in $\mathcal{P}[S]$, then no pair of leaves of this $K_{1,3}$ is resolved by any other vertex in S. Finally, assume that P_4 is a component in $\mathcal{P}[S]$. Then this P_4 is

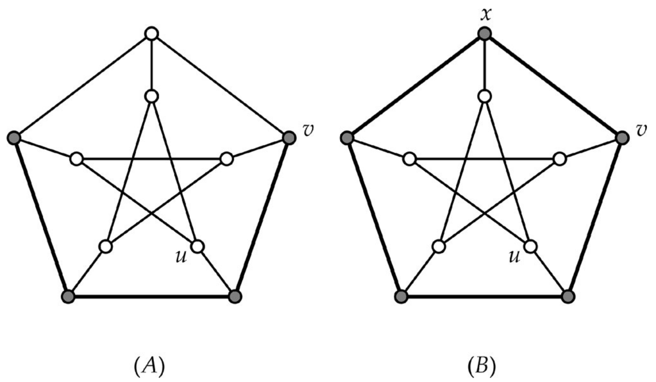


Figure 2. The Petersen graph with an induced P_4 and an induced C_5.

part of a 5-cycle in $\mathcal{P}$. We may take this 5-cycle as the outer cycle in $\mathcal{P}$, see Figure 2(*A*). Here we also have indicated a pair u and v that is not resolved by any other vertex of this P_4. Thus we have shown that no 4-set in $\mathcal{P}$ can be a total resolving set. Hence $dim_t(\mathcal{P}) \geq 5$.

To show that $dim_t(\mathcal{P}) = 5$, we have to exhibit a 5-set that is a *TRS*. A candidate is the 5-cycle in Figure 2(*B*). Note that the pair u, v that was not resolved in the case of the P_4 is now resolved by x. So it is a private pair for x. By symmetry, each vertex on the 5-cycle has a private pair.

To totally resolve a pair, we need to find a vertex that is adjacent to one, but not to the other. We list the different types of pairs that we have to check. There are four types of non-mixed pairs: two adjacent white vertices, two nonadjacent white vertices, two adjacent gray vertices, two nonadjacent gray vertices. There are three types of mixed pairs, one with the white vertex adjacent to the gray vertex, one with the white vertex adjacent to a neighbor of the gray vertex, and finally, one with the white vertex not adjacent to a neighbor of the gray vertex. In each case we easily find a totally resolving gray vertex. □

In Figure 3 we depict the two other 5-sets that form a *TRS* in the Petersen graph: A P_4 with an independent vertex, and a P_5. We omit the proof that these 5-sets are a *TRS*. By only considering the various types of pairs of vertices it is possible to check that they are indeed a *TRS* within reasonable time. In the case of the P_5, it is easier to check all pairs by using a different, fairly common drawing of the Petersen graph, see Figure 3(*B*). It is easier to show that the three other possible 5-sets do not form a *TRS*. In two cases a P_2 or a P_3 appears as component. In the third case, the subgraph induced by the set is connected: it is a tree on five vertices with on vertex of degree 3. The vertex of degree 3 has two leaves, which are not resolved by any other vertex of the tree.

4.4. The grids $P_m \square P_n$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* of these graphs is the graph $G = G_1 \square G_2$ with vertex set $V_1 \times V_2$, where $(u_1, u_2)(v_1, v_2)$ is an edge if and only if either u_1v_1 is an edge in G_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2v_2 is an edge in G_2. The $m \times n$ *grid* is the Cartesian product $P_m \square P_n$ of two paths of orders m and n, respectively, with $m, n \geq 2$. Without loss of generality, we may assume that $m \leq n$.

Note that $P_2 \square P_2 = C_4$, so the 2×2 grid does not have a total resolving set. In this section we determine the total metric dimension of the larger grids.

First we consider the case $m = 2$ and $n \geq 3$. The $2 \times n$ grid, with $n \geq 3$, is also known as the *ladder graph* or the *n-ladder*. It consists of two copies of the path P_n on n vertices, and a matching between corresponding vertices. We draw the two n-paths horizontally from left to right: a *bottom path* and above it a *top path*, so that the matching between the two paths consists of vertical lines. See Figure 4 for an illustration. The vertices of the bottom path will be called *bottom vertices*, and those of the top path *top vertices*. The four vertices of degree 2 are the *corners* of the n-ladder.

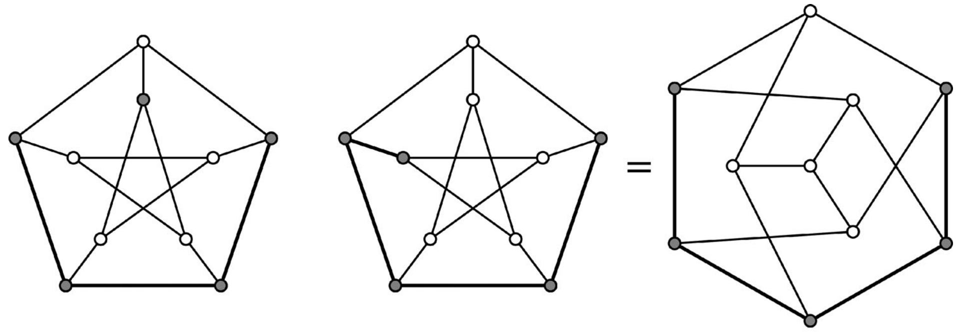
Now we search for total resolving sets S for the ladder graph. Since a total resolving set S is of order at least 3, we may assume, without loss of generalization, that S contains at least two bottom vertices. Let u be the left most bottom vertex in S, and let v be the right most bottom vertex in S. In Figure 4(*A*), the pairs containing u and the pairs contained in a box are the pairs that are not totally resolved by u. We see two types of boxes: an *upward box* consists of a bottom vertex and a top vertex to the right of it, and a *downward box* consists of a bottom vertex and a top vertex to the left of it. Let us call a pair in a box just a *box*. So u does not resolve upward boxes on the left and downward boxes on the right.

Next we also consider v. Now v will resolve downward boxes to the left of v. It also resolves pairs containing u except for the upward box containing u. In Figure 4(*B*), we see all the boxes that are not resolved by the set $\{u, v\}$. The only other pair that is not yet totally resolved is the pair u, v itself.

If S contains other bottom vertices, then these lie between u, being the left most bottom vertex in S, and v, being the right most bottom vertex in S. No such vertex will resolve any of the boxes. It might resolve the pair u, v. Hence, if there are other bottom vertices in S than u and v, then $|S| \geq 4$. In order to totally resolve the boxes, we need top vertices. Any top vertex in a box will not totally resolve that box. The top left corner will not totally resolve the upward boxes, and the top right corner will not totally resolve the downward boxes. So, if a top vertex from a box, or a top corner, is in S, then $|S| \geq 4$.

Any top vertex w between the boxes will totally resolve all boxes. It will totally resolve the pair u, v if and only if $d(w, u) \neq d(w, v)$. In Figure 4(*B*) there is no such top vertex between the boxes. So, any total resolving set S with u and v in S as depicted in Figure 4(*B*) is of order at least 4.

We can do better by taking u to be the left bottom corner, and v to be the right bottom corner, see Figure 5. In this case the middle top vertex between the boxes still does not resolve the pair u, v, but the other two top vertices between the boxes resolve the pair u, v. If we take one of these to be w, then the set $S = \{u, v, w\}$ totally resolves the ladder graph, so that the total metric dimension is 3. We can find such a top vertex w as soon as $n \geq 6$. As long as $n \leq 5$, no single top vertex will simultaneously totally



(*A*) (*B*)

Figure 3. The sets of order 5 in the Petersen graph.

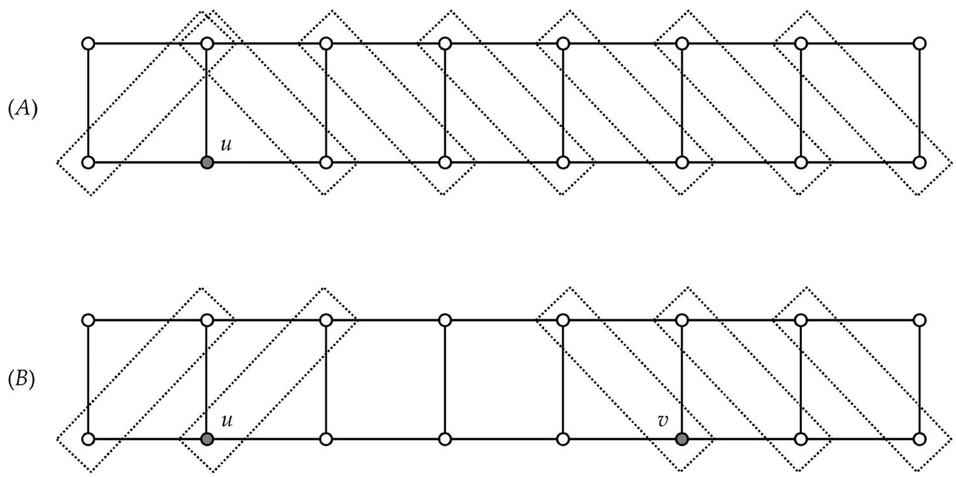


Figure 4. The ladder graph.

resolve the upward box containing the left bottom corner u, the downward box containing the right bottom corner v and the pair u, v. So we have $dim_t(P_2\square P_n) \geq 4$, for $3 \leq n \leq 5$. It is easily deduced from the above considerations that the four corners totally resolve the n-ladder, for any $n \geq 3$.

Thus we have proved the following result.

Theorem 21. *For the n-ladder* $P_2\square P_n$,

$$dim_t(P_2\square P_n) = \begin{cases} 4 & \text{if } 3 \leq n \leq 5 \\ 3 & \text{if } n \geq 6. \end{cases}$$

Next we turn to the grids $P_m\square P_n$ with $3 \leq m \leq n$. We will show that, except for a few small cases, the total metric dimension of such grids is 3. We use Figure 6 to illustrate our arguments, but they apply to all larger grids (except for the three small cases $m = n = 3, 3 = m < n = 4,$ and $4 = m < n = 5$, with which we deal separately).

To construct a *TRS*, we take the bottom left corner u and the bottom right corner v of the grid. In Figure 6 we give two instances. We see two boxes in each of these grids. We call the one containing u the *u-box*, and the one containing v the *v-box*. A *u-pair* is a pair in the u-box containing u, and, similarly, a *v-pair* is a pair in the v-box containing v. Note that v does not resolve any pair in the u-box, but u resolves the pairs of white vertices in the u-box. A similar phenomenon holds for v.

Thus the set $\{u, v\}$ neither totally resolves the pair u, v, nor the u-pairs, nor the v-pairs. All other pairs, so also any pair of white vertices within a box, are totally resolved by the set $\{u, v\}$. Let w be a vertex in the top row of the grid. If w is the top left corner, then it will not resolve the pairs in the u-box. Similarly, if w is the top right corner, then it will not resolve the pairs in the v-box. If w is in the u-box, then it will not resolve the pair u, w. If w is in the v-box, then it will not resolve the pair w, v. Next we consider two cases where w is not in a box and is not a corner.

Proposition 22. *Let* $m \geq 3$ *and let* $n \geq m+2$. *Then* $\dim_t(P_m \square P_n) = 3$.

Proof. This case is illustrated in Figure 6(*B*). Since $n \geq m+2$, the neighbor w of the top left corner is to the left of the boxes. Hence w resolves all pairs in the v-box, in particular the v-pairs. The distance between w and any of the white vertices in the u-box is $m - 1$, whereas $d(w,u) = m+1$. So w resolves all the u-pairs as well.

Finally, we have $d(w,u) = m+1$ and $d(w,v) = (n-1)+m \geq m+4$. So w also resolves the pair u, v. Hence $\{u, v, w\}$ is a *TRS*. □

Proposition 23. *Let* $m = n = 4$ *or* $5 \leq m \leq n \leq m+1$. *Then* $dim_t(P_m\square P_n) = 3$.

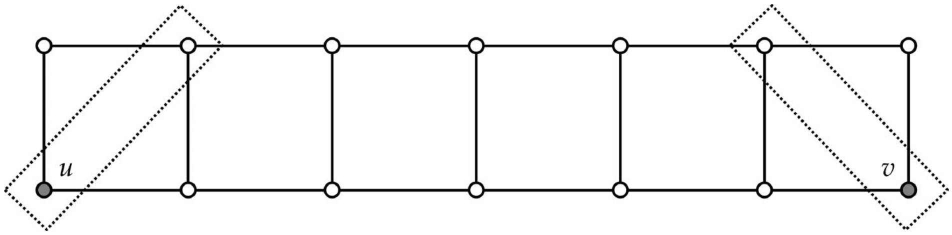
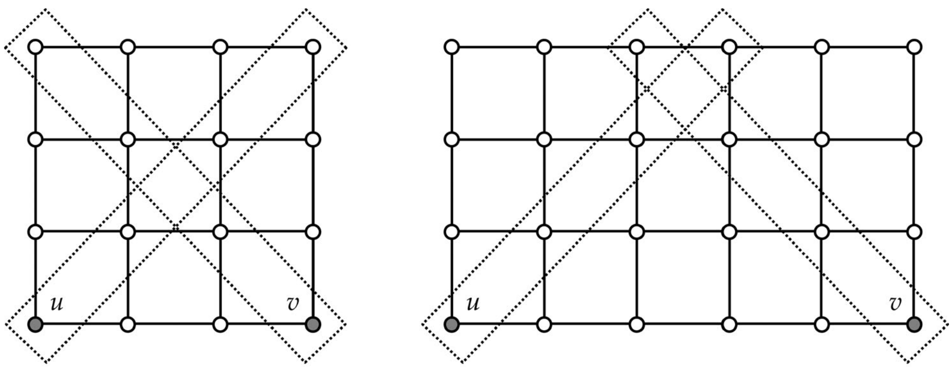


Figure 5. Two corners in the ladder graph.



(A) (B)

Figure 6. The larger grids.

Proof. The case $m = n = 4$ is illustrated in Figure 6(A). In this case, but also in the case $5 \le m \le n \le m + 1$, the top vertex of the v-box is to the left of the top vertex of the u-box, and there are at least two vertices in the top row between the two boxes. At least one of these is a vertex w with $d(w, u) \neq d(w, v)$. So this vertex w resolves the pair u, v. Clearly such a vertex w has smaller distances to the white vertices in the u-box than to u, and also smaller distances to the white vertices in the v-box than to v. Thus the set $\{u, v, w\}$ is a *TRS*. □

The only cases, in which such a non-corner vertex w to the left from the boxes, or between the boxes with different distances to u and v, cannot be found, are when $m = n = 3$, or $m = 3$ and $n = 4$, or $m = 4$ and $n = 5$. We consider these three cases separately.

Lemma 24. $dim_t(P_3 \square P_3) = 4$.

Proof. Assume to the contrary that $P_3 \square P_3$ has a total resolving set S of order 3. Then, by Lemma 14, all three distances between the pairs in S must be different. This implies that S cannot consist of only corners. Also S cannot consist of only non-corners. This is easily checked in Figure 7(A). So S contains at least one corner, without loss of generality, the left bottom corner. Moreover S contains at least one non-corner.

Suppose S contains a neighbor of u, see Figure 7(B), where two pairs are highlighted by boxes. It is straightforward to check that none of the other vertices totally resolves both boxes. So S cannot contain a neighbor of u.

Suppose S contains a middle vertex of the top row or the right most column, say the middle vertex of the top row, see Figure 7(C). To avoid a corner with a neighbor in S, the two top corners are not in S. To have different distances in S, the third vertex in S must be the central vertex of the grid. But now the pair in the box is not totally resolved by S.

Finally, there is only one possible non-corner vertex in S, viz. the central vertex. Any third vertex must now be a corner, and the middle vertex has distance two to both these corners. Again S is not a total resolving set.

Thus we have shown that no 3-set totally resolves the 3×3 grid. In Figure 7(D), a 4-set S of the gray vertices is depicted, which totally resolves $P_3 \square P_3$. This completes the proof. □

Lemma 25. $dim_t(P_3 \square P_4) = 4$.

Proof. Assume to the contrary that there is a total resolving set S of order 3. If S is contained in a 3×3 subgrid, then, by Lemma 24, already four vertices are needed to totally resolve this subgrid. So S must contain a vertex of the left most column and a vertex of the right most column. If S is contained in a 2×4 subgrid, then we get a similar contradiction by Theorem 21, which tells us that there are four vertices needed to totally resolve a 2×4 grid.

So S must contain a vertex from the left most column, from the right most column, from the bottom row, and from the top row. If S does not contain a corner, then already four vertices are needed to satisfy this condition. So S contains a corner, without loss of generality the left bottom corner u. Let v be a vertex in S from the right most column. If v is the top corner on the right, then it is easily seen that no third vertex simultaneously totally resolves all remaining unresolved pairs. The same holds if v is the middle vertex in the right most corner. So v has to be the bottom corner on the right. From our arguments above, it

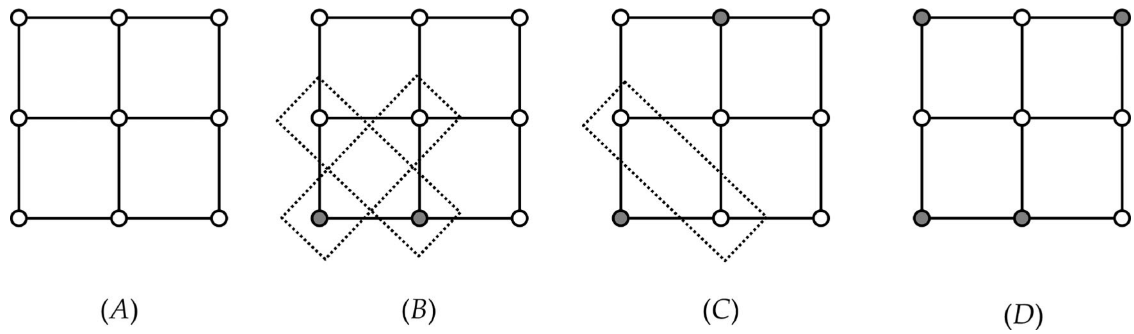


Figure 7. The 3×3 grid.

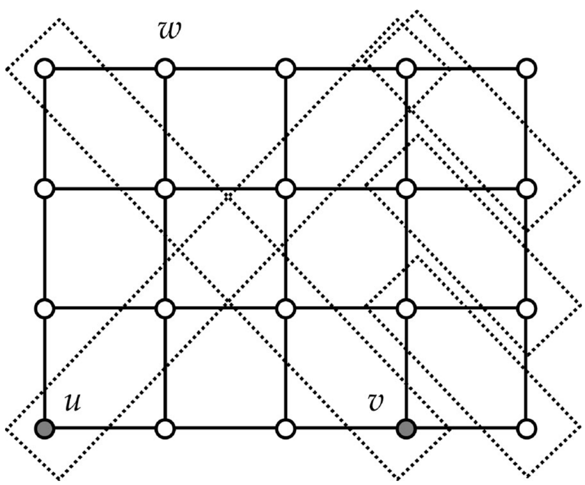


Figure 8. The 4×5 grid.

follows that again there is no third vertex that totally resolves the remaining unresolved pairs.

Finally, the four corners of the 3×4 grid totally resolve the whole grid, by which the proof is complete. □

Lemma 26. $dim_t(P_4 \square P_5) = 3$.

Proof. We choose a vertex u and a vertex v as in Figure 8. In the figure, we see boxes indicating the pairs that are not yet totally resolved by u or v: the usual u-box containing the non-resolved u-pairs, the v-box containing the non-resolved v-pairs, but also three downward boxes on the right containing non-resolved pairs. Obviously, w resolves all these pairs. So $S = \{u, v, w\}$ is a 3-set that totally resolves the 4×5 grid. □

Thus we have proved the following theorem.

Theorem 27. *Let m and n be integers with $3 \le m \le n$. Then*

$$dim_t(P_m \square P_n) = \begin{cases} 4 & \text{if } 3 = m \le n \le 4 \\ 3 & \text{if } m = 3 \text{ and } n \ge 5, \text{ or } 4 \le m \le n. \end{cases}$$

To conclude, we observe that, in all cases, the two bottom corners form a resolving set for the grid. So the metric dimension of any $m \times n$ grid is 2, for $2 \le m \le n$.

5. Concluding remarks

The metric dimension of a graph G of order n is defined in terms of the set of n ordered vectors, $r(w|S) = (d(w, v_1), d(w, v_2), ..., d(w, v_k))$, consisting of the distances of each vertex $w \in V$ to each member of an ordered set $S = \{v_1, v_2, ..., v_k\}$ of k vertices in a graph $G = (V, E)$. The simple condition for a resolving set is that no two of these ordered vectors are identical. This leads one to ask, what other conditions would be reasonable to impose on this set of n vectors?

For example, what if you only considered the set of k distances, with duplications possible, rather than the ordered vector of distances, and you impose the condition that no two of these multisets of distances are identical for two different vertices?

A different type of condition is studied in [6]. Here the requirement for the vectors is that the entries in each vector $r(w|S)$ are distinct. Such sets are called *different-distance sets* in [6]. Note that two vertices v and w now may have the same vector $r(v|S) = r(w|S)$, as long as the entries in this vector are all distinct.

You could also consider situations in which the vectors of distances of the vertices in $V - S$ to the vertices in S must satisfy some condition, but this condition does not apply to the vectors of the vertices in S, similar to the case of total resolving sets.

Notice that we have shown that for paths P_n of order $n \ge 4, dim_t(P_n) = 3$ and for cycles C_n of order $n \ge 6, dim_t(C_n) = 3$. This is a best or smallest possible value of $dim_t(G)$. We say that a graph G of order n is *TRS-min* if $dim_t(G) = 3$, and is *TRS-max* if $dim_t(G) = n$.

Question 8. *Can you characterize the class of TRS-min graphs?*

Question 9. *Can you construct a TRS-max graph?*

Question 10. *Can you characterize the class of TRS-max graphs?*

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