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The k-conversion number of regular graphs

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ABSTRACT

Given a graph $G=(V,E)$ and a set $S_0 \subseteq V$, an irreversible k -threshold conversion process on G is an iterative process wherein, for each $t=1,2,\ldots$, S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. We call the set S_0 the seed set of the process, and refer to S_0 as an irreversible k-threshold conversion set, or a k-conversion set, of G if $S_t = V(G)$ for some $t \geq 0$. The k-conversion number $c_k(G)$ is the size of a minimum k-conversion set of G. A set $X \subseteq V$ is a decycling set, or feedback vertex set, if and only if $G[V-X]$ is acyclic. It is known that k-conversion sets in $(k+1)$-regular graphs coincide with decycling sets. We characterize k-regular graphs having a k-conversion set of size k, discuss properties of $(k+1)$-regular graphs having a k-conversion set of size k, and obtain a lower bound for $c_k(G)$ for $(k+r)$-regular graphs. We present classes of cubic graphs that attain the bound for $c_2(G)$, and others that exceed it—for example, we construct classes of 3-connected cubic graphs H_m of arbitrary girth that exceed the lower bound for $c_2(H_m)$ by at least m.

KEYWORDS

Irreversible k-threshold conversion process; k-conversion number; decycling set; decycling number; cubic graph

AMS SUBJECT CLASSIFICATION 2010

05C99; 05C70; 94C15

1. Introduction

Given a graph $G=(V,E)$ and a set $S_0 \subseteq V$, an *irreversible k-threshold conversion process* on G is an iterative process wherein, for each $t=1,2,\ldots$, S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. We call the set S_0 the *seed set* of the process, and refer to S_0 as an *irreversible k-threshold conversion set*, or simply a *k-conversion set*, of G if $S_t = V(G)$ for some $t \geq 0$. The *k-conversion number* $c_k(G)$ is the size of a minimum k-conversion set of G.

A set $X \subseteq V$ is a *decycling set*, or *feedback vertex set*, if and only if $G[V-X]$ is acyclic. Early research on decycling sets was motivated by applications in logic networks and circuit theory, first in digraphs [8, 23] and later in undirected graphs [15]. More modern applications are given in [16]. The *decycling number* $\phi(G)$ of a graph G is the size of a minimum decycling set of G. Clearly, finding a minimum decycling set of G is equivalent to finding a maximum induced forest. The order of such a forest is called the *forest number* of G, and denoted by $a(G)$. Many authors have derived bounds on $\phi(G)$ and $a(G)$, both for general graphs [3] and for special classes of graphs, including planar graphs [11, 12, 22], cubic graphs [4, 19, 25, 29, 31, 32, 39] and other regular graphs [28, 30].

Dreyer and Roberts [10] have shown that decycling sets in r-regular graphs coincide with $(r-1)$-conversion sets (see Proposition 2.1). Therefore, if G is $(k+1)$-regular, then $c_k(G) = \phi(G)$. A detailed survey of results on k-conversion processes, including results on decycling sets in regular graphs, can be found in [37].

We consider lower bounds on $c_k(G)$ for regular graphs and discuss classes of graphs that meet, or do not meet, the given bound. We begin, in Section 2, by characterizing k-regular graphs having a k-conversion set of size k. In Section 3 we consider $c_k(G)$ for $(k+1)$-regular graphs, first investigating $(k+1)$-regular graphs with $c_k(G)=k$ and then discussing lower bounds on $c_k(G)$. In Section 4 we obtain a lower bound for $c_k(G)$ for $(k+r)$-regular graphs. We restrict our attention to cubic graphs in Section 5 and present classes of cubic graphs that attain the bound for $c_2(G)$, and others that exceed it. It is known that fullerenes and snarks meet the lower bound. We study the 2-conversion number of graphs that have some of the defining properties of snarks in Section 5.1. Our results in this section lead us to study 3-connected cubic graphs in Section 5.2, where we construct classes of 3-connected cubic graphs H_m of arbitrary girth (and other properties) that exceed the lower bound for $c_2(H_m)$ by at least m.

We generally follow the notation of [5]. For graphs G and H, $G+H$ denotes the disjoint union of G and H, and $G \vee H$ denotes the *join* of G and H, obtained by adding all possible edges between G and H. We denote the independence number of G by $\alpha(G)$.

2. k-regular graphs with k-conversion number k

We begin with the straightforward observation that, in order for any conversion to occur in a k-conversion process, the

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seed set must contain at least k vertices. Therefore, k is a trivial lower bound on $c_k(G)$ for any graph G with at least k vertices. More specifically, if G is a graph of order n and maximum degree Δ, then $c_k(G) = n$ if $\Delta < k$ and otherwise $c_k(G) \geq k$. Leaving aside the case where $c_k(G) = n$, we focus on graphs with maximum degree at least k and ask which graphs meet the bound $c_k(G) = k$.

Graphs that meet this bound are easy to find, and exist for any order $k + r$, where $r \geq 1$. (Take, for example, the complete bipartite graph $K_{k,r}$.) Imposing structural constraints on G naturally makes the bound harder to achieve. In Proposition 2.2 we give a complete characterization of the k-regular graphs that meet this bound. In Section 3 we will expand our investigation of the bound to include $(k+1)$-regular graphs.

We first state the following proposition by Dreyer and Roberts for referencing.

Proposition 2.1 [10].

(a) *If G is a k-regular graph, then S is a k-conversion set of G if and only if $V - S$ is independent.*
(b) *If G is a $(k+1)$-regular graph, then S is a k-conversion set of G if and only if $G[V - S]$ is a forest.*

An immediate consequence of Proposition 2.1(a) is that if G is a k-regular graph of order n, then $c_k(G) = n - \alpha(G)$.

Proposition 2.2. *A k-regular graph G has a k-conversion set of size k (that is, $c_k(G) = k$) if and only if $G = H \vee \overline{K_{k\bar{t}}}$, where H is a t-regular graph of order k, and $0 \leq t < k$.*

Proof. Let $G = H \vee \overline{K_{k\bar{t}}}$, where H and t are as above. Each vertex of $\overline{K_{k\bar{t}}}$ has k neighbors in H, so $V(H)$ is a k-conversion set of size k. Since vertices of $\overline{K_{k\bar{t}}}$ have no other neighbors, and each vertex of H has t neighbors in H and $k - t$ neighbors in $\overline{K_{k\bar{t}}}$, G is k-regular. For the converse, let G be a k-regular graph with a k-conversion set S of order k. By Proposition 2.1(a), $V - S$ is independent. Since G is k-regular, $G[S]$ is t-regular for some $0 \leq t < k$ and $|V - S| = k - t$. The result follows with $H = G[S]$. □

3. The k-conversion number of $(k+1)$-regular graphs

In this section we present lower bounds on the k-conversion number of a $(k+1)$-regular graph and determine some properties of the graphs that meet these bounds. We begin with the trivial lower bound $c_k(G) \geq k$, this time applied to $(k+1)$-regular graphs. Proposition 2.1(b) states that a set S is a k-conversion set of a $(k+1)$-regular graph G if and only if $G[V - S]$ is acyclic. In this case S is also known as a decycling set or a feedback vertex set. We rely heavily on this characterization of k-conversion sets in $(k+1)$-regular graphs throughout Section 3.

3.1. *k-conversion sets of size k in $(k+1)$-regular graphs*

If $r \geq 1$ and G is a $(k+r)$-regular graph with a k-conversion set S of size k, then every non-seed vertex has at least r neighbors outside of S. This introduces the possibility that complete conversion of the graph takes more than one time step. For $t \geq 0$, let S_t be the set of vertices that convert at time t, starting from a given seed set $S = S_0$. (It is worth noting that such a graph may still convert in one time step. For example, consider the 4-regular graph $G = \overline{K_3} \vee (K_2 + K_2)$, with 3-conversion set $S = V(\overline{K_3})$.)

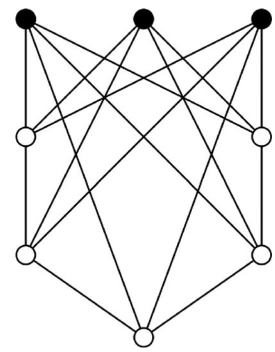


Figure 1. A 4-regular graph with $c_3(G) = 3 = |\cup_{t\geq 2} S_t|$, illustrating sharpness of the bound in Proposition 3.3. This graph also illustrates the construction in Proposition 3.2, with $k = 3$.

In Proposition 3.1 we derive a bound on the number of non-seed vertices in a $(k+1)$-regular graph with a k-conversion set of size k. We use this result to obtain a sharp upper bound on the order of such a graph (Proposition 3.2).

Proposition 3.1. *Let G be a $(k+1)$-regular graph and suppose that S_0 is a k-conversion set of size k. Then $|V(G) - S_0| < \frac{k(k+1)-1}{k-1}$.*

Proof. We begin by deriving a bound on the number of vertices that convert at time $t = 2$ and later. Let $Y = \cup_{t\geq 2} S_t$. We count the edges between Y and S_0 in two ways. First, since G is $(k+1)$-regular and each vertex of S_0 is adjacent to each vertex of S_1, there are at most $k(k+1-|S_1|)$ edges from S_0 to Y. On the other hand, each vertex in Y has at least k neighbors that convert before it. Therefore there are at least $|Y|k$ edges with at least one endpoint in Y. Since $G - S_0$ is a forest with $|Y| + |S_1|$ vertices, at most $|Y| + |S_1| - 1$ have the other endpoint in $Y \cup S_1$. Therefore there are at least $|Y|k - |Y| - |S_1| + 1$ edges from Y to S_0. This gives $|Y|k - |Y| - |S_1| + 1 \leq k(k+1-|S_1|)$. Rearranging, and replacing Y with $\cup_{t\geq 2} S_t$, gives the bound

$$\left| \bigcup_{t\geq 2} S_t \right| \leq \frac{k(k+1) + |S_1|(1-k) - 1}{k-1}. \tag{1}$$

The left side of (1) equals $|V(G) - S_0| - |S_1|$, and the result follows. □

In Proposition 3.2, we use Proposition 3.1 to derive an upper bound on the order of a $(k+1)$-regular graph having a k-conversion set of size k and we prove by construction that the bound is sharp for each $k \geq 2$. The result of the construction for $k = 3$ is illustrated in Figure 1. Let v be a vertex such that $\deg(v) \leq \Delta$. We define the Δ-*deficiency* of v to be $\mathrm{def}_\Delta(v) = \Delta - \deg(v)$.

Proposition 3.2. *If G is a $(k+1)$-regular graph having a k-conversion set of size k, then the order of G is at most $2k+2$.*

Moreover, for every $k \geq 2$, there exists a $(k+1)$-regular graph of order $2k+2$ which has a k-conversion set of size k.

Proof. We obtain the bound for $k=2$ by checking all examples (there are three cubic graphs having a 2-conversion set of size 2: K_4 and the two cubic graphs of order 6). For $k \geq 3$, $\frac{k(k+1)-1}{k-1} > k+3$, so the bound follows from Proposition 3.1.

To prove that the bound is sharp, we construct a $(k+1)$-regular graph of order $2k+2$ which has a k-conversion set of size k. We begin with the graph $K_{2,k}$, where $S_0 = S$ is the set of size k (a k-conversion set) and $S_1 = \{u_1, v_1\}$ is the set of size 2 (the set of vertices that convert at time $t=1$). For each $v \in S_0$ we now have $\mathrm{def}_{k+1}(v) = k-1$ and for each $v \in S_1$ we have $\mathrm{def}_{k+1}(v) = 1$. We will add vertex sets $S_2, S_3, \ldots$ such that the vertices of S_i convert at time $t=i$ from the k-conversion set S_0. To achieve this, for each $i \geq 2$, we must add at least k edges from S_i to $\cup_{j=0}^{i-1} S_j$. Some care is required in choosing the edges, in order to ensure that there will always be at least k distinct vertices available in $\cup_{j=0}^{i-1} S_j$. For $i \geq 2$, if there are still at least $k-1$ vertices in S_0 of deficiency at least 2, let $S_i = \{u_i, v_i\}$. Join u_i to u_{i-1} and to $k-1$ vertices of S_0, beginning with those of highest deficiency. Then join v_i to v_{i-1} and to $k-1$ vertices of S_0, once again beginning with those of highest deficiency. Joining u_i and v_i to u_{i-1} and v_{i-1} at each step means that the vertices of $S_1, \ldots, S_{i-1}$ have degree $k+1$, so the only deficient vertices are the newly added ones and those in S_0. Joining the new vertices first to the vertices of highest deficiency in S_0 guarantees that the deficiencies among the vertices of S_0 are always within 1 of each other. Therefore, the first time there fail to be at least $k-1$ vertices in S_0 with deficiency at least 2, there are either no deficient vertices in S_0 (if k is even) or there are $k-1$ deficient vertices in S_0 and their deficiency is 1 (if k is odd). In the case where k is even, we add vertices u_i and v_i $\frac{k}{2}$ times before we run out of deficient vertices in S_0. That is, the process stops when $i = \frac{k}{2}+1$, and $|\cup_{i=2}^{\frac{k}{2}+1} S_i| = k$. Adding an edge between $u_{\frac{k}{2}+1}$ and $v_{\frac{k}{2}+1}$ yields a simple $(k+1)$-regular graph of order $2k+2$ (including the k vertices of S_0 and the 2 vertices of S_1). In the case where k is odd, we add $\frac{k-1}{2}$ pairs of vertices u_i and v_i before the deficiencies in S_0 become too small. That is, the process stops when $i = \frac{k+1}{2}$ and $|\cup_{i=2}^{\frac{k+1}{2}} S_i| = k-1$. We complete the $(k+1)$-regular graph by adding one final vertex, w, and joining it to $u_{\frac{k+1}{2}}$, $v_{\frac{k+1}{2}}$ and to the $k-1$ vertices of deficiency 1 in S_0. The total number of vertices is now $2k+2$, including the k vertices of S_0 and the 2 vertices of S_1. □

In the proof of Proposition 3.1, we derived the bound (1) on the size of $\cup_{t \geq 2} S_t$ for $(k+1)$-regular graphs with a k-conversion set of size k. Proposition 3.3, below, provides another upper bound on the same quantity. When $|S_1| \geq \frac{2k-1}{k-1}$, the bound provided by (1) is stronger than that of Proposition 3.3. However, the bound of Proposition 3.3 is sharp for small values of $|S_1|$, as shown by the graph in Figure 1.

Proposition 3.3. *Let G be a $(k+1)$-regular graph with a k-conversion set of size k. Then $|\cup_{t \geq 2} S_t| \leq k$.*

Proof. Let $Y = \cup_{t \geq 2} S_t$. By Proposition 2.1(b), $G - S_0$ is a forest F, and its leaves are the vertices in S_1. Therefore, for every $v \in Y$, $\deg_F(v) \leq |S_1|$, and $\deg_G(v) = k+1$, so v has at least $k+1-|S_1|$ neighbors in S_0. Hence there are at least $|Y|(k+1-|S_1|)$ edges between Y and S_0. On the other hand, there are at most $k(k+1-|S_1|)$ edges between S_0 and Y, by the argument given in the proof of Proposition 3.1. Therefore $|Y|(k+1-|S_1|) \leq k(k+1-|S_1|)$. □

3.2. A lower bound on $c_k(G)$ for $(k+1)$-regular graphs

In Sections 2 and 3.1 we began with a fixed seed set size (namely k, the minimum possible size for a nontrivial k-conversion set), and asked which graphs have a k-conversion set of this size. We obtained constraints on the structure and order (respectively) of k- and $(k+1)$-regular graphs with this property. In this section we instead begin with a class of graphs, and ask how small a k-conversion set can be for a graph in this class.

As discussed in Section 2, k is a lower bound on the k-conversion number of any graph with order at least k. While it is possible to have arbitrarily large graphs that attain this bound, for many classes of graphs a k-conversion set of size k can only convert a limited number of vertices. Indeed, we showed in Proposition 3.2 that in the class of $(k+1)$-regular graphs, a k-conversion set of size k can convert at most $2k+2$ vertices. For these graphs, as the order grows beyond the $2k+2$ threshold, we require more than k seed vertices to convert the graph. In this case, k is no longer a good lower bound for the k-conversion number.

Beinecke and Vandell [3, Corollary 1.2] showed that if G is a graph with n vertices, m edges, and maximum degree Δ, then the decycling number of G is at least $\frac{m-n+1}{\Delta-1}$. This generalized the lower bound obtained by Staton [32] on the decycling number of $(k+1)$-regular graphs, which corresponds to the k-conversion number. We present a proof of Staton's result which yields a condition for equality in the bound.

Proposition 3.4. *Let G be a $(k+1)$-regular graph of order n, $k \geq 2$. Then $c_k(G) \geq \lceil \frac{n(k-1)+2}{2k} \rceil$. Moreover, a minimum k-conversion set S of G has size $\frac{n(k-1)+2}{2k}$ if and only if S is independent and $G-S$ is a tree.*

Proof. Let S be a minimum k-conversion set of G, and let $\bar{S} = V(G) - S$. For $X \in \{S, \bar{S}\}$, let n_X and m_X denote the number of vertices and edges, respectively, in $G[X]$. Counting in two ways the number of edges between S and $\bar{S}$ gives the identity

$$(k+1)n_S - 2m_S = (k+1)n_{\bar{S}} - 2m_{\bar{S}}.$$

By Proposition 2.1(b), $G[\bar{S}]$ is a forest; let y be its number of components. Then

$$(k+1)n_S - 2m_S = (k+1)n_{\bar{S}} - 2(n_{\bar{S}} - y).$$

Substituting $n_{\bar{S}} = n - n_S$, and rearranging, this gives

$$n_S = \frac{n(k-1) + 2m_s + 2y}{2k}.$$

Therefore, $c_k(G) = n_S \geq \frac{n(k-1)+2}{2k}$, with equality if and only if S is independent and $G - S$ is a tree. In particular, $c_k(G) \geq \lceil \frac{n(k-1)+2}{2k} \rceil$. □

In the next section we prove a lower bound similar to that of Proposition 3.4 for $(k+r)$-regular graphs.

4. A lower bound on $c_k(G)$ for $(k+r)$-regular graphs

Dreyer and Roberts [10] give a lower bound of $\frac{(k-r)n}{2k}$ on $c_k(G)$ for $(k+r)$-regular graphs of order n, for $0 \leq r < k$. In the case $r = k-1$, where G is a $(2k-1)$-regular graph, Zaker [38] strengthens this bound to $c_k(G) \geq \frac{n+2(k-1)}{2k}$. In this section we improve upon both of these previous bounds by providing, in Proposition 4.3, a new lower bound of $c_k(G) \geq \frac{(k-r)n+(r+1)r}{2k}$, which is sharp for all r, $0 \leq r \leq k-1$.

Proposition 4.1 generalizes Proposition 2.1 by characterizing the k-conversion sets S of $(k+r)$-regular graphs in terms of a condition on $V - S$. For $r \geq 0$, a graph G is *r-degenerate* if every induced subgraph of G has a vertex of degree at most r. We say that G is a *maximal r-degenerate* graph if G is r-degenerate but for every pair of non-adjacent vertices x, y in G, adding the edge xy to $E(G)$ produces a graph that is not r-degenerate. We note that a graph G is 0-degenerate if and only if it has no edges, and it is 1-degenerate if and only if it is acyclic.

We call a nonempty set U of vertices of a graph G *k-immune* if every vertex in U has fewer than k neighbors in $V - U$. It is straightforward to see that $S \subseteq V$ is a k-conversion set of G if and only if $V - S$ does not contain a k-immune set. We use this observation in the proof of Proposition 4.1, and again in Section 5.1.

Proposition 4.1. *Let G be a $(k+r)$-regular graph, with $r \geq 0$. A set S of vertices of G is a k-conversion set if and only if $G[V-S]$ is r-degenerate.*

Proof. Suppose $V - S$ is r-degenerate, so every subgraph H of $V - S$ has a vertex of degree at most r. In other words, some vertex of H has at least k neighbors in $G - H$. Let $H_0 = V - S$ and let S_1 be the set of vertices of degree at most r in H_0. These vertices have at least k neighbors in $G - H_0 = S$, so they convert at time $t = 1$. Let $H_1 = H_0 - S_1$ and let S_2 be the set of vertices of degree at most r in H_1. These vertices have at least k neighbors in $V - H_1 = S \cup S_1$, so they convert at time $t = 2$. Continue this process until some $H_i = \emptyset$. At each step, the set $V - H_j$ is converted, so when $H_i = \emptyset$ the whole graph is converted. Therefore S is a k-conversion set. On the other hand, if $V - S$ is not r-degenerate then there is some subgraph H of $V - S$ in which no vertex has k neighbors outside H. Therefore $V(H)$ is a k-immune set, so S is not a conversion set of G. □

Proposition 4.3 generalizes Proposition 3.4, establishing a lower bound on $c_k(G)$ for $(k+r)$-regular graphs G. The proof technique is the same as for Proposition 3.4, but requires the following lemma, due to Lick and White, bounding the number of edges in an r-degenerate graph.

Lemma 4.2 [24, Proposition 3 and Corollary 1]. *Let G be an r-degenerate graph with $n \geq r$ vertices and m edges. Then $m \leq rn - \binom{r+1}{2}$, with equality if and only if G is maximal r-degenerate.*

Proposition 4.3. *Let G be a $(k+r)$-regular graph of order n, where $0 \leq r < k$. Then*

$$c_k(G) \geq \frac{(k-r)n + (r+1)r}{2k}.$$

Moreover, for $r \geq 1$, a minimum k-conversion set S of G has order $\frac{(k-r)n+(r+1)r}{2k}$ if and only if S is independent and $G[V-S]$ is a maximal r-degenerate graph.

Proof. First suppose $r = 0$. Proposition 2.1(a) implies that $c_k(G) = n - \alpha(G)$. Since G is regular, $\alpha(G) \leq \frac{n}{2}$, and the result follows. Now let $r \geq 1$ and let G be a $(k+r)$-regular graph with $n > k + r$ vertices. Let S be a k-conversion set of G and for $X \in \{S, \bar{S}\}$, let n_X and m_X denote the number of vertices in X and the number of edges in $G[X]$, respectively. Counting in two ways the edges between S and $\bar{S}$ gives

$$(k+r)n_S - 2m_S = (k+r)n_{\bar{S}} - 2m_{\bar{S}}.$$

Applying the bound $m_{\bar{S}} \leq rn_{\bar{S}} - \binom{r+1}{2}$, as provided by Lemma 4.2, and simplifying gives

$$(k+r)n_S - 2m_S \geq (k-r)n_{\bar{S}} + (r+1)r,$$

with equality if and only if $G[\bar{S}]$ is maximal r-degenerate. By substituting $n_{\bar{S}} = n - n_S$ and rearranging, we obtain

$$n_S \geq \frac{(k-r)n + (r+1)r + 2m_S}{2k},$$

with equality if and only if $G[\bar{S}]$ is maximal r-degenerate. The result follows since $m_S \geq 0$ with equality if and only if S is an independent set. □

We note that, by definition of maximal r-degeneracy, in order to determine whether a subgraph H of G (in particular, $H = G[\bar{S}]$) is maximal r-degenerate we must look at all $x, y \in V(H)$ such that $xy \notin E(H)$—regardless of whether $xy \in E(G)$— and determine whether $H + xy$ is still r-degenerate. In other words the maximality of H with respect to r-degeneracy does not depend on whether we can add more vertices or edges of G into H without losing the r-degenerate property, but whether we can add an edge between two non-adjacent vertices of H. In particular, when $H = G[\bar{S}]$, H is an induced subgraph so any additional edge xy under consideration is necessarily absent from G.

5. Cubic graphs

For $k = 2$, Proposition 3.4 gives the lower bound

$$c_2(G) \geq \left\lceil \frac{n+2}{4} \right\rceil \tag{2}$$

for cubic graphs G of order n. In this section we present classes of cubic graphs that attain this bound and others that exceed it. We begin by stating a result by Payan and Sakarovitch [27] that provides a sufficient condition for equality in the bound.

A graph G is *cyclically k-edge connected* (*cyclicallyk-vertex connected*) if at least k edges (vertices) must be removed to disconnect G into two subgraphs that each contain a cycle. A cubic graph $G \notin \{K_4, K_{3,3}\}$ is cyclically 4-edge connected if and only if it is cyclically 4-vertex connected [26], so we simply call such graphs *cyclically* 4-*connected*.

Theorem 5.1 [27]. *If G is a cyclically 4-connected cubic graph of order n, then*

$$c_2(G) = \left\lceil \frac{n+2}{4} \right\rceil.$$

A *fullerene* is a planar cubic graph whose faces, including the outer face, in any plane representation, all have size 5 or 6. Doš lić [9, Theorem 8] proved that all fullerenes are cyclically 4-edge connected, and therefore by Theorem 5.1 they achieve equality in the lower bound (2).

5.1. Snarks and would-be snarks

By Vizing's theorem [5, Theorem 17.2], if G is a graph with maximum degree Δ, then G has chromatic index Δ or $\Delta +$ 1; in the former case, G is of Class 1, and in the latter case, of Class 2.

A *snark* is a connected, bridgeless, Class 2 cubic graph. To avoid degenerate cases, it has long been standard to require snarks to be triangle-free. They have been studied since the 1880's, when Tait [34] proved that the Four Color Theorem is equivalent to the statement that no snark is planar. We refer to such snarks (connected, bridgeless, triangle-free Class 2 cubic graphs) as *Gardner snarks*, as this was the common definition of snarks when Martin Gardner gave them the name "snark" in 1975 [14]. The name, taken from the elusive creature in Lewis Carroll's poem *The Hunting of the Snark*, reflects the scarcity of examples in the years after Tait defined them. The smallest and earliest known example of a snark is the Petersen graph, first mentioned by Alfred Bray Kempe in 1886 [21] and named after the Danish mathematician Julius Petersen, who presented it as counterexample to Tait's claim that all cubic graphs were 3-edge colorable. Due to their connection with the Four Color Theorem (Four Color Conjecture, at the time), much attention was given to the pursuit of new examples of snarks (with the hope of finding a planar one, perhaps), but a second example was not discovered until 1946. Since then, more examples have been discovered, including infinite families.

Interest in snarks has remained steady, due in part to their connection to other important conjectures in graph theory, notably the Cycle Double Cover Conjecture [6, 33]. In 1985, Jaeger [20] proved that a smallest counterexample to the conjecture must be a snark; therefore, if the conjecture is true for snarks, it is true for all graphs.

More recently, more restrictive definitions of snarks have become the standard. It is now common to require snarks to have higher connectivity and larger girth. Some authors use even more restrictive definitions in order to exclude snarks that can be obtained from other snarks. Some require them to be cyclically 4-edge connected, rather than simply triangle-free [17]. We call cyclically 4-edge connected snarks of girth at least five (at least four) *strong (weak) snarks*. A convenient overview of approximately the first century of snark research, including a discussion of modern definitions, can be found in [35].

By Theorem 5.1, strong and weak snarks achieve equality in the lower bound (2). It is therefore natural to ask whether all snarks do. However, we will show in Section 5.2 that there exist infinitely many Gardner snarks that fail to meet the bound.

Theorems 5.2 and 5.3 give well-known sufficient conditions for cubic graphs to be Class 1 and Class 2, respectively, which aids our search for examples in each category. Theorem 5.2 was shown by Tait in 1880 to be equivalent to the Four Color Theorem.

Theorem 5.2 [1, 2, 34]. *Every bridgeless planar cubic graph has chromatic index 3.*

Theorem 5.3 [18]. *Every bridged cubic graph has chromatic index 4.*

Theorem 5.3 allows us to limit our investigation to graphs that are bridgeless or Class 2, since there are no bridged, Class 1 cubic graphs. All other combinations—that is, all allowable combinations—of the three defining characteristics of snarks (bridgeless, Class 2, triangle-free) admit graphs that meet the lower bound and graphs that do not meet the lower bound. Table 1 gives an example of a graph for each type for each of the combinations.

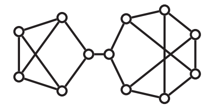
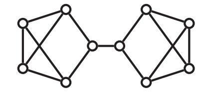
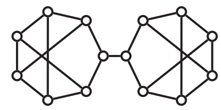
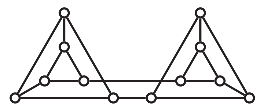
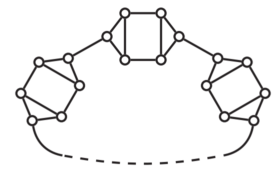
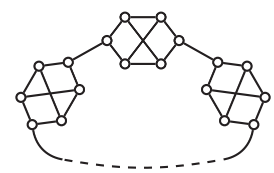
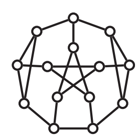
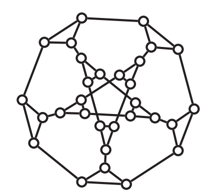
For each combination of properties except bridgeless, Class 2, triangle-free cubic graphs (i.e., Gardner snarks), we now show that the difference between the bound and the 2-conversion number can be arbitrarily large (Propositions 5.6–5.11). We address the remaining category in Section 5.2, where we consider 3-connected cubic graphs with arbitrary girth.

To prove that the difference between the bound and the 2-conversion number can be arbitrarily large for graphs with bridges, we use the following lemma.

Lemma 5.4. *Let G be a cubic graph with a bridge e, and let H_1 and H_2 be the components of $G - e$. Then* $c_2(G) = c_2(H_1) + c_2(H_2)$.

Proof. Clearly, $c_2(G) \leq c_2(H_1) + c_2(H_2)$. To show equality we show that the minimal 2-immune sets of H_1 and H_2 (with respect to containment) are the sets U that induce

Table 1. Combinations of snark properties that permit equality/inequality in the lower bound on $c_2(G)$.

Bridgeless?	Class 2?	Δ-free?	Example with $c_2(G) = \lceil \frac{n+2}{4} \rceil$	Example with $c_2(G) > \lceil \frac{n+2}{4} \rceil$
No	Yes	No		
No	Yes	Yes		Any triangle-free cubic graph of the form (H)—(H) where H has order $n \equiv 1 \pmod 4$
Yes	No	No		
Yes	No	Yes	Q_3, Fullerenes	
Yes	Yes	No		
Yes	Yes	Yes	All strong snarks	Discussion will follow[a]

[a]Examples and discussion are given in Section 5.2.

chordless cycles in G. Let U be a minimal 2-immune set of H_i and let a be the vertex of degree 2 in H_i. First consider the case where $a \notin U$. Then every vertex in U has three neighbors in H_i and, since U is 2-immune, at least 2 of them are in U. By the minimality of U, this implies that $H_i[U]$ is a chordless cycle. Now consider the case where $a \in U$. If $H_i[U]$ does not contain a cycle then it has at least two leaves, and one of these leaves is a vertex of degree 3 in H_i. This is a contradiction, since such a vertex has two neighbors outside U. On the other hand, by minimality, any cycle in $H_i[U]$ contains a (otherwise the cycle is a smaller 2-immune set). Therefore, in both cases, the minimal 2-immune sets of H_i induce chordless cycles. Since G is cubic (and therefore its 2-conversion sets are decycling sets), these are precisely the minimal 2-immune sets of G. Thus U is a minimal 2-immune set of G if and only if it is a minimal 2-immune set of H_1 or H_2. Since H_1 and H_2 are disjoint, the result follows. □

We construct several classes of graphs that exceed the bound from the four graphs H_1, H_2, H_3 and H_4 shown in Figure 2.

Lemma 5.5. *Let H_1, H_2, H_3 and H_4 be as shown in Figure 2, and let G be a cubic graph containing H_i as an induced subgraph, for some $1 \le i \le 4$. Then any minimum 2-conversion set of G contains exactly 2 vertices from each copy of H_i.*

Proof. Figure 2 gives a 2-conversion set of size 2 for each graph H_i. On the other hand, no vertex is on every cycle of H_i, so there is no 2-conversion set of G containing fewer than two vertices from any copy of H_i. □

In Propositions 5.6 and 5.7 we construct bridged, Class 2 cubic graphs with and without triangles, respectively, that exceed the bound.

Proposition 5.6. *Let $m \ge 2$ and let G be the cubic graph constructed from P_m by replacing each leaf with a copy of H_1 and each internal vertex with a copy of H_2, where H_1 and H_2 are as shown in Figure 2. Then*

(a) *G is a bridged, Class 2 cubic graph with triangles, and*
(b) $c_2(G) - \lceil \frac{|V(G)|+2}{4} \rceil = \lfloor \frac{m}{2} \rfloor$.

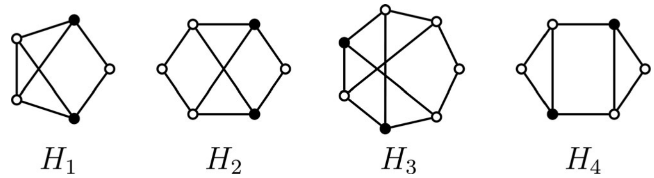


Figure 2. Building blocks for graphs that exceed the bound.

Proof. For (a), the Class 2 property follows from the bridged property by Theorem 5.3. For (b), $|V(G)| = 6m - 2$ and by Lemma 5.5, $c_2(G) = 2m$. □

Proposition 5.7. *Let $m \geq 2$ and let G be the cubic graph constructed from P_m by replacing each leaf with a copy of H_3 and each internal vertex with a copy of H_2, where H_2 and H_3 are as shown in Figure 2. Then*

(a) *G is a bridged, Class 2, triangle-free cubic graph, and*
(b) $c_2(G) - \lceil\frac{|V(G)|+2}{4}\rceil = \lfloor\frac{m}{2}\rfloor - 1.$

Proof. For (a), the Class 2 property follows from the bridged property by Theorem 5.3. For (b), $|V(G)| = 6m + 2$ and by Lemma 5.5, $c_2(G) = 2m$. □

In Proposition 5.8 we construct bridgeless, Class 1 cubic graphs with and without triangles that exceed the bound.

Proposition 5.8. *Let $m \geq 3$ and let H_2 and H_4 be as shown in Figure 2. Let G_1 be the cubic graph constructed from C_m by replacing each vertex with a copy of H_4, and let G_2 be the cubic graph constructed from C_m by replacing each vertex with a copy of H_2. Then*

(a) *G_1 is a bridgeless, Class 1 cubic graph with triangles,*
(b) *G_2 is a bridgeless, Class 1, triangle-free cubic graph, and*
(c) *for $i = 1, 2$, $c_2(G_i) - \lceil\frac{|V(G_i)|+2}{4}\rceil = \lfloor\frac{m-1}{2}\rfloor$.*

Proof. Parts (a) and (b) can be easily verified, using Theorem 5.2 for (a). For part (c), it is clear that $|V(G_i)| = 6m$ and by Lemma 5.5, $c_2(G_i) = 2m$, for $i = 1, 2$. □

We have presented cubic graphs with an arbitrary difference between c_2 and the lower bound for each of the first four categories defined in Table 1. We now describe a construction that produces graphs in the fifth category—bridgeless, Class 2 cubic graphs of girth 3—with an arbitrary difference between c_2 and the bound (2). In fact, the same construction can be used to produce additional examples for any of the girth 3 categories.

To construct girth 3 graphs (which can be bridged or bridgeless and Class 1 or Class 2) with an arbitrary difference between c_2 and the bound (2), we begin with a cubic graph G and replace each vertex with a triangle. We call this operation *triangle replacement* of G and we call the resulting girth 3 graph the *triangle-replaced graph* of G, and denote it by $T(G)$, as in [36]. Lemma 5.9 guarantees that the bridged/bridgeless properties and the Class 1/Class 2 properties are preserved under triangle replacement. Therefore in order to produce a bridgeless, Class 2 cubic graph with triangles, for example, we take the triangle replacement of any bridgeless, Class 2 cubic graph. Figure 3 shows the triangle-replaced graph of the Petersen graph. Since the Petersen graph is bridgeless and Class 2, so is its triangle-replaced graph.

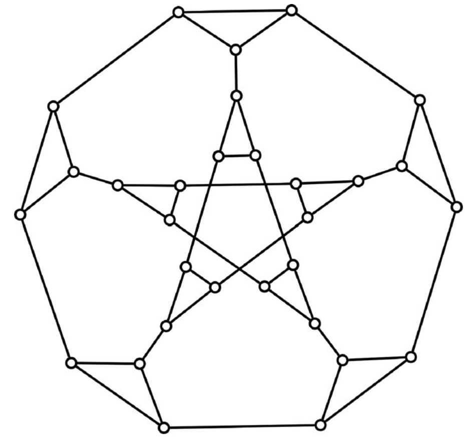


Figure 3. The triangle-replaced graph of the Petersen graph.

Lemma 5.9. For *any cubic graph G, G and $T(G)$ have the same number of bridges and the same chromatic index.*

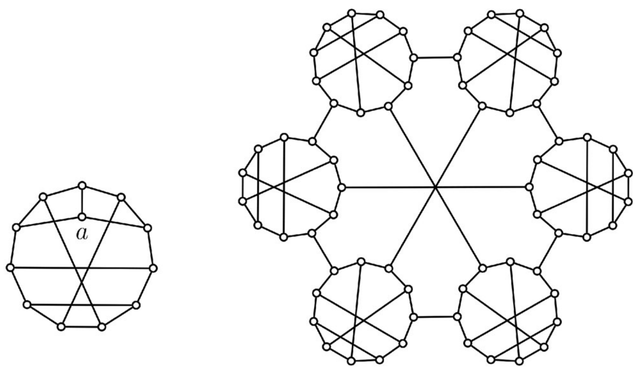
Proof. The first statement is obvious. For the second statement, let $G' = T(G)$ and let $T(v)$ denote the triangle in G' arising from v, for each vertex v of G. We consider $E(G)$ to be a subset of $E(G')$. We show that $\chi'(G) = 3$ if and only if $\chi'(G') = 3$; the result then follows by Vizing's Theorem. Suppose first that G' has a proper 3-edge coloring $f : E(G') \to \{1, 2, 3\}$. Consider three edges incident with a vertex v in G. In a proper 3-edge coloring of G', these edges all have different colors, since each is incident with two of the three edges of $T(v)$. Therefore the coloring of the edges of G obtained by restricting f to $E(G)$ is a proper 3-edge coloring of G. Now suppose G has a proper 3-edge coloring. For each $v \in V(G)$ we extend the coloring f to $T(v)$ such that the edge e of $T(v)$ gets the same color as the edge of $E(G)$ that is incident with the other two edges of $t(v)$. □

Lemma 5.10, which follows immediately from Proposition 2.1(b), gives a lower bound on $c_2(T(G))$, from which we deduce in Proposition 5.11 that there are triangle-replaced graphs $T(G)$ with arbitrary difference between $c_2(T(G))$ and the bound (2).

Lemma 5.10. *Let G be a $(k+1)$-regular graph with a collection of d pairwise disjoint cycles. Then $c_k(G) \geq d$ for all k.*

We are now ready to show that the difference between the 2-conversion number and the bound (2) for triangle-replaced graphs $T(G)$ grows with the order of G. Since there are arbitrarily large graphs G for each feasible category of cubic graphs defined in Table 1, there are arbitrarily large differences between the 2-conversion number and the bound for each category with triangles.

Proposition 5.11. Let *H be a cubic graph of order m and let $G = T(H)$. Then $c_2(G) - \lceil\frac{|V(G)|+2}{4}\rceil \geq \lfloor\frac{m-2}{4}\rfloor$. Moreover, G has the same number of bridges and the same chromatic index as H.*



(a) A cubic graph A of order $4r$

(b) $K_{3,3} \circ (A - a)$

Figure 4. An example of the construction of a cubic graph $G \circ A^{-}$.

Proof. By Lemma 5.10, $c_2(G) \geq m$. The first statement follows, with $|V(G)| = 3m$. The second statement follows from Lemma 5.9. □

For each of the first five categories of cubic graphs defined in Table 1, we have given a construction to produce a graph G with an arbitrarily large difference between $c_2(G)$ and the lower bound $\lceil\frac{|V(G)|+2}{4}\rceil$. However, for all of the triangle-free graphs, while the difference may be large, the ratio $\frac{c_2(G)}{|V(G)|}$ approaches $\frac{1}{4}$, and hence the ratio $\frac{c_2(G)}{\lceil\frac{|V(G)+2}{4}\rceil}$ approaches 1, as $|V(G)|$ becomes large. By contrast, for the girth 3 graphs we have constructed in this section, $\frac{c_2(G)}{|V(G)|}$ approaches $\frac{1}{3}$ as $|V(G)|$ becomes large. In the next section we determine whether this ratio can be greater than $\frac{1}{4}$, asymptotically, for triangle-free graphs.

5.2. Three-edge connected cubic graphs

In the previous section we constructed infinite families of graphs for which the difference between the 2-conversion number and the lower bound (2) could be made arbitrarily large. All of these examples—in fact, all examples we have seen so far that do not meet the lower bound—contain triangles or have connectivity at most 2. We also saw infinite families of graphs for which the ratio $\frac{c_2(G)}{|V(G)|}$ exceeds $\frac{1}{4}$ asymptotically (in $|V(G)|$), but all of these examples have girth 3. These observations lead us to the following two questions.

Question 5.12. *Is there a family of 3-connected, triangle-free cubic graphs G such that* $c_2(G) > \lceil\frac{|V(G)|+2}{4}\rceil$*?*

Question 5.13. *Is there a family of triangle-free cubic graphs such that*

$$\frac{c_2(G)}{|V(G)|} \to r > \frac{1}{4} \text{ as } |V(G)| \to \infty?$$

In this section we answer both questions in the affirmative. In fact, for Question 5.12 we describe a construction for an infinite family of 3-connected graphs of arbitrary girth such that the difference between c_2 and the lower bound (2) increases with order. The same family of graphs provides an answer to Question 5.13.

We begin by defining a graph product that produces an r-regular graph from two smaller r-regular graphs. In this section we use this product with $r = 3$.

Definition 5.14. Let G and A be r-regular graphs, $r \geq 2$, and define $A^{-} = A - a$, for any vertex a. Let $\mathcal{C}$ be the class of graphs that can be obtained by replacing each vertex v of G by a copy A_v^{-} of A^{-} and joining a degree $r-1$ vertex of A_u^{-} to a degree $r-1$ vertex of A_v^{-} if and only if $uv \in E(G)$. We denote by $G \circ A^{-}$ any graph in $\mathcal{C}$.

This construction can yield non-isomorphic graphs depending on a and on how the copies of A^{-} are joined. We will not need to differentiate between different elements of $\mathcal{C}$, as our results hold for any such graph. Figure 4 shows an example of a cubic graph A with vertex a identified, and a graph $K_{3,3} \circ (A - a)$.

Proposition 5.16 asserts that if A is a cubic graph of order $4r$ then $G \circ A^{-}$ exceeds the bound (2). To answer Question 5.12 we then show that the construction can yield 3-edge connected—and therefore 3-connected[1]— graphs of arbitrary girth; this is achieved in Propositions 5.17 and 5.18. We begin with a lemma which guarantees that any 2-conversion set of $G \circ A^{-}$ contains at least r vertices from each copy of A^{-}.

Lemma 5.15. *If A is a cubic graph of order $4r$ and $A^{-} = A - a$ is an induced subgraph of a cubic graph H, then any 2-conversion set of H contains at least r vertices of A^{-}.*

Proof. Suppose H has a 2-conversion set S such that $|S \cap V(A^{-})| < r$. Then $(S \cap V(A^{-}) \cup \{a\}$ is a 2-conversion set of A of cardinality at most r. However, by (2), $c_2(A) \geq \lceil\frac{4r+2}{4}\rceil = \frac{4r+4}{4} = r + 1$. □

Proposition 5.16. For *any cubic graphs G of order $n \geq 6$ and A of order $4r$,*

$$c_2(G \circ A^{-}) - \left\lceil \frac{|V(G \circ A^{-})| + 2}{4} \right\rceil \geq \left\lfloor \frac{n-2}{4} \right\rfloor.$$

Proof. Let S be a 2-conversion set of $G \circ A^{-}$. By Lemma 5.15, S contains at least r vertices of each copy of A^{-}, hence $|S| \geq nr$. The result follows because $V(G \circ A^{-})$ has order $(4r - 1)n$. □

Proposition 5.17. *Let A and G be cubic graphs. Then $G \circ A^{-}$ has girth at least $g(A)$.*

Proof. Let $g(A) = g$ and let C be any cycle in $G \circ A^{-}$. If C is contained in any copy of A^{-}, then C has length at least $g(A)$. If C is not contained in a copy of A^{-}, then for any copy A_v^{-} of A^{-}, $C \cap A_v^{-} = \emptyset$ or $C \cap A_v^{-}$ is a single path, since each copy of A^{-} is joined by only three edges to the rest of $G \circ A^{-}$. Therefore C consists of segments $Q_1, Q_2, \ldots, Q_s$ of paths in distinct copies of A^{-}, together with edges e_i joining Q_i to Q_{i+1}, $i = 1, \ldots, s-1$, and e_s joining Q_s to Q_1. Each Q_i has length at least $g - 2$, otherwise Q_i and the vertex a that was removed from A to form A^- produce a cycle of length less than g in A. Therefore C has length at least $s(g-2)$. Since G has no multiple edges, $s \geq 3$, and the result follows. □

We next show that the product $G \circ A^-$ preserves 3-connectivity.

Proposition 5.18. *Let A and G be 3-connected cubic graphs. Then $G \circ A^-$ is 3-connected.*

Proof. Let x and y be any distinct vertices of $G \circ A^-$, say $x \in V(A_u^-)$ and $y \in V(A_v^-)$, for $u, v \in V(G)$. Let u_i and v_i, $i = 1, 2, 3$, be the vertices of degree 3 in A_u^- and A_v^-, respectively.

First, suppose $u = v$. Since A is 3-connected, A contains three internally disjoint $x - y$ paths, at most one of which contains a. These correspond to three internally disjoint $x - y$ paths in $G \circ A^-$: at least two are contained in A_v^- and the third may contain the vertices v_1 and v_2, say, and a $v_1 - v_2$ path in $(G \circ A^-) - A_v^-$.

Now suppose $u \neq v$. Then in A, x is connected to a by three internally disjoint paths; therefore in A^-, x is connected to the u_i's by three internally disjoint paths. Similarly, in A_v^-, y is connected to the v_i by three internally disjoint paths. Since G is 3-connected, there are, without loss of generality, three internally disjoint paths $u_i - v_i$, $i = 1, 2, 3$. Therefore x is connected to y in $G \circ A^-$ by three internally disjoint paths. □

Together, Lemma 5.16 and Propositions 5.17 and 5.18 imply that if A is a 3-connected cubic graph of order $4r$ and girth g, and G is a 3-connected cubic graph of order $n \geq 6$, then $G \circ A^-$ is a 3-connected cubic graph of girth at least g such that $c_2(G \circ A^-)$ exceeds the bound (2) by at least $\lfloor \frac{n-2}{4} \rfloor$. We note that for $g = 3$, we may use $A = K_4$, and then the graph $G \circ A^-$ is the triangle-replaced graph of G. That is, the 3-connected cubic graphs of girth 3 that we presented in Proposition 5.11 are obtainable from the construction presented in this section.

It remains to show that there exist appropriate cubic graphs A and G for $g \geq 4$. For G, we simply require a 3-connected cubic graph of order at least 6. There are many such graphs; we highlight one example, which will also help us find A.

For $k \geq 2$ and $g \geq 3$, a (k, g)*-cage* is a graph that has the least number of vertices among all k-regular graphs with girth g. Erdös and Sachs [13], as cited in [5], proved that (k, g)-cages exist for all $k \geq 2$ and $g \geq 3$, and Daven and Rodger [7] showed that all (k, g)-cages are 3-connected. Therefore a $(3, g)$-cage is an appropriate choice for G, and if the number of vertices in such a graph is a multiple of 4 then we may use it for A as well. (In fact, we may use a $(3, g_1)$-cage for G, for any $g_1 \geq 3$, and a $(3, g_2)$-cage for A, provided that this graph has order $4r$. The girth of $G \circ A^-$ will then be at least g_2, as shown in Proposition 5.17.)

If, for the specified girth $g \geq 4$, a $(3, g)$-cage B has order $m \equiv 2 \pmod 4$, we can obtain a 3-connected cubic graph of order $4r$ and girth at least g by modifying and joining together two copies of any 3-connected cubic graph of order $4r + 2$ and girth at least g (such as B).

Theorem 5.19. *For every $g \geq 3$ there exists a 3-connected cubic graph of order $4r$ and girth at least g.*

Proof. For every $g \geq 3$ there exists a 3-connected cubic graph with girth g, for example a $(3, g)$-cage. The (3, 3)-cage is K_4, so the statement is true for $g = 3$. Let $g \geq 4$ and suppose B is a 3-connected cubic graph of girth g and order $n \equiv 2 \pmod 4$. Let u and v be two adjacent vertices of B. Since $g \geq 4$, u and v have no common neighbor. Let a and b be the neighbors of u in $B - v$ and let c and d be the neighbors of v in $B - u$. Consider two copies H and H' of $B - \{u, v\}$; for each vertex v in H, we denote its counterpart in H' by v'. Let A be the cubic graph obtained from H and H' by adding edges aa', bb', cd' and dc'. We show that A is 3-edge connected and has girth at least g. Clearly, any cycle in H has length at least g, since it is also a cycle in B. Let C be a cycle in A containing vertices from both H and H' and suppose C has length ℓ. Then, since the vertices a', b', c' and d' are all distinct, $C \cap H$ is a path P of length at most $\ell - 3$ whose endpoints are two of a, b, c and d. If the endpoints of P are a and b then $P + au + ub$ is a cycle in B of length at most $\ell - 1$ in B, so $\ell - 1 \geq g$. If the endpoints of P are a and c, then $P + au + uv + vc+$ is a cycle in B of length at most ℓ, so $\ell \geq g$. It remains to show that A is 3-connected. Let x be any vertex of H. To see that there are three edge-disjoint $x - x'$ paths in A, consider three edge-disjoint $x - v$ paths in B. Without loss of generality, we may assume that one contains the edge au, another contains the edge cv and the third contains the edge dv. Therefore there are paths $x - a$, $x - c$ and $x - d$ in H and paths $a' - x'$, $c' - x'$ and $d' - x'$ in H' which are all edge-disjoint. Adding the edges aa', cd' and dc' produces three edge-disjoint $x - x'$ paths in A. Now let x and y be any two vertices of H. Since B is 3-connected, H is connected. There are two cases to show that there are three edge-disjoint $x - y$ paths in A.

Case 1: Suppose there is only one $x - y$ path P in H. Then u and v are contained in distinct $x - y$ paths of B, one of which contains the subpath $a - u - b$ and the other contains the subpath $c - v - d$. Then H contains edge-disjoint paths $x - a$, $b - y$, $x - c$, $d - y$, each of which is disjoint from P, and these paths are copied in H'. Therefore A contains three edge-disjoint $x - y$ paths, $(x - a) + aa' + (a' - x') + (x' - c') + c'd + (d - y)$, $(x - c) + cd' + (d' - y') + (y' - b') + b'b + (b - y)$, and P.

Case 2: Suppose there are exactly two edge-disjoint $x - y$ paths P_1 and P_2 in H. Then a third such path in B contains u or v (maybe both), and therefore it contains two of a, b, c and d, say a and b (the other cases are similar). Since H' is connected there is a path in H' between any two of a', b', c', d'. Then there is a path $(x - a) + aa' + (a' - b') + b'b + (b' - y)$ in A which is edge-disjoint from P_1 and P_2. Finally, we must show that for any two vertices x, y of H, there are three edge-disjoint $x - y'$ paths in A. Let X be any 2-edge cut in A. Since there are three edge-disjoint $x - y$ paths in A, x and y are in the same component of $A - X$. Likewise, since there are three edge-disjoint $y - y'$ paths in A, y and y' are in the same component of $A - X$. Therefore

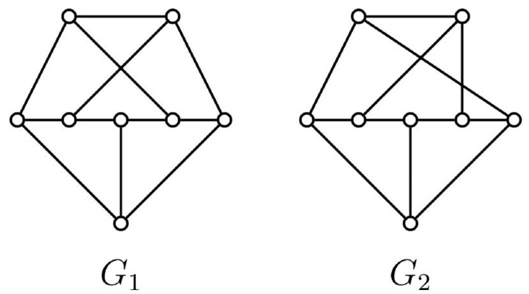


Figure 5. The graphs G_1 and G_2 of Theorem 5.23.

x and y' are in the same component of $A - X$. Since X is any 2-edge cut, there are three edge-disjoint $x - y'$ paths in A. □

We are now ready to answer Question 5.12 by proving the existence of 3-connected cubic graphs of arbitrarily large girth that fail to meet the lower bound (2). However, chromatic index (either 3, corresponding to Class 1, or 4, corresponding to Class 2) was central to our discussion in the previous section, and we have not yet discussed the chromatic index of the graphs we have constructed to answer Question 5.12. In Proposition 5.21 we show that the construction produces a Class 1 graph if and only if G and A are both Class 1. We need a lemma, the proof of which can be found in, e.g., [37, Lemma 4.28].

Lemma 5.20. *If H is a cubic Class 2 graph, then any 4-edge coloring of H contains at least two edges of each color, and $H - v$ is Class 2 for each $v \in V(H)$.*

Proposition 5.21. *For any cubic graphs G and A, the graph $G \circ A^-$ is Class 1 if and only if G and A are Class 1.*

Proof. If A is Class 2, then A^- is Class 2, by Lemma 5.20, and therefore $G \circ A^-$ is Class 2. Hence assume A is Class 1. Say $A^- = A - a$ and let a_1, a_2, a_3 be the vertices of A adjacent to a. Arguing as in the proof of Lemma 5.20, we see that in any 3-edge coloring of A^-, a_1, a_2 and a_3 are incident with edges colored with three different pairs of colors. Assume G is Class 1 and consider any 3-edge colorings of G and A^- in the same colors. Coloring the edges $A_u^- A_v^-$ of $G \circ A^-$ the same color as uv in G and suitably permuting the colors in the copies of A^- produces a 3-edge coloring of $G \circ A^-$. Now assume G is Class 2 and suppose for a contradiction that $G \circ A^-$ has a 3-edge coloring. For any copy A_v^- of A^-, let xa_1, ya_2 and za_3 be the three edges that join A_v^- to the rest of $G \circ A^-$. Since a_1, a_2 and a_3 are incident with edges colored with three different pairs of colors, xa_1, ya_2 and za_3 have three different colors. Contracting each copy of A^- to a single vertex yields G as well as a 3-edge coloring of G, which is a contradiction. □

Theorem 5.22. *For any $g \geq 3$ and $m \in N$, there exists a 3-connected cubic graph $H = G \circ A^-$ of girth at least g such that $c_2(H) - \lceil \frac{|V(H)|+2}{4} \rceil \geq m$. Moreover, H is Class 1 if and only if G and A are Class 1.*

Proof. Theorem 5.19 guarantees the existence of a 3-connected cubic graph of order $4r$ and girth at least g. Let A be such a graph and let G be any 3-connected cubic graph of order at least $4m + 2$. Then by Propositions 5.17 and 5.18, $H = G \circ A^-$ is a 3-connected cubic graph of girth at least g, and by Proposition 5.16, $c_2(H)$ exceeds the lower bound (2) by at least m. The chromatic index of H is given by Proposition 5.21. □

Any Class 2, girth $g \geq 4$ graph $G \circ A^-$ produced by our construction is a Gardner snark. For example, taking A to be the flower snark J_5, a Gardner snark of order 20 and girth 5, and any 3-connected cubic graph G, $G \circ A^-$ is Class 2 (by Proposition 5.21), 3-connected and has girth at least 5. Therefore it is a Gardner snark (in fact it satisfies a more restrictive definition of snarks, since it has girth greater than 4 and connectivity greater than 2).

We now turn our attention to Question 5.13. Consider a 3-connected cubic graph G of order n and a triangle-free 3-connected cubic graph A of order $4r$, as required for our construction of the graph $G \circ A^-$. In Lemma 5.15 we showed that any minimum 2-conversion set of $G \circ A^-$ contains at least r vertices from each copy of A^-. Therefore

$$\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} \geq \frac{rn}{(4r-1)n} = \frac{r}{4r-1} > \frac{1}{4}.$$

For example, taking A to be the graph shown in Figure 4, and G any 3-connected cubic graph, $G \circ A^-$ has $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} = \frac{3}{11}$.

In fact, it follows from the proof of Lemma 5.15 that any 2-conversion set of $G \circ A^-$ contains at least $c_2(A) - 1$ vertices from every copy of A^-, with $c_2(A) \geq r + 1$ by (2). Therefore, if $c_2(A) = r + 1 + s$, $s \geq 0$, then every 2-conversion set of $G \circ A^-$ contains at least $r + s$ vertices from each copy of A^-. Therefore $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} = \frac{r+s}{4r-1}$. That is, by choosing A to be a cubic graph of order $4r$ that does not meet the lower bound (2), we can increase the ratio $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|}$.

Choosing smaller values of r also increases the ratio. For example, if A is a cubic graph of order 8, then $c_2(A) = 3$ (all cubic graphs of order 8 meet the lower bound (2)) and for any cubic graph G, any 2-conversion set of $G \circ A^-$ contains at least two vertices from each copy of A^-. Then $\frac{c_2(G \circ A^-)}{|V(G \circ A^-)|} = \frac{2}{7}$. Examples of 3-connected cubic graphs of order 8 with girth 4—suitable choices for A in the construction of triangle-free 3-connected cubic graphs with ratio $\frac{2}{7}$—are shown in Figure 5.

For comparison we briefly mention some upper bounds on the 2-conversion number of cubic graphs. Let G_1 and G_2 be the graphs in Figure 5 and let $\mathcal{G}$ be the class of cubic graphs obtained from trees, all of whose internal vertices have degree 3, by replacing each internal vertex by a triangle and each leaf by a K_4 in which one edge has been subdivided.

Theorem 5.23. *Let G be a cubic graph of order $n > 4$.*

(a) [4, 25] *If $G \in \mathcal{G}$, then $c_2(G) = \frac{3n+2}{8}$, otherwise $c_2(G) \leq \frac{3n}{8}$.*

(b) [39] *If G is triangle-free and $G \notin \{G_1, G_2\}$, then $c_2(G) \leq \frac{n}{3}$.*

(c) [12] *If G is 2-connected, then $c_2(G) \leq \frac{n+2}{3}$ and this bound is sharp.*

Together, Equation (2) and Theorem 5.23 bound the value of $c_2(G)$ between $\lceil \frac{n+2}{4} \rceil$ and $\lfloor \frac{3n+2}{8} \rfloor$ for cubic graphs G of order $n > 4$. Observe that the ratio $\frac{c_2(G)}{|V(G)|}$ cannot exceed $\frac{1}{3}$ for any triangle-free cubic graph. It also follows from Theorem 5.23 that this ratio is bounded asymptotically by $\frac{3}{8}$ for all cubic graphs, and that the asymptotic bound is attained by the infinite family $\mathcal{G}$. The graphs in $\mathcal{G}$ all have girth 3, so the following question remains open.

Question 5.24. *What is the largest ratio $\frac{c_2(H)}{|V(H)|}$ achievable by an infinite family of 3-connected triangle-free cubic graphs H?*

Note

1. The connectivity of any cubic graph is equal to its edge connectivity [5, Theorem 4.6].

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