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Varieties of Roman domination II

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ABSTRACT
In this work, we continue to survey what has been done on the Roman domination. More precisely, we will present in two sections several variations of Roman dominating functions as well as the signed version of some of these functions. It should be noted that a first part of this survey comprising 9 varieties is published as a chapter book in "Topics in domination in graphs" edited by T.W. Haynes, S.T. Hedetniemi and M.A. Henning. We recall that a function $f: V \to \{0,1,2\}$ is a Roman dominating function (or just RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G.

KEYWORDS
Domination; dominating functions; Roman domination; signed Roman domination; variation of Roman domination

MSC CLASSIFICATION
05C69

1. Introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [31]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G, respectively. The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. A graph G is *regular* or *r-regular* if $\Delta(G) = \delta(G) = r$. A vertex of degree zero is *isolated*, a vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. If $X \subseteq V(G)$, then $G[X]$ is the subgraph induced by X. Let $S \subseteq V(G)$, and let $u \in S$. We say that v is a *private neighbor* of u (with respect to S) if $N[v] \cap S = \{u\}$. Furthermore, we define the private neighbor set of u, with respect to S, to be $\text{pn}[u, S] = \{v : N[v] \cap S = \{u\}\}$. The *complement* of a graph G is denoted by $\bar{G}$. Let P_n, C_n and K_n be the *path*, *cycle* and *complete graph* of order n and $K_{p,q}$ the complete bipartite graph with one partite set of cardinality p and the other of cardinality q. A *star* is a complete bipartite graph of the form $K_{1,q}$. If H and G are graphs, then G is called *H-free* if G does not contain any induced subgraph isomorphic to H. A *claw-free* graph is a $K_{1,3}$-free graph. A graph is a *cactus graph* if every edge belongs to at most one cycle. A cactus graph having one cycle is called a *unicyclic graph*, and a connected graph with no cycles is called a *tree*. A tree is a *double star* if it contains exactly two vertices that are not leaves. A double star with p and q leaves adjacent to each support vertex, respectively, is denoted by $S_{p,q}$. The *corona* of two graphs G_1 and G_2, as defined in [30], is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to each vertex of the ith copy of G_2. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *diameter* of G, denoted by $\text{diam}(G)$, is the maximum value among distances between all pairs of vertices of G.

A subset $S \subseteq V(G)$ is a *dominating set* if every vertex not in S has at least one neighbor in S. The *domination number* $\gamma(G)$ represents the cardinality of a minimum dominating set of G. A subset $S \subseteq V(G)$ is said to be *independent* if $E(G[S]) = \emptyset$. The *independent domination number* of G denoted by $i(G)$ is the size of the smallest maximal independent set in G. A set $S \subseteq V(G)$ is a *2-packing* or *packing* if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The *packing number* $\rho(G)$ is the cardinality of a maximum packing. A *vertex cover* of a graph G is a set of vertices that covers all the edges, and the minimum cardinality of a vertex cover is the *vertex cover number* denoted by $\alpha_0(G)$.

2. Variations of Roman dominating functions

For a positive integer k and a function $f : V \to \{0, 1, \ldots, k\}$, the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$. For every $i \in \{0, 1, \ldots, k\}$, let V_i be the set of vertices assigned the value i under a function f. Note that there is a 1-to-1 correspondence between the functions $f : V \to \{0, 1, \ldots, k\}$ and the

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ordered k-tuple $(V_0, V_1, \ldots, V_k)$ of V, so we will write $f = (V_0, V_1, \ldots, V_k)$.

We would like to point out that nine varieties have already been published as a chapter book in "Topics in domination in graphs" edited by T.W. Haynes, S.T. Hedetniemi and M.A. Henning. More precisely, it is about weak Roman domination, independent Roman domination, Roman k-domination, Roman{2}-domination, double Roman domination, total Roman domination, perfect Roman domination, strong Roman domination and edge Roman domination. In this section, we present nine other varieties.

2.1. Unique response Roman domination

In [49], Rubalcaba and Slater defined a *unique response Roman function* as a function $f = (V_0, V_1, V_2)$ such that (i) if $x \in V_0$, then $|N(x) \cap V_2| \leq 1$; and (ii) if $x \in V_1 \cup V_2$, then $|N(x) \cap V_2| = 0$. A function f is a *unique response Roman dominating function* (URRDF) if it is a unique response Roman function and a Roman dominating function. The *unique response Roman domination number* of G, denoted by $u_R(G)$, is the minimum weight of an URRDF of G. An URRDF on G with weight $u_R(G)$ is called an $u_R(G)$-function. Unique response Roman domination was studied in [29, 34] and elsewhere. We recall that an RDF $f = (V_0, V_1, V_2)$ is an *independent Roman dominating function* (IRDF) if the set $V_1 \cup V_2$ is independent. The *independent Roman domination number* $i_R(G)$ is the minimum weight of an IRDF on G. Independent Roman dominating functions were defined in [15, 23].

It was noticed in Ebrahimi Targhi et al. [29] that $\gamma_R(G) \leq i_R(G) \leq u_R(G)$ for every graph G, and the difference $u_R(G) - \gamma_R(G)$ can be arbitrarily large. For instance, if G is a double star in which any support vertex is of degree $t+2$ for some integer $t \geq 2$, then $u_R(G) = t+4$ while $\gamma_R(G) = 4$. Moreover, if G is a graph with $\gamma_R(G) = u_R(G)$, then every $u_R(G)$-function is a $\gamma_R(G)$-function. But not every $\gamma_R(G)$-function is an $u_R(G)$-function as can be shown by the double star $S_{2,3}$ which admits two $\gamma_R(S_{2,3})$-functions but only one of them is an $u_R(S_{2,3})$-function. This motivates the characterization of graphs G with strong equality between $\gamma_R(G)$ and $u_R(G)$. Jafari Rad and Liu [34] have provided a constructive characterization of trees T with $\gamma_R(T) \equiv u_R(T)$. Recently, Zhao, Li, Zhao and Zhang [63] gavea constructive characterization of trees T with $\gamma_R(T) = u_R(T)$. Also, Zhao et al. [63] have shown that the decision problem corresponding to the problem of computing $u_R(G)$ is NP-complete for triangle-free graphs. Moreover, they proposed a linear algorithm for finding the unique response Roman domination number of trees.

2.1.1 Bounds on u_R

In the first result, we present a lower bound of the unique response Roman domination number in terms of the domination number.

Theorem 2.1 ([29]). *Let G be a graph of order n with $\Delta(G) \geq 1$. Then $u_R(G) \geq \gamma(G) + 1$, with equality if and only if G has a vertex of degree $n - \gamma(G)$.*

Theorem 2.2 ([29]). *For a connected graph G of order n, $u_R(G) = \gamma(G) + 2$ if and only if:*

(i) *G does not have a vertex of degree $n - \gamma(G)$,*
(ii) *either G has a vertex of degree $n - \gamma(G) - 1$, or G has two vertices v, w such that $|N[v] \cup N[w]| = |N[v]| + |N[w]| = n - \gamma(G) + 2$.*

Theorem 2.3 ([29]). *Let G be a graph of order $n \geq 3$, and let m be an integer such that $1 \leq m \leq \frac{n-1}{2}$. If $\delta(G) \geq \frac{n-m}{m+1}$, then $u_R(G) \geq n - m(\Delta(G) - 1)$.*

Theorem 2.4 ([29]). *For any connected graph G of order at least 2, $u_R(G) \leq n - \Delta(G) + 1$, with equality if and only if for any packing set $S = \{a_1, a_2, \ldots, a_t\}$, $\Delta(G) \geq \sum_{i=1}^{t} \deg_G(a_i) - t + 1$.*

As an immediate consequence of Theorem 2.4, we obtain $u_R(G) = n - \Delta(G) + 1$ for every graph G with diameter two. An upper bound on the unique response Roman domination number in terms of the Roman domination number and maximum degree was given as follows.

Theorem 2.5 ([29]). *For every graph G with maximum degree at least three, $u_R(G) \leq i_R(G) + \frac{i_R(G)-2}{2}(\Delta(G) - 2)$, with equality if and only if $i_R(G) = 2i(G)$ and $u_R(G) = 2 + \Delta(G)(i(G) - 1)$.*

Zhao et al. [63] conjectured that $u_R(T) \leq \frac{4}{5}n$ for every tree T of order $n \geq 3$. But the following result shows that this conjecture is false since it states that in the general case there is no better bound for the unique response Roman domination number of a graph G.

Theorem 2.6 ([29]). *For any $t \in (0, 1)$ there are an integer n and a tree T on n vertices such that $u_R(T) > tn$.*

2.2. Maximal Roman domination

In this subsection, we are interested in maximal Roman domination introduced by Abdollahzadeh Ahangar et al. in 2017 [4], and studied also in [5, 6]. A *maximal Roman dominating function* (MRDF) for a graph G is a Roman dominating function $f = (V_0, V_1, V_2)$ such that V_0 is not a dominating set of G. The *maximal Roman domination number* of a graph G, denoted by $\gamma_{mR}(G)$, equals the minimum weight of an MRDF for G. A $\gamma_{mR}(G)$*-function* is a maximal Roman dominating function for G with weight $\gamma_{mR}(G)$. It is shown in [5] that the decision problem corresponding to the problem of computing $\gamma_{mR}(G)$ is NP-complete even when restricted to bipartite or planar graphs. Moreover, as far as we know no linear algorithm has been designed for computing the maximal Roman domination number for any tree.

The motivation for introducing this kind of functions comes from maximal dominating sets defined by Kulli and Janakiram [38] in 1997. A dominating set D is said to be a *maximal dominating set* if $V - D$ is not a dominating set of G. The *maximal domination number* $\gamma_m(G)$ is the minimum cardinality of a maximal dominating set of G. The following observation was given in [5].

Observation 2.7. *Let $f = (V_0, V_1, V_2)$ be a $\gamma_{mR}(G)$-function. Then the following assertions hold:*

(i) *$V_1 \cup V_2$ is a maximal dominating set of G.*
(ii) *Every vertex of V not dominated by V_0 belongs to V_1.*
(iii) *Each vertex of V_2 has a private neighbor in V_0.*

The exact values on the maximal Roman domination number for paths and cycles have been established in [4]. It is shown that for $k \geq 1$, $\gamma_{mR}(P_{3k}) = 2k+1, \gamma_{mR}(P_{3k+1}) = 2k+2$ and $\gamma_{mR}(P_{3k+2}) = 2k+2$. Also, for $k \geq 1, \gamma_{mR}(C_{3k}) = 2k+1$, $\gamma_{mR}(C_{3k+1}) = 2k+2$ and $\gamma_{mR}(C_{3k+2}) = 2k+3$.

2.2.1. Relationships with γ_R and γ_m

Since every MRDF of a graph G is an RDF, it is obvious that $\gamma_R(G) \leq \gamma_{mR}(G)$ for every graph G. Also, it was shown in [4] that for any graph G without isolated vertices, $\gamma_{mR}(G) \leq \gamma_R(G) + \delta(G)$. A characterization of graphs G with $\gamma_R(G) = \gamma_{mR}(G)$ was given in [5] as follows.

Theorem 2.8 ([5]). *Let G be a connected graph of order $n \geq 3$. Then the following assertions are equivalent.*

(i) *$\gamma_{mR}(G) = \gamma_R(G)$.*
(ii) *G contains a weak support vertex u such that $\gamma_R(G) = \gamma_R(G-u)+ 1$.*
(iii) *There exists a γ_R-function for G assigning the value 1 to a weak support vertex and its leaf.*

Since $V_1 \cup V_2$ is a maximal dominating set when $f = (V_0, V_1, V_2)$ is an MRDF, and since assigning a 2 to every vertex of a maximal dominating set provides an MRDF, it was observed that $\gamma_m(G) \leq \gamma_{mR}(G) \leq 2\gamma_m(G)$. The upper bound has been slightly improved by Abdollahzadeh Ahangar et al. [4] who proved that $\gamma_{mR}(G) \leq 2\gamma_m(G) - 1$ for all G. It has also been shown that $\gamma_m(G) = \gamma_{mR}(G)$ if and only if $G = K_n$ or $\overline{K_n}$. Moreover, graphs G with $\gamma_{mR}(G) \in \{\gamma_m(G)+1, \gamma_m(G)+2\}$ have been characterized by Abdollahzadeh Ahangar et al. [4] as follows.

Theorem 2.9 ([4]). *Let G be a connected graph of order $n \geq 3$ different from K_n. Then $\gamma_{mR}(G) = \gamma_m(G) + 1$ if and only if there is a vertex $v \in V$ such that $\deg_G(v) \geq n - \gamma_m(G)$ and $N(v)$ has a subset of size $n - \gamma_m(G)$ that is not a dominating set of G.*

Theorem 2.10 ([4]). *Let G be a connected graph of order $n \geq 7$ different from K_n. Then $\gamma_{mR}(G) = \gamma_m(G) + 2$ if and only if:*

(i) *G does not have a vertex $v \in V$ such that $\deg_G(v) \geq n - \gamma_m(G)$ and $N(v)$ has a subset of size $n - \gamma_m(G)$ that is not a dominating set of G.*
(ii) *either G has a vertex $v \in V$ such that $\deg_G(v) \geq n - \gamma_m(G) - 1$ and $N(v)$ has a subset of size $n - \gamma_m(G) - 1$ that is not a dominating set of G, or G has two vertices u and v such that $|N(u) \cup N(v)| \geq n - \gamma_m(G)$ and $N(u) \cup N(v)$ has a subset of size $n - \gamma_m(G)$ that is not a dominating set of G.*

2.2.2. Graphs with large γ_{mR}

Since $f = (\emptyset, V(G), \emptyset)$ is a maximal Roman dominating function of G, we have $\gamma_{mR}(G) \leq n$. This bound has been slightly improved for connected graphs G with diameter at least four, where it has been shown in this case that $\gamma_{mR}(G) \leq n - \delta(G) - 1$. A characterization of graphs G of order n such that $\gamma_{mR}(G) = n$ was given in [4] as follows.

Theorem 2.11 ([4]). *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{mR}(G) = n$ if and only if $G = P_2, P_3, P_4, C_3, C_4, C_5$ or $G = K_n - M$, where M is a matching of G.*

The characterization of connected graphs G of order n such that $\gamma_{mR}(G) = n - 1$ was the main focus of the paper by Abdollahzadeh Ahangar, Chellali, Kuziak and Samodivkin [6], where a complete characterization has been given for triangle-free graphs as well as connected graphs with girth 3 and either diameter at least 3 or minimum degree at least two. Hence to complete the characterization of all connected graphs with maximal Roman domination number equal to one less their order, it remains to study the case in which G is any connected graph with girth three, diameter two and minimum degree at least three. It is noteworthy that the paper [6] contains an illustration of many graphs for the description of these results that we cannot report here. It is worth mentioning also that a characterization of trees T of order n such that $\gamma_{mR}(T) = n - 2$ was also given in [5].

2.3. Mixed Roman domination

The study of the mixed version of Roman domination was initiated by Abdollahzadeh Ahangar, Haynes and Valenzuela-Tripodoro [12]. A *mixed Roman dominating function* (MRDF) of a graph $G = (V, E)$ is a function $f : V \cup E \rightarrow \{0, 1, 2\}$ such that every element $x \in V \cup E$ for which $f(x) = 0$, is adjacent or incident to at least one element $y \in V \cup E$ with $f(y) = 2$. In other words, an element x for which $f(x) \in \{1, 2\}$ dominates itself, while an element x with $f(x) = 0$ is *dominated* by the mixed Roman function f if it is adjacent or incident to at least one element y for which $f(y) = 2$. The minimum weight, $w(f) = \sum_{x \in V \cup E} f(x)$, of an MRDF is the *mixed Roman domination number* $\gamma^*_R(G)$. A MRDF with minimum weight is called a $\gamma^*_R(G)$-function. Note that each MRDF on G determines a partition of the set $V \cup E = (V_0 \cup E_0) \cup (V_1 \cup E_1) \cup (V_2 \cup E_2)$, where $V_i \cup E_i = \{x \in V \cup E : f(x) = i\}$.

For mixed Roman domination, not only the cities (vertices) that must be protected but also the roadways (edges) must be secured from ambush attacks on travelers. Adopting the same principles as for Roman domination, legions can be placed at a camp on a road as well as stationed in a city, and both cities and roads must be protected. If no legion is stationed on a road, then such a road must be adjacent to a secured road with two legions or incident to a secured city with two legions. In this way any city or road with two legions can deploy a legion to secure any unsecured city or road adjacent or incident to it.

It is shown in [1] that the decision problem corresponding to the problem of computing $\gamma_R^*(G)$ is NP-complete for bipartite graphs. Moreover, as of this writing, a linear algorithm for computing the mixed Roman domination number for any tree has not yet designed.

Recall that if $S \subseteq V \cup E$ is a *mixed dominating set*, then every element in $(V \cup E)\backslash S$ is adjacent or incident to an element in S. The *mixed domination number* $\gamma^*(G)$ of G is the minimum cardinality of any mixed dominating set of G. The following proposition gathers some properties of mixed Roman dominating functions. For any $x \in V \cup E$, we denote by $N_m[x] = \{x\} \cup \{y \in V \cup E : y$ is either adjacent or incident with $x\}$, and let $f[x] = f(N_m[x]) = \sum_{v \in N_m[x]} f(v)$, for all $x \in V \cup E$.

Proposition 2.12 ([12]). *Let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be an MRDF of a graph G. Then the following holds.*

(i) *Every element in $V_0 \cup E_0$ is dominated by some element of $V_2 \cup E_2$.*
(ii) *$V_1 \cup V_2 \cup E_1 \cup E_2$ is a mixed dominating set in G.*
(iii) *$\sum_{v \in V} f[v] + \sum_{e=uw \in E} f[e] = \sum_{v \in V}(2\deg_G(v))f(v) + \sum_{e=uw \in E}(\deg_G(u) + \deg_G(w) + 1)f(uw)$.*

The exact values on the mixed Roman domination number for paths and cycles have been established in [12]. It is shown that $\gamma_R^*(P_n) = \lceil \frac{4n-2}{5} \rceil$ if $n \equiv 0,1,2,3 \pmod 5$, and $\gamma_R^*(P_n) = \lceil \frac{4n-2}{5} \rceil + 1$ if $n \equiv 4 \pmod 5$. Also, $\gamma_R^*(C_n) = \lceil \frac{4n}{5} \rceil$ if $n \equiv 0,2,3,4 \pmod 5$, and $\gamma_R^*(C_n) = \lceil \frac{4n}{5} \rceil + 1$ if $n \equiv 1 \pmod 5$.

2.3.1. Upper and lower bounds on γ_R^*

It is shown in [12] that for every graph G, $\gamma^*(G) \le \gamma_R^*(G) \le 2\gamma^*(G)$, where equality holds in the lower bound if and only if G is an edgeless graph. The upper bound is sharp for complete bipartite graphs $K_{r,s}$, $1 \le r \le s$, where $\gamma_R^*(K_{r,s}) = 2r$ and $\gamma^*(K_{r,s}) = r$. Moreover, graphs with the property $\gamma_R^*(G) = 2\gamma^*(G)$ were called *mixed Roman graphs*. A characterization of mixed Roman graphs has been given in [12].

Proposition 2.13 ([12]). *A graph G is a mixed Roman graph if and only if it has a $\gamma_R^*(G)$-function $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ with $|V_1 \cup E_1| = 0$.*

Additional upper bounds have been obtained by Abdollahzadeh Ahangar et al. [12]. They showed that the mixed Roman domination number of any graph G of order n does not exceed n. The characterization of graphs G of order n such that $\gamma_R^*(G) = n$ was raised in [1], where the following results are obtained.

Theorem 2.14 ([1]). *Let G be a connected graph of odd order n. Then $\gamma_R^*(G) = n$ if and only if $G = K_n$.*

Proposition 2.15 ([1]). *Let G be a connected graph of even ordern. If $\gamma_R^*(G) = n$, then G is a mixed Roman graph having a perfect matching.*

An upper bound on the mixed Roman domination number of a graph in terms of its order, size, and maximum degree was obtained in [12] as well as a characterization of the graphs attaining this bound. Let $G(a,b,c)$ denote the graph obtained from a non-trivial star $K_{1,n-1}$ with center v by adding edges from its complement such that $G(a,b,c) - v = aK_1 \cup bK_2 \cup cP_3$, and for $j \le a$, we let $G_j(a,b,c)$ be the graph obtained from $G(a,b,c)$ by subdividing (once) j pendant edges. Let $\mathcal{H}$ be the family of graphs $\mathcal{H} = \{G(a,b,c), G_j(a,b,c) : a,b,c \ge 0, j \le a\}$ satisfying that if $G \in \mathcal{H}$ and $(b,c) \in \{(0,0),(1,0)\}$, then either $G = G(0,1,0) = K_3$ or $a > j$.

Proposition 2.16 ([12]). *Let G be a connected graph of order $n \ge 2$, size m, and $\Delta(G) \ge 1$. Then $\gamma_R^*(G) \le m + n - 2\Delta(G) + 1$, with equality if and only if $G \in \mathcal{H}$.*

For the class of trees T, two upper bounds on the mixed Roman domination number are obtained in [1], one in terms of the domination number $\gamma(T)$, and the other in terms of the order, number of leaves and support vertices of T. Recall that a set S is an efficient dominating set if S is both a dominating set and a packing of G.

Theorem 2.17 ([1]). *For any nontrivial tree T, $\gamma_R^*(T) \le 3\gamma(T) - 1$, with equality only if T has a unique $\gamma(T)$-set D such that D is efficient and every vertex in $V - D$ has degree at most three.*

If we consider the tree T obtained from a star $K_{1,3}$ by subdividing two edges twice and the remaining edge four times, then one can easily see that T has a unique $\gamma(T)$-set D of size 4 such that D is efficient and every vertex in $V - D$ has degree at most three but $\gamma_R^*(T) < 3\gamma(T) - 1 = 11$. This shows that the converse of Theorem 2.17 is not true.

Theorem 2.18 ([1]). *If T is a tree of order $n \ge 3$ with $\ell(T)$ leaves and $s(T)$ support vertices, then $\gamma_R^*(T) \le n - \ell(T) + s(T)$ with equality if and only if every vertex of T is either a leaf or a support vertex.*

Sharp lower bounds for the mixed Roman domination number of a graph in terms of its order, size, maximum degree and maximum matching are obtained in [12] and [1].

Proposition 2.19 ([12]). *Let G be a graph of order n, size m, and maximum degree $\Delta \ge \delta \ge 1$. Then $\gamma_R^*(G) \ge \lceil \frac{2(m+n)}{2\Delta+1} \rceil$.*

As an immediate consequence of Proposition 2.19, we have $\gamma_R^*(G) \ge \lceil \frac{(r+2)n}{2r+1} \rceil$ for every r-regular graph of order n.

Dehgardi [24] was interested in the relationship between the mixed Roman domination number and the 2-independence number. Recall that the 2-independence number of a graph G, denoted $\beta_2(G)$, is the maximum cardinality of a set of vertices S whose induced subgraph has maximum degree at most one. As shown in [24], in general $\gamma_R^*(G)$ may be smaller or larger than $\beta_2(G)$. However, Dehgardi proved that for every tree T, $\gamma_R^*(T) \le \frac{4}{3}\beta_2(T)$ and provided a constructive characterization of trees attaining this bound.

2.4. Outer-independent (total, double) Roman domination

In [7], Abdollahzadeh Ahangar, Chellali and Samodivkin defined a new variant of Roman domination which they called outer-independent Roman domination. A function $f = (V_0, V_1, V_2)$ is an *outer-independent Roman dominating function* (OIRDF) on G if f is an RDF and V_0 is an independent set. The *outer-independent Roman domination number* $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G. An outer-independent Roman dominating function of weight $\gamma_{oiR}(G)$ is called a $\gamma_{oiR}(G)$-function. If we go back to the defensive strategy of the Roman Empire, a location having no army could be thought of as being vulnerable. If in addition, it has one of its neighbors with no army stationed there, then it will be even more vulnerable. Hence the best situation for a location with no army is to be completely surrounded by locations on which armies are stationed. This leads us to seek a Roman dominating function $f = (V_0, V_1, V_2)$ for which V_0 is independent, that is f is an OIRDF. The following properties of an OIRDF were observed.

Observation 2.20. *Let $f = (V_0, V_1, V_2)$ be an OIRDF of a graph G. Then*

(i) *Every vertex of V_2 has a private neighbor in V_0 with respect to V_2.*
(ii) *$V_1 \cup V_2$ is a dominating set in G.*
(iii) *$V_1 \cup V_2$ is a vertex cover of G.*

It is shown in [7] that the decision problem corresponding to the problem of computing $\gamma_{oiR}(G)$ is NP-complete for bipartite graphs. Moreover, the exact values on the outer-independent Roman domination number for paths and cycles have been established in [7]. Indeed, it is shown that if G is a path or a cycle on $n \geq 3$ vertices, then $\gamma_{oiR}(G) = 3\lfloor \frac{n}{4} \rfloor + i$, where $n \equiv i \pmod 4$ and $i \in \{0, 1, 2\}$, and $\gamma_{oiR}(G) = 3\lfloor \frac{n}{4} \rfloor + 2$ otherwise.

2.4.1. Bounds on γ_{oiR}

Trivially, $\gamma_{oiR}(G) \geq \gamma_R(G)$ for every graph G, since each outer-independent Roman dominating function is a Roman dominating function. So the problem of characterizing the graphs G with $\gamma_{oiR}(G) = \gamma_R(G)$ seems to be interesting. For now, this problem has been considered only for trees, where a constructive characterization was recently done in [45].

The following result that gives a lower bound on the outer-independent Roman domination number for any graph in terms of the order, maximum and minimum degrees.

Proposition 2.21 ([7]). *If G is a connected graph of order $n \geq 2$, then $\gamma_{oiR}(G) \geq \lceil \delta n/(\delta + \Delta) \rceil + 1$.*

By Proposition 2.21, $\gamma_{oiR}(G) \geq \lceil n/2 \rceil + 1$ for all regular graphs of order n. Moreover, a characterization of trees T attaining the lower bound in Proposition 2.21 is given as follows.

Proposition 2.22 ([7]). *Let T be a tree of order n. Then $\gamma_{oiR}(T) = \lceil n/(\Delta + 1) \rceil + 1$ if and only if T is a star or T is obtained from a star by subdividing exactly one of its edges once.*

Since $f = (\emptyset, V(G), \emptyset)$ is an OIDRF of G, we have $\gamma_{oiR}(G) \leq n$. As shown in [7], this bound is sharp if and only if $G = K_n$. For graphs G of order n such that $\gamma_{oiR}(G) = n - 1$, we have the following.

Proposition 2.23 ([7]). *Let G be a connected graph of order $n \geq 2$. Then the following conditions are equivalent:*

(i) *$\gamma_{oiR}(G) = n - 1$.*
(ii) *G is a $(K_{1,3}, 2K_{1,2})$-free graph different from K_n.*
(iii) *G has a $\gamma_{oiR}(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_2| = 1$ and $|V_0| = 2$.*

For the purpose of characterizing all trees T of order n such that $\gamma_{oiR}(T) = n - 2$, let i, j and k be three integers such that $0 \leq i \leq j \leq k$, and let $T_{i,j,k}$ be the tree obtained from a star $K_{1,3}$ by subdividing one edge i times, the second edge j times and the third edge k times. Thus $T_{0,0,0} = K_{1,3}$. Also, let i, j, k and l be four integers and let $T_{i,j,k,l}$ be the tree obtained from a double star $S_{2,2}$ by subdividing two pendant edges incident with one support vertex i and j times and the two other pendant edges k and l times. $T_{0,0,0,0} = S_{2,2}$.

Proposition 2.24 ([7]). *Let T be a tree of order n. Then $\gamma_{oiR}(T) = n - 2$ if and only if:*

(i) *$T \in \{P_7, P_8, P_9, P_{10}\}$, or*
(ii) *$T = T_{i,j,k}$ with $k \leq 3$, where either $i = 0$ and $0 \leq j + k \leq 6$, or $i = 1$ and $2 \leq j + k \leq 5$, or*
(iii) *$T = T_{i,j,k,l}$ with $\max\{i,j,k,l\} \leq 2$, $i + j \leq 3$ and $k + l \leq 3$.*

The next result relates the outer-independent Roman domination number to the vertex cover number for arbitrary graphs without isolated vertices.

Proposition 2.25 ([7]). *If G is a graph without isolated vertices, then $\alpha_0(G) + 1 \leq \gamma_{oiR}(G) \leq 2\alpha_0(G)$.*

Note that the lower bound in Proposition 2.25 is sharp for K_2 and K_3, while the upper bound is sharp for any graph G for which each vertex is either a leaf or a support vertex and each support vertex is a adjacent to at least two leaves. Moreover, graphs G for which $\gamma_{oiR}(G) = 2\alpha_0(G)$ have been called *vertex cover Roman graphs* or *VC-Roman graphs* for short. It is shown in [7] that agraph G is VC-Roman if and only if it has a $\gamma_{oiR}(G)$-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.

Recently, Martínez, Kuziak and Yero [41] have provided a constructive characterization of VC-Roman trees. They showed that no graph with minimum degree at least two is VC-Roman. In addition they proved the following.

Proposition 2.26 ([41]). *Let G be a VC-Roman graph and let $f = (V_0, V_1, V_2)$ be a $\gamma_{oiR}(G)$-function. Then*

(i) *V_0 is a maximum independent set in G.*
(ii) *V_2 is a minimum vertex cover in G.*
(iii) *Every vertex in V_2 has a private neighbor in V_0.*

For the class of trees, Dehgardi and Chellali [25] showed that if T is a tree of order $n \geq 3$ with $s(T)$ support vertices, then $\gamma_{oiR}(T) \leq \min\{\frac{5n}{6}, \frac{3n+s(T)}{4}\}$. Moreover, trees attaining each bound have been characterized.

2.4.2. Outer independent total Roman domination

Recently, Martínez et al. [42] considered the outer-independent property for total Roman dominating functions. We recall that a total Roman dominating function $f = (V_0, V_1, V_2)$ is an RDF such that $V_1 \cup V_2$ induces a subgraph without isolated vertices. Let $\gamma_{oitR}(G)$ denote the minimum weight of an outer-independent total Roman dominating function on G. Martínez et al. [42] showed that the decision problem corresponding to the problem of computing $\gamma_{oitR}(G)$ is NP-complete even when restricted to planar graphs of maximum degree at most 3. Moreover, Li et al. [39] proposed a dynamic programming algorithm to compute $\gamma_{oitR}(T)$ for any tree T.

The parameters $\gamma_{oiR}(G)$ and $\gamma_{oitR}(G)$ are related by the following result given in [42].

Theorem 2.27 ([42]). *Let G be any graph without isolated vertices and let $f = (V_0, V_1, V_2)$ be any $\gamma_{oiR}(G)$-function. Then $\gamma_{oiR}(G) \leq \gamma_{oitR}(G) \leq 2\gamma_{oiR}(G) - |V_2|$. Moreover, $\gamma_{oitR}(G) = 2\gamma_{oiR}(G) - |V_2|$ if and only if G is a star with at least two leaves.*

In addition, Martínez et al. [42] gave upper and lower bounds for $\gamma_{oitR}(G)$ in terms of the vertex cover number $\alpha_0(G)$ and the total co-independent domination number $\gamma_{t,\ coi}(G)$ defined to be the minimum cardinality of a total dominating set D such that $V - D$ is independent and not empty.

Theorem 2.28 ([42]). *Let G be a graph without isolated vertices. Then*

(i) $\alpha_0(G) + 1 \leq \gamma_{oitR}(G) \leq 3\alpha_0(G)$. *Moreover, $\gamma_{oitR}(G) = 3\alpha_0(G)$ if and only if G is a star with at least two leaves.*
(ii) $\gamma_{t,\ coi}(G) + 1 \leq \gamma_{oitR}(G) \leq 2\gamma_{t,coi}(G)$.

According to [42], a graph G is said to be *total co-independent Roman graph* (or *TCI-Roman graph*) if $\gamma_{oitR}(G) = 2\gamma_{t,\ coi}(G)$. Thus, the problem of characterizing all TCI-Roman graphs is naturally posed. However, this problem has been addressed for trees in [42].

2.4.3. Outer independent double Roman domination

The concept of double Roman dominating functions (DRDF) was introduced by Beeler et al. [21]. A DRDF is a function $h : V(G) \to \{0, 1, 2, 3\}$ for which each vertex with label 0 is adjacent to a vertex with label 3 or at least two vertices with label 2, and each vertex with label 1, is adjacent to a vertex with label greater than 1. Outer-independent double Roman dominating functions (OIDRDFs) has recently been investigated in [11], where an OIDRDF is a DRDF $h = (V_0, V_1, V_2, V_3)$ for which V_0 is an independent set. The minimum weight of an OIDRDF of G is the *outer independent double Roman domination number* $\gamma_{oidR}(G)$. It was shown in [11] that the decision problem corresponding to the problem of computing $\gamma_{oidR}(G)$ is NP-complete for bipartite and chordal graphs. Among the various bounds established in [11], it has been shown that for any graph G of order n, $\delta(G) + 2 \leq \gamma_{oidR}(G) \leq n + \gamma(G)$ and $\alpha_0(G) + 2 \leq \gamma_{oidR}(G) \leq 3\alpha_0(G)$. Moreover, if $\delta(G) \geq 2$, then $\gamma_{oidR}(G) \leq 2\alpha_0(G)$. In addition, nontrivial graphs G with $\gamma_{oidR}(G) = \alpha_0(G) + 2$ are characterized as follows.

Theorem 2.29 ([11]). *For each graph G of order $n \geq 2$, $\gamma_{oidR}(G) = \alpha_0(G) + 2$ if and only if one of the following holds:*

(i) *There is a vertex v with $\deg_G(v) = p - 1$;*
(ii) *There are two vertices z_1, z_2 with $V(G) = N(z_1) \cup N(z_2) \cup \{z_1, z_2\}$ and each maximum independent set of $G[N(z_1) \cap N(z_2)]$ is a maximum independent of G.*

Restricted to the class of trees, the authors showed that for each tree T of order $n \geq 3$ with $s(T)$ support vertices, $\gamma_{oidR}(T) \leq p + \frac{s(T)}{2}$. The authors also established Nordhaus-Gaddum bounds for $\gamma_{oidR}(G) + \gamma_{oidR}(\bar{G})$.

Theorem 2.30 ([11]). *For a graph G on $n \geq 2$ vertices,*

(i) $\gamma_{oidR}(G) + \gamma_{oidR}(\bar{G}) \leq 3n + 1$, *with equality if and only if $G \in \{K_n, \overline{K_n}\}$.*
(ii) $\gamma_{oidR}(G) + \gamma_{oidR}(\bar{G}) \leq \lfloor \frac{5n}{2} \rfloor + 2$, *when* $\min\{\delta(G), \delta(\bar{G})\} \geq 1$.
(iii) $\gamma_{oidR}(G) + \gamma_{oidR}(\bar{G}) \geq n + 5$.

2.5. Independent double Roman domination

Independent double Roman dominating functions (IDRDFs) has recently been introduced in [40], where an IDRDF is a DRDF $h = (V_0, V_1, V_2, V_3)$ for which $V_1 \cup V_2 \cup V_3$ is an independent set. The minimum weight of an IDRDF of G is the *independent double Roman domination number* $\gamma_{idR}(G)$. It was shown in [40] that the decision problem corresponding to the problem of computing $\gamma_{idR}(G)$ is NP-complete for bipartite graphs. The parameters $i_{dR}(G)$, $i(G)$, $i_R(G)$ and $i_{\{R2\}}(G)$ are related by the following results.

Theorem 2.31.

(i) [43] *For each graph G,*

$$\max\{2i(G), i(G) + i_{\{R2\}}(G)\} \leq i_{dR}(G) \leq 3i(G).$$

(ii) [40] *For each graph G of order $n \geq 1$, $i_R(G) + 1 \leq i_{dR}(G) \leq 2i_R(G)$ with equality in upper bound if and only if $G = \overline{K_n}$.*
(iii) [43] *For each graph G, $\frac{3}{2}i_{\{R2\}}(G) \leq i_{dR}(G) \leq 2i_{\{R2\}}(G)$.*

Theorem 2.32 ([40]). *For any connected graph G of order $n \geq 3$ with maximum degree $\Delta > 0$,*

$$i_{dR}(G) \geq \frac{2n}{\Delta} + \frac{\Delta - 2}{\Delta} i(G).$$

This bound is sharp for even cycles and paths of order $3k$.

Theorem 2.33 ([43]). *If G is a connected graph of order n with minimum degree δ, then $i_{dR}(G) \le 2n - (2\delta - 1)\rho(G)$.*

Restricted to the class of trees, the authors in [40] showed that for each tree T of order $n \ge 3, \gamma_{idR}(T) \le \frac{5n}{4}$. Mojdeh and Mansouri [43] showed that for each tree T of order $n \ge 2$, $2i(T) + 1 \le i_{dR}(T) \le 3i(T)$. They also proved that any ordered pair (a, b) is realizable as the independent domination and independent double Roman dominations numbers, respectively, of some nontrivial tree if and only if $2a + 1 \le b \le 3a$. Moreover, Rahmouni et al. [47] were interested in trees T with $i_{dR}(T) = 3i(T)$, called independent double Roman trees, where they gave a constructive characterization of such trees.

2.6. Global Roman domination

The study of the global Roman domination was initiated independently by Atapour et al. [20] and Roushini Leely Pushpam and Padmapriea [48]. A *global Roman dominating function* (GRDF) on a graph G is a function $f = (V_0, V_1, V_2)$ such that f is a Roman dominating function for both G and its complement $\bar{G}$. The *global Roman domination number* of G, denoted by $\gamma_{gR}(G)$, is the minimum weight of a GRDF on G. A *$\gamma_{gR}(G)$-function* is a GRDF of G with weight $\gamma_{gR}(G)$. The motivation for introducing this kind of functions comes from global dominating sets defined by Sampathkumar in 1989 [50]. A dominating set of G is *global* if it dominates both G and its complement $\bar{G}$. The *global domination number* $\gamma_g(G)$ is the minimum cardinality of a global dominating set in G. A global dominating set of G with minimum cardinality is called a *$\gamma_g(G)$-set*. The following properties of $\gamma_{gR}(G)$-functions were observed in [48].

Proposition 2.34 ([48]). *Let $f = (V_0, V_1, V_2)$ be any $\gamma_{gR}(G)$-function. Then:*

(i) $|V_2| \neq 1$.
(ii) *For every $v \in V_0$, $V_2 \not\subseteq N(v)$.*
(iii) *V_2 is a $\gamma_g(H)$-set, where $H = G[V_0 \cup V_2]$.*
(iv) *Each $v \in V_2$ has at least two private neighbors in H with respect to V_2.*

2.6.1. Relationships with γ_g and γ_R

Since for every GRDF $f = (V_0, V_1, V_2)$, $V_1 \cup V_2$ is a global dominating set of G, and since assigning a 2 to every vertex of a global dominating set provides a GRDF, it follows that $\gamma_g(G) \le \gamma_{gR}(G) \le 2\gamma_g(G)$ for all G. Roushinin Leely Pushpam and Padmapriea [48] showed that $\gamma_{gR}(G) = \gamma_g(G)$ if and only if G is a complete graph. They also showed that $\gamma_{gR}(G) = \gamma_g(G) + 1$ if and only if G is a complete graph minus an edge, that is $G = K_n - e$.

Moreover, since also every global Roman dominating function of G is an RDF of G and $\bar{G}, \gamma_{gR}(G) \ge \max\{\gamma_R(G), \gamma_R(\bar{G})\}$. A characterization of graphs G such that $\gamma_{gR}(G) = \gamma_R(G)$ was given in [48] as follows.

Proposition 2.35 ([48]). *Let G be any graph. Then $\gamma_{gR}(G) = \gamma_R(G)$ if and only if there exists a $\gamma_{gR}(G)$-function $f = (V_0, V_1, V_2)$ such that for every vertex in V_0 there is a vertex in V_2 such that they are not adjacent.*

Upper bounds on the global Roman domination number of a graph in terms of the Roman domination number were established by Atapour et al. [20].

Proposition 2.36 ([20]). *Let G be a graph of order n. Then*

(i) *If $diam(G) \ge 5$, then $\gamma_{gR}(G) \le \gamma_R(G) + 1$.*
(ii) *If $diam(G) \in \{3, 4\}$, then $\gamma_{gR}(G) \le \gamma_R(G) + 4$.*
(iii) *If $n \ge 4$, then $\gamma_{gR}(G) \le \gamma_R(G) + \delta(G) + 1$.*

By Proposition 2.36-(iii), if T is a tree of order $n \ge 4$, then $\gamma_{gR}(T) \le \gamma_R(T) + 2$. A characterization of trees T such that $\gamma_{gR}(T) = \gamma_R(T) + 2$ or $\gamma_{gR}(T) = \gamma_R(T) + 1$ was given by Atapour et al. [20]. Recall that a *spider* S_t is the graph formed by subdividing j $(j \ge 0)$ edges of a star $K_{1,t}$, for $t \ge 2$. Let $\mathcal{B}_1$ be the family of spiders S_t for some $t \ge 2$ except stars and P_5; $\mathcal{B}_2$ the family of trees T obtained from spiders $S_{r_1}, S_{r_2}, \ldots, S_{r_j}$ with centers $y_1, y_2, \ldots, y_j$ $(j \ge 2)$, where $r_k \ge 2$ for every $k \in \{1, \ldots, j\}$ except P_4 and P_5, by adding a new vertex x and joining x to every y_j; $\mathcal{B}_3$ the family of trees T obtained from a tree in class $\mathcal{B}_2$, say T_1, by adding a new vertex z attached at x, where x is the vertex added to construct the tree T_1 from spiders. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$.

Theorem 2.37 ([20]). *Let T be a tree of order $n \ge 4$. Then $\gamma_{gR}(T) = \gamma_R(T) + 2$ if and only if T is the star $K_{1,t}$ for some $t \ge 3$.*

Theorem 2.38 ([20]). *Let T be a tree of order $n \ge 4$. Then $\gamma_{gR}(T) = \gamma_R(T) + 1$ if and only if $T \in \mathcal{B}$.*

2.6.2. Bounds on γ_{gR}

Obviously $f = (\emptyset, V(G), \emptyset)$ is a GRDF of a graph G and thus $\gamma_{gR}(G) \le n$. It is shown in [48], that $\gamma_{gR}(G) = n$ only if G has diameter at most three. Moreover, they showed that a graph G of order n with diameter three satisfies $\gamma_{gR}(G) = n$ if and only if $G = P_4$ or G is a corona of K_3 or a corona of K_3 minus a leaf. Atapour et al. [20] gave the following necessary and sufficient condition for graphs G of order n such that $\gamma_{gR}(G) = n$.

Proposition 2.39 ([20]). *Let G be a graph of order n. Then $\gamma_{gR}(G) = n$ if and only if for every set S of vertices with $|S| \ge 2$ and each subset B of $N_G(S) - S$ with $|B| > |S|$, there is a vertex $x \in B$ such that $S \subseteq N_G(x)$.*

Additional upper bounds on the global Roman domination number of a graph in terms of the order, maximum and minimum degrees are obtained in [20]. The following result was obtained by using a probabilistic approach.

Theorem 2.40 ([20]). *Let G be a graph of order n and $\delta' = \min\{\delta(G), \delta(\bar{G})\} \ge 1$. Then $\gamma_{gR}(G) \le 2n\left(1 - \frac{\delta'}{(1+\delta')^{1+\frac{1}{\delta'}}}\right)$.*

Proposition 2.41 ([20]). *Let G be a graph of order $n \geq 4$ and $u, v \in V(G)$. If $uv \notin E(G)$, then $\gamma_{gR}(G) \leq n - \deg_G(u) - \deg_G(v) + 2|N(u) \cap N(v)| + 2$.*

Corollary 2.42. *If G is a connected triangle-free graph of order $n \geq 3$, then $\gamma_{gR}(G) \leq \min\{n - \Delta(G) - \delta(G) + 4, \gamma_R(G) + 2\}$.*

2.7. Distance Roman domination

For an integer $k \geq 1$, a *k-distance Roman dominating function* (kDRDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex u with $f(u) = 0$ there is a vertex v at distance at most k from u such that $f(v) = 2$. The *k-distance Roman domination number* of a graph G, denoted by $\gamma_R^k(G)$, equals the minimum weight of a k-distance Roman dominating function on G. A $\gamma_R^k(G)$-function is a k-distance Roman dominating function of G with weight $\gamma_R^k(G)$. It is worth noting that the 1-distance Roman domination number $\gamma_R^1(G)$ is the usual Roman domination number $\gamma_R(G)$. The concept of k-distance Roman domination was introduced in 2013 by Aram et al. [17].

For a vertex $v \in V(G)$, let $N_{k,G}(v) = \{u \in V(G) : u \neq v$ and $d(u, v) \leq k\}$ and $N_{k,G}[v] = N_{k,G}(v) \cup \{v\}$. The *k-degree* of a vertex v is defined as $\deg_{k,G}(v) = |N_{k,G}(v)|$. The *minimum* and *maximum k-degree* of a graph G are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. Let $k \geq 1$ be an integer. A set $D \subseteq V(G)$ is a *k-distance dominating set* of G if every vertex in $V(G) - D$ is within distance k of at least one vertex in D. The *k-distance domination number* $\gamma^k(G)$ is the minimum cardinality among all k-distance dominating sets of G.

Since $V_1 \cup V_2$ is a k-distance dominating set when $f = (V_0, V_1, V_2)$ is a kDRDF, and since assigning a 2 at the vertices of a k-distance dominating set provides a kDRDF, we have $\gamma^k(G) \leq \gamma_R^k(G) \leq 2\gamma^k(G)$. It is shown that $\gamma_R^k(G) = 2\gamma^k(G)$ if and only if G has a $\gamma_R^k(G)$-function $f = (V_0, V_1, V_2)$ with $|V_1| = 0$. The following properties generalize those already obtained for Roman dominating functions.

Proposition 2.43 ([17]). *Let $f = (V_0, V_1, V_2)$ be any $\gamma_R^k(G)$-function of a graph G. Then*

(i) $\Delta_k(G[V_1]) \leq 1$.
(ii) *If $w \in V_1$, then $N_{k,G}(w) \cap V_2 = \emptyset$.*
(iii) *If $u \in V_0$, then $|V_1 \cap N_{k,G}(u)| \leq 2$.*
(iv) *V_2 is a minimum k-dominating set of the induced subgraph $G[V_0 \cup V_2]$.*
(v) *Let $H = G[V_0 \cup V_2]$. Then each vertex $v \in V_2$ with $N_{k,G}(v) \cap V_2 \neq \emptyset$ has at least two private neighbors relative to V_2 in the graph H.*

Additional results have been obtained in [17] and are gathered by the following proposition.

Proposition 2.44 ([17]). *Let $k \geq 1$ be an integer and G a connected graph of order n. Then*

(i) *If $n \geq k + 2$, then $\gamma_R^k(G) \leq 4n/(2k + 3)$, with equality if and only if G is C_{2k+3} or obtained from $\frac{n}{2k+3} P_{2k+3}$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{2k+3} P_{2k+3}$.*
(ii) *If $n - \Delta(G) - k \geq 0$, then $\gamma_R^k(G) \leq n - \Delta(G) - k + 2$.*
(iii) *If $\Delta = \Delta(G) \geq 3$, then $\gamma_R^k(G) \geq \frac{2n(\Delta-2)}{\Delta(\Delta-1)^k - 2}$.*

Sharifi and Jafari Rad [54] presented a probabilistic upper bound for the distance Roman domination number. They also studied distance Roman domination number in Random graphs. The following Nordhaus-Gaddum inequality is given by Aram et al. [17].

Theorem 2.45 ([17]). *Let $k \geq 2$ be an integer, and let G be a graph of order $n \geq 3$. Then $\gamma_R^k(G) + \gamma_R^k(\bar{G}) \leq n + 2$, with equality if and only if G or $\bar{G}$ is isomorphic to $rK_1 \cup sK_2$ for two integers $r, s \geq 0$.*

A variation close to 2-distance Roman domination, namely hop Roman domination, is introduced by Shabani et al. [52] and further studied in [37, 46, 51]. A *hop Roman dominating function* (HRDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex $u \in V$ with $f(u) = 0$ there is a vertex v with $f(v) = 2$ and $d_G(u, v) = 2$. The minimum weight of an HRDF on G is called the *hop Roman domination number* of G and is denoted by $\gamma_{hR}(G)$. An HRDF f is a hop Roman independent dominating function (HRIDF) if for any pair v, w with non-zero labels under f, $d_G(v, w) \neq 2$. The minimum weight of an HRIDF on G is called the *hop Roman independent domination number* of G. It is shown in [37] that the decision problems related to the hop Roman domination and hop Roman independent domination are NP-complete even when restricted to planar bipartite graphs or planar chordal graphs. Moreover, a linear time algorithm to compute the hop Roman domination number in trees is given in [46]. A characterization of graphs G of order n with $\gamma_{hR}(G) \in \{n-1, n\}$ is provided in [51].

2.8. Roman {3}-domination

In [44], Mojdeh and Volkmann defined a variant of double Roman domination, namely Roman {3}-domination or double Italian domination. For a graph G, a *Roman {3}-dominating function* is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that $f(N[u]) \geq 3$ for every vertex $u \in V(G)$ with $f(u) \in \{0, 1\}$. The minimum weight of a Roman {3}-dominating function is the *Roman {3}-domination number*, denoted by $\gamma_{\{R3\}}(G)$. Clearly, $\gamma_{\{R3\}}(G) \leq \gamma_{dR}(G)$, since every double Roman dominating function is a Roman {3}-dominating function. Therefore each upper bound of $\gamma_{dR}(G)$ is also an upper bound of $\gamma_{\{R3\}}(G)$. If T is a tree, then we even have $\gamma_{\{R3\}}(T) = \gamma_{dR}(T)$, and if C_n is a cycle of order n, then $\gamma_{\{R3\}}(C_n) = n$ (see [44]). In addition, the authors in [44] determined the Roman {3}-domination number of any complete r-partite graph for $r \geq 2$.

Observation 2.46 ([44]). *If G is a graph of order $n \geq 2$, then $\gamma_{\{R3\}}(G) \geq 3$, with equality if and only if $\Delta(G) = n - 1$.*

Observation 2.47 ([44]). *If G is a graph of order n with minimum degree $\delta \geq 2$, then $\gamma_{\{R3\}}(G) \leq n + 2 - \delta$.*

Theorem 2.48 ([44]). *If G is a graph, then $\gamma(G)+2 \leq \gamma_{\{R3\}}(G) \leq 3\gamma(G)$.*

Different examples in [44] show that Observation 2.47 and Theorem 2.48 are sharp. Next we present lower bounds on $\gamma_{\{R3\}}(G)$ in terms of order and maximum degree.

Theorem 2.49 ([44]). *If G is a connected graph of order n and maximum degree Δ, then*

$$\gamma_{\{R3\}}(G) \geq \frac{3n}{\Delta+3}.$$

If Δ is great, then the next bound is an improvement of Theorem 2.49

Theorem 2.50 ([44]). *If G is a connected graph of order n and maximum degree Δ, then*

$$\gamma_{\{R3\}}(G) \geq \min\left\{\frac{3n}{\Delta+2}, \frac{2n+\Delta}{\Delta+1}\right\}.$$

In [22], the authors have shown that the decision problem corresponding to the problem of computing $\gamma_{\{R2\}}(G)$ is NP-complete for bipartite graphs. Using this result, the authors in [44] show that the Roman {3}-domination problem is also NP-complete for bipartite graphs.

2.9. Locating Roman domination

Jafari Rad, Rahbani and Volkmann [36] considered Roman dominating functions $f=(V_0,V_1,V_2)$ with a further condition that for each vertex $x \in V_0$ the set $N(x)\cap V_2$ is unique. That is, any two vertices x, y in V_0 are distinguished in the sense that there is a vertex $v \in V_2$ with $|N(v)\cap\{x,y\}|=1$. An RDF $f=(V_0,V_1,V_2)$ is called a *locating Roman dominating function* (or just LRDF) if $N(v)\cap V_2 \neq N(u)\cap V_2$ for any pair u, v of distinct vertices of V_0. The *locating Roman domination number* $\gamma_R^L(G)$ is the minimum weight of an LRDF of G. Note that $\gamma_R^L(G)$ is defined for any graph G, since $(\emptyset, V(G), \emptyset)$ is an LRDF for G. Jafari Rad et al. studied the complexity issue and showed that the decision problem corresponding to the problem of computing $\gamma_R^L(G)$ is NP-complete for bipartite and chordal graphs. They also showed that the locating Roman domination numbers of a graph G and its complement graph $\bar{G}$ differ by one unit, that is, $|\gamma_R^L(G)-\gamma_R^L(\bar{G})| \leq 1$. For paths and cycles, they established that for $n \geq 3, \gamma_R^L(P_n)=\gamma_R^L(C_n)=\lceil\frac{4n}{5}\rceil$.

2.9.1. $\gamma_R^L(G)$ versus $\gamma_R(G), \gamma_L(G)$ or $\gamma_R^L(\bar{G})$

A set D of vertices in a graph $G=(V,E)$ is a *locating-dominating set* if for every two vertices u, v of $V\setminus D$ the sets $N(u)\cap D$ and $N(v)\cap D$ are non-empty and different. The *locating-domination number* $\gamma_L(G)$ is the minimum cardinality of a locating-dominating set. Locating-domination was introduced by Slater [55, 56].

It was observed in [36] that for any connected graph G of order n, $\gamma_R(G) \leq \gamma_R^L(G) \leq n$ and $\gamma_L(G) \leq \gamma_R^L(G) \leq 2\gamma_L(G)$. Moreover, the authors provided a constructive characterization for trees T with $\gamma_R^L(T)=\gamma_R(T)$.

Proposition 2.52 ([36]). *Let G be a graph of order n. Then*

(i) $\gamma_R^L(G)=\gamma_L(G)$ *if and only if* $G=\overline{K_n}$.
(ii) $\gamma_R^L(G)=\gamma_L(G)+1$ *if and only if G is a complete graph or a star of order at least two.*

We will say that a graph G is a *locating Roman graph* if $\gamma_R^L(G)=2\gamma_L(G)$. It was shown that for $n \geq 3$, a path P_n is locating Roman if and only if $n \equiv 0,\ 2$ or $4 \pmod 5$, and a cycle C_n is locating Roman if and only if $n \equiv 0,\ 2$ or $4 \pmod 5$. Jafari Rad et al. [36] gave the following equivalent conditions for locating Roman graphs

Theorem 2.53 ([36]). Let G be a graph. Then the following conditions are equivalent.

(i) *G is a locating Roman graph.*
(ii) *$\gamma_L(G) \leq \gamma_L(G-S)+|S|/2$ for any independent set S.*
(iii) *G has a $\gamma_R^L(G)$-function $f=(V_0,V_1,V_2)$ with $|V_1|=0$.*

Proposition 2.54 ([36]). *If G is a locating Roman graph of order $n \geq 2$, then $\gamma_R^L(G) \geq \max\left\{\lfloor 2\log_2 n\rfloor, \frac{4n}{3+\Delta}\right\}$.*

2.9.2. Bounds on $\gamma_R^L(G)$

Restricted to the class of trees, Jafari Rad and Rahbani [35] obtained upper and lower bounds for the locating Roman domination number.

Theorem 2.55 ([35]). *For any tree T of order $n \geq 2$ with ℓ leaves and s support vertices, $\gamma_R^L(T) \geq (2n+(\ell-s)+2)/3$.*

Corollary 2.56 ([35]). *For any tree T of order $n \geqslant 2, \gamma_R^L(T) \geq (2n+2)/3$.*

Theorem 2.57 ([35]). *For any tree T of order $n \geq 2$, with ℓ leaves and s support vertices, $\gamma_R^L(T) \leq (4n+\ell+s)/5$.*

We note that a constructive characterization of all trees achieving equality in the bounds of Theorem 2.55 or Theorem 2.57 can be found in [35].

3. Signed Roman domination parameters in graphs

In this section, we present different results concerning the signed version of nine types of Roman dominating functions, where for each function negative weight can be assigned to vertices or edges.

3.1. Signed Roman domination number in graphs

Constantine's model (see [23]) did not achieve the desired goal of being both cost effective and of defending the Roman Empire. Following the paper [13], the authors explored an alternative model which would save the Emperor substantial costs of maintaining legions, while still defending the Roman Empire. The 4th century A.D. saw a very large number of new, small legions created, a process which began under Constantine II. In particular, *auxiliary cohortes* (about a tenth the size of a legion) and *auxilia palatina* were formed. Auxiliary troops were mainly recruited from *peregrini*, i.e., free provincial subjects of the Roman Empire who did not hold Roman citizenship, in contrast to the legions, which only admitted Roman citizens. Auxiliary troops were considered second-class soldiers and were looked down on by the elite troops of the *comitatensis* who were paid regularly and were much better equipped. As a cost effective way of securely defending the Roman Empire, Emperor Constantine's strategy would be to minimize the number of legions stationed by placing auxiliary troops at every unsecured location provided that the number of legions stationed at a location and its neighboring location always exceeded the number of auxiliary troop stationed there for every location in the Roman Empire.

In graph theoretic terms, Ahangar, Henning, Löwenstein, Zhao and Samodivkin [13] defined a *signed Roman dominating function* (SRDF) on a graph G as a function $f: V(G) \to \{-1, 1, 2\}$ satisfying the condition that f is a *dominating function* (that is, the sum of the values assigned to a vertex and its neighbors is at least 1 for every vertex), and every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an SRDF f on a graph G is defined by $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed Roman domination number*, denoted $\gamma_{sR}(G)$, is the minimum weight of an SRDF in G. An SRDF of weight $\gamma_{sR}(G)$ is called a $\gamma_{sR}(G)$-function. In the earlier model where a vertex in the graph represents a location in the Roman Empire, an assignment of –1 equates to stationing of an auxiliary cohorte or auxilia palatina at that location, while as before a + 1 and + 2 equates to stationing one or two legions, respectively, at that location.

For an SRDF f on G, let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. An SRDF f can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$. The definitions immediately lead to the first observation.

Observation 3.1 ([13]). *If $f = (V_{-1}, V_1, V_2)$ is an SRDF on a graph G of order n, then the following holds.*

(a) $|V_{-1}| + |V_1| + |V_2| = n$.
(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.
(c) *Every vertex in V_{-1} is dominated by a vertex of V_2.*
(d) *$V_1 \cup V_2$ is a dominating set of G.*

3.1.1. Exact values

In this part, the signed Roman domination numbers of stars, complete graphs, cycles, paths and complete bipartite graphs $K_{p,p}$ are determined. For a star, we assign to the central vertex the value 2 and to as many leaves as possible the value –1 and to the remaining leaves the value 1. This leads to $\gamma_{sR}(K_{1,n-1}) = 1$ if n is even, and $\gamma_{sR}(K_{1,n-1}) = 2$ for odd n.

Proposition 3.2 ([13]). *For $n \geq 1$, we have $\gamma_{sR}(K_n) = 1$, unless $n = 3$ in which case $\gamma_{sR}(K_3) = 2$.*

Proposition 3.3 ([13]). *For the classes of paths P_n and cycles C_n, $\gamma_{sR}(P_n) = \lfloor (2n)/3 \rfloor$ and $\gamma_{sR}(C_n) = \lceil (2n)/3 \rceil$.*

Proposition 3.4 ([33]). *For $p \geq 3$, we have $\gamma_{sR}(K_{p,p}) = 4$.*

3.1.2. Bounds on the signed Roman domination number

Theorem 3.5 ([13]). *If G is an r-regular graph of order n with $r \geq 1$, then $\gamma_{sR}(G) \geq n/(r+1)$.*

Theorem 3.6 ([13]). *If G is a graph of order n, minimum degree δ and maximum degree Δ such that $\delta < \Delta$, then*

$$\gamma_{sR}(G) \geq \left(\frac{-2\Delta + 2\delta + 3}{2\Delta + \delta + 3} \right) n.$$

Theorem 3.7 ([13]). *If G is a graph of order n, then the following holds.*

(a) $\gamma_{sR}(G) \leq n$, *with equality if an only if $G = \overline{K_n}$.*
(b) $\gamma_{sR}(G) \geq 2\gamma(G) - n$, *with equality if an only if $G = \overline{K_n}$.*

For $k \geq 1$, let F_k be the graph obtained from the disjoint union of k stars $K_{1,2k-1}$ by adding all edges between the central vertices of the k stars. Let $\mathcal{F} = \{F_k \mid k \geq 1\}$.

Theorem 3.8 ([13]). *If G is a graph of order n, then $\gamma_{sR}(G) \geq \frac{3}{\sqrt{2}}\sqrt{n} - n$, with equality if and only if $G \in \mathcal{F}$.*

For $k \geq 1$, let L_k be the graph obtained from a graph H of order k by adding $2\deg_H(v) + 1$ pendant edges to each vertex v of H. Let $\mathcal{H} = \{L_k \mid k \geq 1\}$.

Theorem 3.9 ([13]). *If G is a graph of order n and size m without isolated vertices, then $\gamma_{sR}(G) \geq \frac{3n-4m}{2}$, with equality if and only if $G \in \mathcal{H}$.*

For $k \geq 1$, let B_k be the bipartite graph obtained from $K_{k,k}$ by adding $2k + 1$ pendant edges to each vertex of the complete bipartite graph. Let $\mathcal{B} = \{B_k \mid k \geq 1\}$.

Theorem 3.10 ([13]). *If G is a bipartite graph of order n, then $\gamma_{sR}(G) \geq 3\sqrt{n+1} - n - 3$, with equality if and only if $G \in \mathcal{B}$.*

A *signed dominating function* is defined in [28] as a function $f: V(G) \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for all $v \in V(G)$. The *signed domination number*, denoted $\gamma_s(G)$, is the minimum weight of a signed dominating function in G. A signed Roman dominating function combines properties of both a Roman dominating function and a signed dominating function. Some bounds for the signed Roman domination number in terms of the signed domination number are given in [13]. For $k \geq 1$, let G_k be a graph obtained from a bipartite graph H of order $3k$ with partite sets $\mathcal{L}$ and $\mathcal{R}$, where every component of H is isomorphic to P_3 or C_6 and where every vertex in $\mathcal{L}$ has degree 2 in H, by adding edges between vertices of $\mathcal{R}$ in such a way that if $v \in \mathcal{R}$ and v belongs to a P_3-component of H, then $\deg_{G_k}(v) \geq \deg_H(v) + 1 = 2$ and if $v \in \mathcal{R}$ and v belongs to a C_6-component of H, then $\deg_{G_k}(v) \geq \deg_H(v) + 2 \geq 4$. Let $\mathcal{G} = \{G_k \mid k \geq 1\}$.

Theorem 3.11 ([13]). *If G is a graph of order n, then $\gamma_{sR}(G) \leq \gamma_s(G) + \frac{n}{3}$, with equality if and only if $G \in \mathcal{G}$.*

Recently, Volkmann [60] defined the *weak signed Roman dominating function* (WSRDF) of a graph G as a function f :

$V(G) \to \{-1, 1, 2\}$ having the property $f(N[v]) \geq 1$ for each $v \in V(G)$. The weight of a WSRDF is the value $\sum_{u \in V(G)} f(u)$. The *weak signed Roman domination number*, denoted by $\gamma_{wsR}(G)$, is the minimum weight of a WSRDF in G. The definitions lead to $\gamma_{wsR}(G) \leq \gamma_{sR}(G)$. Therefore each lower bound of $\gamma_{wsR}(G)$ is also a lower bound of $\gamma_{sR}(G)$. In [60] it is shown that many lower bounds on $\gamma_{sR}(G)$ are also valid for $\gamma_{wsR}(G)$. In particular, Volkmann [60] proved that Theorems 3.5, 3.6, 3.7 and 3.9 also hold for the weak signed Roman domination number. In addition, the difference $\gamma_{sR}(G) - \gamma_{wsR}(G)$ can be arbitrarily large.

3.2. Signed Roman k-domination in graphs

For every integer $k \geq 1$, the *signed Roman k-dominating function* (SRkDF) on a graph G is defined by Henning and Volkmann in [33] as a function $f : V(G) \to \{-1, 1, 2\}$ such that $f(N[v]) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an SRkDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed Roman k-domination number* $\gamma^k_{sR}(G)$ of G is the minimum weight of an SRkDF on G. The special case $k = 1$ was introduced and investigated in [13] (see Subsection 2.1). A $\gamma^k_{sR}(G)$-function is a signed Roman k-dominating function on G of weight $\gamma^k_{sR}(G)$. For an SRkDF f on G, let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. An SRkDF f can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$. Note that the signed Roman k-domination number exists when $\delta \geq \frac{k}{2} - 1$.

3.2.1. Signed Roman k-domination in special classes of graphs

Below we summarize the results on the signed Roman k-domination in some special classes of graphs.

Proposition 3.12 ([33]). *If $n \geq k \geq 2$ are integers, then $\gamma^k_{sR}(K_n) = k$.*

Proposition 3.13 ([33]). *If $k \geq 1$ and $p \geq k + 2$ are integers, then $\gamma^k_{sR}(K_{p,p}) = 2k + 2$.*

Proposition 3.14 ([33]). *Let $k \geq 1$ and $k - 1 \leq p \leq k + 1$ be integers.*

(a) *If $k \geq 2$, then $\gamma^k_{sR}(K_{k-1,k-1}) = 2k - 2$.*
(b) *$\gamma^1_{sR}(K_{1,1}) = 1$ and if $k \geq 2$, then $\gamma^k_{sR}(K_{k,k}) = 2k$.*
(c) *$\gamma^k_{sR}(K_{k+1,k+1}) = 2k + 1$.*

Proposition 3.15 ([33]). *(1) If $2 \leq n \leq 7$, then $\gamma^2_{sR}(P_n) = n$, and if $n \geq 8$, then $\gamma^2_{sR}(P_n) = \lceil \frac{2n+5}{3} \rceil$.*
(2) For $n \geq 3$, we have $\gamma^2_{sR}(C_n) = \lceil \frac{2n}{3} \rceil + \lceil \frac{n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$.

Proposition 3.16 ([16]). (1) *For $n \geq 4$, we have $\gamma^3_{sR}(P_n) = n + 2$.*
(2) *For $n \geq 3$, we have $\gamma^4_{sR}(P_n) = \lceil \frac{4n}{3} \rceil + 2$.*

3.2.2. Bounds on the signed Roman k-domination number

Theorem 3.17 ([62]). *Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{sR}(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2} - 1$, and each vertex of G is of minimum degree or adjacent to a vertex of minimum degree.*

Theorem 3.18 ([33]). *Let G be a graph of order n with $\delta(G) \geq k - 1$. Then $\gamma^k_{sR}(G) \leq n$. If $\delta(G) \geq k + 2t - 1$ for an integer $t \geq 1$, then $\gamma^k_{sR}(G) \leq n - t$.*

Theorem 3.19 ([33]). *If G is an r-regular graph of order n with $r \geq k - 1$, then $\gamma^k_{sR}(G) \geq \frac{kn}{r+1}$.*

If H is a $(k-1)$-regular graph of order n, then it follows from Theorem 3.19 that $\gamma^k_{sR}(H) \geq n$ and thus $\gamma^k_{sR}(H) = n$, according to Theorem 3.18. This example shows that Theorems 3.18 and 3.19 are both sharp. As an immediate consequence of Theorems 3.18 and 3.19, we obtain $\gamma^3_{sR}(C_n) = n$.

Theorem 3.20 ([33]). *If G is a graph of order n with $\delta(G) \geq k - 1$, then*

$$\gamma^k_{sR}(G) \geq k + 1 + \Delta(G) - n.$$

Proposition 3.12 shows that Theorem 3.20 is sharp.

Theorem 3.21 ([33]). *Let G be a graph of order n, minimum degree $\delta \geq k - 1$ and maximum degree Δ. If $\delta < \Delta$, then*

$$\gamma^k_{sR}(G) \geq \left(\frac{-2\Delta + 2\delta + 3k}{2\Delta + \delta + 3} \right) n.$$

In [33] are given examples which show that Theorem 3.21 is sharp for each $k \geq 1$. Theorems 3.5 and 3.6 are the special case $k = 1$ of Theorems 3.19 and 3.21.

Theorem 3.22 ([33]). *If G is a graph of order n with $\delta(G) \geq k - 1$ and packing number $\rho(G)$, then*

$$\gamma^k_{sR}(G) \geq \rho(G)(k + \delta(G) + 1) - n.$$

In [33] the reader can find examples which show that Theorem 3.22 is sharp. As an application of Theorem 3.19, we next present a Nordhaus-Gaddum type inequality for the signed Roman k-domination number of regular graphs.

Theorem 3.23 ([33]). *If G is an r-regular graph of order n such that $r \geq k - 1$ and $n - r - 1 \geq k - 1$, then*

$$\gamma^k_{sR}(G) + \gamma^k_{sR}(\bar{G}) \geq \frac{4kn}{n+1}.$$

If n is even, then $\gamma^k_{sR}(G) + \gamma^k_{sR}(\bar{G}) \geq 4k(n+1)/(n+2)$.

Let $k \geq 1$ be an odd integer, and let H and $\bar{H}$ be $(k-1)$-regular graphs of order $n = 2k - 1$. As seen above, we have $\gamma^k_{sR}(H) = \gamma^k_{sR}(\bar{H}) = n$. Consequently,

$$\gamma^k_{sR}(H) + \gamma^k_{sR}(\bar{H}) = 2n = \frac{4kn}{n+1}.$$

Thus the Nordhaus-Gaddum bound of Theorem 3.23 is sharp for odd k.

The Dutch-windmill graph, $K_3^{(p)}$, is a graph which consists of p copies of K_3 with one vertex in common.

Theorem 3.24 ([16]). *Let $k \geq 2$ be an integer, and let G be a graph of order n and size m such that $\delta(G) \geq k$. Then*

$$\gamma_{sR}^k(G) \geq \frac{3}{k+7}\left(\frac{5k+8}{3}n - 4m\right).$$

This bound is sharp for the Dutch-windmill graph $K_3^{(p)}$ ($p \geq 2$) when $k=2$.

Theorem 3.25 ([33]). *Let G be a connected cubic graph of order n. Then the following holds.*

(a) $\frac{n}{2} \leq \gamma_{sR}^2(G) \leq \frac{7n}{8}$.
(b) $\frac{3n}{4} \leq \gamma_{sR}^3(G) \leq n$.

Question 1 ([33]). *Is it true that if G is a cubic graph of order n, then $\gamma_{sR}^2(G) \leq 5n/6$?*

For an integer $k \geq 1$, Volkmann [62] recently defined the *weak signed Roman k-dominating function* (WSRkDF) of a graph G as a function $f: V(G) \to \{-1, 1, 2\}$ having the property $f(N[v]) \geq k$ for each $v \in V(G)$. The weight of a WSRkDF is the value $\sum_{u \in V(G)} f(u)$. The *weak signed Roman k-domination number*, denoted by $\gamma_{wsR}^k(G)$, is the minimum weight of a WSRkDF on G. The definitions lead to $\gamma_{wsR}^k(G) \leq \gamma_{sR}^k(G)$. Therefore each lower bound of $\gamma_{wsR}^k(G)$ is also a lower bound of $\gamma_{sR}^k(G)$. In [62] it is shown that many lower bounds on $\gamma_{sR}^k(G)$ are also valid for $\gamma_{wsR}^k(G)$. In particular, Volkmann [62] proved that Theorems 3.19, 3.20, 3.21 and 3.22 also hold for the weak signed Roman k-domination number.

3.2.3. Signed Roman k-domination number in trees

The aim in the subsection is to determine lower and upper bounds on the signed Roman k-domination number of trees in terms of its order for $k = 2, 3, 4$.

Let $\mathcal{T}$ be the family of trees constructed as follows. Let T' be an arbitrary tree of order $n' \geq 2$. For each vertex $v \in V(T')$, add $2\deg_{T'}(v)$ vertex disjoint copies of a star $K_{1,3}$ and join v to a leaf from each of the added $2\deg_{T'}(v)$ stars. Let T be the resulting tree and let $\mathcal{T}$ be the family of all such trees.

Theorem 3.26 ([32]). *If T is a tree of order $n \geq 4$, then $\gamma_{sR}^2(T) \geq \frac{10n+24}{17}$, with equality if and only if $T \in \mathcal{T}$.*

Theorem 3.27 ([16]). *Let T be a tree of order $n \geq 2$. Then*

(a) $\gamma_{sR}^3(T) \leq \frac{3n}{2}$, with equality if and only if T is the corona of some tree T'.
(b) $\gamma_{sR}^4(T) \leq 2n$, with equality if and only if every vertex of T is either a leaf or a support vertex.
(c) $\gamma_{sR}^4(T) \geq n+2$, with equality if and only if $T = P_2$.

Note that Theorem 3.27 (b) is a special case of Theorem 3.17. In [16], one can find the following statement. If T is a tree of order $n \geq 2$, then $\gamma_{sR}^3(T) \geq \frac{4n+7}{5}$, with equality if and only if $T = P_2$. The next example (see [62]) demonstrates that this statement is not valid.

Let $P_n = v_1 v_2 \ldots v_{2p+1}$ be a path of order $2p+1$ with an integer $p \geq 1$. Now attach two pendant edges to v_1 and v_{2p+1} and three pendant edges to v_{2i+1} for $1 \leq i \leq p-1$. The resulting tree T_{5p+2} is of order $5p+2$. Define the function $f: V(T_{5p+2}) \to \{-1, 1, 2\}$ by $f(v_{2i+1}) = 2$ for $0 \leq i \leq p$, $f(v_{2i}) = -1$ for $1 \leq i \leq p$ and $f(x) = 1$ otherwise. Then f is an SR3DF on T_{5p+2} of weight

$$\omega(f) = 2(p+1) - p + 3p + 1 = 4p + 3 = \frac{4n(T_{5p+2}) + 7}{5}.$$

Therefore $\gamma_{sR}^3(T_{5p+2}) \leq \frac{4n(T_{5p+2})+7}{5}$. Since $f(u) + f(v) \geq 3$ if v is a leaf and u its support vertex, it is easy to verify that $\gamma_{sR}^3(T_{5p+2}) = \frac{4n(T_{5p+2})+7}{5}$.

In [62], Volkmann conjectured that the bound $\gamma_{sR}^3(T) \geq \frac{4n+7}{5}$ is really valid for each tree of order $n \geq 2$. However, he only could prove the weaker bound $\gamma_{sR}^3(T) \geq \frac{3n+6}{4}$.

3.3. Nonnegative signed Roman domination in graphs

A *nonnegative signed Roman dominating function* (NNSRDF) on a graph G is defined by Dehgardi and Volkmann in [26] as a function $f: V(G) \to \{-1, 1, 2\}$ such that $f(N[v]) \geq 0$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an NNSRDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *nonnegative signed Roman domination number* $\gamma_{sR}^0(G)$ of G is the minimum weight of an NNSRDF on G.

Observation 3.28 ([26]). *If G is a graph of order n, then $\gamma_{sR}^0(G) \geq \Delta(G) + 1 - n$.*

Proposition 3.29 ([26]). *For $n \geq 1$, we have $\gamma_{sR}^0(K_{1,n}) = 0$ with exception of the cases that $n=1$ or $n=3$, in which cases we have $\gamma_{sR}^0(K_{1,1}) = \gamma_{sR}^0(K_{1,3}) = 1$.*

Proposition 3.30 ([26]). *For $n \geq 1$, we have $\gamma_{sR}^0(K_n) = 1$ when $n = 1, 2, 4$ and $\gamma_{sR}^0(K_n) = 0$ otherwise.*

Propositions 3.29 and 3.30 show that Observation 3.28 is sharp.

Proposition 3.31 ([26]). *(1)For $n \geq 1$, we have $\gamma_{sR}^0(P_n) = 0$ when $n \equiv 0 \pmod 3$ and $\gamma_{sR}^0(P_n) = 1$ otherwise.*
(2) For $n \geq 3$, we have $\gamma_{sR}^0(C_n) = 0$ when $n \equiv 0 \pmod 3$, $\gamma_{sR}^0(C_n) = 2$ when $n \equiv 1 \pmod 3$ and $\gamma_{sR}^0(C_n) = 1$ when $n \equiv 2 \pmod 3$.
(3) For $n \geq m \geq 2$, we have $\gamma_{sR}^0(K_{m,n}) = 3$ when $m=3$ and $\gamma_{sR}^0(K_{m,n}) = 2$ otherwise.

In the following we present some sharp upper and lower bounds on the nonnegative signed Roman domination number.

Theorem 3.32 ([26]). *If G is a graph of order n, then $\gamma_{sR}^0(G) \leq n$, with equality if and only if $G = \overline{K_n}$.*

Theorem 3.33 ([26]). *If G is a regular graph, then $\gamma_{sR}^0(G) \geq 0$.*

Proposition 3.30 demonstrates that Theorem 3.33 is sharp.

Theorem 3.34 ([26]). *If G is a graph of order n, minimum degree δ and maximum degree Δ such that $\delta < \Delta$, then*

$$\gamma_{sR}^{0}(G) \geq \frac{2n(\delta - \Delta)}{2\Delta + \delta + 3}.$$

In [26], the reader can find examples which show that Theorem 3.34 is sharp.

Let $\mathcal{F}_k = \{F_k \mid k \geq 1\}$ be a family of graphs defined as follows. Let X be the vertex set of the complete graph K_k, and let F_k be the graph obtained from K_k by adding $2k$ new vertices to each vertex of the complete graph such that for each new vertex x, $1 \leq \deg(x) \leq 2$ and for every vertex $u \in X, \deg(u) = 3k - 1$. Let $\mathcal{F} = \cup_{k \geq 1} \mathcal{F}_k$.

Theorem 3.35 ([26]). *If G is a graph of order n, then $\gamma_{sR}^{0}(G) \geq \frac{3}{4}\left(\sqrt{8n+1} - 1\right) - n$, with equality if and only if $G \in \mathcal{F}$.*

Theorem 3.36 ([26]). *If G is a connected graph of order $n \geq 2$ and size m, then $\gamma_{sR}^{0}(G) \geq \frac{8n-12m}{7}$.*

Examples in [26] show that Theorem 3.36 is sharp.

Let $\mathcal{B}_k = \{B_k \mid k \geq 1\}$ be a family of bipartite graphs defined as follows. Let X and Y be the partite sets of the complete bipartite graph $K_{k,k}$, and let B_k be the bipartite graph obtained from $K_{k,k}$ by adding $2k+2$ new vertices to each vertex of the complete bipartite graph such that for each new vertex x, $1 \leq \deg(x) \leq 2$ and for every vertex $u \in X \cup Y, \deg(u) = 3k + 2$. Let $\mathcal{B} = \cup_{k \geq 1} \mathcal{B}_k$.

Theorem 3.37 ([26]). *If G is a bipartite graph of order n, then $\gamma_{sR}^{0}(G) \geq \frac{3}{2}\left(\sqrt{4n+9} - 3\right) - n$, with equality if and only if $G \in \mathcal{B}$.*

3.4. Signed total Roman k-domination in graphs

If $k \geq 0$ is an integer, then the *signed total Roman k-dominating function* (STRkDF) on a graph G is defined by Volkmann [59] as a function $f : V(G) \to \{-1, 1, 2\}$ such that $f(N(v)) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRkDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed total Roman k-domination number* $\gamma_{stR}^{k}(G)$ of G is the minimum weight of an STRkDF on G. A $\gamma_{stR}^{k}(G)$-function is a signed total Roman k-dominating function on G of weight $\gamma_{stR}^{k}(G)$. For an STRkDF f on G, let $V_i = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A signed total Roman k-dominating function $f : V(G) \to \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$.

The signed total Roman k-domination number exists when $\delta(G) \geq \frac{k}{2}$. The special cases $k = 0$ and $k = 1$ were introduced and investigated in [27] and [57].

3.4.1. Signed total Roman k-domination in special classes of graphs

We summarize below the signed total Roman k-domination numbers of some special classes of graphs.

Observation 3.38 ([27, 57]). *For $n \geq 1$, we have $\gamma_{stR}^{0}(K_{1,n}) = 2$. For $n \geq 2$, we have $\gamma_{stR}^{1}(K_{1,n}) = 3$.*

Proposition 3.39 ([27, 57, 59]). *For $n \geq k + 2$, we have $\gamma_{stR}^{k}(K_n) = k + 2$.*

Proposition 3.40 ([57, 59]). *If $k \geq 1$ and $p \geq k$ are integers, then $\gamma_{stR}^{k}(K_{p,p}) = 2k$, with exception of the case that $k = 1$ and $p = 3$, in which case $\gamma_{stR}^{1}(K_{3,3}) = 4$.*

Proposition 3.41 ([27]). *For $n \geq 2$, we have $\gamma_{stR}^{0}(K_{2,n}) = 2$ when $n = 2$ or $n = 4$ and $\gamma_{stR}^{0}(K_{2,n}) = 1$ otherwise.*

Proposition 3.42 ([27]). *For $n \geq m \geq 3$, we have $\gamma_{stR}^{0}(K_{m,n}) = 2$ when $n = m = 4$, $\gamma_{stR}^{0}(K_{m,n}) = 1$ when $m = 3$ and $n = 4$ or $m = 4$ and $n \geq 5$ and $\gamma_{stR}^{0}(K_{m,n}) = 0$ otherwise.*

Proposition 3.43 ([27]). *For $n \geq 3$, we have $\gamma_{stR}^{0}(C_n) = \lceil \frac{n}{2} \rceil$ when $n \equiv 0, 1, 3 \pmod 4$ and $\gamma_{stR}^{0}(C_n) = \frac{n}{2} + 1$ when $n \equiv 2 \pmod 4$.*

Proposition 3.44 ([57]). *For $n \geq 3$, we have $\gamma_{stR}^{1}(C_n) = \frac{n}{2}$ when $n \equiv 0 \pmod 4$ and $\gamma_{stR}^{1}(C_n) = \frac{n+3}{2}$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stR}^{1}(C_n) = \frac{n+6}{2}$ when $n \equiv 2 \pmod 4$.*

Proposition 3.45 ([27]). *For $n \geq 3$, we have $\gamma_{stR}^{0}(P_n) = \lceil \frac{n}{2} \rceil$ when $n \equiv 0, 1, 3 \pmod 4$ and $\gamma_{stR}^{0}(P_n) = \frac{n}{2} + 1$ when $n \equiv 2 \pmod 4$.*

Proposition 3.46 ([57]). *For $n \geq 3$, we have $\gamma_{stR}^{1}(P_n) = \frac{n}{2}$ when $n \equiv 0 \pmod 4$ and $\gamma_{stR}^{1}(P_n) = \lceil \frac{n+3}{2} \rceil$ otherwise.*

3.4.2. Bounds on the signed total Roman k-domination number

We will present below upper and lower bounds on the signed total Roman k-domination number.

Proposition 3.47 ([27, 57, 59]). *If G is a graph of order n with $\delta(G) \geq k$, then $\gamma_{stR}^{k}(G) \leq n$.*

Theorem 3.48 ([58]). *If G is a graph of order n with $\delta(G) \geq 1$ and $\Delta(G) \geq 3$, then $\gamma_{stR}^{1}(G) \leq n - 1$.*

As a simple consequence of Theorem 3.48, Propositions 3.44 and 3.46, we obtain the next result.

Corollary 3.49 ([58]). *Let G be a graph of order n with $\delta(G) \geq 1$. Then $\gamma_{stR}^{1}(G) = n$ if and only if the components of G are K_2, K_3, P_3 or C_6.*

Theorem 3.50 ([27, 59]). *If G is a graph of order $n \geq 2$ with minimum degree $\delta \geq k + 2$, then*

$$\gamma_{stR}^{k}(G) \leq n + 1 - 2\left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

Theorem 3.51 ([27, 57, 59]). *If G is an r-regular graph of order n with $r \geq \max\{1, k\}$, then $\gamma^k_{stR}(G) \geq \frac{kn}{r}$.*

For the special case $k=1$ and $r=3$, the following improvement of Theorem 3.51 is valid.

Theorem 3.52 ([58]). *If G is a 3-regular graph of order n, then $\gamma^1_{stR}(G) \geq \frac{2n}{3}$, with equality if and only if $|V_{-1}| = |V_1| = |V_2|$ for every $\gamma^1_{stR}(G)$-function $f = (V_{-1}, V_1, V_2)$ on G.*

Theorem 3.53 ([27, 57, 59]). *Let G be a graph of order n, minimum degree $\delta \geq k$ and maximum degree Δ. If $\delta < \Delta$, then*

$$\gamma^k_{stR}(G) \geq \frac{(2\delta + 3k - 2\Delta)n}{2\Delta + \delta}.$$

We note that Theorems 3.51 and 3.53 are sharp as it is shown in [27, 57, 59].

Theorem 3.54 ([57, 59]). *Let $k \geq 1$ be an integer. If G is an r-regular graph of order n such that $r \geq k$ and $n - r - 1 \geq k$, then*

$$\gamma^k_{stR}(G) + \gamma^k_{stR}(\bar{G}) \geq \frac{4kn}{n-1}.$$

Theorem 3.55 ([57]). *If G is a graph of order $n \geq 3$ with $\delta(G) \geq 1$, then*

$$\gamma^1_{stR}(G) \geq \frac{3}{2}\left(1 + \sqrt{2n+1}\right) - n.$$

This bound is sharp.

Next we determine a similar result for the signed total Roman 0-domination number. For this purpose, we define a family of graphs. Let $\mathcal{F}_p = \{F_p \mid p \geq 2\}$ be the following family of graphs. Let X be the vertex set of the complete graph K_p, and let F_p be the graph obtained from K_p by adding $2p - 2$ new vertices to each vertex of the complete graph such that for each new vertex x, $1 \leq \deg(x) \leq 3$ and for every $u \in X, \deg(u) = 3(p-1)$. Let $\mathcal{F} = \cup_{p \geq 2} \mathcal{F}_p$.

Theorem 3.56 ([27]). *If G is a graph of order $n \geq 2$ with $\delta(G) \geq 1$, then*

$$\gamma^0_{stR}(G) \geq \frac{3}{4}\left(1 + \sqrt{8n+1}\right) - n,$$

with equality if and only if $G \in \mathcal{F}$.

Theorem 3.57 ([27, 57]). *If G is a connected graph of order $n \geq 3$ and size m, then $\gamma^1_{stR}(G) \geq \frac{11n-12m}{4}$ and $\gamma^0_{stR}(G) \geq \frac{10n-12m}{5}$.*

Theorem 3.58 ([27, 57]). *If G is a bipartite graph of order $n \geq 3$ with $\delta(G) \geq 1$, then $\gamma^1_{stR}(G) \geq 3\sqrt{n} - n$ and $\gamma^0_{stR}(G) \geq \frac{3}{2}\left(\sqrt{4n+1} - 1\right) - n$.*

Examples in [27] and [57] demonstrate that all the bounds in Theorems 3.57 and 3.58 are sharp.

Recently, Volkmann [61] introduced the *signed total Italian dominating function* (STIDF) of a graph G as function $f : V(G) \to \{-1, 1, 2\}$ having the property that $\sum_{x \in N(v)} f(x) \geq 1$ for each $v \in V(G)$ and if $f(u) = -1$, then the vertex must have a neighbor v with $f(v) = 2$ or two neighbors w and z with $f(w) = f(z) = 1$. The weight of an STIDF f is $\sum_{v \in V(G)} f(v)$. The *signed total Italian domination number* of G, denoted by $\gamma_{stI}(G)$, is the minimum weight of an STIDF in G. The definitions lead to $\gamma_{stI}(G) \leq \gamma_{stR}(G) \leq n(G)$. Therefore each lower bound of $\gamma_{stI}(G)$ is also a lower bound of $\gamma_{stR}(G)$. In [61] it is shown that many lower bounds on $\gamma_{stR}(G)$ are also valid for $\gamma_{stI}(G)$. In particular, Volkmann [60] proved that Theorems 3.51 and 3.53 in the case $k=1$ also hold for the signed total Italian domination number. In addition, the difference $\gamma_{stR}(G) - \gamma_{stI}(G)$ can be arbitrarily large.

For $k \geq 2$, let L_k be the graph obtained from a connected graph H of order k by adding $2\deg_H(v) - 1$ pendant edges to each v of H. Let $\mathcal{F} = \{L_k | k \geq 2\}$. As a generalization of the first bound in Theorem 3.57, Volkmann [61] proved the following result.

Theorem 3.59 ([61]). *If G is a connected graph of order $n \geq 3$ and size m, then*

$$\gamma_{stI}(G) \geq \frac{11n - 12m}{4},$$

with equality if and only if $G \in \mathcal{F}$.

Furthermore, the next sharp bound in terms of the total domination number and the order is valid.

Theorem 3.60 ([61]). *If G is a graph of order n with $\delta(G) \geq 1$, then $\gamma_{stI}(G) \geq 2\gamma_t(G) - n$, with equality if an only if $G = sK_2$ for an integer $s \geq 1$.*

3.5. Signed double Roman domination in graphs

A *signed double Roman dominating function* (SDRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii)every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f[v] = f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$ holds for any vertex v. The *signed double Roman domination number* $\gamma_{sdR}(G)$ is the minimum weight of an SDRDF on G. Signed double Roman domination have been mainly studied in [9, 10]. It is shown in [9] that the decision problem corresponding to the problem of computing $\gamma_{sdR}(G)$ is NP-complete for bipartite and chordal graphs. However, in special classes of graphs, including complete graphs, paths, cycles and complete bipartite graphs, exact values of the signed double Roman domination number have been well established. Below the results obtained.

Proposition 3.61 ([10]). *For $n \geq 2$, $\gamma_{sdR}(K_n) = \begin{cases} 2 & \text{if } n = 2 \text{ or } 4. \\ 1 & \text{otherwise.} \end{cases}$*

Proposition 3.62 ([9]). *For $n \geq 2$,*

$$\gamma_{sdR}(P_n) = \begin{cases} n/3 & \text{if } n \equiv 0 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

Proposition 3.63 ([10]). *For $n \geq 3$,*

$$\gamma_{sdR}(C_n) = \begin{cases} n/3 & \text{if} \quad n \equiv 0 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 & \text{if} \quad n \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 1 & \text{if} \quad n \equiv 2 \pmod 3). \end{cases}$$

Proposition 3.64 ([10]). *For $2 \leq m \leq n$,*

$$\gamma_{sdR}(K_{m,n}) = \begin{cases} 3 & \text{if} \ m = 2 \text{ and } n \geq 3, \\ 4 & \text{if} \ m \geq 4 \text{ or } m = n = 2, \\ 5 & \text{if} \ m = 3. \end{cases}$$

3.5.1. Bounds on γ_{sdR}

Before presenting some sharp bounds on the signed double Roman domination number in graphs, we point out, as seen in [9], that there exist graphs with signed double Roman domination numbers which are positive or negative. In particular, it was shown that for every integer $k \geq 0$, there exists a tree T_{k+2} such that $\gamma_{sdR}(T_{k+2}) \leq -k$. The example of the tree T_{k+2} that was given is obtained from $k+2$ stars $K_{1,4}$ by adding a new vertex attached to each center vertex of the star. Consequently, the problem that may arise is to characterize the graphs G for which $\gamma_{sdR}(G) \geq 0$. It should be noted that the authors [9] have shown that there is no forbidden induced subgraph characterization of such graphs.

Proposition 3.65 ([10]). *If G is a graph of order n, minimum degree δ and maximum degree Δ, then*

$$\gamma_{sdR}(G) \geq \left(\frac{-3\Delta + 3\delta + 4}{3\Delta + \delta + 4} \right) n.$$

Proposition 3.66 ([10]). *For any graph G, $\gamma_{sdR}(G) \geq \Delta(G) + 2 - n$. This bound is sharp for complete graphs except K_4.*

For $k \geq 1$, let M_k be the graph obtained from a graph H of order k by adding $3\deg_H(v) + 2$ pendant edges to each vertex v of H. Note that $M_1 = K_{1,2}$. Let $\mathcal{S} = \{M_k | k \geq 1\}$.

Theorem 3.67 ([10]). *Let G be a graph of order n and size m with no isolated vertex. Then*

$$\gamma_{sdR}(G) \geq \frac{19n - 24m}{9}$$

with equality if and only if $G \in \mathcal{S}$.

For $k \geq 1$, let F_k be the graph obtained from the complete graph K_k by adding $3k - 1$ pendant edges at each vertex and let $A(F_k)$ be the family of graphs obtained from F_k by adding edges (possibly none) between the leaves of F_k so that to be independent. Let $\mathcal{F} = \cup_{k \geq 1} A(F_k)$.

Theorem 3.68 ([10]). *Let G be a graph of order n. Then $\gamma_{sdR}(G) \geq 4\sqrt{\frac{n}{3}} - n$, with equality if and only if $G \in \mathcal{F}$.*

The next result relates the signed double Roman domination number to the double Roman domination and domination numbers of any graph.

Proposition 3.69 ([10]). *For every graph G of order n, $\gamma_{dR}(G) - \gamma_{sdR}(G) + \gamma(G) \leq n$.*

Using a result of Beeler et al. [21] stating that $\gamma_{dR}(G) \geq 2\gamma(G)$ for any graph G, we derive from Proposition 3.69 the following corollary.

Corollary 3.70. *For any graph G, $\gamma_{sdR}(G) \geq 3\gamma(G) - n$.*

Clearly, by the previous corollary $\gamma_{sdR}(G) \geq 0$ for all graphs G of order n with $\gamma(G) \geq n/3$. Which gives a partial answer to the problem concerning a characterization of graphs G for which $\gamma_{sdR}(G) \geq 0$.

3.5.2. Signed double Roman domination in trees

Restricted to the class of trees, lower and upper bound on the signed double Roman domination number have been obtained. Moreover, a characterization of trees attaining each bound is provided.

For any tree T, let F_T be the tree obtained from T by adding $3\deg_T(v) + 2$ pendant edges at v for each $v \in V(T)$. Assume that $\mathcal{T} = \{F_T | T \text{ is a tree}\}$.

Theorem 3.71 ([9]). *Let T be a tree of order $n \geq 2$. Then*

$$\gamma_{sdR}(T) \geq \frac{-5n + 24}{9}$$

with equality if and only if $T \in \mathcal{T}$.

The double Roman domination number that may be greater than the order of a graph G (see [8] and [21]), but as observed in [9], it seems that it is not the same for the signed double Roman domination number. The authors conjectured that $\gamma_{sdR}(G) \leq n$ holds for every connected graph G of order $n \geq 2$. This conjecture has been proven for trees.

Theorem 3.72 ([9]). *Let T be a tree of order $n \geq 2$. Then*

$$\gamma_{sdR}(T) \leq n$$

with equality if and only if $T = P_2$.

As an immediate consequence to Theorem 3.72 we have the following corollary.

Corollary 3.73. *If T is a tree of order $n \geq 3$, then $\gamma_{sdR}(T) \leq n - 1$, and this bound is sharp.*

The sharpness of the bound in Corollary 3.73 can be seen for the corona of $K_{1,2}$ and $K_{1,3}$, where $\gamma_{sdR}(\text{cor}(K_{1,2})) = 5$ and $\gamma_{sdR}(\text{cor}(K_{1,3})) = 7$.

3.5.3. Signed total double Roman domination

In [14], Abdollahzadeh Ahangar et al. studied the total version of signed double Roman dominating functions, defined as follows: a *signed total double Roman dominating function* (STDRDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(v) = \sum_{u \in N(v)} f(u) \geq 1$ holds for any vertex v. The *signed total double Roman domination number* $\gamma^t_{sdR}(G)$ is the minimum weight of a STDRDF on G. It is shown in [14] that the decision problem corresponding to the problem of computing $\gamma^t_{sdR}(G)$ is NP-complete even when restricted to bipartite and chordal graphs. Among the various lower and upper bounds established in [14, 53] on γ^t_{sdR}, we give the following.

Theorem 3.74 ([53]). *Let G be a graph of order n and size m with no isolated vertex. Then*

$$\gamma^t_{sdR}(G) \geq \frac{11n - 12m}{3}.$$

The bound of Theorem 3.74 is sharp for the following infinite family of graphs. For $t \geq 2$, let F_t be the graph obtained from a connected graph F of order t by attaching at each vertex v of F, $3\deg_F(v) - 1$ new vertices.

Theorem 3.75 ([53]). *Let G be a graph of order n. Then*

$$\gamma^t_{sdR}(G) \geq \left\lceil 3\sqrt{\frac{n}{2} + \frac{1}{2}} \right\rceil - n + 1.$$

This bound is sharp for K_2, K_3.

Recall that the *matching number* $\alpha'(G)$ of a graph G is the size of a maximum independent edge set in G.

Theorem 3.76 ([14]). *For any graph G of order $n \geq 2$ with $\delta(G) \geq 1$, then*

$$\gamma^t_{sdR}(G) \leq n + \alpha'(G)$$

This bound is sharp for $G = mK_2$ $(m \geq 1)$.

3.6. Signed Roman edge domination in graphs

The *open neighborhood* $N_G(e) = N(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* $N_G[e] = N[e] = N(e) \cup \{e\}$. In [2], Ahangar, Amjadi, Sheikholeslami, Volkmann and Zhao defined a *signed Roman edge dominating function* (SREDF) on a graph G as a function $f : E(G) \to \{-1, 1, 2\}$ satisfying $f(N[e]) \geq 1$ for each edge e and every edge e for which $f(e) = -1$ is adjacent to an edge e' for which $f(e') = 2$. The *signed Roman edge domination number*, denoted $\gamma'_{sR}(G)$, is the minimum weight of an SREDF in G.

The following observation follows immediately from the definition.

Observation 3.77 ([2]). *For any nonempty graph G of order $n \geq 2$, we have $\gamma'_{sR}(G) = \gamma_{sR}(L(G))$, where $L(G)$ is the line graph of G.*

Using Observation 3.77 and Propositions 3.2 and 3.3, we obtain the next results immediately.

Proposition 3.78 ([2]). *For $n \geq 1$, we have $\gamma'_{sR}(K_{1,n}) = 1$, unless $n = 3$ in which case $\gamma'_{sR}(K_{1,3}) = 2$.*

For $n \geq 3$, we have $\gamma'_{sR}(C_n) = \lceil (2n)/3 \rceil$.
For $n \geq 3$, we have $\gamma'_{sR}(P_n) = \lfloor (2(n-1))/3 \rfloor$.

Observation 3.79 ([2]). *If G is a graph of size m, maximum degree Δ and minimum degree δ, then $\gamma'_{sR}(G) \geq \Delta + \delta - m$. Furthermore, this bound is sharp for stars $K_{1,r}$ with $r \neq 3$.*

Let $\mathcal{F}$ be the family of trees obtained from a subdivided star $S(K_{1,r})$, $r \geq 1$, by adding $2r - 2$ pendant edges at the center of $K_{1,r}$.

Theorem 3.80 ([2]). *Let T be a tree of order $n \geq 3$. Then $\gamma'_{sR}(T) \geq \frac{7-n}{4}$, with equality if and only if $T \in \mathcal{F}$.*

Theorem 3.81 ([2]). *If T is a tree of order $n \geq 3$, then $\gamma'_{sR}(T) \leq \frac{2(n-1)}{3}$.*

Since $\gamma'_{sR}(P_n) = \lfloor (2(n-1))/3 \rfloor$, according to Proposition 3.78, we see that Theorem 3.81 is sharp.

Theorem 3.82 ([2]). *If G is a graph of size m, minimum degree $\delta \geq 1$ and maximum degree Δ, then $\gamma'_{sR}(G) \geq \frac{2m\delta}{2\Delta - 1} - m$.*

Theorem 3.83 ([2]). *Let G be a connected graph of size $m \geq 2$. Then $\gamma'_{sR}(G) \leq m - 1$, and $\gamma'_{sR}(G) = m - 1$ if and only if $G \in \{P_3, P_4, C_3, C_4, C_5, K_{1,3}\}$.*

3.7. Signed total Roman edge domination in graphs

In [19], Asgharsharghi and Sheikholeslami defined a *signed total Roman edge dominating function* (STREDF) on a graph G as a function $f : E(G) \to \{-1, 1, 2\}$ satisfying $f(N(e)) \geq 1$ for each edge e and every edge e for which $f(e) = -1$ is adjacent to an edge e' for which $f(e') = 2$. The *signed total Roman edge domination number*, denoted $\gamma'_{stR}(G)$, is the minimum weight of an STREDF in G.

Observation 3.84 ([19]). *For any connected graph G of order $n \geq 3$, we have $\gamma'_{stR}(G) = \gamma_{stR}(L(G))$.*

Using Observation 3.84 and Propositions 3.39, 3.44 and 3.46, the next results follow immediately.

Proposition 3.85 ([19]). *For $n \geq 2$, we have $\gamma'_{stR}(K_{1,n}) = 2$ when $n = 2$ and $\gamma'_{stR}(K_{1,n}) = 3$ otherwise.*

For $n \geq 3$, we have $\gamma'_{stR}(C_n) = \frac{n}{2}$ when $n \equiv 0 \pmod 4$ and $\gamma'_{stR}(C_n) = \frac{n+3}{2}$ when $n \equiv 1, 3 \pmod 4$ and $\gamma'_{stR}(C_n) = \frac{n+6}{2}$ when $n \equiv 2 \pmod 4$.

For $n \geq 4$, we have $\gamma'_{stR}(P_n) = \frac{n-1}{2}$ when $n \equiv 1 \pmod 4$ and $\gamma'_{stR}(P_n) = \lceil \frac{n+2}{2} \rceil$ otherwise.

Let r be a positive integer, and let T_r be the tree obtained from the star $K_{1,3r+1}$ with central vertex x and leaves $x_1, x_2, \ldots, x_{3r+1}$ by adding two pendant edges at x_i for each $1 \leq i \leq r + 2$. Let $\mathcal{T} = \{T_r \mid r \geq 1\}$.

Theorem 3.86 ([19]). *Let T be a tree of order $n \geq 4$. Then $\gamma'_{stR}(T) \geq \frac{17-2n}{5}$, with equality if and only if $T \in \mathcal{T}$.*

Theorem 3.87 ([19]). *If G is a graph of size m, minimum degree δ and maximum degree $\Delta \geq 2$, then $\gamma'_{stR}(G) \geq \frac{m(2\delta - 1)}{2(\Delta - 1)} - m$. This bound is sharp.*

For regular graphs there is the following Nordhaus-Gaddum type inequality.

Theorem 3.88 ([19]). *If G is an r-regular graph with $r \geq 2$ of order $n \geq 4$ such that G and $\bar{G}$ are connected and $r \leq \frac{n-1}{2}$, then*

$$\gamma'_{stR}(G) + \gamma'_{stR}(\bar{G}) \geq \frac{rn}{n-3}.$$

Theorem 3.89 ([19]). *Let G be a connected graph of order $n \geq 3$ and size m. Then $\gamma'_{stR}(G) \leq m$, and $\gamma'_{stR}(G) = m$ if and only if $G \in \{P_3, P_4, C_3, C_4, C_6, K_{1,3}\}$.*

Problem 1 ([19]). *Prove or disprove: For any tree T of order $n \geq 3$, $\gamma'_{stR}(T) \leq \lceil \frac{n+2}{2} \rceil$.*

3.8. Signed mixed Roman domination in graphs

For an element $x \in V(G) \cup E(G)$, the *mixed open neighborhood* of x is the set $N_m(x) = \{y \in V(G) \cup E(G) \mid y$ is adjacent or incident to $x\}$. The *mixed closed neighborhood* of x is $N_m[x] = N_m(x) \cup \{x\}$.

In [3], Ahangar, Asgharsharghi, Sheikholeslami and Volkmann defined a *signed mixed Roman dominating function* (SMRDF) on a graph G as a function $f : V(G) \cup E(G) \to \{-1, 1, 2\}$ satisfying $f(N_m[x]) \geq 1$ for each $x \in V(G) \cup E(G)$ and every element $x \in V(G) \cup E(G)$ for which $f(x) = -1$ is adjacent or incident to an element $y \in V(G) \cup E(G)$ for which $f(y) = 2$. The *signed mixed Roman domination number*, denoted $\gamma^*_{sR}(G)$, is the minimum weight of an SMRDF in G.

Proposition 3.90 ([3]). *For $n \geq 1$, we have $\gamma^*_{sR}(K_{1,n}) = 2$.*

Proposition 3.91 ([3]). *For $n \geq 3$, we have $\gamma^*_{sR}(C_n) = \lceil \frac{2n}{5} \rceil$ when $n \equiv 0, 1 \pmod 5$, $\gamma^*_{sR}(C_n) = \lceil \frac{2n}{5} \rceil + 1$ when $n \equiv 2, 3 \pmod 5$ and $\gamma^*_{sR}(C_n) = \lceil \frac{2n}{5} \rceil + 2$ when $n \equiv 4 \pmod 5$.*

Proposition 3.92 ([3]). *For $n \geq 3$, we have $\gamma^*_{sR}(P_n) = \lceil \frac{2n+1}{5} \rceil$ when $n \equiv 0 \pmod 5$, $\gamma^*_{sR}(P_n) = \lceil \frac{2n+1}{5} \rceil + 2$ when $n \equiv 1, 2 \pmod 5$ and $\gamma^*_{sR}(P_n) = \lceil \frac{2n+1}{5} \rceil + 1$ when $n \equiv 3, 4 \pmod 5$.*

Observation 3.93 ([3]). *If G is a connected graph of order $n \geq 2$ and size m, then $\gamma^*_{sR}(G) \leq m + n - 1$, with equality if and only if $G = K_2$.*

For $\Delta \geq 2$, Observation 3.93 can be improved.

Observation 3.94 ([3]). *If G is a connected graph of order $n \geq 2$, size m and $\Delta(G) \geq 2$, then $\gamma^*_{sR}(G) \leq m + n - 2\Delta(G) + 1$.*

Proposition 3.90 shows that Observation 3.94 is sharp.

Theorem 3.95 ([3]). *If G is an r-regular graph of order n and size m with $r \geq 2$, then $\gamma^*_{sR}(G) \geq \frac{n+m}{2r+1}$.*

If G is 1-regular, then Proposition 3.90 implies that $\gamma^*_{sR}(G) = n$.

Theorem 3.96 ([3]). *Let G be a graph of order n, size m, minimum degree δ and maximum degree Δ. If $2 \leq \delta < \Delta$, then*

$$\gamma^*_{sR}(G) \geq \frac{-4\Delta + 4\delta + 3}{4\Delta + 2\delta + 3}(n + m).$$

Theorem 3.97 ([3]). *If G is a graph of order n and size m, then*

$$\gamma^*_{sR}(G) \geq 2\left\lceil \frac{-1}{4} + \sqrt{\frac{1}{16} + n + m} \right\rceil + 1 - n - m.$$

3.9. Signed strong Roman domination in graphs

The defensive strategy of signed Roman domination is based on the fact that every place in which there is established a Roman legion (a label 1) is able to protect itself under external attacks; and that every place with an auxiliary troop (a label -1) must have at least a stronger neighbor (a label 2). In that way, if an unsecured place (a label -1) is attacked, then a stronger neighbor could send one of its two legions in order to defend the weak neighbor vertex (label -1) from the attack. If several simultaneous attacks to weak places are developed, then the only stronger place will not be able to defend its neighbors efficiently. With this motivation in mind, Asgharsharghi, Khoeilar and Sheikholeslami [18], introduced the concept of signed strong Roman dominating functions as follows. For this purpose, they consider that a strong place should be able to defend itself and at least half of its weak neighbors.

In graph theoretic terms, a *signed strong Roman dominating function* (SStRDF) on a graph G is defined in [18] as a function $f : V(G) \to \{-1, 1, 2, 3, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ satisfying the conditions that $f(N[v]) \geq 1$ for each $v \in V(G)$ and every vertex v for which $f(v) = -1$ is adjacent to at least one vertex u for which $f(u) \geq 1 + \lceil \frac{1}{2}|N(u) \cap V_{-1}| \rceil$, where $V_{-1} = \{x \in V(G) \mid f(x) = -1\}$. The *signed strong domination number*, denoted $\gamma_{ssR}(G)$, is the minimum weight of an SStRDF in G. First the authors note that $\gamma_{ssR}(G) = \gamma_{sR}(G)$ for each graph G with $\Delta(G) \leq 2$. Therefore Proposition 3.3 immediately implies that for $n \geq 3$, $\gamma_{ssR}(C_n) = \lceil (2n)/3 \rceil$ and $\gamma_{ssR}(P_n) = \lfloor (2n)/3 \rfloor$. Signed strong Roman domination numbers in some special classes of graphs are determined as follows.

Proposition 3.98 ([18]). *(1) For $n \geq 4$, we have $\gamma_{ssR}(K_n) = 1$.*
(2) For $n \geq 3$, we have $\gamma_{ssR}(K_{1,n}) = 1$.
(3) For $p \geq 3$, we have $\gamma_{ssR}(K_{p,p}) = 4$.

A few bounds on the signed strong domination number have been established in [18] which are summarized in the following result.

Theorem 3.99 ([18]). *If G is a connected graph of order n, then the following holds.*

(a) $\gamma_{ssR}(G) \geq \Delta(G) + 2 - n$.
(b) $\gamma_{ssR}(G) \leq n$, *with equality if and only if* $G = \overline{K_n}$.
(c) $\gamma_{ssR}(G) \geq 2\gamma(G) - n$, *with equality if and only if* $G = \overline{K_n}$.
(d) *If* $n \geq 4$, *then* $\gamma_{ssR}(G) \geq 3 - \lfloor n/2 \rfloor$.
(e) *If G is a tree and $n \geq 3$, then $\gamma_{ssR}(G) \leq \frac{2n}{3}$, with equality if and only if $T = P_{3t}$.*

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