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# Modeling and Stability Analysis of Nonlinear Sampled-Data Systems with Embedded Recovery Algorithms

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# MODELING AND STABILITY ANALYSIS OF NONLINEAR SAMPLED-DATA SYSTEMS WITH EMBEDDED RECOVERY ALGORITHMS

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ABSTRACT

## MODELING AND STABILITY ANALYSIS OF NONLINEAR SAMPLED-DATA SYSTEMS WITH EMBEDDED RECOVERY ALGORITHMS

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Computer control systems for safety critical systems are designed to be fault tolerant and reliable, however, soft errors triggered by harsh environments can affect the performance of these control systems. The soft errors of interest which occur randomly, are nondestructive and introduce a failure that lasts a random duration. To minimize the effect of these errors, safety critical systems with error recovery mechanisms are being investigated. The main goals of this dissertation are to develop modeling and analysis tools for sampled-data control systems that are implemented with such error recovery mechanisms. First, the mathematical model and the well-posedness of the stochastic model of the sampled-data system are presented. Then this mathematical model and the recovery logic are modeled as a dynamically colored Petri net (DCPN). For stability analysis, these systems are then converted into piecewise deterministic Markov processes (PDP). Using properties of a PDP and its relationship to discrete-time Markov chains, a stability theory is developed. In particular, mean square equivalence between the sampled-data and its associated discrete-time system is proved. Also conditions are given for stability in distribution to the delta Dirac measure and mean square stability for a linear sampled-data system with recovery logic.

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| | | |
|---|---|---|
| $\mathcal{C}^n$ | Subspace of continuous functions | 5 |
| $\mathcal{PC}^n$ | Subspace of piecewise-continuous functions | 6 |
| $\mathcal{S}^n$ | Space of sequences | 6 |
| $(\Omega, \mathcal{F}, \Pr)$ | Underlying probability space | 6 |
| $\boldsymbol{\theta}$ | Random process modeling the switching | 6 |
| $\boldsymbol{x}_p$ | State of the plant | 6 |
| $\boldsymbol{x}_c$ | State of the controller | 6 |
| $\mathsf{T}$ | Sample period | 7 |
| $\mathcal{S}_{\mathsf{T}}$ | Sampling operator | 7 |
| $\mathcal{H}_{\mathsf{T}}$ | Hold operator | 7 |
| $\Sigma_N$ | Nominal closed-loop system | 8 |
| $\Sigma_R$ | Recovery closed-loop system | 8 |
| $\beta_o(\mathcal{S}^n)$ | $\sigma$-algebra generated by the open sets in the space of sequences | 9 |
| $\beta_o(\mathcal{C}^n)$ | $\sigma$-algebra generated by the open sets in the subspace of continuous functions | 9 |
| $\beta_o(\mathcal{PC}^n)$ | $\sigma$-algebra generated by the open sets in the subspace of piecewise functions | 10 |
| $\mathbb{R}^{n_{x_p}+n_{x_c}}$ | Real vector space of dimension $n_{x_p} + n_{x_c}$ | 12 |
| $\boldsymbol{z}$ | Initial condition of a stochastic motion | 12 |
| $\mathbf{z}$ | Stochastic motion | 12 |
| $\mathbb{S}$ | Stochastic dynamic system | 12 |
| $\boldsymbol{V}$ | Lyapunov function | 17 |
| DCPN | Dynamically Colored Petri Net | 21 |
| $\mathcal{C}(P(A_{\mathrm{in}}(T)))$ | Color of the input places of the transition $T$ | 24 |
| $\mathcal{D}$ | Set of delay transitions | 24 |
| $\mathsf{F}$ | Set of firing measures | 24 |
| PDP | Piecewise Deterministic Markov Process | 41 |
| $\boldsymbol{\chi}_t$ | Sampled-data PDP | 61 |
| $\mathbb{E}$ | Hybrid state space | 62 |
| $C_0(\mathbb{E})$ | Space of real functions defined on $\mathbb{E}$ | 62 |
| $\mathcal{L}(C_0(\mathbb{E}))$ | Space of bounded linear operators | 62 |
| $L_n$ | Bounded linear operator | 62 |

# CHAPTER I

# INTRODUCTION

A sampled-data system consists of a continuous-time plant connected with a discrete-time controller. A very important result is the equivalence between a sampled-data and its discretized version, where equivalence means that under some conditions the sampled-data is stable if and only if its discretized version is stable. When the discrete-time controller is subject to stochastic upsets a recovery mechanism is considered such that, these upsets randomly switch the control law between a nominal control algorithm and a recovery algorithm. This class of systems will be called sampled-data system with stochastic upsets. Since the upsets can randomly switch the control law between a nominal control algorithm and a recovery algorithm, the effect of the random switching on the stability of the closed-loop system needs to be understood. To analyze these effects an objective of this dissertation is to develop an appropriate mathematical model of the random switching, the discrete-time jump linear controller and the nonlinear continuous-time plant. A second objective is to determine if an equivalence between the stability of sampled-data system with stochastic upsets and its discretized version still holds[1].

In [40, 48] models of sampled-data system with stochastic upsets were derived. The stochastic upsets were modeled with a homogeneous discrete-time Markov chain (DTMC) and the random switching process between the nominal and recovery algorithms with a dynamical transformation of this DTMC. The transformation was realized either as a finite state machine (FSM) or as a stochastic FSM (SFSM). To simplify the analysis of the discrete-time closed loop system, a joint process, consisting of the state of the FSM or SFSM and the DTMC was formed. It was shown that this joint process was also a homogeneous DTMC and the transition probability matrix was derived. It was observed in applications of this theory that the matrices needed in the analysis could have very large dimensions, which could introduce serious numerical issues. This possible problem motivated the work presented here: to derive an alternate model and analysis methodology, where the matrices needed for analysis are not as large. In particular, it is necessary that the dimension of the

[1]The format in this dissertation follows IEEE Transactions.

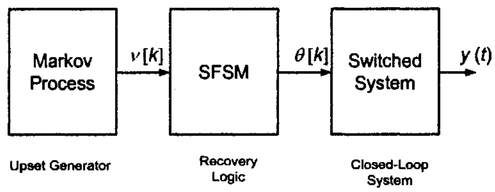


FIG. 1: A simplified block diagram of a closed-loop system switched by the output of a stochastic finite state machine with a Markov process input.

on the recovery algorithm duration. Another open problem that will be studied is the relation between stability of a sampled-data system with stochastic upsets and its discrete-time version.

A piece-wise deterministic Markov process (PDP) is a stochastic process that study the behavior of a continuous-time system subject to stochastic jumps. In particular PDP give a relation between a continuous-time system and the dynamics of its stochastic jumps. To model a continuous-time system with a recovery algorithm that is randomly trigged, a Dynamically Colored Petri Net, (DCPN) is introduced. A DCPN is a bipartite directed graph with two types of nodes: places and transitions. Directed arcs connect the places to transitions and the transitions to nodes. A very useful DCPN property is that it can be mapped into a PDP [20] for which several analysis tools are available. Therefore, two main problems are going to be studied in this dissertation. First, establish the relation between the stability properties of a sampled-data system with stochastic upsets and its associated discrete-time version using results from the theory of PDP. Second, develop the tools to study the stability of sampled-data system with stochastic upsets that use smaller dimension matrices than in other techniques..

## I.1 LITERATURE SURVEY AND MOTIVATION

A motivation for this dissertation is that safety critical control systems operating in harsh environments are affected by common mode faults [25, 48]. When the safety critical control systems have error recovery mechanisms, it is important to study the effect of the stochastic switching triggered by these faults. To study the stability and performance of closed-loop safety-critical systems, a closed-loop model that includes the interference of the error recovery logic driven by common-mode faults is needed. In [40] the closed-loop model was modeled as a jump linear system driven by a FSM with a DTMC input. In [48], the FSM was replaced with a SFSM as shown in FIG. I to investigate the effects of the following recovery logic: As long as there is no upset, the system operates in the normal mode; as soon as an upset occurs, the recovery logic switches to the recovery mode. The closed-loop system remains in the recovery mode for a random number of sample periods or frames, and then returns to the normal mode. During recovery, the control input is held constant and detection of new faults is disabled. This simple recovery logic was based on a series of simulated neutron irradiation experiments of a prototype flight control computer with error recovery capabilities conducted at the NASA Langley Research Center. In these experiments, 80% of the recovery periods lasted six frames and 20% lasted five frames. For analysis, these systems have been represented as discrete-time Markovian jump linear systems. Thus, their stability and performance can be analyzed by generalizing the theory, for example as in [12, 13, 15, 40, 48]. The main stability result is a necessary and sufficient condition for the mean square stability of a discrete-time Markov jump linear system. This stability result requires the calculation of the spectral radius of a specific transition matrix. The dimensions of this matrix increase as the duration of the recovery increases. To reduce the dimensions of this matrix, dynamically colored Petri net models will be investigated.

To date, DCPN models have not been applied to fault tolerant sampled-data systems. But other models have been used to study interconnected digital and continuous-time systems. Sampled-data switching systems, where the digital system drives the switching at discrete transitions, depending on whether a condition on the continuous or discrete states holds or not, were analyzed in [1, 34, 42, 45]. Continuous-time systems switched by a digital system driven by a Markov process over the sampling interval were studied in [5, 32]. An application where continuous-time dynamics are controlled by a digital device in biological systems was developed in [9, 31].

## I.2 DISSERTATION OUTLINE

The dissertation is organized as follows. A mathematical descriptions of sampled-data system with stochastic upsets is introduced in Chapter II. In this chapter it is also shown the equivalence between the stability of the sampled-data systems and their discretized version. In Chapter III, a sampled-data system including the recovery logic is modeled as a stochastically and dynamically colored Petri net. For simplicity this model is switched by a continuous-time process. This model is then mapped to a PDP. In particular, the PDP is used to characterize the switching process produced by the recovery logic. In Chapter IV a model of the sampled-data system with stochastic upsets as a PDP is presented that is switched at the sampling instants by a discrete-time process. An associated discrete-time system of the PDP model is presented for stability analysis. Specifically, a testable sufficient condition for the convergence to the delta Dirac measure is given. Finally, in Chapter V, the conclusions and future research directions are given.

# CHAPTER II

# MODELING AND ANALYSIS OF SAMPLED-DATA SYSTEMS WITH STOCHASTIC UPSETS

## II.1 INTRODUCTION

This chapter analyzes the stability of a sampled-data system consisting of a deterministic, nonlinear, time-invariant, continuous-time plant and a stochastic, discrete-time, jump linear controller. To analyze stability, appropriate topologies are introduced for the signal spaces of the sampled-data system. These topologies are used to form measurable spaces and to show that the ideal sampling and zero-order-hold operators are measurable maps. This chapter shows that the known equivalence between the stability of a deterministic, linear sampled-data system and its associated discrete-time representation, as well as between a nonlinear sampled-data system and a linearized representation, holds even in a stochastic framework. The equivalence between the stability of a deterministic sampled-data system and its associated discrete-time system when the continuous-time plant is linear time-invariant (LTI), linear time-varying (LTV), or nonlinear is well known (see, for example, [24, 30, 33]). In this chapter it is shown that a similar equivalence is possible when the plant is a deterministic LTI continuous-time system and the controller is a stochastic jump linear system, that is, the stochastic stability of the sampled-data system is equivalent to the stochastic stability of its associated discrete-time jump linear system. In addition, when the plant is nonlinear and the underlying stochastic process has finite states, it is shown that if the origin of the linearized sampled-data system is $2^{\text{nd}}$-moment stable then the origin of the nonlinear sampled-data system is also $2^{\text{nd}}$-moment stable. The results in this dissertation enable the design of sampled-data jump linear controllers by considering only the associated discrete-time representation of the plant. Thus, the known results in the discrete-time jump linear system literature such as [8, 12, 22] are given the appropriate foundation to validate their application in the analysis and design of sampled-data systems.

The following notation is similar to that which appears in sampled-data papers such as [7, 35, 44]. Let $'$ denote transposition. $\mathcal{C}^n$ denotes the subspace of continuous, $\mathbb{R}^n$-valued functions that map the nonnegative reals, $\mathbb{R}^+ = [0, \infty)$, into $\mathbb{R}^n$ and are bounded on compact subsets of $\mathbb{R}^+$ and right continuous at the origin. Similarly, let $\mathcal{PC}^n$ denote the subspace of piecewise-continuous, $\mathbb{R}^n$-valued functions that map $\mathbb{R}^+$ into $\mathbb{R}^n$, are bounded on compact subsets of $\mathbb{R}^+$, are continuous from the right, and have limits from the left on half-open intervals of the form $[t_0, t_1)$ ($t_0, t_1 \in \mathbb{R}$, where $t_0 \geq 0$ and $t_1$ could be finite or infinite). Let $\mathcal{S}^n$ denote the space of bounded $\mathbb{R}^n$-valued sequences that map the non-negative integers, $\mathbb{Z}^+$, into $\mathbb{R}^n$. The arguments of functions in $\mathcal{C}^n$ or $\mathcal{PC}^n$ will be denoted between parentheses and those in $\mathcal{S}^n$ between square brackets. A continuous, real-valued function $\Upsilon$ belongs to class $K$ if $\Upsilon \in \mathcal{C}$, $\Upsilon(0) = 0$, and $\Upsilon$ is strictly increasing on $\mathbb{R}^+$. A continuous, real-valued function $\Upsilon$ belongs to class $K_\infty$ if $\Upsilon \in K$ and $\lim_{r\to\infty} \Upsilon(r) = \infty$.

## II.2 A STOCHASTIC SAMPLED-DATA SYSTEM

The sampled-data system under consideration is shown in FIG.2, where the A/D and D/A conversions are performed by ideal sampling and zero-order-hold operators, respectively. Quantization is not considered. Let $(\Omega, \mathcal{F}, \Pr)$ be the underlying probability space, and let $\{\boldsymbol{\theta}[k]; k \in \mathbb{Z}^+\}$ be a finite-state, discrete-time random process that drives the switching of the jump linear controller. The continuous-time plant is represented by

$$\begin{aligned}\dot{\boldsymbol{x}}_p(t) &= f\big(\boldsymbol{x}_p(t)\big) + B_p\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= C_p\boldsymbol{x}_p(t),\end{aligned}$$

where $f : \mathcal{D} \to \mathbb{R}^{n_{xp}}$, $\mathcal{D} \subset \mathbb{R}^{n_{xp}}$ a ball centered at the origin, $f(0) = 0 \in \mathbb{R}^{n_{xp}}$, $B_p \in \mathbb{R}^{n_{xp}\times m_c}$, $C_p \in \mathbb{R}^{m_p\times n_{xp}}$, and $t > 0$. The deterministic initial condition is $x(0) = x_0$. Assuming that $f$ is continuously differentiable on $\mathcal{D}$, then the linearized state equation of the plant around $x = 0$, $u = 0$ is

$$\begin{aligned}\dot{\boldsymbol{x}}_p(t) &= A_p\boldsymbol{x}_p(t) + B_p\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= C_p\boldsymbol{x}_p(t),\end{aligned}$$

where $A_p \triangleq \left[\frac{\delta f}{\delta \boldsymbol{x}_p}\right]_{\boldsymbol{x}_p=0,\boldsymbol{u}=0}$ and $g\big(\boldsymbol{x}_p(t)\big) \triangleq f\big(\boldsymbol{x}_p(t)\big) - A_p\boldsymbol{x}_p(t)$. The jump linear discrete-time controller is represented by

$$\begin{aligned}\boldsymbol{x}_c[k+1] &= A_{\boldsymbol{\theta}[k]}\boldsymbol{x}_c[k] + B_{\boldsymbol{\theta}[k]}\boldsymbol{\eta}[k]\\ \boldsymbol{\psi}[k] &= F_c\boldsymbol{x}_c[k],\end{aligned}$$

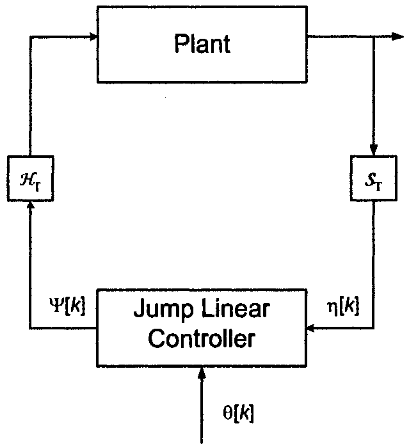


FIG. 2: A sampled-data system with ideal sample and zero-order-hold operators, $\mathcal{S}_{\mathrm{T}}$ and $\mathcal{H}_{\mathrm{T}}$, respectively.

$A_{\theta[k]} \in \mathbb{R}^{n_{x_c} \times n_{x_c}}$, $B_{\theta[k]} \in \mathbb{R}^{n_{x_c} \times m_p}$, and $F_c \in \mathbb{R}^{m_c \times n_{x_c}}$. The sample paths of the plant's output are sampled at uniformly spaced sampling instants, T, by the sampling operator

$$\begin{aligned} \mathcal{S}_{\mathrm{T}} : \mathcal{C}^{m_p} &\longrightarrow \mathcal{S}^{m_p} \\ y &\longmapsto \eta = \mathcal{S}_{\mathrm{T}} y \triangleq \lim_{t \to (k\mathrm{T})^-} y(t), \end{aligned}$$

where $\mathcal{S}^{m_p} \triangleq \{\kappa | \kappa : \mathbb{N} \to \mathbb{R}^{m_p}\}$ and $\mathcal{C}^{m_p} \triangleq \{x | x : \mathbb{R}^+ \to \mathbb{R}^{m_p}\}$. The sample paths of the plant's input are a zero-order-hold transformation of the controller's output sequence. The zero-order-hold operator is given by

$$\begin{aligned} \mathcal{H}_{\mathrm{T}} : \mathcal{S}^{m_c} &\longrightarrow \mathcal{PC}^{m_c} \\ \psi &\longmapsto u = \mathcal{H}_{\mathrm{T}} \psi, \end{aligned}$$

where $\mathcal{PC}^{m_c} = \{u | u : \mathbb{R}^+ \to \mathbb{R}^{m_c}\}$, $u(t) = \psi[k]$ for all $t \in [k\mathrm{T}, (k+1)\mathrm{T})$, $k \in \mathbb{Z}^+$. Now the state-space representations of the closed-loop sampled data system result in hybrid representations that include both continuous-time and discrete-time dynamics. For example, in [24, 35] the hybrid systems are given in terms of periodic system representations. One such hybrid nonlinear system representation is

$$\begin{bmatrix} \dot{\boldsymbol{x}}_p(t) \\ \boldsymbol{x}_c[k+1] \end{bmatrix} = \begin{bmatrix} f(\boldsymbol{x}_p(t)) + B_p\mathcal{H}_{\mathrm{T}}F_c\boldsymbol{x}_c[k] \\ B_{\boldsymbol{\theta}[k]}\mathcal{S}_{\mathrm{T}}C_p\boldsymbol{x}_p(t) + A_{\boldsymbol{\theta}[k]}\boldsymbol{x}_c[k] \end{bmatrix}$$
$$\boldsymbol{y}(t) = \begin{bmatrix} C_p & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_p(t) \\ \boldsymbol{x}_c[k] \end{bmatrix}, \tag{1}$$

where $k \in \mathbb{Z}^+$ and $k\mathrm{T} \leq t < (k+1)\mathrm{T}$. The hybrid system representation linearized around $x_p = 0$, $u = 0$ is

$$\begin{bmatrix} \dot{\boldsymbol{x}}_p(t) \\ \boldsymbol{x}_c[k+1] \end{bmatrix} = \begin{bmatrix} A_p & B_p\mathcal{H}_{\mathrm{T}}F_c \\ B_{\boldsymbol{\theta}[k]}\mathcal{S}_{\mathrm{T}}C_p & A_{\boldsymbol{\theta}[k]} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_p(t) \\ \boldsymbol{x}_c[k] \end{bmatrix}$$
$$\boldsymbol{y}(t) = \begin{bmatrix} C_p & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_p(t) \\ \boldsymbol{x}_c[k] \end{bmatrix}. \tag{2}$$

For simplicity, two sampled-data system modes of operation are considered, nominal and upset, as defined next.

**Definition 1.** A *nominal mode*, represented by $\Sigma_N$, is a closed-loop system interconnection of the continuous-time plant and the nominal controller. A *recovery mode*, denoted by $\Sigma_R$, is a closed-loop system interconnection of the continuous-time plant and the upset control law.

The next section presents a framework to guarantee that suitable probability measures are properly induced throughout the closed-loop system.

## II.3 THE RANDOM ELEMENTS OF SAMPLED-DATA SYSTEMS WITH STOCHASTIC UPSETS

In this section the hybrid system representation introduced for deterministic, discontinuous systems in [46,47] and extended to the stochastic setting in [29] will be used. To guarantee that probability measures can be induced throughout the closed-loop system, the signals in the sampled-data system will be assumed to be in specific measurable spaces, and it will then be shown that this choice results in the sampling and zero-order-hold operators being measurable maps.

The sampled-data system has one source of randomness: the upset process $\boldsymbol{\theta}[k]$. The sampled-data system is well posed in the probabilistic sense if all the induced probability measures are well defined. A *probability measure is well defined* on a measurable space if it is a finitely additive measure satisfying the usual properties [41]. To present the $\sigma$-algebras, where the elements of the sampled-data system with stochastic upsets have very well defined probability measures, the following definitions are needed.

**Definition 2.** Let $(\mathcal{E}, d)$ be a metric space, where $d$ is a metric on the set $\mathcal{E}$. The intersection of all $\sigma$-algebras generated by open sets in the metric topology is the smallest $\sigma$-algebra containing them. It is called a *Borel $\sigma$-algebra* and it is denoted by $\beta_o(\mathcal{E})$.

The following definition introduces a key mathematical concept used throughout this dissertation.

**Definition 3.** Let $(\Omega, \mathcal{F})$ and $(E, \beta_o(E))$ be measurable spaces. A function $\boldsymbol{X}$ defined on $\Omega$ and taking values in $E$ is said to be a *random element* if $\{\omega \in \Omega : \boldsymbol{X}(\omega) \in B\} \in \mathcal{F}$ for every $B \in \beta_o(E)$. $\boldsymbol{X}$ is also said to be a $\mathcal{F}/\beta_o(E)$-measurable function.

Random elements induce well defined probability measure as shown next.

**Lemma 1.** *Let $(\Omega, \mathcal{F})$ and $(E, \beta_o(E))$ be measurable spaces. If* Pr *is a probability measure in $(\Omega, \mathcal{F})$ then the probability measure induced by $\boldsymbol{X}$ is*

$$\Pr_{\boldsymbol{X}}(B) = \Pr(\omega : \boldsymbol{X}(\omega) \in B)$$

*for all $B \in \beta_o(E)$.*

*Proof.* See [41], Theorem 7, pp. 196-197. □

Thus, to guarantee that the induced probability measures are well defined, it is necessary and sufficient to check that the transformations of the sampled-data system with stochastic upsets are random elements. To define the measurable spaces, Borel $\sigma$-algebras are introduced for each signal space.

**Definition 4.** The measurable space for the signal space of sequences is $(\mathcal{S}^n, \beta_o(\mathcal{S}^n))$, where $\beta_o(\mathcal{S}^n)$ is the smallest $\sigma$-algebra generated by the open sets $B_\epsilon(x, y) = \{y \in \mathcal{S}^n : d_\infty(x, y) < \epsilon\}$ and $d_\infty(x, y) = \sum_{k=1}^{\infty} 2^{-k}|x[k] - y[k]|$.

**Definition 5.** The measurable space for the signal space of continuous functions is $(\mathcal{C}^n, \beta_o(\mathcal{C}^n))$, where $\beta_o(\mathcal{C}^n)$ is the smallest $\sigma$-algebra generated by the open sets $B_\epsilon(x, y) = \{y \in \mathcal{C}^n : d(x, y) < \epsilon\}$ and $d_1(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min\left(\sup_{0 \le t \le n} |x(t) - y(t)|, 1\right)$.

**Definition 6.** The measurable space for the signal space of piecewise continuous functions is $(\mathcal{PC}^n, \beta_o(\mathcal{PC}^n))$, where $\beta_o(\mathcal{PC}^n)$ is the smallest $\sigma$-algebra generated by the open sets $B_\epsilon(x, y) = \{y \in \mathcal{PC}^n : d_2(x, y) < \epsilon\}$ and $d_2(x, y)$ is the Skorohod metric [2,18]. The Skorohod metric is defined as follows: Let $\Xi$ be the collection of increasing functions $\Upsilon$ mapping $[0, \infty)$ onto $[0, \infty)$ (in particular $\Upsilon(0) = 0$, $\lim_{t\to\infty} \Upsilon(t) = \infty$ and $\Upsilon$ is continuous). Let $\tilde{\Xi}$ be the subset of Lipschitz continuous functions $\tilde{\Xi} \subset \Xi$ such that $\gamma(\Upsilon) \triangleq \sup_{s>t\geq 0} |\log \frac{\Upsilon(s)-\Upsilon(t)}{s-t}|$ is well defined. The Skorohod metric is defined as: $d_2(x, y) = \inf_{\Upsilon\in\tilde{\Xi}} \max\{\gamma(\Upsilon), \int_0^\infty e^{-u} d(x, y, \Upsilon, u)du\}$, where $d(x, y, \Upsilon, u) \triangleq \sup_{t\geq 0} \min\{|x(\min\{t, u\}) - y(\min\{\Upsilon(t), u\})|, 1\}$.

The elements of the $\sigma$-algebras $\beta_o(\mathcal{PC}^n)$ and $\beta_o(\mathcal{C}^n)$ are sets of functions. A very important property is that these sets of functions can be represented as a projection of the functions as indicate in the following lemmas.

**Lemma 2** ( [18], pp. 127). *For each $t \geq 0$, define $\Pi_t : \mathcal{PC}^n \longrightarrow \mathbb{R}^n$ by $\Pi_t(x) = x(t)$. Then $\beta_o(\mathcal{PC}^n) = \sigma\big(\boldsymbol{x}(t) : \boldsymbol{x}(t) = \Pi_t(\boldsymbol{x}),\ t \in D \text{ and } \boldsymbol{x} \in \mathcal{PC}^n\big)$, where $D$ is any dense subset of $[0, \infty)$, and $\sigma(\Pi_t : t \in D)$ is the smallest $\sigma$-algebra generated by the random variables $\Pi_t(\boldsymbol{x})$.*

**Lemma 3** ( [41], pp. 150). *For $A \in \beta_o(\mathcal{C}^n)$ there is a countable set of points $\{t_1, t_2, ...\}$ and a set $B \in \beta_o(\mathcal{S}^n)$ such that*

$$A = \{\boldsymbol{x} \in \mathcal{C}^n : (\boldsymbol{x}_{t_1}, \boldsymbol{x}_{t_2}, ...) \in B\} \in \beta_o(\mathcal{C}^n).$$

These lemmas show that the $\sigma$-algebras $\beta_o(\mathcal{PC}^n)$ and $\beta_o(\mathcal{C}^n)$ are equal to the $\sigma$-algebras generated by the countable projections $\Pi_t$ of the coordinate random variables in each space. Hence, any event in $\beta_o(\mathcal{PC}^n)$ and $\beta_o(\mathcal{C}^n)$ can be characterized by countable specifications in time. With this characterization of the measurable spaces and associated $\sigma$-algebras, the next lemma shows that the sampling and zero-order-hold operators are random elements.

**Theorem 1.** *The sampling operator, $\mathcal{S}_T$, and zero-order-hold operator, $\mathcal{H}_T$, are random elements between the following measurable subspaces:*

$$\begin{aligned} \mathcal{S}_T &: (\mathcal{C}^{m_p}, \beta_o(\mathcal{C}^{m_p})) \longrightarrow (\mathcal{S}^{m_p}, \beta_o(\mathcal{S}^{m_p})) \\ \mathcal{H}_T &: (\mathcal{S}^{m_c}, \beta_o(\mathcal{S}^{m_c})) \longrightarrow (\mathcal{PC}^{m_c}, \beta_o(\mathcal{PC}^{m_c})). \end{aligned}$$

*Proof.* The sampling operator is a random element if the inverse image of any event in $\beta_o(\mathcal{S}^{m_p})$ is an event in $\beta_o(\mathcal{C}^{m_p})$, that is, $\{y \in \mathcal{C}^{m_p} : \mathcal{S}_T y \in B\} \in \beta_o(\mathcal{C}^{m_p})$ for every

$B \in \beta_o(\mathcal{S}^{m_p})$. From Lemma 3, it is known that every event in $\beta_o(\mathcal{C}^{m_p})$ is determined by restrictions imposed on the functions $y \in \mathcal{C}^{m_p}$ on at most a countable set of points. Thus, $\{y \in \mathcal{C}^{m_p} : \mathcal{S}_{\mathrm{T}} y \in B\}$ is an event in $\beta_o(\mathcal{C}^{m_p})$, since every element of $\{y \in \mathcal{C}^{m_p} : \mathcal{S}_{\mathrm{T}} y \in B\}$ has restrictions given by the countable sequence $\mathcal{S}_{\mathrm{T}} y$.
The zero-order-hold operator is a random element if the inverse image of any event in $\beta_o(\mathcal{PC}^{m_c})$ is an event in $\beta_o(\mathcal{S}^{m_c})$, that is $\{\psi \in \mathcal{S}^{m_c} : \mathcal{H}_{\mathrm{T}}\psi \in B\} \in \beta_o(\mathcal{S}^{m_c}), B \in \beta_o(\mathcal{PC}^{m_c})$. From Lemma 2, if $D = \mathbb{Q}^+$ then every event in $\beta_o(\mathcal{PC}^{m_c})$ is determined by restrictions imposed on the functions $\mathcal{H}_{\mathrm{T}}\psi$ on at most a countable set of points. Since the action of the zero-order-hold operator is to keep constant the value for each $t \in [k\mathrm{T}, (k+1)\mathrm{T})$ when $k \in \mathbb{Z}^+$, this means that its restrictions are in $\beta_o(\mathcal{S}^{m_c})$. But the countable restrictions of $\mathcal{H}_{\mathrm{T}}\psi$ are $\psi$, this implies that $\{\psi \in \mathcal{S}^{m_c} : \mathcal{H}_{\mathrm{T}}\psi \in B\}$ is an event in $\beta_o(\mathcal{S}^{m_c})$. $\square$

It is assumed that the state and output of the plant and the jump linear controller are random elements, in the following sense. The state $\boldsymbol{x}_p$ and the output $\boldsymbol{y}$ of the plant are random elements from $(\Omega, \mathcal{F})$ to $(\mathcal{C}^{n_{x_p}}, \beta_o(\mathcal{C}^{n_{x_p}}))$ and $(\mathcal{C}^{m_p}, \beta_o(\mathcal{C}^{m_p}))$, respectively. The state $\boldsymbol{x}_c$ and the output $\boldsymbol{\psi}$ of the jump linear controller are random elements from $(\Omega, \mathcal{F})$ to $(\mathcal{S}^{n_{x_c}}, \beta_o(\mathcal{S}^{n_{x_c}}))$ and $(\mathcal{S}^{m_c}, \beta_o(\mathcal{S}^{m_c}))$, respectively. Now, since the composition of random elements is a random element and applying Theorem 1, it follows that the input to the plant, $\mathcal{H}_{\mathrm{T}}\boldsymbol{\psi} = \mathcal{H}_{\mathrm{T}} F \boldsymbol{x}_c$, and the input to the controller, $\mathcal{S}_{\mathrm{T}}\boldsymbol{y} = \mathcal{S}_{\mathrm{T}} C \boldsymbol{x}_p$, are random elements. Thus, it follows directly that all the signals in sampled-data systems with stochastic upsets are random elements.

A *random process* is a set of random variables. Random processes allow one to define the dynamics of the variables of the sampled-data system subject to stochastic upsets. These random variables can be modeled as solutions of differential or difference equations. The main conclusion of Lemmas 2 and 3 is that every random element defined from $(\Omega, \mathcal{F})$ to $(\mathcal{PC}^n, \beta_o(\mathcal{PC}^n))$ and from $(\Omega, \mathcal{F})$ to $(\mathcal{C}^n, \beta_o(\mathcal{C}^n))$ is a random process. This demonstrates then that the random elements of the sampled-data system are random processes. This allows one to study the dynamics of a sampled-data system with stochastic upsets as developed in the next section.

## II.4 THE DYNAMICS OF SAMPLED-DATA SYSTEMS WITH STOCHASTIC UPSETS

In the previous section, it was shown that the states of the plant and the controller are random elements, which are random processes with sample paths in $\mathcal{C}^{n_{x_p}}$ and $\mathcal{S}^{n_{x_c}}$, respectively. Stacking them gives a hybrid state vector $\left[\boldsymbol{x}'_p(t)\ \boldsymbol{x}'_c[k]\right]'$ for the sampled-data system. Since the stochastic processes are families of random variables, stability of the closed-loop can be analyzed from the coordinate random variables $\left[\boldsymbol{x}'_p(t)\ \boldsymbol{x}'_c[k]\right]' \in \mathcal{X} \subset \mathbb{R}^{n_{x_p}+n_{x_c}}$, where $k \in \mathbb{Z}^+$ and $k\mathrm{T} \leq t < (k+1)\mathrm{T}$. Since these random variables take values $\mathcal{X}$, a subset of $\mathbb{R}^{n_{x_p}+n_{x_c}}$, a $p$-norm can be used to analyze stability. The next subsections introduce concepts similar to those in [29] for the sampled-data system in (1).

### II.4.1 Stochastic Motions in Sampled-data Systems

The purpose of this subsection is to characterize the dynamics of the nominal, $\Sigma_N$, and upset, $\Sigma_R$, modes of the sampled-data system. Because the variables involved are random processes, it is possible to define the random process

$$\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) \triangleq \left[\boldsymbol{x}'_p(t)\ \boldsymbol{x}'_c[k]\right]' \in \mathcal{X}$$

with the initial condition $\boldsymbol{z}[k] \triangleq \left[\boldsymbol{x}'_p[k]\ \boldsymbol{x}'_c[k]\right]'$, where $\boldsymbol{\theta}[k]$ is a discrete-time Markov chain with finite state space, $k \in \mathbb{Z}^+$, $\boldsymbol{x}_c[k] \triangleq \boldsymbol{x}_c[k\mathrm{T}]$, and $k\mathrm{T} \leq t < (k+1)\mathrm{T}$. The random process $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])$ depends of the initial conditions $\boldsymbol{z}[k]$ and $\boldsymbol{\theta}[k]$. The following definitions from [29] will be used to characterize the modes.

**Definition 7.** Let $(\mathcal{X}, \|\cdot\|^p)$ be a $p$-normed space with $\mathcal{X} \subset \mathbb{R}^{n_{x_p}+n_{x_c}}$. Let $\boldsymbol{z}[k] \in \mathsf{A} \subset \mathcal{X}$ be an initial state of the system at the initial time $k\mathrm{T}$. A stochastic process $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])$, $t \in [k\mathrm{T}, (k+1)\mathrm{T})$ taking values in $\mathcal{X}$ is called *stochastic motion* if $\mathbf{z}(k\mathrm{T}, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) = \boldsymbol{z}[k]$ for all $\omega \in \Omega$.

**Definition 8.** Let $\mathbb{S}$ be a family of stochastic motions taking values in $\mathcal{X}$ such that

$$\mathbb{S} \subset \{\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) : \mathbf{z}(k\mathrm{T}, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) = \boldsymbol{z}[k],\ \forall k \in \mathbb{Z}^+, \omega \in \Omega, t \in [k\mathrm{T}, (k+1)\mathrm{T})\}.$$

Then $\mathbb{S}$ is called a *stochastic dynamical system.*

The stochastic motions $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])$ model the sample response of the closed-loop system. A characterization of stochastic motions for a sampled-data system with stochastic upsets is given in the following theorem.

**Theorem 2.** *The dynamics of the stochastic motions of the hybrid system* (1) *are given by*

$$\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) = N(kT, t)\boldsymbol{z}[k] + \boldsymbol{m}(t), \tag{3}$$

*and those of* (2) *are given by*

$$\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) = N(kT, t)\boldsymbol{z}[k], \tag{4}$$

*where* $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) = \left[\boldsymbol{x}_p'(t)\ \boldsymbol{x}_c'[k]\right]'$, $k \in \mathbb{Z}^+$, $kT \leq t < (k+1)T$, $N(kT, t)$ *is the nonsingular and bounded matrix*

$$N(kT, t) = \begin{bmatrix} e^{A_p(t-kT)} & \int_{kT}^t e^{A_p(t-s)} ds B_p F_c \\ 0 & I \end{bmatrix}$$

*and*

$$\boldsymbol{m}(t) = \begin{bmatrix} \int_{kT}^t e^{A_p(t-s)} g(\boldsymbol{x}_p(s)) ds \\ 0 \end{bmatrix}.$$

*Proof.* In (1), the sample path responses of the hybrid system due to any initial states $\boldsymbol{z}[k]$ and $\boldsymbol{\theta}[k], k \in \mathbb{Z}^+$ result in the following usual convolution response for the random process $\{\boldsymbol{x}_p(t), k\mathrm{T} \leq t < (k+1)\mathrm{T}\}$

$$\boldsymbol{x}_p(t) = e^{A_p(t-k)\mathrm{T}} \boldsymbol{x}_p(k\mathrm{T}) + \int_{k\mathrm{T}}^t e^{A_p(t-s)} ds\ B_p \boldsymbol{u}(s) + \int_{k\mathrm{T}}^t e^{A_p(t-s)} g(\boldsymbol{x}_p(s)) ds.$$

Because $\boldsymbol{u}(s) = \mathcal{H}_{\mathrm{T}} F_c \boldsymbol{x}_c[k]$ it follows that

$$\begin{aligned} \boldsymbol{x}_p(t) &= e^{A_p(t-k\mathrm{T})} \boldsymbol{x}_p(k\mathrm{T}) + \int_{k\mathrm{T}}^t e^{A_p(t-s)} ds\ B_p F_c \boldsymbol{x}_c[k] + \int_{k\mathrm{T}}^t e^{A_p(t-s)} g(\boldsymbol{x}_p(s)) ds. \\ \boldsymbol{x}_c[k] &= \boldsymbol{x}_c[k]. \end{aligned}$$

The desired result follows by expressing these equations in matrix form. □

The stochastic motion $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])$ has as initial conditions $\boldsymbol{z}[k]$ and $\boldsymbol{\theta}[k]$, for every $k \in \mathbb{N}$, and evolves according to the equations (3) or (4).

To analyze the stability of the hybrid linearized system (2), its associated discrete-time system needs to be characterized. It is characterized by interconnecting the jump linear discrete-time controller to the zero-order-hold equivalent model of the plant as seen from the input/output channels of the controller. The dynamics of the associated discrete-time system $\{\boldsymbol{z}[k]; k \in \mathbb{Z}^+\}$ are given by the following theorem.

**Theorem 3.** *The associated discrete-time system, $\{\boldsymbol{z}[k]; k \in \mathbb{Z}^+\}$, for* (1) *is*

$$\boldsymbol{z}[k+1] = M_{\boldsymbol{\theta}[k]}\boldsymbol{z}[k] + \boldsymbol{m}[k], \tag{5}$$

*and the associated discrete-time system for* (2) *is*

$$\boldsymbol{z}[k+1] = M_{\boldsymbol{\theta}[k]}\boldsymbol{z}[k], \tag{6}$$

*where* $\boldsymbol{z}'[k] = \begin{bmatrix}\boldsymbol{x}_p'[k] & \boldsymbol{x}_c'[k]\end{bmatrix}'$,

$$M_{\boldsymbol{\theta}[k]} = \begin{bmatrix} e^{A_p T} & \int_0^T e^{A_p(T-s)} ds B_p F_c \\ B_{\boldsymbol{\theta}[k]} C_p & A_{\boldsymbol{\theta}[k]} \end{bmatrix},$$

*and*

$$\boldsymbol{m}[k] = \begin{bmatrix} \int_{kT}^{(k+1)T} e^{A_p(kT+T-s)} g(\boldsymbol{x}_p(s)) ds \\ 0 \end{bmatrix}.$$

*Proof.* The solution of the differential equation of the continuous-time plant for $t \in [k\mathrm{T}, (k+1)\mathrm{T})$ is

$$\boldsymbol{x}_p(t) = e^{A_p(t-k\mathrm{T})}\boldsymbol{x}_p(k\mathrm{T}) + \int_{k\mathrm{T}}^{t} e^{A_p(t-s)} ds\; B_p F_c \boldsymbol{x}_c[k] + \int_{k\mathrm{T}}^{t} e^{A_p(t-s)} g(\boldsymbol{x}_p(s)) ds.$$

The sampling operator gives $\boldsymbol{x}_p[k] \triangleq \lim_{t \to (k\mathrm{T})^-} \boldsymbol{x}_p(t)$. Thus,

$$\boldsymbol{x}_p[k+1] = e^{A_p \mathrm{T}}\boldsymbol{x}_p[k] + \int_{k\mathrm{T}}^{(k+1)\mathrm{T}} e^{A_p(k\mathrm{T}+\mathrm{T}-s)} ds\; B_p F_c \boldsymbol{x}_c[k] + \int_{k\mathrm{T}}^{(k+1)\mathrm{T}} e^{A_p(k\mathrm{T}+\mathrm{T}-s)} g(\boldsymbol{x}_p(s)) ds.$$

The desired result follows by expressing this equation and the expression $\boldsymbol{x}_c[k+1] = A_{\boldsymbol{\theta}[k]}\boldsymbol{x}_c[k] + B_{\boldsymbol{\theta}[k]} C_p \boldsymbol{x}_p[k]$ in matrix form. □

Now that the dynamics of the motions of the sampled-data have been defined, the $p^{\text{th}}$-moment stability of the sampled-data system can be analyzed.

### II.4.2 Stochastic Motions and $p^{\text{th}}$-moment Stability

In this section the equivalence between $p^{\text{th}}$-moment stability of the stochastic motion representation of the linearized hybrid stochastic system (4) and the $p^{\text{th}}$-moment stability of the discrete stochastic system in (6) is shown. The following additional definitions from [29] are needed.

**Definition 9.** Let $\mathbb{S}$ be a stochastic dynamical system. A set $M \subset \mathbb{R}^{n_{xp}+n_{xc}}$ is said to be *invariant* with respect to $\mathbb{S}$ if $\forall k \in \mathbb{Z}^+$, $\boldsymbol{z}[k] = a \in M$ implies that

$$\Pr\left(\mathbf{z}(t, \boldsymbol{z}[k] = a, \boldsymbol{\theta}[k]) \in M,\ \forall t \in [k\mathrm{T}, (k+1)\mathrm{T})\right) = 1$$

for all $\boldsymbol{\theta}[k]$.

**Definition 10.** The vector $a \in \mathbb{R}^{n_{xp}+n_{xc}}$ is called an *equilibrium* of the stochastic dynamical system $\mathbb{S}$ if the set $\{a\}$ is invariant with respect to $\mathbb{S}$.

Notice that the origin of $\mathbb{R}^{n_{xp}+n_{xc}}$ is an equilibrium of the stochastic dynamical system (2) since $\mathbf{z}(t, \boldsymbol{z}[k] = 0, \boldsymbol{\theta}[k]) = 0$, $\forall k \in \mathbb{Z}^+$, $t \in [k\mathrm{T}, (k+1)\mathrm{T})$, and $\forall \omega \in \Omega$. Thus, $\Pr\big(\mathbf{z}(t, \boldsymbol{z}[k] = 0, \boldsymbol{\theta}[k]) = 0,\ t \in [k\mathrm{T}, (k+1)\mathrm{T})\big) = 1\ \forall k \in \mathbb{Z}^+$. The $p^{\text{th}}$-moment stability of the equilibrium state at the origin is described next.

**Definition 11.** The equilibrium state $0 \in \mathbb{R}^{n_{xp}+n_{xc}}$ of the stochastic dynamical system $\mathbb{S}$ is *stable in the $p^{th}$-moment* if $\forall \epsilon > 0,\ \exists\, \delta = \delta(\epsilon, k) > 0$ such that $\|z[0]\| < \delta$ implies $E\{\|\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])\|^p\} < \epsilon\ \forall k \in \mathbb{Z}^+,\ t \in [k\mathrm{T}, (k+1)\mathrm{T})$. The origin is said to be *asymptotically stable in the $p^{th}$-moment* if in addition to being stable in the $p^{\text{th}}$-moment, $E\{\|\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])\|^p\} \to 0$ as $k \to \infty$, $t \in [k\mathrm{T}, (k+1)\mathrm{T})$.

The following theorem shows the equivalence between the $p^{\text{th}}$-moment stability of the stochastic motions, $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])$, and its associated discrete-time system $\boldsymbol{z}[k]$.

**Theorem 4.** *The origin of the stochastic dynamic system representation,* (4)*, of a sampled-data system with stochastic upsets is asymptotically stable in the $p^{th}$-moment if and only if the origin of the discrete-time system* (6) *is asymptotically stable in the $p^{th}$-moment.*

*Proof.* Let the origin of the discrete stochastic system (6) be asymptotically stable in the $p^{\text{th}}$-moment. The stochastic motions (4) satisfy the following inequalities for $k \in \mathbb{Z}^+,\ t \in [k\mathrm{T}, (k+1)\mathrm{T})$, and $\omega \in \Omega$:

$$\begin{aligned} \|\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])\|^p &\leq \|N(k\mathrm{T}, t)\|^p \|\boldsymbol{z}[k]\|^p \\ E\{\|\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])\|^p\} &\leq \|N(k\mathrm{T}, t)\|^p E\{\|\boldsymbol{z}[k]\|^p\}. \end{aligned}$$

Since $N(k\mathrm{T}, t)$ is bounded, there exist $N \in \mathbb{R}^+$ such that $||N(k\mathrm{T}, t)||^p < N$ for all $k$ and $t$, and therefore, $E\{\|\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])\|^p\} \leq NE\{\|\boldsymbol{z}[k]\|^p\}$. Thus, if $E\{\|\boldsymbol{z}[k]\|^p\} \to 0$ then $E\{\|\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k])\|^p\} \to 0$. A similar proof follows when it is assumed that the origin of the stochastic motions (2) is asymptotically stable in the $p^{\text{th}}$-moment. Since $N(k\mathrm{T}, t)$ is nonsingular $\forall k \in \mathbb{Z}^+$ and $t \in [k\mathrm{T}, (k+1)\mathrm{T})$, the following inequalities are satisfied:

$$\begin{aligned} \|z[k]\|^p &\leq \|N^{-1}(k\mathrm{T}, t)\|^p \cdot \|\mathbf{z}(t, z[k], \boldsymbol{\theta}[k])\|^p \\ E\{\|z[k]\|^p\} &\leq \|N^{-1}(k\mathrm{T}, t)\|^p \cdot E\{\|\mathbf{z}(t, z[k], \boldsymbol{\theta}[k])\|^p\}. \end{aligned}$$

Thus, if $E\{\|\mathbf{z}(t, z[k], \boldsymbol{\theta}[k])\|^p\} \to 0$ then $E\{\|z[k]\|^p\} \to 0$, since $N^{-1}(k\mathrm{T}, t)$ is bounded. □

Therefore, to determine $p^{\text{th}}$-moment stability of the hybrid system (2), it is necessary and sufficient to check the $p^{\text{th}}$-moment stability of the associated discrete-time system (6).

### II.4.3 Mean Square Stability of Markovian Nonlinear Sampled-data Systems

Stability of the nonlinear sampled-data (2) is analyzed when the stochastic process, $\boldsymbol{\theta}[k]$, is a discrete-time Markov chain. Definition 12 defines a Markov chain [6].

**Definition 12.** Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space with a nondecreasing family $(\mathcal{F}_n)$ of $\sigma$-algebras, $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$. A stochastic process, $\boldsymbol{\theta}[k]$, taking values in a measurable space $(E, \mathbb{E})$ is called a *Markov chain* with respect to the measure Pr if

$$\Pr(\boldsymbol{\theta}[k] \in B | \mathcal{F}_m) = \Pr(\boldsymbol{\theta}[k] \in B | \boldsymbol{\theta}[m]),$$

for all $k > m$ and every event $B \in \mathbb{E}$.

The theory of Markov processes shows that a fundamental role is played by the one-step transition probabilities $\Pr(\boldsymbol{\theta}[k+1] \in B|\theta[k])$ and by the functions $p_{k+1}(B|x)$. The latter are called *transition functions* and are defined such that for every $x \in E$, $p_{k+1}(\cdot|x)$ is a probability measure on $(E, \mathbb{E})$ and for every event $B \in \mathbb{E}$, $p_{k+1}(B|\cdot)$ is a measurable function on $E$. The relationship between the transition probabilities and the transition functions is

$$\Pr(\boldsymbol{\theta}[k+1] \in B|\theta[k]) = p_{k+1}(B|\theta[k]) \text{ almost everywhere.}$$

If the transition functions have the following property

$$p_2(B|\theta[1]) = p_3(B|\theta[2]) = \cdots = p_{k+1}(B|\theta[k]), \; k \geq 1,$$

then the corresponding Markov chain is called a *time homogeneous* Markov chain. When $E$ is a finite set, the transition functions are denoted by $\pi_{ij} \triangleq p(\boldsymbol{\theta}[k+1] = j|\boldsymbol{\theta}[k] = i)$, and the matrix $\Pi \triangleq [\pi_{ij}]$ is called the *transition probability matrix.*

In this section, it is assumed that the stochastic process $\boldsymbol{\theta}[k]$ that drives the discrete-time controller is a homogeneous, discrete-time Markov chain with initial distribution $\pi(0)$, transition probability matrix $\Pi = [\pi_{ij}]$, and finite state space $\{1, \ldots, \eta_\theta\}$. Under this assumption, it is possible to show that the $2^{\mathrm{nd}}$-moment stability of the origin of (2) implies the $2^{\mathrm{nd}}$-moment stability of the origin of (1). First, Theorems 5 and 6 are introduced. They are special cases of known results.

**Theorem 5.** *[29] Let $\mathbb{S}$ be a stochastic dynamical system with an equilibrium at the origin and sample paths of the stochastic motions given by $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\theta}[k]) \triangleq \mathbf{z}(t,k) \in \mathcal{X} \subset \mathbb{R}^{n_{x_p}+n_{x_c}}$, $k \in \mathbb{Z}^+$, $t \in [kT, (k+1)T)$. The origin of $\mathbb{S}$ is $2^{nd}$-moment stable if there exists a function $\boldsymbol{V} : \mathcal{X} \longrightarrow \mathbb{R}^+$ satisfying:*

(a) $c_1\|\mathbf{z}(t,k)\|^2 \leq \boldsymbol{V}(\mathbf{z}(t,k)) \leq c_2\|\mathbf{z}(t,k)\|^2$ *for all* $\boldsymbol{z} \in \mathcal{X}$ *and* $t \in \mathbb{R}^+$ *where* $c_i \in \mathbb{R}^+$, $i = 1, 2$*;*

(b) $E\{\boldsymbol{V}(\boldsymbol{z}[k+1])\} - E\{\boldsymbol{V}(\boldsymbol{z}[k])\} \leq 0$ *for* $k = 1, 2, \ldots$*; and*

(c) *there exists a continuous real positive function* $h$ *with* $h(0) = 0$ *such that:* $E\{\boldsymbol{V}(\mathbf{z}(t,k))\} \leq h(E\{\boldsymbol{V}(\boldsymbol{z}[k])\})$ *for* $k \in \mathbb{Z}^+$ *and* $t \in [kT, (k+1)T)$.

**Theorem 6.** *[12] The origin of (6) is $2^{nd}$-moment stable if and only if for any given set of complex positive definite matrices $\{W_i, i = 1, 2, \ldots, \eta_\theta\}$, there exists a set of complex positive definite matrices, $\{L_i, i = 1, 2, \ldots, \eta_\theta\}$, satisfying*

$$L_i - \sum_{j=1}^{\eta_\theta} \pi_{ij} M_i' L_j M_i = W_i. \tag{7}$$

Now, the main result of this section is given.

**Theorem 7.** *Let $\boldsymbol{\theta}[k]$ be a homogeneous, aperiodic, discrete Markov chain. If the origin of the linearized hybrid system (2) is $2^{nd}$-moment stable then the origin of the nonlinear hybrid system (1) is $2^{nd}$-moment stable.*

*Proof.* Mean square stability of the origin of (1) will be shown using Theorem 5. Consider the following function $\boldsymbol{V}(\mathbf{z}(t,k)) = \mathbf{z}'(t,k) L_{\boldsymbol{\theta}[k]} \mathbf{z}(t,k)$ for $k \in \mathbb{Z}^+$, $t \in [k\mathrm{T}, (k+1)\mathrm{T})$, $\theta[k] \in \{1, 2, \ldots, \eta_\theta\}$ and $L_i > 0$ for $i \in \{1, 2, \ldots, \eta_\theta\}$. The three properties of $\boldsymbol{V}$ in Theorem 5 will now be verified.

To show the first property notice that for each positive definite $L_i$, $i \in \{1, 2, \ldots, \eta_\theta\}$ there exists $\lambda_i, \lambda^i \in (0, \infty)$ satisfying

$$\lambda_i \|\mathbf{z}(t,k)\|^2 \leq \mathbf{z}'(t,k) L_i \mathbf{z}(t,k) \leq \lambda^i \|\mathbf{z}(t,k)\|^2.$$

Thus,

$$\begin{aligned}\sum_{i=1}^{\eta_\theta} \lambda_i \|\mathbf{z}(t,k)\|^2 1_{\{\theta[k]=i\}} &\leq \sum_{i=1}^{\eta_\theta} \boldsymbol{V}(\mathbf{z}(t,l)) 1_{\{\theta[k]=i\}} \\ \sum_{i=1}^{\eta_\theta} \lambda^i \|\mathbf{z}(t,k)\|^2 1_{\{\theta[k]=i\}} &\geq \sum_{i=1}^{\eta_\theta} \boldsymbol{V}(\mathbf{z}(t,k)) 1_{\{\theta[k]=i\}}.\end{aligned}$$

This shows that property $(a)$ is satisfied with $c_1 = \min_i\{\lambda_i\}$ and $c_2 = \max_i\{\lambda^i\}$.

Now consider the difference

$$\begin{aligned}\Delta V(z,k) &= E\{\boldsymbol{V}(\boldsymbol{z}[k+1])\} - E\{\boldsymbol{V}(\boldsymbol{z}[k])\} \\ &= E\{\boldsymbol{z}'[k+1] L_{\theta[k+1]} \boldsymbol{z}[k+1]\} - E\{\boldsymbol{z}'[k] L_{\theta[k]} \boldsymbol{z}[k]\}.\end{aligned}$$

The difference expands using Theorem 5 to the following four terms

$$\begin{aligned}\Delta V(z,k) = {} & E\big\{\boldsymbol{z}'[k](M'_{\theta[k]} L_{\theta[k+1]} M_{\theta[k]} - L_{\theta[k]})\boldsymbol{z}[k]\big\} \\ & + E\big\{\boldsymbol{z}'[k] M'_{\theta[k]} L_{\theta[k+1]} \boldsymbol{m}[k]\big\} \\ & + E\big\{\boldsymbol{m}'[k] L_{\theta[k+1]} M_{\theta[k]} \boldsymbol{z}[k]\big\} + E\big\{\boldsymbol{m}'[k] L_{\theta[k+1]} \boldsymbol{m}[k]\big\}.\end{aligned} \tag{8}$$

Each of the four terms on the right hand side (RHS) of (8) can be simplified as follows. Since the linearized hybrid system (2) is $2^{\text{nd}}$-moment stable then by Theorem 4 the associated discrete-time system (6) is also $2^{\text{nd}}$-moment stable. By Theorem 6 the set of positive definite matrices $\{L_i, i = 1, 2, \ldots, \eta_\theta\}$ is a solution of (7) for a given set of positive definite matrices $\{W_i, i = 1, 2, \ldots, \eta_\theta\}$. Thus, the first term on the RHS can be shown to be $-\sum_{i=1}^{\eta_\theta} E\{\boldsymbol{z}'[k] W_i \boldsymbol{z}[k] 1_{\{\theta[k]=i\}}\}$. Arbitrarily choosing $W_i = I$ for every $i = 1, 2, \ldots, \eta_\theta$ simplifies this first term to $-E\|\boldsymbol{z}[k]\|^2$. The next three terms on the RHS of (8) can be simplified with the following argument. Since $f$ in (1) is continuously differentiable on a neighborhood of the origin, it follows that $\lim_{\|x_p\| \to 0} \frac{\|g(x_p)\|}{\|x_p\|} = 0$. Alternatively, $\forall \epsilon > 0\ \exists\ \delta > 0$ such that $\|x_p\| \leq \delta$ implies $\|g(x_p)\| \leq \epsilon \|x_p\|$. Thus, the second and third terms on the RHS of (8) are less than $\alpha\epsilon E\|\boldsymbol{z}[k]\|^2$, where $\alpha \in \mathbb{R}^+$. The fourth term on the RHS can be shown to be less than $\beta\epsilon^2 E\|\boldsymbol{z}[k]\|^2$, where $\beta \in \mathbb{R}^+$. Hence, $\Delta V(z,k) \leq -(1 - 2\alpha\epsilon - \beta\epsilon^2) E\|\boldsymbol{z}[k]\|^2$, which is less than zero for appropriate $\epsilon$. So property $(b)$ is satisfied.

The proof of the third property is based on the following inequality that holds almost everywhere $\|x_p(t)\| \le (\|x_p[k]\| + \|x_c[k]\|)H_1 e^{H_2}$, where $H_i, i = 1, 2$ are positive constants [30]. In this stochastic framework, for each $k \in \mathbb{Z}^+$ and $t \in [k\mathrm{T}, (k+1)\mathrm{T})$ the inequality is $\|\boldsymbol{x}_p(t)\| \le (\|\boldsymbol{x}_p[k]\| + \|\boldsymbol{x}_c[k]\|)H_1 e^{H_2}$. Since $\|\boldsymbol{z}[k]\|^2 = \|\boldsymbol{x}_p[k]\|^2 + \|\boldsymbol{x}_c[k]\|^2$ the previous expression simplifies to $\|\boldsymbol{x}_p(t)\|^2 \le 2H_1^2 e^{2H_2}\|\boldsymbol{z}[k]\|^2$. Combining this inequality with the first property yields: $\boldsymbol{V}(\mathbf{z}(t,k)) \le \max_i\{\lambda^i\}\left(\|\boldsymbol{x}_p(t)\|^2 + \|\boldsymbol{x}_c[k]\|^2\right)$, that is,

$$\boldsymbol{V}(\mathbf{z}(t,k)) \le \max_i\{\lambda^i\}\left(2H_1^2 e^{2H_2} + 1\right)\|\boldsymbol{z}[k]\|^2,$$

where $\|\boldsymbol{z}[k]\|^2 \le \frac{\boldsymbol{V}(\boldsymbol{z}[k])}{\min_i\{\lambda_i\}}$. Therefore,

$$E\{\boldsymbol{V}(\mathbf{z}(t,k))\} \le \frac{\max_i\{\lambda^i\}\left(2H_1^2 e^{2H_2} + 1\right)}{\min_i\{\lambda_i\}} E\{\boldsymbol{V}(\boldsymbol{z}[k])\}.$$

Thus, property $(c)$ is satisfied. Since a function $\boldsymbol{V}$ was found that satisfied the three properties of Theorem 5 for (1) then the origin of (1) is MSS. □

The proof for Theorem 7 also has shown the following corollary.

**Corollary 1.** *If the origin of the linearized discrete-time system* (6) *is* $2^{nd}$*-moment stable then the nonlinear hybrid system* (1) *is* $2^{nd}$*-moment stable.*

The summary and main contributions of this chapter are given next.

## II.5 CHAPTER SUMMARY

This chapter has introduced a mathematical framework that makes the sampling and zero-order-hold measurable mappings, thus making it possible to induce probability measures throughout a closed-loop sampled-data system. Equivalence was then shown between the $p^{\text{th}}$-moment stability of a stochastic, linearized hybrid dynamical system and the $p^{\text{th}}$-moment of its corresponding discrete-time representation. This equivalence also holds for two more stability definitions: stability in the mean and in the mean square sense. In addition, for stochastic, nonlinear, hybrid systems with an underlying Markovian process, $2^{\text{nd}}$-moment stability of the linearized discrete-time system was shown to imply $2^{\text{nd}}$-moment stability of the nonlinear system.

# CHAPTER III

# DYNAMICALLY COLORED PETRI NET (DCPN) REPRESENTATION OF SAMPLED-DATA SYSTEMS WITH EMBEDDED RECOVERY ALGORITHMS

## III.1 INTRODUCTION

To study the stability and performance properties of a closed loop system, an appropriate mathematical model of the recovery logic, the discrete-time jump linear controller, the nonlinear continuous-time plant and the stochastic upsets is needed. It is important to characterize the switching between the nominal and upsets modes, while taking into account the uncertainty in the number of recovery cycles and uncertainty in the switching time between nominal and upset modes. In [48] a model of a discrete-time version of a sampled-data system was studied. The theoretical model was a stochastic finite state machine (SFSM) modeling the recovery mechanism and a discrete-time Markov chain (DTMC) modeling the faults induced in the controller. Basically the theoretical model was a SFSM with a DTMC as an input. Using this model the switching mechanism was modeled as a DTMC, and its transition matrix was derived. A problem with this approach is that as the number of recovery cycles increases the size of the transition matrix increases significantly. To avoid possible numerical problems associated with large matrices, a goal of this chapter is to introduce a model such that the dimension of the transition probability matrix of the switching process does not depend on the number of recovery cycles. The idea is to model the mathematical description of the sampled-data system in Chapter II and the recovery logic as a dynamically colored Petri net (DCPN), where the possibility of inducing a failure in the controller is modeled as a continuous-time Markov chain (CTMC). This model is intuitively more appealing than a DTMC because in real systems the faults can occur at any time. In this chapter, the recovery logic in Definition 13 from [48] is assumed.

**Definition 13.** *Basic Error Recovery Algorithm* As long as there is no upset, the sampled-data closed-loop system operates in the nominal mode, $\Sigma_N$. As soon as an upset occurs, the error recovery logic switches the operation of the closed-loop system to the recovery mode, $\Sigma_R$, where it remains for a random number of sample periods, $n_a$ or $n_b$, with probability $p_{n_a}$ and $p_{n_b} = 1 - p_{n_a}$, respectively. At the end of the random chosen duration, the nominal dynamics are restored. The arrival of any additional upsets events during a recovery is disregarded.

The effect of random upsets is interference on closed-loop performance produced by the recovery logic. During an upset state, a recovery takes place such that the control input is held constant and the detection of new faults is disabled. A recovery is modeled with a different difference equation then for the nominal mode. This simple recovery logic is based on a series of simulated neutron irradiation experiments of a prototype flight control computer with error recovery capabilities conducted at the NASA Langley Research Center. In these experiments, 80% of the recovery periods lasted six frames and 20% lasted five. For analysis, this class of closed-loop systems with error recoveries triggered by random upsets was represented as discrete-time Markovian jump linear system [40, 48]. Thus, their performance and stability was analyzed using the theory developed, for example, in [14]. The main stability result is a necessary and sufficient condition for the mean square stability of a discrete-time Markov jump linear system, which requires the calculation of the spectral radius of a specific matrix. In this chapter another way to model this problem is investigated. The interconnection of the recovery logic and the closed-loop system is represented using a type of Petri net called a Dynamically Colored Petri Net (DCPN) [4, 20]. A Petri net (cf. [6]) is a type of graph, consisting primarily of places (possible conditions or discrete modes) and transitions (possible switches). Each place can have zero or more tokens. At each time instant, the number of tokens in each place defines the state of the Petri net. A vector that indicates the number of tokens in each place is called the marking of the Petri Net. In a DCPN, at each time instant, the tokens have a value corresponding to the solution of a differential equation. It is a simple extension to let the tokens have a value corresponding to the solution of a difference equation or a system of difference equations. The transitions in a DCPN can be partitioned into three classes: immediate, delay and guard transitions.

The main goal of this chapter is to model a closed-loop sampled-data system, where the controller is implemented in a computer with error recovery capabilities that are triggered by stochastic upsets, as a DCPN. Since the dynamics of the sampled-data are represented with motions, consisting of augmented differential and

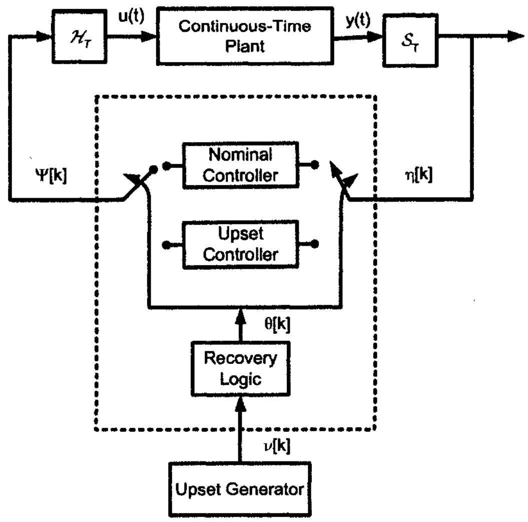


FIG. 3: A sampled-data system with a stochastic upset process $\nu[k]$.

difference equation solutions, the resulting DCPN will be referred to as a sampled-data DCPN. It will then be shown that stability can be determined by mapping the sampled-data DCPN into a Piecewise Deterministic Markov Process (PDP) [16,17,20] using the results from [28].

A general block-diagram of the sampled-data system of interest is shown in FIG. 3. The random process $\boldsymbol{\theta}[k]$ is the output of the recovery logic. The upset generator, $\boldsymbol{\nu}[k]$, is a discrete-time Markov chain with two states, which models the upset or no upset conditions.

The main concepts of a DCPN are presented in Section III.2. In Section III.3, the sampled-data system with a recovery algorithm will be represented as a DCPN. In Section III.4 the PDP model of the sampled-data will be presented. Finally, the summary is given in Section III.5.

### III.2 DCPN

This section first presents the definition and basic properties of DCPNs [4,20]. The following definition introduces the basic concept and notation of a DCPN.

**Definition 14.** A DCPN is an 11-tuple, $(\mathcal{P}, \mathcal{T}, \mathcal{A}, \mathcal{N}, \mathcal{S}, \mathcal{C}, \mathsf{z}, \mathcal{G}_T, \mathcal{D}, \mathsf{F}, \mathcal{I})$, together with firing rules $R_0 - R_4$: The 11-tuple consists of a set of places $\mathcal{P}$, a set of transitions $\mathcal{T}$, a set of arcs $\mathcal{A}$, a node function $\mathcal{N}$, a set of color types $\mathcal{S}$, a set of token color functions $\mathcal{C}$, a set of motions z, a set of transition guards $\mathcal{G}$, a set of delay transition enabling rate functions $\mathcal{D}$, a set of firing measures $\mathcal{F}$, and an initialization function $\mathcal{I}$, respectively.

The firing rules are going to be presented after the description of the DCPN elements. The firing rules $R_0 - R_4$ are used to resolve firing ambiguities. A brief description of the DCPN elements together with definitions of the notation is given next.

- $\mathcal{P}$ is a finite set of places. The places are labeled as follows $\mathcal{P} = \{P_1, ..., P_{|\mathcal{P}|}\}$, where $|\mathcal{P}|$ is the number of places. Each place can have a nonnegative number of tokens. In this dissertation it is assumed that each place has no more than one token. A token denotes which places are active.

- $\mathcal{T}$ is a finite set of transitions. $\mathcal{T}$ is partitioned into the set of guard transitions, $\mathcal{T}_G$, the set of immediate transitions, $\mathcal{T}_I$, and the set of delay transitions, $\mathcal{T}_D$.

- $\mathcal{A}$ is a finite set of arcs connecting a place to or from a transition. The endpoints of an arc are referred to as the nodes of the arc. In addition to the ordinary or simple arcs, there are enable arcs, which do not reduce the number of tokens in their input places.

- The node function is a mapping, $\mathcal{N} : \mathcal{A} \to (\mathcal{P} \times \mathcal{T}) \bigcup (\mathcal{T} \times \mathcal{P})$, that maps each arc $A \in \mathcal{A}$ to an ordered pair. The place associated with an arc is denoted by $P(A)$. The transition associated with an arc $A$ is denoted by $T(A)$. If $P(A)$ is a source node, $\mathcal{N}(A) = (P(A), T(A))$, and if it is a sink node $\mathcal{N}(A) = (T(A), P(A))$. Also, let $A(T) = \{A \in \mathcal{A} : T(A) = T\} \subset \mathcal{A}$ with the partition $A(T) = A_{\text{in}}(T) \bigcup A_{\text{out}}(T)$, where $A_{\text{in}}(T)$ and $A_{\text{out}}(T)$ denote the sets of input and output arcs of $T$, respectively. By composition, $P(A(T))$ is the set of places connected to $T$ by the set of arcs $A(T)$.

- The finite set of Euclidean spaces associated with each place is $\mathcal{S}$, that is, $\mathcal{S} = \{\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_{|\mathcal{P}|}}\}$, where $n_i$, $i = 1, \ldots, |\mathcal{P}|$, is the dimension of the $i$-th place, and $\mathbb{R}^0 = \varnothing$. Each Euclidean space is called a *color type*.

- The color function is a mapping $\mathcal{C} : P \rightarrow \mathcal{S}$ that assigns to each place a specific color type. For example, $\mathcal{C}(P(A_{\text{in}}(T)))$ is the color type of the input places of the transition $T$. If $P(A_{\text{in}}(T))$ contains more than one place, suppose $m$ places, then $\mathcal{C}(P(A_{\text{in}}(T))) \triangleq \mathcal{C}(P_{i_1}(A_{\text{in}}(T))) \times \cdots \times \mathcal{C}(P_{i_m}(A_{\text{in}}(T)))$, where $i_1 < \ldots < i_m$. The color types identify the Euclidean spaces of the solutions of the differential or difference equations associated with the tokens in each place.

- In each place $P$, a map is defined whenever a token exists as follows: $\mathbf{z}^i_{st} : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathcal{C}(P_i) \rightarrow \mathcal{C}(P_i)$, where, at time $t$ the value is $\mathbf{z}^i_{st}(z^i)$, and $z^i$ is the initial condition at time $s$, satisfying $\mathbf{z}^i_{ss}(z^i) = z^i$. This map has the following property $\mathbf{z}^i_{st}(z^i) = \mathbf{z}^i_{ut}(\mathbf{z}^i_{su}(z^i)), 0 \leq s \leq u \leq t$. This map is called a *motion token function*. This could be the solution of a differential or difference equation. It is assumed to exist and to be unique.

- Associated with each $T \in \mathcal{T}_G$ there is an open set $G_T \subset \mathcal{C}(P(A_{\text{in,OE}}(T)))$. For this case suppose that there is only one input place, where $A_{\text{in,OE}}(T)$ is the set of input enable, (E), and ordinary arcs, (O). Associated with each $T \in T_G$ there is also an indicator function $\mathcal{G}_T : \mathcal{C}(P(A_{\text{in,OE}}(T))) \rightarrow \{0, 1\}$ such that $\mathcal{G}_T(\mathbf{z}_{st}(z)) = 0$ for all $\mathbf{z}_{st}(z) \in G_T$ and $\mathcal{G}_T(\mathbf{z}_{st}(z)) = 1$ for all $\mathbf{z}_{st}(z) \in \partial G_T$. $\partial G_T$ denote the boundary of an open subset $G_T$ for $s, t \in \mathbb{R}^+$. $G_T$ is an open subset of $\mathcal{C}(P(A_{\text{in,OE}}(T)))$. If $\mathcal{C}(P(A_{\text{in,OE}}(T))) = \varnothing$, $G_T$ and $\partial G_T$ will also be empty.

- The set of delay transition enabling rate functions is defined as follows: $\mathcal{D} = \{\delta_T : T \in \mathcal{T}_D\}$ and $\delta_T : \mathcal{C}(P(A_{\text{in,OE}}(T))) \rightarrow \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of non-negative real numbers. When there is only one input place, the rate function $\delta_T$ is a mapping that specifies the parameter of an exponential distribution of the delay time random variable $\boldsymbol{D}_T(\tau)$. If $\mathbf{z}_{\tau s}(z) \in \mathcal{C}(P(A_{\text{in,OE}}(T)))$ then a sample of delay time, $\boldsymbol{D}_T(\tau)$, is given by $D_T(\tau) = \inf\{t | e^{-\int_\tau^t \delta_T(\mathbf{z}_{\tau s}(z))ds} \leq u\}$, where $u$ is a uniform random number. If the color type is empty, $\delta_T$ is a constant function. It is the parameter of a exponential distribution.

- The set of firing measures is $\mathsf{F} = \{\mathsf{F}_T : T \in \mathcal{T}\}$. For each transition $T$ the firing measure $\mathsf{F}_T$ is a transition kernel.

- The initialization function specifies the initial marking of the DCPN. It specifies the initial number of tokens in each place and a color or a Euclidean space point associated with each token. It is a mapping, $\mathcal{I} : \mathcal{P} \rightarrow \mathbb{N}^{|\mathcal{P}|} \times \mathcal{C}(\mathcal{P})_{ms}$, where

$\mathbb{N}$ denotes the natural numbers, and $\mathcal{C}(P)_{ms}$ selects a specific Euclidean space point for each token in each place according to the dimension of the color types in each place.

Next, to define the state of a DCPN, the token and color states are introduced. At time $t$ the numbers of tokens in each place is denoted by the marking vector $\boldsymbol{\theta}(t) = (\boldsymbol{v}_1(t), \ldots, \boldsymbol{v}_{|\mathcal{P}|}(t))$ of length $|\mathcal{P}|$, where $\boldsymbol{v}_i(t)$ denotes the numbers of tokens in place $P_i$. This dissertation considers only the case of a 0 or 1 token. This vector will be called the **token state**. The value taken by the vector $\theta(t)$ will be denoted by $\theta_j$, where $j \in \{1, 2, \ldots, 2^{|\mathcal{P}|}\}$, or simply by $\theta$. The colors associated with each token are determined by the token color functions associated with each place. Suppose that each place only has one token and that the last transition fired at time $s$. Then let $\mathsf{z}^i_{st}(z_i) \in \mathcal{C}(P_i)$ be the token color function associated with the $i$-th place $P_i$. The token color functions for $t > s$ are aggregated into a single object denoted by $x_s(t) = (\mathsf{z}^1_{st}(z_1), \ldots, \mathsf{z}^{|\mathcal{P}|}_{st}(z_{|\mathcal{P}|}))$, which will be called the **color state**. The value of $x_s(t)$ will be denoted by $x$. Now, the state space of the DCPN is $(\theta, x_s(t))$ taking values in the space

$$\bigcup_{j \in 2^{|\mathcal{P}|}} \left\{ \{\theta_j\} \times \mathbb{R}^{n(\theta_j)} \right\},$$

where $n(\theta_j) \triangleq \sum_{i=1}^{|\mathcal{P}|} v_i \times n_i$, $\theta_j = (v_1, ..., v_{|\mathcal{P}|})$, and $n_i$ is the color dimension of the $i$-th place. The $\sigma$-algebra defined on this space is denoted by $\beta\big(\bigcup_{j\in 2^{|\mathcal{P}|}} \left\{\{\theta_j\} \times \mathbb{R}^{n(\theta_j)}\right\}\big)$. This state space together with its $\sigma$-algebra will be called the *measurable space* of the DCPN.

When a transition fires, it takes a token from its input place or places and puts it into the transition's output place or places. These transformations, called firing measures, are characterized for each transition with a transition kernel. The set of firing measures is $\mathsf{F} = \{\mathrm{F}_T(\theta', C|\theta, x) : T \in \mathcal{T}\}$. For each transition $T$ the firing measure $\mathrm{F}_T(\theta', C|\theta, x)$ is a transition kernel, that is,

- for every event $(\theta', C) \in \beta\big(\bigcup_{j\in 2^{|\mathcal{P}|}} \left\{\{\theta_j\} \times \mathbb{R}^{n(\theta_j)}\right\}\big)$, $\mathrm{F}_T(\theta', C|\cdot, \cdot)$ is a nonnegative measurable function on $\bigcup_{j\in 2^{|\mathcal{P}|}} \left\{\{\theta_j\} \times \mathbb{R}^{n(\theta_j)}\right\}$, and
- for each $(\theta, x) \in \bigcup_{j\in 2^{|\mathcal{P}|}} \left\{\{\theta_j\} \times \mathbb{R}^{n(\theta_j)}\right\}$, $\mathrm{F}_T(\cdot, \cdot|\theta, x)$ is a probability measure on $\beta\big(\bigcup_{j\in 2^{|\mathcal{P}|}} \left\{\{\theta_j\} \times \mathbb{R}^{n(\theta_j)}\right\}\big)$.

Thus, this transition kernel, $\mathrm{F}_T$, gives a probability to the event $(\theta', C)$ given that the state of the DCPN is $(\theta, x)$. Note that the entries of $(\theta, x)$ are the token and color states that pre-enabled the transition $T$. The entries $(\theta', C)$ are the new token and color states of the output places of transition $T$. A transition $T$ is *pre-enabled* if it has at least one token per incoming arc in each of its input places.

For modeling purposes the elements of the DCPN help to get the model development. To determine if a DCPN model captures the desired behavior of a system, the execution of a DCPN needs to be described. But first, the following definitions are needed, the time when $T$ is pre-enabled is denoted by

$$\tau_1^{pre} = \inf\{t | T \text{ is pre-enabled at time } t\}.$$

A transition $T$ is *enabled* if it is pre-enabled and a second requirement holds. For an immediate transition the second requirement holds immediately. For the delay transition, the second requirement holds $D(\tau_1^{pre})$ units after $\tau_1^{pre}$. For guard transitions the second requirement holds when $\mathcal{G}_T(\mathrm{z}_{\tau_1^{pre}t}(z)) = 1$, whenever $\mathrm{z}_{\tau_1^{pre}t}(z) \in \partial G_T$. If the transition $T$ is enabled, then the transition $T$ *fires*. This means that $T$ moves token and color states from its input places, $(\theta, x)$, to its output places according to its firing measure $\mathrm{F}_T(\cdot, \cdot | \theta, x)$.
Now the firing rules are introduced:

$R_0$: Immediate transitions fire with a higher priority than guard and delay transitions.

$R_1$: If one transition becomes enabled by two or more sets of input tokens at exactly the same time, and the firing of any one set will not disable one or more of the other sets, then it will fire these sets of tokens independently, at the same time.

$R_2$: If one transition becomes enabled by two or more sets of input tokens at exactly the same time, and the firing of any one set disables the other sets, then the set that is fired is selected randomly. Each set of inputs tokens has the same probability of firing.

$R_3$: If two or more transitions become enabled at exactly the same time, and the firing of any of one transition will not disable the other transitions, then they fire at the same time.

$R_4$: If two or more transitions become enabled at exactly the same time and the firing of the transitions disables another transition, then the transition that will fire is selected randomly. The transitions have the same probability of firing.

For the application considered in this dissertation it is only necessary to illustrate the execution when only one delay condition and one guard condition are pre-enabled. The execution in this case is as follows.

(a) Using the initialization function, $\mathcal{I}$, the token and color states, $(\theta(0), x(0))$, of the DCPN at time $\tau_0 \triangleq 0$ are chosen and as well as the motions related to the token distribution $\theta(0)$.

(b) With these initial tokens and colors states, a set of transitions are pre-enabled and the motions are evolved.

(c) Suppose that a delay transition $T_D$ and a guard transition $T_G$ are pre-enabled. Using the exponential distribution suppose $\tau_1$ is chosen, and at time $\tau_2$ the motion reach the guard condition.

(d) Now, without loss of generality, suppose $\tau_1 < \tau_2$. Then the motion $x_{\tau_0}(t)$ evolves from $\tau_0$ to $\tau_1^-$, and $T_D$ is enabled and fires using the firing measure $\mathrm{F}_{T_D}(\cdot, \cdot|\theta(\tau_1^-), x_{\tau_0}(\tau_1^-))$, where $\theta(\tau_1^-) = \theta(0)$ and $x(\tau_1^-) = \lim_{t \to \tau_1} x_{\tau_0}(t)$. With this firing measure the new token and color states, $(\theta(\tau_1), x_{\tau_1}(\tau_1))$, are chosen.

(e) At time $\tau_1$ suppose that the pre-enabled transitions are $T_D'$ and $T_G$, and that the guard transition is enabled at time $\tau_2$ before the delay transition $T_D'$ is enabled.

(f) At time $\tau_2$ the transition $T_G$ fires using the firing measure $\mathrm{F}_{T_G}(\cdot, \cdot|\theta(\tau_2^-), x_{\tau_1}(\tau_2^-))$.

(g) Finally, the process repeats from the beginning.

From the execution of the DCPN the following elements are generated: the stopping times $\tau_k$ and states of the DCPN $(\theta(t), x_{\tau_k}(t))$, where $\theta(t) = \theta(\tau_{k-1})$. $x_{\tau_{k-1}}(t)$ is the value of the color at time $t$ with initial condition $x_{\tau_{k-1}}(\tau_{k-1})$ and $t \in [\tau_{k-1}, \tau_k)$. Each execution of the DCPN generates a sample path which is generated randomly by the exponential distribution of the delay transitions and firing measures of the transitions. Next, the nature of the stochastic process corresponding to these sample paths is presented.

### III.2.1 The Stochastic Process Nature of a DCPN

The execution of a DCPN generates a sample path from a stochastic process which is uniquely defined as follows. The process state at time $t$ is defined by the numbers of tokens in each place and the colors of these tokens. To make this characterization of a DCPN equivalent to a stochastic process, define the set of random vectors as follows.

**Definition 15.** The *stochastic process generated by the execution of a DCPN* is the set of random vectors $(\boldsymbol{\theta}(t), \boldsymbol{x}_s(t))$ indexed by $s, t \in \mathbb{R}^+$ such that

$$(\boldsymbol{\theta}(t), \boldsymbol{x}_s(t)) : \Omega \to \bigcup_{j \in 2^{|\mathcal{P}|}} \left\{ \{\theta_j\} \times \mathbb{R}^{n(\theta_j)} \right\}.$$

The sample of random variable $\boldsymbol{\theta}(t)$ at time $t$ is $\theta(t) = (v_1(t), ..., v_{|\mathcal{P}|}(t))$ and the sample of random variable $\boldsymbol{x}_s(t)$ is $x_s(t) = (\mathsf{z}_{st}^1(\boldsymbol{z}_1), ..., \mathsf{z}_{st}^{|\mathcal{P}|}(\boldsymbol{z}_{|\mathcal{P}|}))$.

For analysis, it is important to characterize the stochastic nature of a DCPN. In particular, its probability measure, $\Pr(\boldsymbol{\theta}(t), \boldsymbol{x}_s(t))$, needs to be characterized. There is no compact representation of this probability measure. It is determined through the delay time, guard condition and the transition kernel $Q_t$. The transition kernel, $Q_t$, is characterized using the *reachability graph* of a DCPN. The reachability graph is a graph with two elements, the nodes, which are the token states $\theta$, and the arrows, connecting the nodes. These arrows are labeled by the transitions that allow the transfer between nodes.

The transition kernel $Q_t(\theta', C|\theta, x)$ has the following two properties:

- For each event $(\theta', C) \in \beta\left(\bigcup_{j \in 2^{|\mathcal{P}|}} \left\{ \{\theta_j\} \times \mathbb{R}^{n(\theta_j)} \right\}\right)$, $Q_t(\theta', C|\cdot, \cdot)$ is a non-negative measurable function on $\bigcup_{j \in 2^{|\mathcal{P}|}} \left\{ \{\theta_j\} \times \mathbb{R}^{n(\theta_j)} \right\}$.
- For each $(\theta, x) \in \bigcup_{j \in 2^{|\mathcal{P}|}} \left\{ \{\theta_j\} \times \mathbb{R}^{n(\theta_j)} \right\}$, $Q_t(\cdot, \cdot|\theta, x)$ is a probability measure on $\beta\left(\bigcup_{j \in 2^{|\mathcal{P}|}} \left\{ \{\theta_j\} \times \mathbb{R}^{n(\theta_j)} \right\}\right)$.

The transition kernel $Q_t$ is characterized in terms of the DCPN, $(\mathcal{P}, \mathcal{T}, \mathcal{A}, \mathcal{N}, \mathcal{S}, \mathcal{C}, \mathsf{z}, \mathcal{G}, \mathcal{D}, \mathsf{F}, \mathcal{I})$, as follows.

(a) **Determine the set of pre-enabled transitions**. For every state $(\theta(t), x_s(t))$ determine the set of transitions that are pre-enabled using the reachability graph. The possible transitions from a node $\theta$ are given by the labels of the output arrows, which correspond to the pre-enabled transitions. Let $B_\theta$ denote the set of this pre-enabled transitions for each $\theta$.

(b) **Determine the probability that the pre-enabled transitions become enabled.** All the immediate transitions in $B_\theta$, that is, those in $B_\theta \cap \mathcal{T}_I$, become enabled with probability one. Let

$$B_\theta^G \triangleq \{B \in B_\theta |\ \forall T \in B_\theta \cap \mathcal{T}_G,\ x_s(t) \in \partial G_T\}.$$

If $B_\theta^G$ is a unitary set then the probability that the guard transition becomes enabled and fires, $P_B(\theta, x)$, is equal to one when the guard condition is true. If $B_\theta^G = \varnothing$ then only delay transitions can be pre-enabled in $(\theta, x)$. Consider the set $B_\theta^D \triangleq \{B \in B_\theta | \forall T \in B \cup \mathcal{T}_D\}$. If $B \in B_\theta^D$ consists only of one delay transition, then the probability that it becomes enabled and fires is equal to

$$p_B(\theta, x) \triangleq \frac{\delta_B(x(t))}{\sum\limits_{T \in B_\theta^D} \delta_T(x(t))}. \tag{9}$$

In this dissertation, the cases where $B_\theta^G$ and $B_\theta^D$ have more than one elements are not considered.

(c) **Determine for each pre-enabled transition whether its firing can lead to token state $\theta'$.** In the reachability graph, consider the possible ways of going from $\theta$ to $\theta'$. First, the firing of only one transition takes $\theta$ to $\theta'$. Second, the firing at the same time of $m$ transitions, $T_1, T_2, \cdots, T_m$, take $\theta$ to $\theta'$. This is denoted by $T_1 + T_2 + \cdots + T_m$. It is assumed that guard and delay transitions do not fire at the same time. Third, the firing of one or more transitions, $T_1 + T_2 + \cdots + T_m$, enables an immediate transition, $T$, which when fired takes $\theta$ to $\theta'$. This is denoted by $T_1 + T_2 + \cdots + T_m \circ T$. The set of transitions that allow one to move from $\theta$ to $\theta'$ is denoted by $\mathcal{L}_{\theta\theta'}$.

(d) **Characterization of $Q_t$.** The transition kernel $Q_t$ is given by

$$Q_t(\theta', C|\theta, x) = \sum_{L \in \mathcal{L}_{\theta\theta'}} p_{\theta', C|\theta, x, L}(\theta', C|\theta, x, L) \times p_{L|\theta, x}(L|\theta, x) \tag{10}$$

for every $\mathcal{L}_{\theta\theta'}$, where:

- The probability of enabling a transition $L$, $p_{L|\theta,x}(L|\theta, x)$, is found as follows

$$p_{L|\theta,x}(L|\theta, x) \triangleq \frac{p_L(\theta, x)}{\sum\limits_{B \in \mathcal{L}_{\theta\theta'}} p_B(\theta, x)}, \text{ if } \mathcal{L}_{\theta\theta'} \subset B_\theta. \tag{11}$$

Now if $L_1$ is an immediate transition and $L \circ L_1$ then

$$p_{L_1 \circ L|\theta,x}(L_1 \circ L|\theta, x) \triangleq p_{L|\theta,x}(L|\theta, x).$$

The others cases are not consider in this dissertation.

- The firing measure of transition $L$, $p_{\theta',C|\theta,x,L}(\theta', A|\theta, x, L)$, is defined as follows

$$p_{\theta',C|\theta,x,L}(\theta', A|\theta, x, L) = \mathrm{F}_L(\theta', C|\theta, x). \tag{12}$$

If the transitions $L_1, L_2$ fire simultaneously, denoted by $L_1 + L_2$, then

$$p_{\theta',C|\theta,x,L_1+L_2}(\theta', C|\theta, x, L_1 + L_2) =$$
$$\int 1_{(\theta',C)}(\theta'_1, \theta'_2, C_1, C_2)\mathrm{F}_{L_1}(\theta'_1, C_1|\theta, x)\mathrm{F}_{L_2}(\theta'_2, C_2|\theta, x)d(\theta'_1, \theta'_2)d(C_1, C_2),$$

where $1_{(\theta',C)}(\theta'_1, \theta'_2, C_1, C_2)$ takes the value of one if $(\theta'_1, \theta'_2) = \theta'$ and $(C_1, C_2) \in C$ otherwise zero.

If more than one pair of transitions fire simultaneously the procedure is the same. If the firing of transition $L_1$ enables the immediate transition $L$, denoted by $L_1 \circ L$, then

$$p_{\theta',C|\theta,x,L_1 \circ L}(\theta', C|\theta, x, L_1 \circ L) \triangleq$$
$$\int_{\bigcup_{j\in 2^{|\mathcal{P}|}}\{\{\theta_j\}\times\mathbb{R}^{n(\theta_j)}\}} \mathrm{F}_L(\theta', C|a)\mathrm{F}_{L_1}(a|\theta, x)d(a).$$

Next, an example is given where a DCPN model of a continuous-time Markov chain is developed and its transition matrix $Q_t$ is obtained. First, a continuous-time Markov chain (CTMC) is defined.

**Definition 16.** Let $\{\boldsymbol{x}(t), t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \Pr)$ with states in a set $E$. Then for every set of times $t_0 \leq t_1 \leq \cdots \leq t_k \leq t_{k+1}$ the following equality holds:

$$\Pr(\boldsymbol{x}(t_{k+1}) = x_{k+1}|\boldsymbol{x}(t_k) = x_k, \boldsymbol{x}(t_{k-1}) = x_{k-1}, ..., \boldsymbol{x}(t_0) = x_0) =$$
$$\Pr(\boldsymbol{x}(t_{k+1}) = x_{k+1}|\boldsymbol{x}(t_k) = x_k),\ k \geq 0.$$

In order to characterize the probability measure of the continuous-time Markov chain, it is necessary to have an initial state probability measure $\nu_0(x)$, $x \in E$ and a

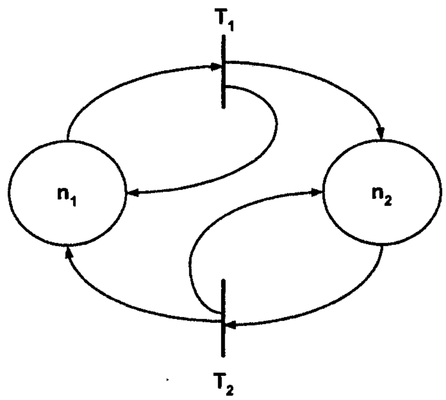


FIG. 4: DCPN model of a CTMC.

transition matrix $\mathbf{P}(t)$, where the $(i,j)$ entry, $p_{x_i x_j}(t) = \Pr(\boldsymbol{x}(t) = x_j | \boldsymbol{x}(0) = x_i)$, is the probability of transition from $x_i$ to $x_j$ within time interval of duration $t$. Finally, the transition matrix, $\mathbf{P}(t)$, is a solution of the Chapman-Kolmogorov equation $\dot{\mathbf{P}} = \mathbf{P}(t)\Lambda$, where $\Lambda$ is the transition rate matrix, that is $\mathbf{P}(t) = e^{\Lambda t}$. The following example shows how to model a continuous-time Markov chain as a DCPN.

**Example 1.** Consider the continuous time Markov chain (CTMC) with two states $\{n_1, n_2\}$, transition rate matrix $\Lambda = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$, $\lambda_1$, $\lambda_2 \in \mathbb{R}^+$, and transition matrix

$$\mathbf{P}(t) = \begin{bmatrix} \frac{\lambda_2 + \lambda_1 e^{-t(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2} & -\lambda_1 \frac{e^{-t(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2} \\ -\lambda_2 \frac{e^{-t(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2} & \frac{\lambda_1 + \lambda_2 e^{-t(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2} \end{bmatrix}.$$

Its DCPN model, shown in FIG. 4, consists of two places, $\mathcal{P} = \{n_1, n_2\}$, and two delay transitions $\mathcal{T} = \{T_1, T_2\}$. Since there are no motions associated with the CTMC states, the set of colors is empty; the initialization function is only the initial distribution $\nu$, which assigns a token to either place $n_1$ or $n_2$; the transition enabling rate functions are $\delta_{T_1} : \mathcal{C}(P(A_{\text{in}}(T_1))) \to \mathbb{R}^+$, $\delta_{T_2} : \mathcal{C}(P(A_{\text{in}}(T_2))) \to \mathbb{R}^+$, which give the parameters $\delta_{T_1}(\emptyset) \triangleq \lambda_1$ and $\delta_{T_2}(\emptyset) \triangleq \lambda_2$ of the exponential distributed random variables $\boldsymbol{D}_{T_1}$ and $\boldsymbol{D}_{T_2}$, respectively. There is not a color state. The token state, $\theta$, is just the marking of each place, i.e., $\theta(t) = (v_1(t), v_2(t))$. Note that the token state at each time $t$ takes values in $\{(1,0),(0,1)\}$. Let $\theta_1 \triangleq (1,0)$ and $\theta_2 \triangleq (0,1)$. The measurable space of the DCPN model of the Markov chain is $(\{\theta_1,\theta_2\}, \beta(\{\theta_1,\theta_2\}))$. The transitions fire according to the following transition kernels

$$\mathrm{F}_T : \{\theta_1,\theta_2\} \times \beta(\{\theta_1,\theta_2\}) \quad \rightarrow \quad [0,1],$$

where $i = 1,2$. At time $t$, the state $\theta(t)$ enables the transition $T$. In particular, for transitions $T_1$ and $T_2$, $\mathrm{F}_{T_1}(\theta_i|\theta_1(t)) \triangleq p_{n_1n_i}(t)$ and $\mathrm{F}_{T_2}(\theta_i|\theta_2(t)) \triangleq p_{n_2n_i}(t)$, respectively, where $p_{n_1n_1}(t) \triangleq \frac{\lambda_2+\lambda_1 e^{-t(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2}$, $p_{n_1n_2}(t) \triangleq 1 - p_{n_1n_1}(t)$, $p_{n_2n_1}(t) \triangleq -\lambda_2\frac{e^{-t(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2}$, and $p_{n_2n_2}(t) \triangleq 1 - p_{n_2n_1}(t)$.

The execution of the DCPN model of the continuous-time Markov chain is as follows:

(a) A random number is generated with the initial distribution $\nu$ to place a token in $n_1$ or $n_2$. Suppose that the token is given to place $n_1$.

(b) The token distribution is $\theta_1$ and transition $T_1$ is pre-enabled.

(c) Using the exponential distribution with parameter $\lambda_1$ a time $\sigma_1$ is chosen.

(d) The token state remains at $\theta_1$ until $\tau_1 = \sigma_1$ when the transition $T_1$ is enabled. Using the measure $\mathrm{F}_{T_1}(\theta_i|\theta_1(\tau_1)) = p_{n_1n_i}(\tau_1)$, the token state is chosen. Suppose $\theta_2$ is chosen.

(e) The transition $T_2$ is pre-enabled, and using the exponential distribution with parameter $\lambda_2$ a time $\sigma_2$ is chosen and $\tau_2 = \tau_1 + \sigma_2$.

(f) At time $\tau_2$ the transition $T_2$ is fired with firing measures $\mathrm{F}_{T_2}(\theta_i|\theta_2(\sigma_2)) = p_{n_2n_i}(\sigma_2)$, and the execution repeats from the beginning.

TABLE I is used to form the transition kernel $Q_t : \{\theta_1,\theta_2\} \times \beta(\{\theta_1,\theta_2\}) \rightarrow [0,1]$ as follows:

(a) **Determine which transitions are pre-enabled**. The pre-enabled transitions depend on the token state. Since there are two possible states, there are two pre-enabled delay transitions $B^D_{\theta_1} = \{T_1\}$ and $B^D_{\theta_2} = \{T_2\}$.

| | $\theta_1$ | $\theta_2$ |
|---|---|---|
| $\theta_1$ | $T_1$ | $T_1$ |
| $\theta_2$ | $T_2$ | $T_2$ |

TABLE I: Reachability table for a CTMC

(b) **Determine the probability that the pre-enabled transitions becomes enabled.** Let $B^D_{\theta_1} = \{T_1\}$ and $B^D_{\theta_2} = \{T_2\}$. Then using Equation (9) the probabilities are

$$p_{T_1}(\theta_1) = \frac{\delta_{T_1}(\emptyset)}{\sum\limits_{T \in B^D_{\theta_1}} \delta_T(\emptyset)} = \frac{\lambda_1}{\lambda_1} = 1$$

and $p_{T_2}(\theta_2) = 1$.

(c) **Determine for each pre-enabled transition whether its firing can possibly lead to a token state.** Since from $\theta_1$ and $\theta_2$ it is possible to reach $\theta_1$ and $\theta_2$, then $\mathcal{L}_{\theta_1\theta_1} = \mathcal{L}_{\theta_1\theta_2} = \{T_1\}$ and $\mathcal{L}_{\theta_2\theta_1} = \mathcal{L}_{\theta_2\theta_2} = \{T_2\}$.

(d) **To characterize $Q_t$, use the previous two steps and the set of firing measures.** Using Equation (10) and the fact that there is no color, it follows that

$$Q_t(\theta_i|\theta_j) = \sum_{L \in \mathcal{L}_{\theta_j\theta_i}} p_{\theta_i|\theta_j,L}(\theta_i|\theta_j, L) \cdot p_{L|\theta_j}(L|\theta_j).$$

Using Equation (11), the probabilities $p_{L|\theta_j}(L|\theta_j)$ for the $L$ elements of the set are

$$\begin{aligned} p_{T_1|\theta_1}(T_1|\theta_1) &= \frac{p_{T_1}(\theta_1)}{\sum\limits_{B \in \mathcal{L}_{\theta_1\theta_1}} p_B(\theta_1)} = \frac{p_{T_1}(\theta_1)}{p_{T_1}(\theta_1)} = 1 \\ p_{T_2|\theta_2}(T_2|\theta_2) &= 1. \end{aligned}$$

Using Equation (12), the probabilities $p_{\theta_i|\theta_j,L}(\theta_i|\theta_j, L)$ are

$$\begin{aligned} p_{\theta_1|\theta_1,T_1}(\theta_1|\theta_1, T_1) &= \mathrm{F}_{T_1}(\theta_1|\theta_1(t)) = p_{n_1n_1}(t) = \frac{\lambda_2 + \lambda_1 e^{-t(\lambda_1+\lambda_2)}}{\lambda_1 + \lambda_2} \\ p_{\theta_2|\theta_1,T_1}(\theta_2|\theta_1, T_1) &= 1 - p_{\theta_1|\theta_1,T_1}(\theta_1|\theta_1, T_1) \end{aligned}$$

$$\begin{aligned} p_{\theta_1|\theta_2,T_2}(\theta_1|\theta_2,T_2) &= \mathrm{F}_{T_2}(\theta_1|\theta_2(t)) = p_{n_2n_1}(t) = \lambda_2 \frac{1-e^{-t(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2} \text{ and} \\ p_{\theta_2|\theta_2,T_2}(\theta_2|\theta_2,T_2) &= 1 - p_{\theta_1|\theta_2,T_2}(\theta_1|\theta_2,T_2). \end{aligned}$$

Using these probabilities, it follows that

$$Q_t \triangleq [Q_t(\theta_i|\theta_j)] = \begin{bmatrix} \frac{\lambda_2+\lambda_1 e^{-t(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2} & -\lambda_1 \frac{e^{-t(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2} \\ -\lambda_2 \frac{e^{-t(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2} & \frac{\lambda_1+\lambda_2 e^{-t(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2} \end{bmatrix}.$$

This example provides a DCPN representation of a continuous-time Markov chain. It shows a stochastic process characterization of the DCPN, including the methodology for obtaining the transition matrix of a continuous-time Markov chain.

Next a DCPN model for a sampled-data system will be presented.

## III.3 DCPN REPRESENTATION OF SAMPLED-DATA SYSTEMS WITH RECOVERY ALGORITHMS

The DCPN representation of the stochastically switched sampled-data system in FIG. 3 is analyzed in this section, The advice from [3, 19] was very important for the development of this section. The closed-loop system is formed by a continuous-time plant interconnected with a discrete-time controller. Because the system is in a harsh environment, the controller is subject to fail, and when the controller fails, it is in an upset mode. This behavior is modeled by a DCPN as shown in FIG. 5. This DCPN has the following elements:

(a) The set of places $\mathcal{P} = \{n_1, n_2, \Sigma_N, \Sigma_R\}$ model the upset, no upset, nominal mode and upset modes, respectively.

(b) The set of transitions $\mathcal{T} = \{T_G, T_1, T_2, T_3\}$, where $T_G$ is a guard transition, modeling the recovery frames, $T_i$, $i = 1, 2$ are delay transitions, modeling the time when an upset or no upset occurs, and $T_3$ is an immediate transition, modeling the jump from nominal to upset mode.

(c) The set of arcs, $\mathcal{A}$, and the set of nodes functions, $\mathcal{N}$, are shown in FIG. 5.

(d) The set of colors is $\mathcal{S} = \{\mathbb{R}^{n_{x_p}+n_{x_c}+1}, \mathbb{R}^{n_{x_p}+n_{x_c}+2}, \varnothing\}$, where $n_{x_p}$ and $n_{x_c}$ are the dimensions of the state of the plant and the state of the controller, respectively.

(e) The color function $\mathcal{C} : \mathcal{P} \to \mathcal{S}$ is defined as follows: To the place $\Sigma_N$ assign the color $\mathbb{R}^{n_{xp}+n_{xc}+1}$, to the place $\Sigma_R$ assign the color $\mathbb{R}^{n_{xp}+n_{xc}+2}$, and the places $n_1$ and $n_2$ have no colors.

(f) The motions are defined as follows:

$$\begin{aligned} \mathcal{C}(\Sigma_N) &\mapsto \mathbf{z}^1_{st}(z) \in \mathbb{R}^{n_{xp}+n_{xc}+1} \\ \mathcal{C}(\Sigma_R) &\mapsto \mathbf{z}^2_{st}(z) \in \mathbb{R}^{n_{xp}+n_{xc}+2}, \end{aligned}$$

where $\mathbf{z}^1_{st}(z)$ is the motion modeling the nominal closed-loop system in a cross product with a sojourn time $\sigma \in \mathbb{R}^+$. $\mathbf{z}^2_{st}(z)$ is the motion modeling the upset closed-loop system in a cross product with two sojourn times as follows. $\mathbf{z}^2_{st}(z)$ take the values of $z(s, n_a), z(s, n_b)$ or $z(s, 0)$, which model the motions of the recovery closed-loop system with two sojourn times $(n_a, \sigma)$, $(n_b, \sigma)$ and $(0, \sigma)$, respectively. $\mathbf{z}^2_{st}(z)$ take the values of $\mathbf{z}^2_{st}(z(s, n_a))$ or $\mathbf{z}^2_{st}(z(s, n_b))$, which model the motions of the recovery closed-loop system with initial conditions $z(s, n_a)$, $z(s, n_b)$ and with two sojourn times $(n_a - (t-s), \sigma - (t-s))$, and $(n_b - (t - s), \sigma - (t-s))$, respectively. The sojourn time $\sigma$ is a sample of a exponential distribution with parameter $\lambda_1$ or $\lambda_2$, the sojourn times $n_a, n_b$ are the number of recovery cycles. Finally the places $n_1$ and $n_2$ have no motions.

(g) The guard condition is defined as follows: $\mathcal{G}_{T_G} : \mathcal{C}(P(A_{\text{in}}(T_G))) = \mathbb{R}^{n_{xp}+n_{xc}+2} \to \{0, 1\}$, $\mathcal{G}_{T_G}(\mathbf{z}^2_{st})$ evaluates to 1 if $\mathbf{z}^2_{st} \in \partial G_{T_G} = \{(y'\ t)' | y \in \mathbb{R}^{n_{xp}+n_{xc}+1} \text{ and } t = 0\}$, and $G_{T_G} = \{(y'\ t)' | y \in \mathbb{R}^{n_{xp}+n_{xc}+1} \text{ and } t \in \mathbb{R}^+\}$. In others words, when the sojourn time reach zero the guard condition is true.

(h) The transition enabling rate functions are $\delta_{T_1} : \mathcal{C}(P(A_{\text{in}}(T_1))) \to \mathbb{R}^+$, $\delta_{T_2} : \mathcal{C}(P(A_{\text{in}}(T_2))) \to \mathbb{R}^+$, where $\delta_{T_1}(\emptyset) \triangleq \lambda_1$ and $\delta_{T_2}(\emptyset) \triangleq \lambda_2$, respectively.

(i) The initialization function $\mathcal{I}$ puts a token in the places $\Sigma_N$ and $n_1$, and the colors are $\mathcal{C}(\Sigma_N) = \mathbf{z}^1_{st}(z)$ and $\mathcal{C}(n_1) = \emptyset$.

The token state is $\theta(t) = \{v_1(t), v_2(t), v_3(t), v_4(t)\}$, where $\theta(t)$ denotes the tokens in places $n_1, n_2, \Sigma_N$, and $\Sigma_R$, respectively. The possible values of $\theta(t)$ are $\theta_1 = (1, 0; 1, 0), \theta_2 = (1, 0; 0, 1), \theta_3 = (0, 1; 1, 0)$ and $\theta_4 = (0, 1; 0, 1)$. The color state is $x(t) = (\mathbf{z}^1_{st}, \mathbf{z}^2_{st})$, where $\mathbf{z}^1_{st} \in \mathbb{R}^{n_{xp}+n_{xc}+1}$ and $\mathbf{z}^2_{st} \in \mathbb{R}^{n_{xp}+n_{xc}+2}$. $\mathbf{z}^2_{st}$ can take the values $z(s, n_a), z(s, n_b), z(s, 0), \mathbf{z}^2_{st}(z(s, n_a))$, and $\mathbf{z}^2_{st}(z(s, n_b))$.

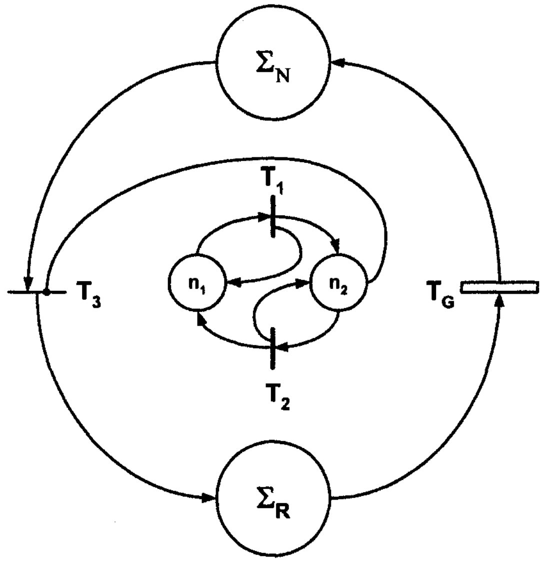


FIG. 5: DCPN model of a sampled-data system.

The measurable space of the DCPN is $\left(\bigcup_{i=1}^{4}\{\{\theta_i\}\times\mathbb{R}^{n(\theta_i)}\},\beta\left(\bigcup_{i=1}^{4}\{\{\theta_i\}\times\mathbb{R}^{n(\theta_i)}\}\right)\right)$, where $n(\theta_1)=n(\theta_3)=n_{x_p}+n_{x_c}+1$, $n(\theta_2)=n(\theta_4)=n_{x_p}+n_{x_c}+2$ and the $\sigma$-algebra defined on this space is denoted as $\beta\left(\bigcup_{i=1}^{4}\{\{\theta_i\}\times\mathbb{R}^{n(\theta_i)}\}\right)$. From the description of the sampled-data system with recovery algorithm, it follows that the states of the DCPN are as follows:

- No upset and nominal closed-loop system is denoted by $a_1 \triangleq (\theta_1, \mathbf{z}_{st}^1(z))$.
- Upset, recovery closed-loop system and sojourn time $n_a$ is denoted by $a_2 \triangleq (\theta_4, z(s, n_a))$.
- Upset, recovery closed-loop system and sojourn time $n_b$ is denoted by $a_3 \triangleq (\theta_4, z(s, n_b))$.
- No upset, recovery closed-loop system and sojourn time evolving from $n_b$ to zero is denoted by $a_4 \triangleq (\theta_2, \mathbf{z}_{st}^2(z(s, n_b)))$.

- No upset, recovery closed-loop system and sojourn time evolving from $n_a$ to zero is denoted by $a_5 \triangleq (\theta_2, \mathbf{z}^2_{st}(z(s,n_a)))$.
- Upset, recovery closed-loop system and sojourn time evolving from $n_a$ to zero is denoted by $a_6 \triangleq (\theta_4, \mathbf{z}^2_{st}(z(s,n_a)))$.
- Upset, recovery closed-loop system and sojourn time evolving from $n_b$ to zero is denoted by $a_7 \triangleq (\theta_4, \mathbf{z}^2_{st}(z(s,n_b)))$.
- No upset, recovery closed-loop system and 0 sojourn time is denoted by $a_8 \triangleq (\theta_2, z(s,0))$.
- Finally, upset, recovery closed-loop system and 0 sojourn time, $a_9 \triangleq (\theta_4, z(s,0))$.

Now that DCPN states have been identified the firing measures are described as follows. If the token states are $\theta_1$ and $\theta_2$, then the firing measure of $T_1$ is defined as follows

$$\begin{aligned}
\mathrm{F}_{T_1} : \bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\} \times \beta(\bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\}) &\rightarrow [0,1] \\
\mathrm{F}_{T_1}(\theta_j, \mathbf{z}^1_{\tau t}(z))|\theta_1, \mathbf{z}^1_{s\tau}(z) &\triangleq p_{n_1\cdot}(\tau), \\
\mathrm{F}_{T_1}(\theta_j, \mathbf{z}^2_{\tau t}|\theta_2, z_2'(\tau)) &\triangleq p_{n_1\cdot}(\tau),
\end{aligned}$$

where $z_2'(\tau)$ take values $\mathbf{z}^2_{s\tau}(z(s,n_a))$ and $\mathbf{z}^2_{s\tau}(z(s,n_b))$, and $\sum_{i=1}^{2} p_{n_1 n_i}(\tau) = 1$. When the token state is $\theta_4$, then the transition is $T_2$ with firing measure given by

$$\begin{aligned}
\mathrm{F}_{T_2} : \bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\} \times \beta(\bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\}) &\rightarrow [0,1] \\
\mathrm{F}_{T_2}(\theta_2, \mathbf{z}^2_{\tau t})|\theta_4, z_2(\tau)) &\triangleq p_{n_2 n_1}(\tau) \\
\mathrm{F}_{T_2}(\theta_4, \mathbf{z}^2_{\tau t})|\theta_4, z_2(\tau)) &\triangleq p_{n_2 n_2}(\tau),
\end{aligned}$$

where $p_{n_2 n_1}(\tau) + p_{n_2 n_2}(\tau) = 1$, $z_2(\tau)$ takes the values $z(\tau, n_a), z(\tau, n_b), \mathbf{z}^2_{s\tau}(z(s,n_a))$, and $\mathbf{z}^2_{s\tau}(z(s,n_b))$. When the token state is $\theta_3$, the pre-enabled transition is $T_3$. Its firing measures is

$$\begin{aligned}
\mathrm{F}_{T_3} : \bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\} \times \beta(\bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\}) &\rightarrow [0,1] \\
\mathrm{F}_{T_3}(\theta_4, z(\tau, n_a)|\theta_3, \mathbf{z}^1_{s\tau}(z)) &\triangleq p_{n_a}, \\
\mathrm{F}_{T_3}(\theta_4, z(\tau, n_b)|\theta_3, \mathbf{z}^1_{s\tau}(z)) &\triangleq p_{n_b},
\end{aligned}$$

where $p_{n_a} + p_{n_b} = 1$. Finally, when the token states are $\theta_2$ and $\theta_4$, the pre-enabled transition is $T_G$. Its firing measure is

$$\begin{aligned}
\mathrm{F}_{T_G} : \bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\} \times \beta(\bigcup_{i=1}^{4}\{\{\theta_i\} \times \mathbb{R}^{n(\theta_i)}\}) &\rightarrow [0,1] \\
\mathrm{F}_{T_G}(\theta_1, \mathrm{z}^1_{\tau t}(z)|\theta_2, z(\tau,0)) &\triangleq 1, \\
\mathrm{F}_{T_G}(\theta_3, \mathrm{z}^1_{\tau t}(z)|\theta_4, z(\tau,0)) &\triangleq 1.
\end{aligned}$$

Next the execution of the DCPN model of the sampled-data system is presented.

### III.3.1 DCPN Execution of Sampled-data Systems with Recovery Algorithms

The execution of a DCPN yields a series of increasing stopping times. The token and color states are as follows.

- The initialization function $\mathcal{I}$ with probability one gives a token to the places $\Sigma_N$ and $n_1$. This means that the initial DCPN state is $a_1 = (\theta_1, \mathrm{z}^1_{0t}(z(0)))$.
- Because the DCPN is in state $a_1$, the delay transition $T_1$ is pre-enabled.
- Using the exponential distribution with parameter $\lambda_1$, a time $\tau_1$ is chosen.
- The DCPN remains at state $a_1$ until $t = \tau_1$, when the transition $T_1$ is enabled and fires using the measure $\mathrm{F}_{T_1}(\theta_j, \mathrm{z}^1_{\tau_1 t}(z)|\theta_1, \mathrm{z}^1_{0\tau_1}(z(0)))$. Suppose the state $(\theta_1, \mathrm{z}^1_{\tau_1 t}(z))$ is chosen.
- At time $\tau_1$ the delay transition $T_1$ is pre-enabled, and using an exponential distribution with parameter $\lambda_1$ a time $\sigma_1$ is chosen.
- The DCPN remains at state $a_1$ until $\tau_2 = \tau_1 + \sigma_1$, when the transition $T_1$ is enabled and fires with firing measure $\mathrm{F}_{T_1}(\theta_j, \mathrm{z}^1_{\tau_2 t}(z_1)|\theta_1, \mathrm{z}^1_{\tau_1\tau_2}(z))$. Suppose $\theta_3$ is chosen. Then the immediate transition $T_3$ is enabled and fires with firing measure $\mathrm{F}_{T_3}(\theta_4, z(\tau_2, \cdot)|\theta_3, \mathrm{z}^1_{\tau_2\tau_2}(z_1))$. Suppose that the state $a_2 = (\theta_4, z(\tau_2, n_a))$ is selected.
- At this moment $T_G$ and $T_2$ are pre-enabled. The transition $T_G$ is enabled after $n_a$ sojourn times, and $T_2$ is enabled after $\sigma_2$, which is chosen using the exponential distribution with parameter $\lambda_2$.

- The DCPN remains at state $a_2$ until $\tau_3 = \tau_2 + \sigma_2$, when the delay transition $T_2$ is enabled and fires with firing measure $\mathrm{F}_{T_2}(\theta_j, C|\theta_4, \mathbf{z}^2_{\tau_2\tau_3}(z(\tau_2, n_a)))$. Suppose $a_5$ is chosen.

- This process go on until time $\tau_2 + n_a$, when the guard transition is enabled and fires with firing measures $\mathrm{F}_{T_G}(\theta_1, \mathbf{z}^1_{(\tau_2+n_a)t}(z)|\theta_2, \mathbf{z}^2_{\tau_2(\tau_2+n_a)}(z(\tau_2, n_a))) = 1$ or $\mathrm{F}_{T_G}(\theta_3, \mathbf{z}^1_{(\tau_2+n_a)t}(z)|\theta_4, \mathbf{z}^2_{\tau_2(\tau_2+n_a)}(z(\tau_2, n_a))) = 1$, which depends on whether the DCPN state is in $\theta_2$ or $\theta_4$.

- Finally, the process repeats from the beginning.

Now that a sample has been defined, the transition matrix, $Q_t$, can be characterized as follows:

(a) When the token states are $\theta_i$ for $i = 1, 2, 3, 4$, the following transitions are pre-enabled

$$B_{\theta_1} = \{T_1\},\ B_{\theta_2} = \{T_1, T_G\},\ B_{\theta_3} = \{T_2, T_3\},\ B_{\theta_4} = \{T_2, T_G\}.$$

(b) Determine for each pre-enabled transition the probability with which it is enabled and fires. Let $B^D_{\theta_1} = \{T_1\}$, $B^D_{\theta_2} = \{T_1\}$, $B^D_{\theta_3} = \{T_2\}$ and $B^D_{\theta_4} = \{T_2\}$. Using Equation (9), the probabilities are then

$$P_{T_1}(\theta_1) = \frac{\lambda_1}{\lambda_1} = 1,\ P_{T_1}(\theta_2) = P_{T_2}(\theta_3) = P_{T_2}(\theta_4) = 1.$$

Let $B^G_{\theta_1} = \varnothing$, $B^G_{\theta_2} = \{T_G : z(s,0) \in \partial G_{T_G}\}$, $B^G_{\theta_3} = \varnothing$, $B^G_{\theta_4} = \{T_G : z_2(s,0) \in \partial G_{T_G}\}$. Using the fact that these guard condition became true, it follows that

$$P_{T_G}(\theta_2, z(s,0)) = 1,\ P_{T_G}(\theta_4, z(s,0)) = 1.$$

(c) Determine the sets of pre-enabled transitions that lead to another token state.

$$\begin{aligned}
&\mathcal{L}_{\theta_1\theta_1} = \{T_1\},\ \mathcal{L}_{\theta_1\theta_2} = \emptyset,\ \mathcal{L}_{\theta_1\theta_3} = \emptyset,\ \mathcal{L}_{\theta_1\theta_4} = \{T_3 \circ T_1\},\\
&\mathcal{L}_{\theta_2\theta_1} = \{T_G\},\ \mathcal{L}_{\theta_2\theta_2} = \{T_1\},\ \mathcal{L}_{\theta_2\theta_3} = \emptyset,\ \mathcal{L}_{\theta_2\theta_4} = \{T_1\},\\
&\mathcal{L}_{\theta_4\theta_1} = \emptyset,\ \mathcal{L}_{\theta_4\theta_2} = \{T_2\},\ \mathcal{L}_{\theta_4\theta_3} = \emptyset,\ \mathcal{L}_{\theta_4\theta_4} = \{T_2, T_3 \circ T_G\}.
\end{aligned}$$

(d) To characterize $Q_\tau$, use the previous two steps, the set of firing measures and Equation (10). According to the execution, it is possible to go from the state

$a_1$ to the states $a_1, a_2$ and $a_3$. Using Equation (10), it follows that

$$\begin{aligned}
Q_\tau(\theta_1, \mathrm{z}^1_{\tau t}|\theta_1, \mathrm{z}^1_{s\tau}) &= \sum_{L\in\mathcal{L}_{\theta_1\theta_1}} P_{\theta_1,C|\theta_1,\mathrm{z}^1_{s\tau},L}(\theta_1, A|\theta_1, \mathrm{z}^1_{s\tau}, L) \\
&\quad \times P_{L|\theta_1,\mathrm{z}^1_{s\tau}}(L|\theta_1, \mathrm{z}^1_{s\tau}) \\
Q_\tau(\theta_1, \mathrm{z}^1_{\tau t}|\theta_1, \mathrm{z}^1_{s\tau}) &= P_{\theta_1,\mathrm{z}^1_{\tau t}|\theta_1,\mathrm{z}^1_{s\tau},T_1}(\theta_1, \mathrm{z}^1_{\tau t}|\theta_1, \mathrm{z}^1_{s\tau}, T_1) \times P_{T_1|\theta_1,\mathrm{z}^1_{s\tau}}(T_1|\theta_1, \mathrm{z}^1_{s\tau}) \\
Q_\tau(\theta_1, \mathrm{z}^1_{\tau(t)}|\theta_1, \mathrm{z}^1_{s\tau}) &= \mathrm{F}_{T_1}(\theta_1, \mathrm{z}^1_{\tau t}|\theta_1, \mathrm{z}^1_{s\tau}) \times \frac{P_{T_1}(\theta_1, \mathrm{z}^1_{s\tau})}{P_{T_1}(\theta_1, \mathrm{z}^1_{s\tau})} \\
Q_\tau(\theta_1, \mathrm{z}^1_{\tau t}|\theta_1, \mathrm{z}^1_{s\tau}) &= p_{n_1n_1}(\tau) \\
Q_\tau(\theta_4, z(\tau, n_a)|\theta_1, \mathrm{z}^1_{s\tau}) &= P_{\theta_4,z(\tau,n_a)|\theta_1,\mathrm{z}^1_{s\tau}),T_3\circ T_1}(\theta_4, z(\tau, n_a)|\theta_1, \mathrm{z}^1_{s\tau}, T_3 \circ T_1) \\
&\quad \times P_{T_3\circ T_1|\theta_1,\mathrm{z}^1_{s\tau}}(T_3 \circ T_1|\theta_1, \mathrm{z}^1_{s\tau}) \\
Q_\tau(\theta_4, z_2(\tau, n_a)|\theta_1, \mathrm{z}^1_{s\tau})) &= \mathrm{F}_{T_3}(\theta_4, z_2(\tau, n_a)|\theta_3, \mathrm{z}^1_{\tau\tau})\mathrm{F}_{T_1}(\theta_3, \mathrm{z}^1_{\tau\tau}|\theta_1, \mathrm{z}^1_{s\tau}) \\
&\quad \times \frac{P_{T_1}(\theta_1, \mathrm{z}^1_{s\tau})}{P_{T_1}(\theta_1, \mathrm{z}^1_{s\tau})} \\
Q_\tau(\theta_4, z_2(\tau, n_a)|\theta_1, \mathrm{z}^1_{s\tau}) &= p_{n_a}p_{n_1n_2}(\tau) \\
Q_\tau(\theta_4, z_2(\tau, n_b)|\theta_1, \mathrm{z}^1_{s\tau}) &= p_{n_b}p_{n_1n_2}(\tau).
\end{aligned}$$

It is possible to go from state $a_2$ to either $a_5$ or $a_6$. Similarly, it is possible to go from $a_3$ to either $a_4$ or $a_7$. Thus,

$$\begin{aligned}
Q_\tau(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) &= P_{\theta_2,\mathrm{z}^2_{\tau t}|\theta_4,z_2(\tau),T_2}(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau), T_2) \\
&\quad \times P_{T_2|\theta_4,z_2(\tau)}(T_2|\theta_4, z_2(\tau)) \\
Q_\tau(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) &= \mathrm{F}_{T_2}(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) = p_{n_2n_1}(\tau), \\
Q_\tau(\theta_4, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) &= \mathrm{F}_{T_2}(\theta_4, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) = p_{n_2n_2}(\tau),
\end{aligned}$$

and it is possible to go from state $a_6$ either $a_5$ or $a_6$. Similarly, it is possible to go from $a_7$ to either $a_5$ or $a_7$. Therefore,

$$\begin{aligned}
Q_\tau(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) &= P_{\theta_2,\mathrm{z}^2_{\tau t}|\theta_4,z_2(\tau),T_2}(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau), T_2) \\
&\quad \times P_{T_2|\theta_4,z_2(\tau)}(T_2|\theta_4, z_2(\tau)) \\
Q_\tau(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) &= \mathrm{F}_{T_2}(\theta_2, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) = p_{n_2n_1}(\tau), \\
Q_\tau(\theta_4, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) &= \mathrm{F}_{T_2}(\theta_4, \mathrm{z}^2_{\tau t}|\theta_4, z_2(\tau)) = p_{n_2n_2}(\tau),
\end{aligned}$$

where $z_2(\tau)$ take the values $z(\tau, n_a), z(\tau, n_b), \mathrm{z}^2_{s\tau}(z(s, n_a))$ and $\mathrm{z}^2_{s\tau}(z(s, n_b))$.

It is possible to go from state $a_5$ to either $a_5$ or $a_6$.

$$\begin{aligned}
Q_\tau(\theta_2, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) &= P_{\theta_2,\mathrm{z}^2_{\tau t}|\theta_2,\mathrm{z}^2_{s\tau}(z(s,n_a)),T_1}(\theta_2, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a)), T_1) \\
&\quad \times P_{T_1|\theta_2,\mathrm{z}^2_{s\tau}(z(s,n_a))}(T_1|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) \\
Q_\tau(\theta_2, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) &= \mathrm{F}_{T_1}(\theta_2, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) = p_{n_1 n_1}(\tau) \\
Q_\tau(\theta_4, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) &= P_{\theta_4,\mathrm{z}^2_{\tau t}|\theta_2,\mathrm{z}^2_{s\tau}(z(s,n_a)),T_1}(\theta_4, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a)), T_1) \\
&\quad \times P_{T_1|\theta_2,\mathrm{z}^2_{s\tau}(z(s,n_a))}(T_1|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) \\
Q_\tau(\theta_4, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) &= \mathrm{F}_{T_1}(\theta_4, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) = p_{n_1 n_2}(\tau).
\end{aligned}$$

Similarly, for going from $a_4$ to either $a_4$ or $a_7$, it follows that

$$\begin{aligned}
Q_\tau(\theta_2, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) &= p_{n_1 n_1}(\tau), \\
Q_\tau(\theta_4, \mathrm{z}^2_{\tau t}|\theta_2, \mathrm{z}^2_{s\tau}(z(s,n_a))) &= p_{n_1 n_2}(\tau).
\end{aligned}$$

It is possible to go from $a_8$ to $a_1$. Thus,

$$\begin{aligned}
Q_\tau(\theta_1, \mathrm{z}^1_{\tau t}|\theta_2, z(\tau,0)) &= P_{\theta_1,\mathrm{z}^1_{\tau t}|\theta_2,z(\tau,0),T_G}(\theta_1, \mathrm{z}^1_{\tau t}|\theta_2, z(\tau,0), T_G) \\
&\quad \times P_{T_G|\theta_2,z(\tau,0)}(T_G|\theta_2, z(\tau,0)) \\
Q_\tau(\theta_1, \mathrm{z}^1_{\tau t}|\theta_2, z(\tau,0)) &= \mathrm{F}_{T_G}(\theta_1, \mathrm{z}^1_{\tau t}|\theta_2, z(\tau,0))\frac{P_{T_G}(\theta_2, z(\tau,0))}{P_{T_G}(\theta_2, z(\tau,0))} = 1.
\end{aligned}$$

Finally, it is possible to go from $a_9$ to either $a_2$ or $a_3$. Therefore,

$$\begin{aligned}
Q_\tau(\theta_4, z(\tau,n_a)|\theta_4, z(\tau,0)) &= P_{\theta_4,z(\tau,n_a)|\theta_4,z(\tau,0),T_3\circ T_G}(\theta_4, z(\tau,n_a)|\theta_4, z(\tau,0), T_3 \circ T_G) \\
&\quad \times P_{T_G|\theta_4,z(\tau,0)}(T_G|\theta_4, z(t,0)) \\
Q_\tau(\theta_4, z(\tau,n_a)|\theta_4, z(\tau,0)) &= \mathrm{F}_{T_3}(\theta_4, z(\tau,n_a)|\theta_3, z(\tau,0))\mathrm{F}_{T_G}(\theta_3, z(\tau,0)|\theta_4, z(\tau,0)) \\
Q_\tau(\theta_4, z(\tau,n_a)|\theta_4, z(\tau,0)) &= p_{n_a} \\
Q_\tau(\theta_4, z(\tau,n_b)|\theta_4, z(\tau,0)) &= p_{n_b}.
\end{aligned}$$

From this it is possible to define a matrix $Q_\tau$ whose elements are $Q_\tau(a_j|a_i)$, where $a_i$, $a_j$ are the states of the DCPN.

## III.4 MAPPING A DCPN INTO A PIECEWISE DETERMINISTIC MARKOV PROCESS (PDP)

Next the DCPN model is mapped into a piecewise deterministic Markov process (PDP). First a PDP is defined as in [16,17]. Let $K$ be a countable set denoting the possible modes of operation and consider a mapping $d : K \to \mathbb{N}$. If $\{E_\theta \subseteq \mathbb{R}^{d(\theta)} : \theta \in K\}$ is a collection of open subsets then the state space of the PDP is defined by

$$\mathbb{E} \triangleq \bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}.$$

For each $\theta \in K$ there is a flow $\Phi_\theta : \mathbb{R}^+ \times E_\theta \to E_\theta$ with initial condition $\Phi_\theta(0, \omega) = \omega$, $\omega \in E_\theta$, satisfying the semigroup property $\Phi_\theta(t+s, \omega) = \Phi_\theta(t, \Phi_\theta(s, \omega))$ for every $0 < t + s < \infty$. For each $\theta \in K$ and $\Phi_\theta(t, \omega) \in E_\theta$ there is a function

$$\begin{aligned} \lambda : \bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\} &\to \mathbb{R}^+ \\ (\theta, \Phi_\theta(t, \omega)) &\mapsto \lambda(\theta, \Phi_\theta(t, \omega)) \in \mathbb{R}^+. \end{aligned}$$

Let $\beta\big(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}\big)$ denote the $\sigma$-algebra defined on $\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}$. For each $\theta \in K$ and $\Phi_\theta(t, \omega) \in E_\theta$ there is a transition kernel

$$\begin{aligned} \mathrm{R}_t : \bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\} \times \beta\Big(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}\Big) &\to [0,1] \\ ((\theta, \Phi_\theta(t, \omega)), (\tilde{\theta}, C)) &\mapsto \mathrm{R}_t(\tilde{\theta}, C | \theta, \Phi_\theta(t, \omega)), \end{aligned}$$

where $(\tilde{\theta}, C) \in \beta\big(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}\big)$.

To start the definition of the PDP consider the following random variables $\{\boldsymbol{\tau}_k\}_{k \in \mathbb{N}}$, where $\boldsymbol{\tau}_k$ is a mapping from $(\Omega, \mathcal{F}, \Pr)$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, and random vectors $\{(\boldsymbol{\theta}(\tau_k), \boldsymbol{\omega}(\tau_k))\}_{k \in \mathbb{N}}$, where $(\boldsymbol{\theta}(\tau_k), \boldsymbol{\omega}(\tau_k))$ is a mapping from $(\Omega, \mathcal{F}, \Pr)$ to $(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}, \beta(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}))$. These two sets of random variables and vectors are defined in two iterative steps: Define $\boldsymbol{\tau}_0 \triangleq 0$ and let for $k = 0$, $(\boldsymbol{\theta}(\boldsymbol{\tau}_0), \boldsymbol{\omega}(\boldsymbol{\tau}_0))$ be the initial state. Then for $k \geq 1$: The random variables $\boldsymbol{\tau}_k$ are recursively defined as $\boldsymbol{\tau}_k \triangleq \boldsymbol{\tau}_{k-1} + \boldsymbol{\sigma}_k$, where the random variables $\boldsymbol{\sigma}_k : \Omega \to \mathbb{R}^+$ have the following density function

$$1_{\left(t - \tau_{k-1} < t_*\left(\theta(\tau_{k-1}), \omega(\tau_{k-1})\right)\right)} e^{-\int_{\tau_{k-1}}^{t} \lambda\left(\theta(\tau_{k-1}), \Phi_{\theta(\tau_{k-1})}(s - \tau_{k-1}, \omega(\tau_{k-1}))\right) ds}$$

with $1_{\left(t - \tau_{k-1} < t_*\left(\theta(\tau_{k-1}), \omega(\tau_{k-1})\right)\right)}$ take the value of 1 if $t - \tau_{k-1} < t_*\big(\theta(\tau_{k-1}), \omega(\tau_{k-1})\big)$, 0 other cases and $t_*\big(\theta(\tau_{k-1}), \omega(\tau_{k-1})\big) \triangleq \inf\{t > 0 : \Phi_{\theta_{\tau_{k-1}}}(t, \omega(\tau_{k-1})) \in \partial E_{\theta_{k-1}}\}$ and $\partial E_{\theta_{k-1}}$ is the boundary of the set $E_{\theta_{k-1}}$.

The transition measure for going from $(\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1}), \Phi_{\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})}(s - \boldsymbol{\tau}_k, \boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})))$ to $(\boldsymbol{\theta}(\boldsymbol{\tau}_k), C)$ is

$$\mathrm{R}_{\boldsymbol{\tau}_k}(\cdot | \boldsymbol{\theta}(\boldsymbol{\tau}_{k-1}), \Phi_{\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})}(\boldsymbol{\tau}_k - \boldsymbol{\tau}_{k-1}, \boldsymbol{\omega}(\boldsymbol{\tau}_{k-1})).$$

Now that the random variables $\boldsymbol{\tau}_k$. $(\boldsymbol{\theta}(\boldsymbol{\tau}_k), \boldsymbol{\omega}(\boldsymbol{\tau}_k))$ and their probabilities are defined, the PDP definition is given next.

**Definition 17.** [17] A stochastic process $(\boldsymbol{\theta}(t), \boldsymbol{\omega}(t))$ mapping from $(\Omega, \mathcal{F})$ to $(\bigcup_{\theta_K} \{\{\theta\} \times E_\theta\}, \beta(\bigcup_{\theta_K} \{\{\theta\} \times E_\theta\}))$ is a *piecewise deterministic Markov process* (PDP) if and only if

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(\tau_{k-1}) \text{ and } \boldsymbol{\omega}(t) = \Phi_{\boldsymbol{\theta}(\tau_{k-1})}(t - \tau_{k-1}, \boldsymbol{\omega}(\boldsymbol{\tau}_{k-1}))$$

for every $t \in [\boldsymbol{\tau}_{k-1}, \boldsymbol{\tau}_k)$, where $\boldsymbol{\tau}_k = \boldsymbol{\tau}_{k-1} + \boldsymbol{\sigma}_k$, $\boldsymbol{\sigma}_k$ has a density function

$$1_{\left(t-\tau_{k-1}<t_*\left(\theta(\tau_{k-1}),\omega(\tau_{k-1})\right)\right)} e^{-\int_{\tau_{k-1}}^{t} \lambda\left(\theta(\tau_{k-1}),\Phi_{\theta(\tau_{k-1})}(s-\tau_{k-1},\omega(\tau_{k-1}))\right)ds},$$

and the transition measure from $(\boldsymbol{\theta}(\tau_{k-1}), \Phi_{\theta_{k-1}}(s - \boldsymbol{\tau}_k, \boldsymbol{\omega}(\boldsymbol{\tau}_{k-1})))$ to $(\boldsymbol{\theta}(\tau_k), C)$ is

$$\mathrm{R}_{\boldsymbol{\tau}_k}(\cdot|\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1}), \Phi_{\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})}(\boldsymbol{\tau}_k - \boldsymbol{\tau}_{k-1}, \boldsymbol{\omega}(\boldsymbol{\tau}_{k-1})).$$

The elements $\Phi_\theta(t-\tau, \omega)$, $\lambda(\theta, \Phi_\theta)$ and $\mathrm{R}_t(\cdot|\theta, \Phi_\theta)$ are called the *local characteristics* of the PDP.

Describing a process as a PDP also implies the following standard conditions.

(a) If $|\Phi_\theta(t-\tau, \omega(\tau))| \to \infty$ as time $t \to t_\infty$, then $t_\infty$ is called explosion time. The explosion time of a PDP is equal to $\infty$ whenever $t_*(\theta, \Phi) = \infty$.

(b) The function $\lambda(\theta, \Phi_\theta(t, \omega))$ is a measurable function and integrable in the variable $t \in [0, \varsigma)$, where $\varsigma$ is a time depending on $\theta$ and $\Phi_\theta(t, \omega)$.

(c) $\mathrm{R}_t$ is a transition kernel such that $\mathrm{R}_t(\theta, \{\Phi\}|\theta, \Phi) = 0$ for each $(\theta, \Phi) \in \mathbb{E}$.

(d) The number of jumps is finite, which means that for each $t \in \mathbb{R}^+$ $E\{\sum_{k\in\mathbb{N}} 1_{(t\geq\tau_k)}\} < \infty$.

A very useful concept associated with a PDP is its execution. The execution of a PDP is defined as follows. The initial state is $\xi(0) \triangleq (\theta(0), \omega(0))$ at initial time $\tau_0 \triangleq 0$. If no jump occurs, the process state at time $t$ is given by $\xi(t) = (\theta(t), \omega(t)) = (\theta(0), \Phi_{\theta(0)}(t, \omega(0)))$. The density function for the time of the first jump is given by

$$G_{\xi(0),t-\tau_0} \triangleq 1_{(t-\tau_0<t_*(\theta(0,\omega(0)))} \cdot e^{-\int_{\tau_0}^{t}(\lambda(\theta(0),\Phi_{\theta(0)}(s-\tau_0,\omega(0)))ds}.$$

The value of the state to which the jump is made is generated by using the transition kernel $\mathrm{R}_t$. The algorithm to determine a sample path for the PDP $(\boldsymbol{\theta}(t), \boldsymbol{\omega}(t))$

starting with initial state $(\theta(0), \omega(0))$ is as follows: Define $\tau_0 \triangleq 0$ and let for $k = 0$ $\xi(\tau_k) \triangleq (\theta(\tau_k), \omega(\tau_k))$ be the initial state. Then for $k \in \mathbb{N}$ :

Step 1: Draw a sample $\sigma_k$ using the density function $G_{\xi(\tau_{k-1}), t-\tau_{k-1}}$. Then the time $\tau_k$ of the $k$th jump is $\tau_k = \tau_{k-1} + \sigma_k$. The sample path up to the $k$th jump is given by

$$\xi(t) = (\theta(\tau_{k-1}), \Phi_{\theta(\tau_{k-1})}(t - \tau_{k-1}, \omega(\tau_{k-1}))) \text{ for } t \in [\tau_{k-1}, \tau_k).$$

Step 2: Select the sample $\xi(\tau_k)$ using the transition measure

$$\mathrm{R}_{\tau_k}(\cdot|\theta(\tau_{k-1}), \Phi_{\theta(\tau_{k-1})}(\tau_k - \tau_{k-1}, \omega(\tau_{k-1}))). \tag{13}$$

The PDP definition is illustrated with the following example.

**Example 2.** Let the general probability space be $(\Omega, \mathcal{F}, \mathrm{Pr})$, $\theta_1 = (1,0)$ and $\theta_2 = (0,1)$. The random variables $\boldsymbol{\theta}(t) : \Omega \to \{\theta_1, \theta_2\}$ and $\boldsymbol{w}(t) : \Omega \to \mathbb{R}^+$ define the PDP $(\boldsymbol{\theta}(t), \boldsymbol{w}(t)) : \Omega \to \bigcup_{i=1}^{2} \left\{\{\theta_i\} \times \mathbb{R}^+\right\}$ with the following local characteristics:

$$\begin{aligned}
\lambda : \bigcup_{i=1}^{2} \{\{\theta_i\} \times \mathbb{R}^+\} &\to \mathbb{R}^+ \\
(\theta_1, \Phi_{\theta_1}(t, \varrho_1)) &\mapsto \lambda(\theta_1, \Phi_{\theta_1}(t-\tau, \varrho_1)) = \lambda_1 \\
(\theta_2, \Phi_{\theta_2}(t, \varrho_2)) &\mapsto \lambda(\theta_2, \Phi_{\theta_2}(t-\tau, \varrho_2)) = \lambda_2
\end{aligned}$$

$$\begin{aligned}
\Phi_\theta : \mathbb{R}^+ \times \mathbb{R}^+ &\to \mathbb{R}^+ \\
(t-\tau, \varrho) &\mapsto \Phi_{\theta_i}(t-\tau, \varrho_j) = \varrho_j - (t-\tau),
\end{aligned}$$

where $\tau \in \mathbb{R}^+$ is the initial time, and $\varrho_j$ is a sample of the exponential distribution with parameter $\lambda_j$ for $j = 1, 2$. This will be denoted by $\varrho_j \backsim \exp(\lambda_j)$. Finally the transition kernels are as follows:

$$\begin{aligned}
\mathrm{R}_t(\theta_1, \varrho_1|\theta_1, 0) &= \frac{\lambda_2}{\lambda_1 + \lambda_2}, \ \varrho_1 \backsim \exp(\lambda_1) \\
\mathrm{R}_t(\theta_2, \varrho_2|\theta_1, 0) &= \frac{\lambda_1}{\lambda_1 + \lambda_2}, \ \varrho_2 \backsim \exp(\lambda_2) \\
\mathrm{R}_t(\theta_1, \varrho_1|\theta_2, 0) &= \frac{\lambda_2}{\lambda_1 + \lambda_2}, \ \varrho_1 \backsim \exp(\lambda_1) \\
\mathrm{R}_t(\theta_2, \varrho_2|\theta_2, 0) &= \frac{\lambda_1}{\lambda_1 + \lambda_2}, \ \varrho_2 \backsim \exp(\lambda_2).
\end{aligned}$$

The execution of this PDP is described next. Suppose at time $\tau_0 = 0$ the initial state is $\theta(0) = \theta_1$, and a sample $\varrho_1$ is chosen with an exponential distribution with parameter $\lambda_1$. Then the next stopping time is $\tau_1 = \tau_0 + \varrho_1$, and for $t \in [\tau_0, \tau_1)$ the state of the PDP is $(\theta(t) = \theta_1, w(t) = \varrho_1 - t)$. Observe that at time $\tau_1$, the state of the PDP is $(\theta_1, 0)$. Two possible events can be chosen with the measure $\mathrm{R}_t(\cdot|\theta_1, 0)$: $(\theta_1, \varrho_2)$, where $\varrho_2$ is chosen with an exponential distribution having parameter $\lambda_2$, and $(\theta_2, \varrho_2)$, where $\varrho_1$ is chosen with an exponential distribution having parameter $\lambda_1$. Suppose $(\theta_1, \varrho_2)$ is chosen then $\tau_2 = \tau_1 + \varrho_2$, and for every $t \in [\tau_1, \tau_2)$ the state of the PDP is $(\theta(t) = \theta_1, w(t) = \varrho_2 - t + \tau_1)$, and the execution follows as the beginning. $\square$

Having defined the execution of the DCPN and PDP, it is now possible to present the following theorem.

**Theorem 8.** *[21] For the execution of a stochastic and dynamically colored Petri net satisfying $R_0$ through $R_4$, there exists an equivalent process, which is the execution of a PDP if the following conditions are satisfied:*

*D1) There are no explosions, i.e., the time at which a token color equals $+\infty$ or $-\infty$ approaches infinity whenever the time until the first guard transition enabling moment approaches infinity.*

*D2) After a transition firing (or after a sequence of firings that occur at the same time instant), at least one place must contain a different number of tokens, or the color of at least one token, must have jumped.*

*D3) In a finite time interval, each transition is expected to fire a finite numbers of times, and as $t \to \infty$, the number of tokens remains finite.*

*D4) In the initial marking, no immediate transition is enabled.*

*Proof.* The proof is in [20, 21]. $\square$

In order to represent the DCPN as a PDP, the elements $K$, $d(\theta)$, $\omega(0)$, $\Phi_\theta(t-\tau, \omega)$, $\partial E_\theta$, $\lambda, Q_t$ and the PDP conditions $C_1 - C_4$ need to be characterized in terms of the DCPN [4, 20]. This is done below.

(a) $K$: The set $K$ are the nodes of the reachability graph $\theta$. The nodes are vectors $\theta(t) = (v_1(t), ..., v_{|\mathcal{P}|}(t))$, where $v_i(t)$ equals the number of tokens in place $P_i$ at time $t$.

(b) The color of a token in a place $P$ is an element of $\mathcal{C}(P) = \mathbb{R}^{n(P)}$, therefore $d(\theta) = \sum_{i=1}^{|\mathcal{P}|} v_i \times n(P_i)$ with $\theta = (v_1, ..., v_{|\mathcal{P}|}) \in K$.

(c) The initial distribution $\theta(0)$ and $\omega(0)$ are obtained from the initialization function of the DCPN.

(d) The map $\Phi_\theta(t-\tau,\omega)$ for each mode $\theta$ is a vector formed by the color of the places that are pre-enable in $\theta$.

(e) For each token distribution $\theta$, the boundary $\partial E_\theta$ of subsets $E_\theta$ is determined from the transition guards that under token distribution $\theta$ are pre-enable. For the case where the state $\theta$ has only one guard transition pre-enabled, it follows that $\partial E_\theta = \partial E_{T_G}$. The other cases are not consider in this dissertation.

(f) For each token distribution, $\theta$, the jump rate $\lambda(\theta,\cdot)$ is determined from the transition delays under which the token distribution, $\theta$, is pre-enabled. In other words, if the transition $T_1$ is pre-enabled then $\lambda(\theta,\cdot) = \delta_{T_1}(\cdot)$. The other cases are not considered in this dissertation.

(g) The transition kernel, $\mathrm{R}_t(\theta', C|\theta, x)$, of the PDP is equal to the transition kernel, $Q_t(\theta', C|\theta, x)$, of the DCPN.

The mapping from the sampled-data DCPN to the PDP is developed next.

### III.4.1 A PDP Model of Sampled-Data Systems

In this section, the mapping from the sampled-data system DCPN to a PDP is developed. It is possible to see that the DCPN model of the sampled-data system meets the conditions of Theorem 8 as follows. Condition $D1$ holds because the dynamics of the nominal and recovery system do not approach to infinity in a finite time. Condition $D2$ holds because after a transition fires, the state of the DCPN always change from sojourn time zero to the value $\sigma_k$. Condition $D3$ holds because the transitions $T_1, T_2$ and $T_3$ fire a finite number times in a finite interval, and the numbers of tokens is always equal to two. Finally, $D4$ holds because the system begins in a nominal mode with no upset, which means that no immediate transition is enabled. Now the elements of the PDP model of the sampled-data are obtained from the DCPN model as follows.

(a) The set, $K$, of modes of operation of the PDP is $\{\theta_1, \theta_2, \theta_4\}$, where $\theta_i$, $i = 1, 2, 3$ are the token states of the DCPN.

(b) The map $d$ is defined as $d(\theta_1) = n_{x_p} + n_{x_c} + 1$ and $d(\theta_2) = d(\theta_4) = n_{x_p} + n_{x_c} + 2$.

(c) The initial distribution, with probability one, gives a token to the places $\Sigma_N$, $n_1$ and the color $\mathbb{R}^{n_{x_p}+n_{x_c}+1}$ to $\Sigma_N$.

(d) The map $\Phi_\theta(\cdot,\cdot)$ has two states. The nominal closed-loop system, represented by $\Phi_{\theta_1} = \mathbf{z}^1$, and the recovery closed-loop system represented by $\Phi_{\theta_2} = \Phi_{\theta_4} = \mathbf{z}^2$.

(e) The boundary for the modes are $\partial E_{\theta_1} = \emptyset$ and $\partial E_{\theta_2} = \partial E_{\theta_4} = \partial G_{T_G}$.

(f) The jump rates are $\lambda(\theta_1, \Phi_{\theta_1}) = \lambda(\theta_2, \Phi_{\theta_2}) = \lambda_1$ and $\lambda(\theta_4, \Phi_{\theta_4}) = \lambda_2$.

(g) In Section III.3.1, the transition kernel, $Q_t$, of the DCPN was obtained. Now this transition kernel is also the transition kernel, $\mathrm{R}_t$, of the PDP. The transition kernel has the following matrix notation

$$\begin{bmatrix}
p_{n_1n_1}(t) & p_{n_a}p_{n_1n_2}(t) & p_{n_b}p_{n_1n_2}(t) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{n_2n_1}(t) & p_{n_2n_2}(t) & 0 & 0 & 0 \\
0 & 0 & 0 & p_{n_2n_1}(t) & 0 & 0 & p_{n_2n_2}(t) & 0 & 0 \\
0 & 0 & 0 & p_{n_1n_1}(t) & 0 & 0 & p_{n_1n_2}(t) & 0 & 0 \\
0 & 0 & 0 & 0 & p_{n_1n_1}(t) & p_{n_1n_2}(t) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{n_2n_1}(t) & p_{n_2n_2}(t) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_{n_2n_1}(t) & 0 & p_{n_2n_2}(t) & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{n_a} & p_{n_b} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

where the entries $\mathrm{R}_t(a_i|a_j)$ represent the probability of going from the DCPN state $a_j$ to DCPN state $a_i$ for $i,j = 1,2,...,9$, $p_{n_1n_1}(t) = \frac{\lambda_2+\lambda_1 e^{-t(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2}$, $p_{n_1,n_2}(t) = 1 - p_{n_1n_1}(t)$, $p_{n_2n_1}(t) = -\lambda_2\frac{e^{-t(\lambda_1+\lambda_2)}-1}{\lambda_1+\lambda_2}$, and $p_{n_2,n_2}(t) = 1 - p_{n_2n_1}(t)$.

This shows that the sampled-data DCPN can be mapped to a PDP. Having the PDP model, in the next section the execution of the PDP model is characterized.

### III.4.2 Execution of the PDP Model of Sampled-Data Systems

The initial state of the sampled-data system is $(\theta(0), x(0)) = a_1$, which represents the nominal closed-loop system and the no upset state. The distribution function for the time of the first jump does not have a guard condition. It has an exponential distribution with parameter $\lambda_1$, which is used to choose $\tau_1$. At time $\tau_1$ the new state is chosen using the transition measure $\mathrm{R}_{\tau_1}(\cdot|\theta(0), \Phi_{\theta(0)}(\tau_1-\tau_0, x(0))$. The possible states are $a_2$ or $a_3$. Suppose $\xi(\tau_1) = a_2$ is chosen, then the density function for the jump is $1_{(t-\tau_1<n_a)} \cdot e^{-\lambda_2(t-\tau_1)}$. This density function is used to choose $\sigma_2$. At time $\tau_2 = \tau_1+\sigma_2$,

| | |
|---|---|
| $a_1$ | Nominal closed-loop system $\Sigma_N$ |
| $a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$ | Recovery closed-loop system $\Sigma_R$ |

TABLE II: Switching rule of the nominal and recovery closed-loop system.

using the transition measure $R_{\tau_2}(\cdot|\theta(\tau_1), \Phi_{\theta(\tau_1)}(\tau_2 - \tau_1, \omega(\tau_1)))$, the state $a_5$ or $a_6$ is chosen. Suppose $\xi(\tau_2) = a_5$ is chosen. For this state, the density function for the jump is $1_{(t-\tau_2<n_a-\sigma_2)} \cdot e^{-\lambda_1(t-\tau_2)}$. This density function is used to choose $\sigma_3$. At time $\tau_3 = \tau_2 + \sigma_3$, the state is chosen with the measure $R_{\tau_3}(\cdot|\theta(\tau_2), \Phi_{\theta(\tau_2)}(\tau_3 - \tau_2, \omega(\tau_2)))$, the only possible states are $a_5$ or $a_6$. This process go on until the density function for the time is $1_{(t-\tau_{n_a}<0)} \cdot e^{-\tau_j(t-\tau_{n_a})}$, where $j$ could be 1 or 2. At time $\tau_{n_a}$, the indicator function is zero and the system jumps to the nominal closed-loop system, and the execution of the PDP repeats from the beginning. The stochastic process generated by this execution is

$$\boldsymbol{\xi}(t) = (\boldsymbol{\theta}_{\boldsymbol{\tau}_{k-1}}, \boldsymbol{\Phi}_{\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})}(t - \boldsymbol{\tau}_{k-1}, \omega(\boldsymbol{\tau}_{k-1}))),$$

where $t \in [\boldsymbol{\tau}_{k-1}, \boldsymbol{\tau}_k)$ and $\tau_k$ is chosen using the exponential distribution with parameter $\lambda_1$ or $\lambda_2$. The samples of $\boldsymbol{\xi}(t)$ taking values in $a_i$ with $i \in \{1, 2, 3, ..., 9\}$ were chosen using the transition kernel $R_t$ evaluated at $\tau_k - \tau_{k-1}$ and the $\Phi_{\theta(\tau_{k-1})}(t - \tau_{k-1}, \omega(\tau_{k-1}))$. $\Phi_{\theta(\tau_k)}$ is the dynamic of the nominal closed-loop system or the recovery closed-loop system according to the values of the state $\xi(t)$, as shown in the TABLE II. From [4], it is known that the stochastic process $\boldsymbol{\xi}(t)$ is a continuous-time Markov chain, and $\boldsymbol{\xi}(\tau_k)$ is a discrete-time Markov chain with transition matrix $R_{\tau_k - \tau_{k-1}}$.

## III.5 CHAPTER SUMMARY

In this chapter, the problem of modeling a continuous-time plant in closed-loop with a discrete-time controller subject to random upsets was addressed as follows: The theory of DCPNs model was briefly reviewed. Then a DCPN model of the sampled-data system was obtained. For analysis purposes, this DCPN model was mapped to a PDP. This PDP characterization of the system shows the Markov property of the system. More importantly, a transition kernel that models the behavior of the system was obtained. This transition kernel will play a very important role in the next chapter.

# CHAPTER IV

# SAMPLED-DATA PIECEWISE DETERMINISTIC MARKOV PROCESS

## IV.1 INTRODUCTION

The purpose of this chapter is to study the stochastic nature of the switching between nominal and recovery modes when the sampling time is $\mathtt{T}$. From Section III.4.1, it is known that the interconnection of recovery system and disturbances in the sampled-data system is a PDP. This will be called the sampled-data PDP model. It only takes into account stopping times generated by the disturbances and the recovery algorithm. Now the objective is to also take into account the sampling instances $k\mathtt{T}$, $k \in \mathbb{N}$ in the sampled-data model. The main objective of this chapter is to characterize the state process of this sampled-data PDP.

The PDP model of the sampled-data is an abstraction of the behavior of the system. It does not take into account the sampling instants $k\mathtt{T}$. It will be shown that in order to include the sampling instants in the PDP model, the PDP model needs an appropriate time scale, i.e. This is a time scaling operation. A particular concern for the time scaling operation is preserves the Markov property of a PDP model. It will also be important to determine the relationship between the transition matrix of the Markov chain representation of the PDP model before and after the time scaling operation is applied.

This chapter also gives initial results on the stability analysis of the PDP model representation of sampled-data systems with a jump linear controller. The jumps are modeled as a Markov chain, where the jumps are related to the real time of the sampled-data system. In addition, results of the PDP literature will be used to establish a relation between the invariant measures of sampled-data systems driven by a stochastic process and the invariant measures of the associated discrete-time representation. As an application, when the plant is linear with no external input, a sufficient testable condition is given for the convergence in distribution to the invariant delta Dirac measure.

There are two specific objectives in this chapter. In Section IV.2, a methodology to embed the sampling instants, $k\mathtt{T}$, in the PDP model in such a way that the

Markov property is preserved is presented. In addition, the transition matrix of this embedded Markov chain is given. In Section IV.3, the sampled-data PDP which is a stochastic model of the PDP model that takes into account the sampling instants $k\mathrm{T}$, is provided and then used in stability analysis.

## IV.2 PIECEWISE DETERMINISTIC MARKOV PROCESS

Piecewise deterministic Markov processes are a class of continuous-time stochastic models that have found wide applicability since they were introduced in [16] and thoroughly investigated in [17]. PDP's are a class of stochastically switched systems consisting of a family of Markov processes with motions between random jump times. The definition of PDP's as presented in [16, 17] is given next. Let $K$ be a countable set denoting the possible modes of operation. If $\{E_\theta \subseteq \mathbb{R}^{d(\theta)} : \theta \in K,\ d(\theta) \in \mathbb{N}\}$ is a collection of open subsets then the state space of the PDP is defined by

$$\mathbb{E} \triangleq \bigcup_{\theta \in K} \big\{\{\theta\} \times E_\theta\big\}.$$

For each $\theta \in K$ there is a flow $\Phi_\theta : \mathbb{R}^+ \times E_\theta \to E_\theta$ with initial condition $\Phi_\theta(0, \omega) = \omega$, $\omega \in E_\theta$ that satisfies the semigroup property $\Phi_\theta(t + s, \omega) = \Phi_\theta(t, \Phi_\theta(s, \omega))$ for every $0 < t + s < \infty$. For each $\theta \in K$ and $\Phi_\theta(t, \omega) \in E_\theta$ there is a function

$$\begin{aligned} \lambda : \bigcup_{\theta \in K} \big\{\{\theta\} \times E_\theta\big\} &\rightarrow \mathbb{R}^+ \\ (\theta, \Phi_\theta(t, \omega)) &\mapsto \lambda(\theta, \Phi_\theta(t, \omega)) \in \mathbb{R}^+. \end{aligned}$$

Denoted by $\beta\big(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}\big)$ the $\sigma$-algebra defined on $\bigcup_{\theta \in K} \big\{\{\theta\} \times E_\theta\big\}$. For each $\theta \in K$ and $\Phi_\theta(t, \omega) \in E_\theta$ there is a transition kernel

$$\begin{aligned} \mathrm{R}_t : \bigcup_{\theta \in K} \big\{\{\theta\} \times E_\theta\big\} \times \beta\big(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}\big) &\rightarrow [0, 1] \\ ((\theta, \Phi_\theta(t, \omega)), (\tilde{\theta}, C)) &\mapsto \mathrm{R}_t(\tilde{\theta}, C | \theta, \Phi_\theta(t, \omega)), \end{aligned}$$

where $(\tilde{\theta}, C) \in \beta\big(\bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}\big)$.

Now the definition of PDP is given next.

**Definition 18.** [17] A stochastic process $(\boldsymbol{\theta}(t), \boldsymbol{w}(t))$ mapping $(\Omega, \mathcal{F})$ to $\big(\bigcup_{\theta_K} \{\{\theta\} \times E_\theta\}, \beta(\bigcup_{\theta_K} \{\{\theta\} \times E_\theta\})\big)$ is a *piecewise deterministic Markov process* if and only if

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}(\tau_{k-1}) \text{ and } \boldsymbol{w}(t) = \Phi_{\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})}(t - \tau_{k-1}, \boldsymbol{w}(\boldsymbol{\tau}_{k-1}))$$

for every $t \in [\boldsymbol{\tau}_{k-1}, \boldsymbol{\tau}_k)$. The elements $\Phi_\theta(t-\tau, w)$, $\lambda(\theta, \Phi_\theta)$ and $Q_t(\cdot|\theta, \Phi_\theta)$ are called the *local characteristics* of the PDP, because these are enough to characterize a PDP.

Describing a process as a PDP will also imply the following standard conditions.

(a) There is an explosion if $|\Phi_\theta(t-\tau, w(\tau))| \to \infty$ as time $t \to t_\infty$. In this case $t_\infty$ is called the explosion time. The explosion time of a PDP is equal to $\infty$ whenever $t_*(\theta, \Phi_\theta) = \infty$.

(b) The function $\lambda(\theta, \Phi_\theta(t, w))$ is a measurable function and integrable in the variable $t \in [0, \varsigma)$, where $\varsigma \in \mathbb{R}^+$ is a time that depend on $\theta$ and $\Phi_\theta(t, w)$.

(c) $\mathrm{R}_t$ is a transition kernel such that $\mathrm{R}_t(\cdot, \cdot|\theta, \Phi) = 0$ for each $(\theta, \Phi) \in \mathbb{E}$.

(d) $E\{\sum_{k \in \mathbb{N}} 1_{(t \geq \tau_k)}\} < \infty$ for each $t \in \mathbb{R}^+$.

Next the strong Markov property of a PDP is presented. Let $D_\mathbb{E}$ denote the set of right-continuous functions on $\mathbb{R}^+$ with values in $\mathbb{E} \triangleq \bigcup_{\theta \in K} \{\{\theta\} \times E_\theta\}$. Let $\mathcal{H}_t^0 = \beta\big(h(s) : s \leq t\big)$ denote the natural filtration generated by the function $h(t)$ of $h \in D_\mathbb{E}$. The PDP can be interpreted as a random measure $\boldsymbol{\Psi}_{(\theta,\Phi)} : \Omega \to D_\mathbb{E}$ for every initial condition $(\theta, \Phi) \in \mathbb{E}$. These random elements induce the probability measures $P_{(\theta,\Phi)}(A) \triangleq \Pr(\omega : \boldsymbol{\Psi}_{(\theta,\Phi)}(\omega) \in A)$, which are used to form a family of measures $\{P_{(\theta,\Phi)}, (\theta, \Phi) \in \mathbb{E}\}$. Now, let $\mathcal{H}_t^\nu$ be the completion of $\mathcal{H}_t^0$ and define $\mathcal{H}_t = \bigcap_{\nu \in \mathtt{W}(\mathbb{E})} \mathcal{H}_t^\nu$, where $\mathtt{W}(\mathbb{E})$ is the set of probability measures on $\mathbb{E}$. The $\mathcal{H}_t$-*stopping times* $\boldsymbol{\tau}_k$, $k \in \mathbb{N}$ on a filtered probability space are random variables taking values in $\mathbb{R}^+$ such that $(\boldsymbol{\tau}_k \leq t) \in \mathcal{H}_t$. Thus, given $\mathcal{H}_t$ one knows whether $\boldsymbol{\tau}_k$ has happened by time $t$ or not. The following theorem shows that a PDP is a strong Markov process [17].

**Theorem 9.** *The process $(\boldsymbol{\theta}(t), \boldsymbol{w}(t))$ is a homogeneous strong Markov process, i.e, for any $(\theta, \Phi) \in \mathbb{E}$, the $\mathcal{H}_t$-stopping time $\boldsymbol{\tau}_k$ satisfies for any bounded measurable function $f$*

$$E_{(\theta,\Phi)}\{f(\boldsymbol{\theta}(\boldsymbol{\tau}_k + s), \boldsymbol{w}(\boldsymbol{\tau}_k + s))|\mathcal{H}_{\boldsymbol{\tau}_k}\} = E_{(\theta(\tau_k), w(\tau_k))}\{f(\boldsymbol{\theta}(s), \boldsymbol{w}(s))\}$$

*where $E_{(\theta,\Phi)}$ denotes integration with respect to the measure $P_{(\theta,\Phi)}$.*

From the construction of a PDP and Theorem 9, the stochastic process $\{(\boldsymbol{\theta}(\boldsymbol{\tau}_k), \boldsymbol{w}(\boldsymbol{\tau}_k))\}_{k\in\mathbb{N}}$ is a Markov process with stationary transition probability

$$\begin{aligned} p(C|\theta, w) &\triangleq P_{(\theta,w)}(\boldsymbol{\theta}(\boldsymbol{\tau}_1), \boldsymbol{w}(\boldsymbol{\tau}_1) \in C) \\ &= \int_0^{t_*(\theta,w)} \mathrm{R}_t(C|\theta, \Phi_\theta(t,w))\lambda(\theta, \Phi_\theta(t,w))e^{-\Lambda(t,\theta,w)}dt \\ &\quad + e^{-\Lambda(t_*(\theta,w),\theta,w)}\mathrm{R}_t(C|\theta, \Phi_\theta(t_*(\theta,w),w)), \end{aligned} \tag{14}$$

where $\Lambda(t,\theta,w) = \int_0^t \lambda(\theta, \Phi_\theta(s,w))ds$ and $t_*(\theta,w) = \inf\{t > 0 : \Phi_\theta(t-\tau,w) \in \partial E_\theta\}$. Equation (14) gives the transition matrix of the discrete-time Markov chain associated with a PDP. The next example calculates the discrete-time Markov chain associated with the PDP of Example 2.

**Example 3.** Consider the PDP in Example 2. Note that $t_*(\theta,w) = \infty$ because there is no boundary condition. The discrete-time Markov chain, $(\boldsymbol{\theta}(\tau_k), \boldsymbol{w}(\tau_k))$, associated with this PDP, $(\boldsymbol{\theta}(t), \boldsymbol{w}(t))$, has the following transition probability

$$\begin{aligned} p(\theta_1, \sigma|\theta_1, 0) &= \int_0^\infty \mathrm{R}_t(\theta_1, \sigma|\theta_1, 0)\lambda(\theta_1, \Phi_{\theta_1}(t,w))e^{-\int_0^t \lambda(\theta_1, \Phi_{\theta_1}(s,w))ds}dt \\ &= \int_0^\infty \frac{\lambda_2}{\lambda_1+\lambda_2}\lambda_1 e^{-\lambda_1 t}dt \\ &= \frac{\lambda_2}{\lambda_1+\lambda_2}. \end{aligned}$$

The resulting transition probability matrix is

$$Q \triangleq \begin{bmatrix} \frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_1}{\lambda_1+\lambda_2} \\ \frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_1}{\lambda_1+\lambda_2} \end{bmatrix}.$$

□

Example 3 shows the procedure to obtain the discrete-time Markov chain associated with a PDP. Next, a procedure to construct a CTMC from a discrete-time Markov chain is presented. The construction uses the set of nonnegative real numbers $q_0 = \{q_0(s),\ s \in E\}$, which associate with each state $s \in E$ a real number $q_0(s)$, a stochastic matrix $W_0 = \{W_0(s'|s),\ s', s \in E\}$ and a probability distribution $\nu(s)$. To start the construction, define the random variables $\boldsymbol{t}_n$ mapping $(\Omega, \mathcal{F}, \Pr)$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and the random variables $\boldsymbol{y}_n$ mapping $(\Omega, \mathcal{F}, \Pr)$ to $(E, \beta(E))$ such that the stochastic process $\{\boldsymbol{y}_n\}$ is a discrete-time Markov chain with transition matrix $W_0$ and initial distribution $\nu$, and the random variables $\boldsymbol{t}_n$, $n \geq 0$ are mutually independent. Assume each $\boldsymbol{t}_n$ has an exponential distribution with parameter $q_0(y_n)$, and $y_n$ is a sample of $\boldsymbol{y}_n$. Define a stochastic process $\boldsymbol{x}(t)$ which take values in a finite state space $E$ as follows

$$\boldsymbol{x}(t) \triangleq \boldsymbol{y}_{\boldsymbol{N}_t}, \tag{15}$$

where $\boldsymbol{N}_t = \sup\{n \geq 0 : \sum_{i=0}^{n-1} \boldsymbol{t}_i \leq t\}$ and $\boldsymbol{N}_t < \infty$. The following theorem shows that $\boldsymbol{x}(t)$ is a CTMC [27].

**Theorem 10.** *The stochastic process $\{\boldsymbol{x}(t),\ t \geq 0\}$ in (15) is a time-homogeneous CTMC with initial distribution $\nu$. The transition matrix $W$ for the embedded jump chain is given by $W(s'|s) = W_0(s'|s)/(1 - W_0(s|s))$ for $s', s \in E$ with $s' \neq s$. The infinitesimal generator matrix $\Lambda = [\Lambda(s'|s)]_{s's \in E}$ of the CTMC is*

$$\begin{aligned}
\Lambda(s'|s) &= q_0(s)(1 - W_0(s|s))W(s'|s), \text{ for } s \neq s' \\
\Lambda(s|s) &= -q_0(s)(1 - W_0(s|s)).
\end{aligned}$$

In Example 3, the associated discrete-time Markov chain associated with the PDP $(\boldsymbol{\theta}(t), \boldsymbol{w}(t))$ was obtained. Now Example 4 will show how to get a CTMC from the discrete-time Markov chain $(\boldsymbol{\theta}(\tau_k), \boldsymbol{\omega}(\tau_k))$ using Theorem 10.

**Example 4.** Consider the Markov chain $\boldsymbol{\theta}(t)$ taking values in $\{\theta_1, \theta_2\}$ from Example 3 with transition matrix $Q$ and initial distribution $\nu$. Also, consider the random variables $\boldsymbol{t}_n$ exponentially distributed with parameters $q_0(\theta_1)$ or $q_0(\theta_2)$. Then the stochastic process $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_{\boldsymbol{N}_t}$, where $\boldsymbol{N}_t = \sup\{n \geq 0 : \sum_{i=0}^{n-1} \boldsymbol{t}_i \leq t\}$ is a CTMC and

$$\begin{aligned}
W(\theta_2|\theta_1) &= \frac{Q(\theta_2|\theta_1)}{1 - Q(\theta_1|\theta_1)} = \frac{\frac{\lambda_1}{\lambda_1+\lambda_2}}{1 - \frac{\lambda_2}{\lambda_1+\lambda_2}} = 1 \\
W(\theta_1|\theta_2) &= \frac{Q(\theta_1|\theta_2)}{1 - Q(\theta_2|\theta_2)} = 1.
\end{aligned}$$

The entries of the infinitesimal generator matrix $\Lambda$ are

$$\begin{aligned}
\Lambda(\theta_1|\theta_1) &= -q_0(\theta_1)\frac{\lambda_1}{\lambda_1 + \lambda_2} \\
\Lambda(\theta_2|\theta_1) &= q_0(\theta_1)\frac{\lambda_1}{\lambda_1 + \lambda_2} \\
\Lambda(\theta_2|\theta_2) &= -q_0(\theta_2)\frac{\lambda_2}{\lambda_1 + \lambda_2} \\
\Lambda(\theta_1|\theta_2) &= q_0(\theta_2)\frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

If the parameters are $q_0(\theta_1) = q_0(\theta_2) = \lambda_1 + \lambda_2$ then

$$\Lambda = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}.$$

This shows that $\boldsymbol{\theta}(t)$ is a CTMC with infinitesimal generator matrix $\Lambda$ and stopping times chosen by the exponential distributions with parameters $\lambda_1$ and $\lambda_2$. Note that an important step in deriving a CTMC from a discrete-time Markov chain is the relation $q_0(\theta_1) = q_0(\theta_2) = \lambda_1 + \lambda_2$, the motivation of this example came from [26]. □

Now a methodology for converting a CTMC into a DTMC is presented. This methodology is called *uniformization* of a Markov chain, [6, 37]. Let a CTMC $\boldsymbol{x}(t)$ take values in a finite set $E$ with infinitesimal generator matrix $\Lambda = [\lambda_{ij}]$. If $\max_i\{-\lambda_{ii}\} < \infty$, the CTMC is said to be uniformizable, which means that it is possible to get an associated DTMC. The methodology is as follows. Fix $q \geq \sup_i\{-\lambda_{ii}\}$. The probability distribution $\Pr(t)$ of the random variable $\boldsymbol{x}(t)$ then satisfies the Chapman-Kolmogorov equation

$$\begin{aligned} \frac{d\Pr(t)}{dt} &= \Pr(t)\Lambda \\ &= -\Pr(t)q[I - \Pi], \end{aligned}$$

where $\Pi \triangleq [I + \frac{1}{q}\Lambda]$. The transition matrix is

$$\Pr(t) = \Pr(0)\sum_{k=0}^{\infty} e^{-tq}\frac{(tq)^k}{k!}\Pi^k.$$

The transition matrix $\Pi$ and the initial distribution $\Pr(0)$ induce a DTMC $\boldsymbol{x}(k)$. The probability distribution $\Pr(k)$ of $\boldsymbol{x}(k)$ is $\Pr(k) = \Pr(0)\Pi^k$. Consider the Poisson process $\boldsymbol{K}(t)$ with rate $q$ and independent of the chain $\boldsymbol{x}(k)$. The probability distribution of the Poisson process is $\Pr(\boldsymbol{K}(t) = k) = \frac{(qt)^k}{k!}e^{-tq}$. Let $\boldsymbol{T}_k$ be the first time that $\boldsymbol{K}(t) = k$. Then it is possible to define the following process

$$\hat{\boldsymbol{x}}(t) \triangleq \boldsymbol{x}(k) \;\forall t \in [\boldsymbol{T}_k, \boldsymbol{T}_{k-1}).$$

Note that $\{\boldsymbol{T}_k - \boldsymbol{T}_{k-1}\}_{k\in\mathbb{N}}$ are iid random variables with probability density $\Pr(\boldsymbol{T}_k - \boldsymbol{T}_{k-1} = t) = qe^{-qt}$ since $\boldsymbol{T}_k - \boldsymbol{T}_{k-1}$ are exponentially distributed. The CTMC, $\boldsymbol{x}(t)$, can be interpreted as a DTMC, $\boldsymbol{x}(k)$, where its stopping times are chosen according to a exponential distribution with parameter $q$. To see that the processes $\{\boldsymbol{x}(t)\}_{t\in\mathbb{R}^+}$

and $\{\boldsymbol{x}(k)\}_{k\in\mathbb{N}}$ have the same probability distribution in each $k \in \mathbb{N}$, observe by Bayes's rule that for every $n \in \mathbb{N}$ the following holds

$$\begin{aligned}\Pr(\boldsymbol{x}(k)=s) &= \Pr(\hat{\boldsymbol{x}}(t)=s)\\ &= \sum_{i=0}^{\infty}\Pr(\boldsymbol{x}(i)=s)\Pr(\boldsymbol{K}(t)=i)\\ &= \sum_{i=0}^{\infty}\Pr(\boldsymbol{x}(i)=s)\frac{(qt)^i}{i!}e^{-tq}\\ &= \Pr(\boldsymbol{x}(t)=s). \end{aligned} \tag{16}$$

This methodology is illustrated in the following example.

**Example 5.** Consider the CTMC $\boldsymbol{\theta}(t)$ in Example 4 with infinitesimal generator matrix $\Lambda$. If $q = \lambda_1 + \lambda_2$ then the transition matrix of the DTMC $\boldsymbol{\theta}(k)$ is

$$\begin{aligned}\Pi &= I + \frac{1}{\lambda_1+\lambda_2}\begin{bmatrix}-\lambda_1 & \lambda_1\\ \lambda_2 & -\lambda_2\end{bmatrix},\\ \Pi &= \begin{bmatrix}\frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_1}{\lambda_1+\lambda_2}\\ \frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_1}{\lambda_1+\lambda_2}\end{bmatrix}.\end{aligned}$$

This gives a DTMC with stopping times chosen according to a exponential distribution with parameter $q = \lambda_1 + \lambda_2$. □

These two techniques to convert a DTMC to a CTMC and vice-versa yield what are known in the literature as *subordinated* processes, [10,23]. Next, the DTMC of a sampled-data PDP model is going to be obtained.

## IV.3 SAMPLED-DATA PDP

The PDP model of a sampled-data system captures the stochastic nature of the switching rule, but the switching rule evolves in continuous-time. Since the mode switches occur at specific sampling time instants $k\mathrm{T}$, $k \in \mathbb{N}$, the sampled-data PDP parameters $\lambda_1$ and $\lambda_2$ need to be properly selected. Thus, the relationship between the switching rule modeled as a PDP and the sampling times $k\mathrm{T}$ is analyzed in this section. Recall that the interconnection of the upset generator and the recovery logic was modeled as a DCPN from which a PDP model was obtained. The execution of the PDP model gives the stopping times and the mode of operation for all time. So the relationship between the stopping times from the PDP model and the sampling times needs to be determined. From Chapter III it is known that at the stopping times the state of the sampled-data system has the Markov property. It is desired that at sampling times the state of the sampled-data PDP also has the Markov property. It is also important to determine the transition matrix for the Markov process. This is one of the main objectives of this section.

### IV.3.1 PDP Model of the Switching Rule of Sampled-Data Systems

In Section III.4.1 the mapping from a sampled-data system DCPN to a PDP was developed, where the PDP $(\boldsymbol{\theta}(t), \boldsymbol{\Phi}_{\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})}(t-\boldsymbol{\tau}_{k-1}, \boldsymbol{w}(\boldsymbol{\tau}_{k-1})))$ has the state space

$$\mathbb{E} \triangleq \bigcup_{i=1,2,4} \{\{\theta_i\} \times E_{\theta_i}\}.$$

The set of modes is $K = \{\theta_1, \theta_2, \theta_4\}$. The collections of open subsets of the $\mathbb{R}^{n_{xp}+n_{xc}}$ space are $E_{\theta_1} \subseteq \mathbb{R}^{n_{xp}+n_{xc}}$, $E_{\theta_2} \subseteq \mathbb{R}^{n_{xp}+n_{xc}}$ and $E_{\theta_4} \subseteq \mathbb{R}^{n_{xp}+n_{xc}}$. The motions associated with each mode are

$$\begin{aligned}
\Phi_{\theta_1} : \mathbb{R}^+ \times E_{\theta_1} &\rightarrow E_{\theta_1} \times \mathbb{R}^+ \\
(\sigma, z) &\mapsto \mathbf{z}^1_{\tau_{k-1}t}(\sigma, z) = \Phi_{\theta_1}(t-\tau_{k-1}, \sigma, z) \\
\Phi_{\theta_i} : \mathbb{R}^+ \times \mathbb{R}^+ \times E_{\theta_i} &\rightarrow E_{\theta_i} \times \mathbb{R}^+ \times \mathbb{R}^+ \\
(n_a, \sigma, z) &\mapsto \mathbf{z}^2_{st}(n_a, \sigma, z) = \Phi_{\theta_i}(t-\tau_{k-1}, \sigma, n_a, z) \\
(n_b, \sigma, z) &\mapsto \mathbf{z}^2_{st}(n_b, \sigma, z) = \Phi_{\theta_i}(t-\tau_{k-1}, \sigma, n_b, z),
\end{aligned}$$

where $i = 2, 4$ (Section III.4). For each mode $\theta_i$ and flow $\Phi_{\theta_i}$ there exists the following function

$$\begin{aligned}
\lambda : \bigcup_{i=1,2,4} \{\{\theta_i\} \times E_{\theta_i}\} &\rightarrow \mathbb{R}^+ \\
(\theta_1, \Phi_{\theta_1}) &\mapsto \lambda(\theta_1, \Phi_{\theta_1}) = \lambda_1 \\
(\theta_2, \Phi_{\theta_2}) &\mapsto \lambda(\theta_2, \Phi_{\theta_2}) = \lambda_1 \\
(\theta_2, \Phi_{\theta_4}) &\mapsto \lambda(\theta_4, \Phi_{\theta_4}) = \lambda_2.
\end{aligned}$$

The transition kernel is

$$\mathrm{R}_t : \bigcup_{i=1,2,4} \{\{\theta_i\} \times E_{\theta_i}\} \times \beta\Big(\bigcup_{i=1,2,4} \{\{\theta_i\} \times E_{\theta_i}\}\Big) \rightarrow [0,1].$$

The transition probability matrix associated with $\mathrm{R}_t$ is

$$Q_t = \begin{bmatrix} p_{n_1n_1} & p_{n_a}p_{n_1n_2} & p_{n_b}p_{n_1n_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{n_2n_1} & p_{n_2n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{n_2n_1} & 0 & 0 & p_{n_2n_2} & 0 & 0 \\ 0 & 0 & 0 & p_{n_1n_1} & 0 & 0 & p_{n_1n_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{n_1n_1} & p_{n_1n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{n_2n_1} & p_{n_2n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{n_2n_1} & 0 & p_{n_2n_2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{n_a} & p_{n_b} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $p_{n_1n_1} = \frac{\lambda_2}{\lambda_1+\lambda_2}$, $p_{n_1n_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $p_{n_2n_1} = \frac{\lambda_2}{\lambda_1+\lambda_2}$, and $p_{n_2,n_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$. The entries $Q_t(a_i|a_j)$ are the transition probabilities for going from $a_j$ to $a_i$, for $i, j = 1, 2, ..., 9$. The PDP $(\boldsymbol{\theta}(t), \boldsymbol{\Phi}_{\boldsymbol{\theta}(\boldsymbol{\tau}_{k-1})}(t - \boldsymbol{\tau}_{k-1}, \boldsymbol{w}(\boldsymbol{\tau}_{k-1})))$ has a discrete-time Markov chain $(\boldsymbol{\theta}(\tau_k), \boldsymbol{w}(\boldsymbol{\tau}_{k-1}))$. The transition probabilities of this Markov chain are obtained using the following equation

$$\begin{aligned} p(C|\theta, w) &\triangleq P_{(\theta,w)}((\boldsymbol{\theta}(\boldsymbol{\tau}_1), \boldsymbol{w}(\boldsymbol{\tau}_1)) \in C) \\ &= \int_0^{t_*(\theta,w)} Q_t(C|\theta, \Phi_\theta(t,w))\lambda(\theta, \Phi_\theta(t,w))e^{-\Lambda(t,\theta,w)}dt \\ &\quad + e^{-\Lambda(t_*(\theta,w),\theta,w)}Q_t(C|\theta, \Phi_\theta(t_*(\theta,w),w)), \end{aligned} \tag{17}$$

where $C$ is an event, $\Lambda(t,\theta,w) = \int_0^t \lambda(\theta, \Phi_\theta(s,w))ds$ and $t_*(\theta,w) = \inf\{t > 0 : \Phi_\theta(t,w) \in \partial E_\theta\}$. For the nine DCPN states, $t_*(\theta,w) = t_*(a_i)$ yields.

$$\begin{aligned} t_*(a_1) &= \inf\{t > 0 : \; \Phi_{\theta_1}(t-\tau,\sigma,\omega) \in \partial E_{\theta_1}\} = +\infty \\ t_*(a_2) &= \inf\{t > 0 : \; \Phi_{\theta_4}(t-\tau,\sigma,n_a,\omega) \in \partial E_{\theta_4}\} = n_a \\ t_*(a_3) &= \inf\{t > 0 : \; \Phi_{\theta_4}(t-\tau,\sigma,n_b,\omega) \in \partial E_{\theta_4}\} = n_b \\ t_*(a_8) &= \inf\{t > 0 : \; \Phi_{\theta_2}(t-\tau,\sigma,n_b,\omega) \in \partial E_{\theta_2}\} = 0 \\ t_*(a_9) &= \inf\{t > 0 : \; \Phi_{\theta_4}(t-\tau,\sigma,n_b,\omega) \in \partial E_{\theta_4}\} = 0, \end{aligned}$$

and $t_*(a_4) = t_*(a_7) = n_b$ and $t_*(a_5) = t_*(a_6) = n_a$. The parameter of the exponential distributions corresponding to the states are

$$\begin{aligned} \lambda(a_1) &= \lambda(a_4) = \lambda(a_5) = \lambda(a_8) = \lambda_1, \\ \lambda(a_2) &= \lambda(a_3) = \lambda(a_6) = \lambda(a_7) = \lambda(a_9) = \lambda_2. \end{aligned}$$

The transition probabilities of the DTMC, $(\boldsymbol{\theta}(\tau_k), \boldsymbol{w}(\boldsymbol{\tau}_{k-1}))$, are obtained using Equation (17) as follows

$$
\begin{aligned}
p(a_1|a_1) &= \int_0^\infty Q_t(a_1|a_1)\lambda(a_1)e^{-\int_0^t \lambda(a_1)ds}dt, \\
&= \int_0^\infty p_{n_1n_1}\lambda_1 e^{-\lambda_1 t}dt = p_{n_1n_1}.
\end{aligned}
$$

Similarly,

$$
p(a_2|a_1) = p_{n_a}p_{n_1n_2} \text{ and } p(a_3|a_1) = p_{n_b}p_{n_1n_2},
$$

while the probabilities for going from $a_1$ to the other states are zero. From the state $a_2$ it is possible to go to $a_5$, $a_6$, $a_2$ and $a_3$.

$$
\begin{aligned}
p(a_5|a_2) &= \int_0^{n_a} Q_t(a_5|a_2)\lambda(a_2)e^{-\int_0^t \lambda(a_2)ds}dt + e^{-\int_0^{na}\lambda(a_2)ds}Q_t(a_5|a_9), \\
&= \int_0^{n_a} p_{n_2n_1}\lambda_2 e^{-\lambda_2 t}dt = p_{n_2n_1}(1-e^{-\lambda_2 n_a}), \\
p(a_6|a_2) &= p_{n_2n_2}(1-e^{-\lambda_2 n_a}), \\
p(a_2|a_2) &= \int_0^{n_a} Q_t(a_2|a_2)e^{-\lambda_2 t}dt + e^{\int_0^{na}\lambda_2 dt}Q_t(a_2|a_9) = e^{-n_a\lambda_2}p_{n_a}, \\
p(a_3|a_2) &= \int_0^{n_a} Q_t(a_3|a_2)e^{-\lambda_2 t}dt + e^{\int_0^{na}\lambda_2 dt}Q_t(a_3|a_9) = e^{-n_a\lambda_2}p_{n_b}.
\end{aligned}
$$

From the state $a_3$ it is possible to go to $a_4$, $a_7$, $a_2$ and $a_3$ and again using Equation (17) follows.

$$
\begin{aligned}
p(a_4|a_3) &= \int_0^{n_b} Q_t(a_4|a_3)\lambda(a_3)e^{-\int_0^t \lambda(a_3)ds}dt + e^{-\int_0^{na}\lambda(a_3)}Q_t(a_4|a_9), \\
&= \int_0^{n_b} p_{n_2n_1}\lambda_2 e^{-\lambda_2 t}dt = p_{n_2n_1}(1-e^{-\lambda_2 n_a}), \\
p(a_7|a_3) &= p_{n_2n_2}(1-e^{-n_b\lambda_2}), \\
p(a_2|a_3) &= \int_0^{n_b} Q_t(a_2|a_3)e^{-\lambda_2 t}dt + e^{-\int_0^{n_b}\lambda_2 dt}Q_t(a_2|a_9), \\
&= e^{-\int_0^{n_b}\lambda_2 dt}Q_t(a_2|a_9) = e^{-\lambda_2 n_b}p_{n_a}, \\
p(a_3|a_3) &= e^{-\lambda_2 n_b}p_{n_b}.
\end{aligned}
$$

Similarly from $a_4$ it is possible to go to $a_4$, $a_7$ and $a_1$.

$$\begin{aligned}
p(a_4|a_4) &= \int_0^{n_b} Q_t(a_4|a_4)e^{-\lambda_1 t}dt + e^{-\int_0^{n_b}\lambda_1 dt}Q_t(a_4|a_8),\\
&= \int_0^{n_b} p_{n_1n_1}e^{-\lambda_1 t}dt = p_{n_1n_1}(1-e^{-\lambda_1 n_b}),\\
p(a_7|a_4) &= p_{n_1n_2}(1-e^{-\lambda_1 n_b}),\\
p(a_1|a_4) &= \int_0^{n_b} Q_t(a_1|a_4)e^{-\lambda_1 t}dt + e^{-\lambda_1 n_b}Q_t(a_1|a_8) = e^{-\lambda_1 n_b}Q_t(a_1|a_8) = e^{-\lambda_1 n_b}.
\end{aligned}$$

From $a_5$ it is possible to go to $a_5$, $a_6$.

$$\begin{aligned}
p(a_5|a_5) &= \int_0^{n_a} Q_t(a_5|a_5)e^{-\lambda_1 t}dt + e^{-\lambda_1 n_a}Q_t(a_5|a_8),\\
&= \int_0^{n_a} p_{n_1n_1}e^{-\lambda_1 t}dt = p_{n_1n_1}(1-e^{-\lambda_1 n_a}),\\
p(a_6|a_5) &= p_{n_1n_2}(1-e^{-\lambda_1 n_a}),\\
p(a_1|a_5) &= \int_0^{n_a} Q_t(a_1|a_5)e^{-\lambda_1 t}dt + e^{-\lambda_1 n_a}Q_t(a_1|a_8) = e^{-\lambda_1 n_a}Q_t(a_1|a_8) = e^{-\lambda_1 n_a}.
\end{aligned}$$

From the state $a_6$ it is possible to go to $a_5$, $a_6$, $a_2$ and $a_3$.

$$\begin{aligned}
p(a_5|a_6) &= \int_0^{n_a} Q_t(a_5|a_6)\lambda_2 e^{-\lambda_2 t}dt + e^{-n_a\lambda_2}Q_t(a_5|a_9),\\
&= \int_0^{n_a} p_{n_2n_1}\lambda_2 e^{-\lambda_2 t}dt = p_{n_2n_1}(1-e^{-\lambda_2 n_a}),\\
p(a_6|a_6) &= p_{n_2n_2}(1-e^{-\lambda_2 n_a}),\\
p(a_2|a_6) &= \int_0^{n_a} Q_t(a_2|a_6)\lambda_2 e^{-\lambda_2 t}dt + e^{-n_a\lambda_2}Q_t(a_2|a_9) = e^{-n_a\lambda_2}p_{n_a},\\
p(a_3|a_6) &= e^{-n_a\lambda_2}p_{n_b}.
\end{aligned}$$

From $a_7$ it is possible to go to $a_5$, $a_7$, $a_2$ and $a_3$.

$$\begin{aligned}
p(a_5|a_7) &= \int_0^{n_b} Q_t(a_5|a_7)\lambda_2 e^{-\lambda_2 t}dt + e^{-n_b\lambda_2}Q_t(a_5|a_9),\\
&= \int_0^{n_b} p_{n_2n_1}\lambda_2 e^{-\lambda_2 t}dt = p_{n_2n_1}(1-e^{-\lambda_2 n_b}),\\
p(a_7|a_7) &= p_{n_2n_2}(1-e^{-\lambda_2 n_b}),\\
p(a_2|a_7) &= \int_0^{n_b} Q_t(a_2|a_7)\lambda_2 e^{-\lambda_2 t}dt + e^{-n_b\lambda_2}Q_t(a_2|a_9),\\
&= e^{-n_b\lambda_2}Q_t(a_2|a_9) = e^{-n_b\lambda_2}p_{n_a},\\
p(a_3|a_7) &= e^{-n_b\lambda_2}p_{n_b}.
\end{aligned}$$

From $a_8$ it is possible to go to $a_1$.

$$p(a_1|a_8) = \int_0^0 Q_t(a_1|a_8)\lambda_1 e^{-\lambda_1 t}dt + e^{-\int_0^0 \lambda_1 ds}Q_t(a_1|a_8) = 1.$$

From $a_9$ it is possible to go to $a_2$ and $a_3$.

$$p(a_2|a_9) = \int_0^0 Q_t(a_2|a_9)\lambda_2 e^{-\lambda_2 t}dt + e^{-\int_0^0 \lambda_2 ds}Q_t(a_2|a_9) = p_{n_a},$$

$$p(a_3|a_9) = \int_0^0 Q_t(a_3|a_9)\lambda_2 e^{-\lambda_2 t}dt + e^{-\int_0^0 \lambda_2 ds}Q_t(a_3|a_9) = p_{n_b}.$$

The transition matrix of the DTMC $(\boldsymbol{\theta}(\tau_k), \boldsymbol{w}(\boldsymbol{\tau}_k))$ associated with the sampled-data PDP is $\Pi = [p(a_i|a_j)]_{i,j}$ where $i, j \in \{1, 2, ..., 9\}$.

$$\Pi = \begin{bmatrix} p_{n_1n_1} & p_{n_a}p_{n_1n_2} & p_{n_b}p_{n_1n_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-n_a\lambda_2}p_{n_a} & e^{-n_a\lambda_2}p_{n_b} & 0 & p_{n_2n_1}c1 & p_{n_2n_2}c1 & 0 & 0 & 0 \\ 0 & e^{-n_b\lambda_2}p_{n_a} & e^{-n_b\lambda_2}p_{n_b} & p_{n_2n_1}c2 & 0 & 0 & p_{n_2n_2}c2 & 0 & 0 \\ e^{-n_b\lambda_1} & 0 & 0 & p_{n_1n_1}d1 & 0 & 0 & p_{n_1n_2}d1 & 0 & 0 \\ e^{-n_a\lambda_1} & 0 & 0 & 0 & p_{n_1n_1}d2 & p_{n_1n_2}d2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{n_2n_1} & p_{n_2n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{n_2n_1} & 0 & p_{n_2n_2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{n_a} & p_{n_b} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $c1 = 1 - e^{-n_a\lambda_2}$, $c2 = 1 - e^{-n_b\lambda_2}$, $d1 = 1 - e^{-n_b\lambda_1}$ and $d2 = 1 - e^{-n_a\lambda_1}$.

In a sampled-data system, the upset times and the recovery durations are an integer multiple of the sample period. The DTMC associated with the sampled-data PDP has stopping times which are exponentially distributed with parameters $\lambda_1$ and $\lambda_2$. Next the relationship between the sample period and the parameters of the exponential distribution is developed. A CTMC is first constructed from the DTMC $(\boldsymbol{\theta}(\tau_k), \boldsymbol{w}(\boldsymbol{\tau}_k))$. To start the construction, define the random variables $\boldsymbol{y}_n \triangleq (\boldsymbol{\theta}(\tau_n), \boldsymbol{w}(\boldsymbol{\tau}_n))$ mapping $(\Omega, \mathcal{F}, \Pr)$ to $(\bigcup_{i=1}^{9}\{a_i\}, \beta(\bigcup_{i=1}^{9}\{a_i\}))$ and $\boldsymbol{t}_n$ mapping the space $(\Omega, \mathcal{F}, \Pr)$ to $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. The stochastic process $\{\boldsymbol{y}_n\}_{n\in\mathbb{N}}$ is a DTMC with transition matrix $\Pi = [p(a_i|a_j)]_{i,j}$, where $i, j \in \{1, 2, ..., 9\}$, with arbitrary initial distribution $\nu$. The random variables $\{\boldsymbol{t}(n)\}$ are mutually independent and each $\boldsymbol{t}(n)$ has an exponential distribution with intensity $q_0(y_n)$, where $y_n$ is a sample of $\boldsymbol{y}_n$. Define the stochastic process $\Theta(t) \triangleq \boldsymbol{y}_{\boldsymbol{N}_t}$, where $\boldsymbol{N}_t = \sup\{n \geq 0 : \sum_{i=0}^{n-1} \boldsymbol{t}(i) \leq t\}$ and $\boldsymbol{N}_t < \infty$. From Theorem 10 it follows that $\Theta(t)$ is a CTMC with infinitesimal generator

$$\begin{aligned} \Lambda(a_j|a_i) &= q_0(a_i)p(a_j|a_i) \text{ for every } i \neq j, \\ \Lambda(a_i|a_i) &= -q_0(a_i)(1 - p(a_i|a_i)). \end{aligned}$$

Now that the CTMC has been characterized, the uniformization technique is used to obtain a DTMC, $\Theta[k]$, as follows. Let $q = \max\{-\Lambda(a_i|a_i);\ i = 1, 2, ..., 9\}$. The transition matrix of the DTMC is

$$\Pi = I + \frac{1}{q}\Lambda,$$

where

$$\begin{aligned} \Pi_{ii} &= 1 - \frac{q_0(a_i)}{q}(1 - p(a_i|a_i)) \\ \Pi_{ij} &= \frac{q_0(a_i)}{q} p(a_j|a_i), \text{ for every } i \neq j. \end{aligned}$$

The continuous-time Markov chain, $\Theta(t)$, is characterized by the parameters $q_0(a_i)$. The discrete-time Markov chain, $\Theta[k]$, is characterized by the parameter $q$. One way to related the time behavior of the upset and recovery algorithm with the sample instant so that the Markov property is preserved is to choose $q_0(a_i) = \lambda_1 + \lambda_2$ for every $i = 1, 2, ..., 9$. This implies

$$\begin{aligned} \Pi_{ii} &= 1 - \frac{q_0(a_i)}{q}(1 - p(a_i|a_i)) = p(a_i|a_i), \\ \Pi_{ij} &= \frac{q_0(a_i)}{q} p(a_j|a_i) = p(a_j|a_i), \end{aligned}$$

where $q \triangleq \frac{1}{\mathrm{T}} \triangleq \lambda_1 + \lambda_2$. From this one obtains the transition matrix, $\Pi = [\Pi_{ij}]_{i,j\in\{1,2,...,9\}}$, of a DTMC, $\Theta[k]$, observed at the time instants $\{0, \mathrm{T}, 2\mathrm{T}, ...\}$. This DTMC takes values $a_i$, where each $a_i$ indicates the mode of operation of the sampled-data system as shown in Table II. The main conclusion of this section is that the switching rule between the nominal and recovery closed-loop system can be modeled as a DTMC, $\Theta[k]$, with transition matrix $\Pi$. Now that the switching rule is characterized, the PDP model of the sampled-data is described next.

### IV.3.2 PDP Model of Sampled-Data Systems

Since the switching rule is characterized by the Markov chain $\Theta[k]$, the sampled-data system associated with a jump linear controller in FIG. 3 can be represented as a PDP in several ways depending on the problem of interest. For sampled-data systems with a jump linear controller, however, it is possible to define a single sampled-data PDP that is useful for analysis. This PDP consists of the closed-loop system's hybrid stochastic process $\chi_t \triangleq (\Theta[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ which aggregates the continuous states of the plant, $\boldsymbol{x}_p(t)$, and the discrete states of the controller, $\boldsymbol{x}_c[k]$, with the states of the switching rule $\boldsymbol{\Theta}[k]$ taking values in $\{a_i : i = 1, 2, ..., 9\}$. The sample paths of the aggregated plant and controller states take values in an open subset $E \subseteq \mathbb{R}^{n_{x_p}+n_{x_c}}$. Thus, the state space of the sampled-data PDP is $\mathbb{E} \triangleq \bigcup_{i=1}^{9} \big\{\{a_i\} \times E\big\}$ with the following local characteristics:

(a) The stochastic motion for $t \in [k\mathrm{T}, (k+1)\mathrm{T})$ corresponding to the mode $\boldsymbol{\Theta}[k]$ with initial condition given by $(\boldsymbol{z}[k] = [\boldsymbol{x}_p'[k], \boldsymbol{x}_c'[k]]')$ is $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\Theta}[k]) = N(kT, t)\boldsymbol{z}[k] + \boldsymbol{m}(t)$ (c.f. Equation (3), Chapter II).

(b) Since the sampled-data PDP state can only jump at $t = k\mathrm{T}$, $k \in \mathbb{Z}^+$, the distribution of the jumping times can be trivially modeled by $F(t; \Theta[k], x_p[k], x_c[k]) = 1_{\{t<(k+1)\mathrm{T}\}}$ for $t > k\mathrm{T}$. Observe that the jump rate $\lambda$ is set to zero so $\Lambda(t; \boldsymbol{\Theta}[k], (\boldsymbol{x}_p[k], \boldsymbol{x}_c[k])) = 0$.

(c) The transition measure $\mathrm{D}(A|\Theta, x_p, x_c)$ has the properties that for each event $A \in \beta\big(\bigcup_{i=1}^{9} \big\{\{a_i\}\times E\big\}\big)$, $\mathrm{D}(A|\cdot)$ is measurable on $\bigcup_{i=1}^{9} \big\{\{a_i\}\times E\big\}$, and for each $(\Theta, x_p, x_c) \in \bigcup_{i=1}^{9} \big\{\{a_i\} \times E\big\}$, $\mathrm{D}(\cdot|\Theta, x_p, x_c)$ is a measure on $\beta\big(\bigcup_{i=1}^{9} \big\{\{a_i\} \times E\big\}\big)$.

Now that the sampled-data PDP has been defined, it is important to determine the local characteristic D. This will be resolved in the remainder of this section. D is characterized using a Feller semigroup, which is introduced next. The *hybrid state space* of the stochastic motion of the sampled-data system with stochastic upsets is $\mathbb{E}$. Assume that $\mathbb{E}$ is a locally compact separable metric space with respect to one of the suitable topologies, and let $\hat{\mathbb{E}} = \mathbb{E} \cup \{\Delta\}$ be the one point compactification of $\mathbb{E}$, where $\Delta$ is the point at infinity. The $\sigma$-algebra defined on $\hat{\mathbb{E}}$ is denoted by $\mathcal{B}(\hat{\mathbb{E}})$. Let $C_0(\mathbb{E})$ be the space of continuous functions $h : \mathbb{E} \to \mathbb{R}$ with $h(\zeta) \to 0$ as $\zeta$ approaches $\infty$, using a suitable metric. The space $(C_0(\mathbb{E}), ||\cdot||)$ is a Banach space if $||h|| \triangleq \sup_\zeta |h(\zeta)|$, $\zeta \in \mathbb{E}$. This is the Banach space of continuous functions on $\mathbb{E}$ that vanish at infinity. Let $\mathcal{L}(C_0(\mathbb{E})) \triangleq \mathcal{L}(C_0(\mathbb{E}), C_0(\mathbb{E}))$ be the space of bounded linear operators. The following definitions help to induce a Markov process using bounded linear operators [36, pp 314].

**Definition 19.** The family of bounded linear operators $\{L_n\} \triangleq \{L_n \in \mathcal{L}(C_0(\mathbb{E})) : n \in \mathbb{Z}^+\}$ is said to be an *operator semigroup* if the composition of the operators satisfies

$$L_{n+m} = L_n L_m,$$

where $L_0 = I$ is the identity operator. The family of bounded linear operators is called a *contraction semigroup* if $||L_n|| \leq 1, \forall n \geq 0$, and it is called a *positive semigroup* if $\forall n \geq 0, \forall \zeta \in \mathbb{E}$, $L_n(h(\zeta)) \geq 0$ whenever $h(\zeta) \geq 0$, $h \in C_0(\mathbb{E})$.

**Definition 20.** The family of bounded linear operators $\{L_n\}$ on $C_0 \triangleq \mathcal{L}(C_0(\mathbb{E}))$ is a *Feller semigroup* if $\{L_n\}$ is a positive contraction semigroup of operators with the following properties:

(a) $L_n C_0 \subset C_0$, $n \geq 0$,

(b) $L_n h(\zeta) \longrightarrow h(\zeta)$ as $n \longrightarrow 0$, $\forall h \in C_0$, $\forall \zeta \in \mathbb{E}$.

**Definition 21.** A mapping $\mu : \hat{\mathbb{E}} \times \mathcal{B}(\hat{\mathbb{E}}) \longrightarrow \mathbb{R}^+$ is called a *stochastic kernel* if the function $\mu(s, B)$ is measurable for every event $B \in \mathcal{B}(\hat{\mathbb{E}})$, and it is a probability measure for every $s \in \hat{\mathbb{E}}$.

Under certain conditions, the existence of a Feller semigroup $\{L_n\}$ implies the existence of a family of Markov processes. The following lemma provides such a necessary and sufficient condition [36, Theorem 7.4 and Proposition 17.14].

**Lemma 4.** *For every Feller semigroup $\{L_n\}$ and for any probability measure $\kappa$ on $\hat{\mathbb{E}}$ there exists a Markov process $\boldsymbol{X}^\kappa$ with initial distribution $\kappa$ and Markov transition kernels $\mu_n$ on $\hat{\mathbb{E}}$ satisfying $L_n h(\zeta) = \int h(y)\mu_n(\zeta, dy)$, $\forall h \in C_0$, $\forall \zeta \in \hat{\mathbb{E}}$, and $\mu_n(\Delta, \{\Delta\}) = 1$ $\forall n \in \mathbb{Z}^+$.*

It is now possible to introduce the operator $L \in C_0$ for the sampled-data PDP. This definition is adapted from [39].

**Definition 22.** The family of operators $\{L_n\}$ is defined recursively by $L_0 h = h$ and $L_{n+1}h = L(L_n h)$, $n \geq 0$, where $L$ is the following bounded linear operator in $C_0$

$$Lh(\Theta, x_p, x_c) = \int_{\hat{\mathbb{E}}} h(\tilde{\Theta}, \tilde{x}_p, \tilde{x}_c) p(d\tilde{x}_p|\tilde{x}_c, x_p) p(d\tilde{x}_c|\tilde{\Theta}, x_c) p(d\tilde{\Theta}|\Theta), \tag{18}$$

where the transition kernels in the integrals are characterized by the following probabilities:

$$\begin{aligned} p(d\tilde{\Theta}|\Theta[k]) &= \Pi(\boldsymbol{\Theta}[k+1] \in d\tilde{\Theta}|\Theta[k]) \\ p(d\tilde{x}_c|\Theta[k], x_c[k]) &= \Pr(\boldsymbol{x}_c[k+1] \in d\tilde{x}_c|\Theta[k], x_c[k]) \\ p(d\tilde{x}_p|x_c[k], x_p[k]) &= \Pr(\boldsymbol{x}_p[k+1] \in d\tilde{x}_p|x_c[k], x_p[k]). \end{aligned}$$

This family of operators is a Feller semigroup. The following theorem shows that $(\boldsymbol{\Theta}[k], \boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$ is a Markov chain.

**Theorem 11.** *The Feller semigroup $\{L_n\}$ in Definition 22, equation (18), induces a unique Markov process satisfying*

$$\begin{aligned} L_n h(\Theta, x_p, x_c) &= E^{(\Theta, x_p, x_c)} h(\boldsymbol{\Theta}[n], \boldsymbol{x}_p[n], \boldsymbol{x}_c[n]) \\ &= \int h(\tilde{\Theta}, \tilde{x}_p, \tilde{x}_c) \cdot \mu_n(\Theta, x_p, x_c; d\tilde{\Theta}, d\tilde{x}_p, d\tilde{x}_c), \end{aligned}$$

*where $E^{(\Theta, x_p, x_c)} h(\boldsymbol{\Theta}[n], \boldsymbol{x}_p[n], \boldsymbol{x}_c[n])$ is the expected value with respect to the measure $\mu_n(\Theta, x_p, x_c, d\tilde{\Theta}, d\tilde{x}_p, d\tilde{x}_c)$.*

*Proof.* Because the family of operators $\{L_n\}$ with $L$ in (18) is a Feller semigroup, it is possible to apply Lemma 4. This implies the existence of a Markov chain $(\boldsymbol{\Theta}[k], \boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$ with initial distributions on $\hat{\mathbb{E}}$. □

One very important consequence of this theorem is the existence of a discrete-time Markov process $(\boldsymbol{\Theta}[k], \boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$ with transition kernel $L1_A(\Theta, x_p, x_c) = \int_A p(d\tilde{x}_p|\tilde{x}_c, x_p)p(d\tilde{x}_c|\tilde{\Theta}, x_c)p(d\tilde{\Theta}|\Theta)$, where $A \in \mathcal{B}(\hat{\mathbb{E}})$. Obviously this is a probability measure for every $(\Theta, x_p, x_c)$ and a measurable function for every event $A$. The following theorem gives a characterization of the sampled-data PDP.

**Theorem 12.** *The sampled-data PDP has the following local characteristics: the closed-loop dynamics for each sample period $T \in \mathbb{R}^+$ are given by $\mathbf{z}(t, \boldsymbol{z}[k], \boldsymbol{\Theta}[k]) = N(kT, t)\boldsymbol{z}[k] + \boldsymbol{m}(t)$ for $t \in [kT, (k+1)T)$, the survival function is $F(t, \Theta[k], x_p[k], x_c[k]) = 1_{\{t<(k+1)T\}}$ for $t > kT$, and the transition kernel is* $\mathrm{D}(A|\Theta, x_p, x_c) = L1_A(\Theta, x_p, x_c)$.

*Proof.* The only task is to characterize the kernel $\mathrm{D}(A|\Theta, x_p, x_c)$ of the sampled-data PDP. Since the operator $L$ in (18) is defined on $\hat{\mathbb{E}}$, from Theorem 11 it follows that $\mathrm{D}(A|\Theta, x_p, x_c) = L1_A(\Theta, x_p, x_c) = \int_A p(d\tilde{x}_p|\tilde{x}_c, x_p)p(d\tilde{x}_c|\tilde{\Theta}, x_c)p(d\tilde{\Theta}|\Theta)$. □

The PDP derived in Lemma 12 is a sampled-data PDP as defined in [28]. Now that the sampled-data PDP is characterized, it is possible to consider the invariant measure of the sampled-data with stochastic upsets. This is developed next.

## IV.4 STATIONARY DISTRIBUTIONS

Knowing that the sampled-data system has a PDP representation, it is possible to use some of the known PDP results, such as the existence and uniqueness of an invariant measure. This is helpful in stability analysis.

**Definition 23.** Let $(W, \mathbf{W})$ be a measurable space, and consider a stochastic kernel $p(A|x)$, such that for every $x \in W$, $p(A|\cdot)$ is a measurable function, and for every event $A \in \mathbf{W}$, $p(\cdot|x)$ is a probability measure. A measure $\nu$ is an *invariant measure* if for every event $A$

$$\nu(A) = \int_W p(A|x)\nu(dx).$$

In this section a relationship is presented between the sets $\Pi_{PDP}$ and $\Pi_{MC}$ of invariant measures for the processes $(\Theta[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ and the embedded Markov chain $(\Theta[k], \boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$, respectively. Let $\Pi^*_{MC}$ be the set of finite invariant measures $\pi \in \Pi_{MC}$. The relationship between the invariant measure $\mu$ for the process $(\Theta[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ and the invariant measure $\pi$ for the chain $(\Theta[k], \boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$ is given by the following theorem from [11, Theorem 2], [17, Theorem 34. 31].

**Theorem 13.** *If $\pi \in \Pi^*_{MC}$ then $\mu$ belongs to $\Pi_{PDP}$, with*

$$\mu(A) = \frac{\int_{\mathbb{E}} \int_0^{t_*(x)} 1_A(\Phi_\theta(t,x)) e^{-\Lambda(t,x)} dt \pi(dx)}{\int_{\mathbb{E}} \int_0^{t_*(x)} e^{-\Lambda(t,x)} dt \pi(dx)},$$

*where $t_*(x) = \inf\{t > 0 : \Phi_\theta(t,w) \in \partial E_\theta\}$ and $\pi(dx)$ denotes that the integration is over the variable $x$ using the measure $\pi$.*

For the case of the sampled-data PDP consider the follow theorem.

**Theorem 14.** *The invariant measure $\mu$ of $(\Theta[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ for the sampled-data PDP is*

$$\mu(A) = \frac{\int_{\hat{\mathbb{E}}} \int_0^T \mathbf{1}_A(\Theta, \mathbf{z}(t, \boldsymbol{z}[k], \Theta[k])) dt \nu(d\Theta, x_p, x_c) \eta(dx_p, dx_c)}{T \int_{\hat{\mathbb{E}}} \nu(\Theta, x_p, x_c) \eta(dx_p, dx_c)},$$

*where $\eta$ is an invariant measure of $(\boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$, and the invariant measure $\pi \in \Pi^*_{MC}$ with respect to the stochastic kernel $\nu(d\Theta, x_p, x_c)$ satisfies*

$$\pi(A_\Theta, A_{x_p}, A_{x_c}) = \int_{A_{x_p} \times A_{x_c}} \nu(A_\Theta, x_p, x_c) \eta(dx_p, dx_c),$$

*where $A_\Theta \in \beta(\{a_i : i = 1, 2, ..., 9\})$, $A_{x_p} \in \mathcal{B}(\mathbb{R}^{n_{x_p}})$, and $A_{x_c} \in \mathcal{B}(\mathbb{R}^{n_{x_c}})$.*

*Proof.* Since $\Lambda(t,x) = \int_0^t \lambda(\mathbf{z}(s,\boldsymbol{z}[k],\Theta[k]))ds = 0$, implies that $t_*(\Theta, x_p, x_c) = \mathtt{T}$, then from Theorem 13 it follows that

$$\mu(A) = \frac{\int_{\hat{\mathbb{E}}}\int_0^{\mathtt{T}} \mathbf{1}_A(\Theta, \mathbf{z}(t,\boldsymbol{z}[k],\Theta[k]))dt\pi(d\Theta, dx_p, dx_c)}{\int_{\hat{\mathbb{E}}} \mathtt{T}\cdot\pi(d\Theta, dx_p, dx_c)}.$$

Observing that $\pi(d\Theta, dx_p, dx_c) = \nu(d\Theta, x_p, x_c)\eta(dx_p, dx_c)$ completes the proof. □

Theorem 14 gives a formula to obtain an invariant measure of $(\boldsymbol{\Theta}[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ from an invariant measure of $(\boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$. In the next section, a condition is given to obtain the invariant measure of $(\boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$.

## IV.5 STABILITY ANALYSIS OF SAMPLED-DATA SYSTEMS AS A PIECEWISE DETERMINISTIC MARKOV PROCESS

### IV.5.1 Invariant Measure of Sampled-Data Systems

Stability analysis of a sampled-data system can be performed by analyzing the convergence of the distributions over time to an invariant measure. From Theorem 14, it was concluded that an invariant measure $\mu$ of $(\boldsymbol{\Theta}[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ can be recovered from an invariant measure of $(\boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$. In this section the invariant measure of $(\boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$ is analyzed. In Chapter II, Theorem 3 the difference equation for the linearized discrete-time system $\boldsymbol{z}'[k] = \left[\boldsymbol{x}_p'[k]\ \boldsymbol{x}_c[k]\right]'$ was shown to be

$$\boldsymbol{z}[k+1] = M_{\boldsymbol{\Theta}[k]}\boldsymbol{z}[k], \tag{19}$$

where $M_{\boldsymbol{\Theta}[k]} = \begin{bmatrix} e^{A_p\mathtt{T}} & \int_0^{\mathtt{T}} e^{A_p(\mathtt{T}-s)}dsB_pF_c \\ B_{\boldsymbol{\Theta}[k]}C_p & A_{\boldsymbol{\Theta}[k]} \end{bmatrix}$. The function $M_{\boldsymbol{\Theta}[k]}$ is a measurable mapping between $(\mathbb{E}, \beta(\mathbb{E}))$ and $(\mathbb{R}^{n_{x_p}+n_{x_c}}, \beta_o(\mathbb{R}^{n_{x_p}+n_{x_c}}))$ called the *random transformation* of system (19). To study the evolution of the probability measure of $(\boldsymbol{\Theta}[k], \boldsymbol{z}[k])$, it is necessary to define the inner product $\langle g(z), \mu_n(\omega,\cdot)\rangle = \int_{\mathcal{I}_N^\infty}\int_E g(z)\mu_n(\omega, dz)\nu(d\omega)$, where $g \in C$, $C$ is the set of continuous functions from $\mathbb{R}^{n_{x_p}+n_{x_c}}$ to $\mathbb{R}$ with compact support, $\mathcal{I}_N = \{a_i : i = 1, 2, ..., N\}$ (here $N = 9$), $E \subseteq \mathcal{I}_N^n \times \mathbb{R}^{n_{x_p}+n_{x_c}}$, $\mathcal{I}_N^n \triangleq \mathcal{I}_N \times \mathcal{I}_N \times \cdots \mathcal{I}_N$ ($n$ times), $\mathcal{I}_N^\infty \triangleq \mathcal{I}_N \times \mathcal{I}_N \times \cdots$, $\omega \in \mathcal{I}_N^\infty$, and $\mu_n \in \mathcal{M}$, with $\mathcal{M}$ denoting the set of locally finite measures on $E$. Finally, Let $M_{\Theta[k]}$ be a random transformation of system (19), and let $M_{\omega_{n-1}} \cdots M_{\omega_0}(z)$ denote the composition $M_{\omega_{n-1}} \circ \cdots \circ M_{\omega_0}(z)$. The following definitions are taken from [38].

**Definition 24.** The operator $\mathbf{P} : \mathcal{M} \longrightarrow \mathcal{M}$ given by $\langle g(z), \mathbf{P}\mu_n(\omega,\cdot)\rangle = \langle g(M_{\omega_n}(z)), \mu_n(\omega,\cdot)\rangle$ is called the *Foias operator* corresponding to the dynamical system (19).

The following lemma characterizes the evolution of the Foias operator $\mathbf{P}$ in terms of a random transformation $M_{\Theta[k]}$.

**Lemma 5.** *The Foias operator corresponding to the dynamical system* (19) *satisfies the following property*

$$\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mathbf{P}^n\mu_0(dx)\nu(d\omega) = \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_{n-1}}..M_{\omega_0}(x))\mu_0(dx)\nu(d\omega),$$

*for every $n \geq 1$, initial distribution $\mu_0$ and $g \in C$.*

*Proof.* The proof is by mathematical induction. First, prove that is true for $n = 1$. Using the definition of the Foias operator its follows that

$$\begin{aligned}\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mathbf{P}\mu_0(dx)\nu(d\omega) &= \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mu_1(\omega_0,dx)\nu(d\omega)\\ &= \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_0}(x))\mu_0(dx)\nu(d\omega).\end{aligned}$$

Now, prove it is true for $n = 2$:

$$\begin{aligned}&\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mathbf{P}^2\mu_0(dx)\nu(d\omega)\\ =& \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mathbf{P}\mu_1(\omega_0,dx)\nu(d\omega)\\ =& \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mu_2(\omega_0,\omega_1,dx)\nu(d\omega)\\ =& \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_1}(x))\mu_1(\omega_0,dx)\nu(d\omega)\\ =& \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_1}M_{\omega_0}(x))\mu_0(dx)\nu(d\omega).\end{aligned}$$

The above analysis twice applied the Foias operator definition. Now suppose the claim is true for $n-1$. Then

$$\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mathbf{P}^{n-1}\mu_0(dx)\nu(d\omega) = \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_{n-2}}\cdots M_{\omega_0}(x))\mu_0(dx)\nu(d\omega).$$

One needs to prove for $n$ that

$$\begin{aligned}
&\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mathbf{P}^n\mu_0(dx)\nu(d\omega)\\
=&\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(x)\mu_n(\omega_0,.,\omega_{n-1},dx)\nu(d\omega)\\
=&\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_{n-1}}(x))\mu_{n-1}(\omega_0,.,\omega_{n-2},dx)\nu(d\omega)\\
=&\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_{n-1}}(x))\mathbf{P}^{n-1}\mu_0(dx)\nu(d\omega)\\
=&\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_{n-1}}\cdots M_{\omega_1}M_{\omega_0}(x))\mu_0(dx)\nu(d\omega).
\end{aligned}$$

Again the Foias operator definition was applied $n$ times. Hence, the identity holds for all $n \geq 1$. □

This result can be interpreted as follows. The mean of the random variable $g(\boldsymbol{z}[n])$ is equal to the mean of $g \circ M_{\omega_{n-1}} \circ \ldots \circ M_{\omega_0}(\boldsymbol{z}[0])$. In other words, even if we do not know the measure of the random variable $\boldsymbol{z}[n]$, one can compute the mean of $g(\boldsymbol{z}[n])$ from the initial distribution $\mu_0$ and the distribution of the jumps $\boldsymbol{\Theta}$. Now the definition of an invariant measure is given in terms of the Foias operators.

**Definition 25.** A measure $\mu_* \in \mathcal{M}$ is called *invariant with respect to the Foias operator* $\mathbf{P}$ if $\langle g(z), \mathbf{P}\mu_*\rangle = \langle g(z), \mu_*\rangle$ for every $g \in C$.

The next theorem shows that system (19) has an invariant measure [28]. The delta Dirac measure defined by: $\delta(A) = 1$ if $0 \in A$ and $\delta(A) = 0$ if $0 \notin A$ for every event $A \in \beta_o(\mathbb{R}^{n_{xp}+n_{xc}})$.

**Theorem 15.** *If $M_{\boldsymbol{\Theta}[k]}$ is a random transformation of system* (19) *having the Foias operator* $\mathbf{P}$ *and $M_j(0) = 0$ for every $j \in \mathcal{I}_N$, then the delta Dirac measure is invariant.*

*Proof.* Consider

$$\begin{aligned}
\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(z)\mathbf{P}^n\delta(dz)\nu(d\omega) &= \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(M_{\omega_{n-1}}\cdots M_{\omega_1}M_{\omega_0}(z))\delta(dz)\nu(d\omega)\\
&= \int_{\mathcal{I}_N^\infty} g(M_{\omega_{n-1}}\cdots M_{\omega_1}M_{\omega_0}(0))\nu(d\omega).
\end{aligned}$$

Since $M_{\omega_i}(0) = 0$ it follows that

$$\int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{xp}+n_{xc}}} g(z)\mathbf{P}^n\delta(dz)\nu(d\omega) = \int_{\mathcal{I}_N^\infty} g(0)\nu(d\omega) = g(0).$$

Hence, $\mathbf{P}^n\delta = \delta$ for every $n \geq 0$. $\square$

The stability analysis of the sampled-data PDP is studied using the following definition of stability in terms of the Foias operator.

**Definition 26.** The dynamical system (19) is *weakly asymptotically stable* to the invariant measure $\mu_*$ if for every initial measure $\mu_0$ it follows that

$$\lim_{n\longrightarrow\infty}\langle g(z), \mathbf{P}^n\mu_0\rangle = \langle g(z), \mu_*\rangle$$

for every $g \in C$.

The next two lemmas are used in the proof of Theorem 16 and can be found in [38, pp. 422].

**Lemma 6.** *Let* $\mathbf{P}$ *be a Foias operator of system* (19). *The dynamical system* (19) *is weakly asymptotically stable if it has an invariant distribution and if*

$$\lim_{n\to\infty}\langle g, \mathbf{P}^n\mu_0 - \mathbf{P}^n\kappa_0\rangle = 0 \tag{20}$$

*for* $g \in C$ *and* $\kappa_0, \mu_0 \in \mathcal{M}$.

**Lemma 7.** *Let* $C_* \subset C$ *be a dense subset. If condition* (20) *holds for every* $g \in C_*$ *and* $\mu_0$, $\kappa_0$ *with bounded supports, then it is satisfied for arbitrary* $g \in C$ *and* $\kappa_0, \mu_0 \in \mathcal{M}$.

The next theorem gives a sufficient condition for the weak convergence of the dynamical system (19) to the delta Dirac measure.

**Theorem 16.** *Let* $\mathbf{P}$ *be a Foias operator corresponding to the dynamical system* (19). *If* $\mathbf{\Theta}[k]$ *is a Markov chain with transition probability matrix* $\Pi$ *taking values in* $\mathcal{I}_N$, $M_i(0) = 0$, $||M_i(z) - M_i(w)|| \leq l_i|z - w|$ *for every* $i \in \mathcal{I}_N$, *and the matrix*

$$\Gamma = \begin{bmatrix} l_1\Pi_{11} & \cdots & l_N\Pi_{1N} \\ l_1\Pi_{12} & \cdots & l_N\Pi_{2N} \\ \vdots & & \vdots \\ l_1\Pi_{1N} & \cdots & l_N\Pi_{NN} \end{bmatrix}$$

*has spectral radius less than 1 then the dynamical system* (19) *is weakly asymptotically stable to the delta Dirac measure, where* $\Pi_{ij}$ *are the transition probabilities of the Markov chain* $\mathbf{\Theta}[k]$.

*Proof.* Observe that

$$\begin{aligned}
\langle g(M_{\omega_{n-1}}\cdots M_{\omega_0}(z)),\mu_0\rangle &= \int_{\mathcal{I}_N^\infty}\int_{\mathbb{R}^{n_{x_p}+n_{x_c}}} g(M_{\omega_{n-1}}\cdots M_{\omega_0}(z))\mu_0(dz)\nu(d\omega) \\
&= \int_{\mathcal{I}_N^\infty} g(M_{\omega_{n-1}}\cdots M_{\omega_0}(\overline{z}))\nu(d\omega),
\end{aligned}$$

where the mean value theorem guarantees the existence of $\overline{z}\in E$. This implies that for any two initial measures the following is satisfied

$$\begin{aligned}
& \left|\langle g(z),\mathbf{P}^n\mu_0\rangle - \langle g(z),\mathbf{P}^n\kappa_0\rangle\right| \\
= & \left|\int_{\mathcal{I}_N^\infty} g(M_{\omega_{n-1}}\cdots M_{\omega_0}(z_1))\nu(d\omega) - \int_{\mathcal{I}_N^\infty} g(M_{\omega_{n-1}}\cdots M_{\omega_0}(z_2))\nu(d\omega)\right| \\
\leq & \int_{\mathcal{I}_N^\infty} |g(M_{\omega_{n-1}}\cdots M_{\omega_0}(z_1)) - g(M_{\omega_{n-1}}\cdots M_{\omega_0}(z_2))|\nu(d\omega),
\end{aligned}$$

where the existent of $z_1$ and $z_2$ are guarantee by the mean value theorem. Now consider a subset of Lipschitz functions $C_*$ of $C$. If $g\in C_*$ then it follows that

$$\begin{aligned}
& \left|\langle g(z),\mathbf{P}^n\mu_0\rangle - \langle g(z),\mathbf{P}^n\kappa_0\rangle\right| \\
\leq & \ \overline{v}\int_{\mathcal{I}_N^\infty} ||M_{\omega_{n-1}}\cdots M_{\omega_0}(z_1) - M_{\omega_{n-1}}\cdots M_{\omega_0}(z_2)||\nu(d\omega) \\
= & \ \overline{v}E\{||M_{\omega_{n-1}}\cdots M_{\omega_0}(z_1) - M_{\omega_{n-1}}\cdots M_{\omega_0}(z_2)||\} \\
\leq & \ \overline{v}\sum_{i_0,\cdots,i_{n-1}=1}^{N} E\{||M_{i_{n-1}}\cdots M_{i_0}(z_1) - M_{i_{n-1}}\cdots M_{i_0}(z_2)|| \\
& \cdot 1_{[\omega_0=i_0,\cdots,\omega_{n-1}=i_{n-1}]}\} \\
\leq & \ \overline{v}\sum_{i_0,\cdots,i_{n-1}=1}^{N}\{l_{i_0}\cdots l_{i_{n-1}}|z_1-z_2|\Pr(\omega_0=a_{i_0},\cdots,\omega_{n-1}=a_{i_{n-1}})\} \\
\leq & \ \overline{v}[l_1\cdots l_N]\Gamma^{n-1}[p_0(1)\cdots p_0(N)]'|z_1-z_2|,
\end{aligned}$$

where $\overline{v}>0$, $\overline{v}\in\mathbb{R}$ is the constant value obtained from the Lipschitz continuity of $g\in C$. This implies that $\lim_{n\to\infty}\langle g,\mathbf{P}^n\mu_0-\mathbf{P}^n\kappa_0\rangle = 0$ for every $g$ satisfying the Lipschitz condition. By Lemma 7, the dynamical system (19) is then weakly asymptotically stable to the delta Dirac measure. $\square$

The main result of this section is the stability analysis of the sampled-data system with a stochastic upset. Now using the sufficient condition for convergence to the delta Dirac for $(\boldsymbol{x}_p[k],\boldsymbol{x}_c[k])$ and the relationship of the invariant measures developed in Section IV.4 for $(\Theta[k],\boldsymbol{x}_p(t),\boldsymbol{x}_c[k])$ and $(\Theta[k],\boldsymbol{x}_p[k],\boldsymbol{x}_c[k])$, the following result for convergence to the invariant distribution of the sampled-data system with stochastic upsets is given.

**Theorem 17.** *If the matrix*

$$\boldsymbol{A} = \begin{bmatrix} ||M_1||\Pi_{11} & \cdots & ||M_N||\Pi_{N1} \\ ||M_1||\Pi_{12} & \cdots & ||M_N||\Pi_{42} \\ \vdots & & \vdots \\ ||M_1||\Pi_{1N} & \cdots & ||M_N||\Pi_{NN} \end{bmatrix}$$

*has spectral radius less than* 1 *then the sampled-data PDP* $(\Theta[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ *of a linear time invariant plant with stochastic upsets converges in distribution to the invariant measure* $\delta(0,0,\Theta)$, *where*

$$M_i = \begin{bmatrix} e^{A_p T} & \int_0^T e^{A_p(T-s)} ds B_p F_c \\ B_i C_p & A_i \end{bmatrix},$$

$\delta(0,0,\Theta) = \delta(0,0)\delta(\Theta)$ *takes value one for* $(0,0,\Theta)$ *and zero for other cases. The* $\Pi_{ij}$*'s are the transition probabilities of the Markov chain* $\boldsymbol{\Theta}[k]$ *with transition matrix* $\Pi = [\Pi_{ij}]$.

*Proof.* Since A has spectral radius less than one, it follows from Theorem 16 for $(\boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$ that the system (19) converges to the invariant measure $\delta(0,0)$. By Theorem 14, the invariant measure of the sampled-data PDP is

$$\mu(A) = \frac{\int_{\mathbb{E}} \int_0^{\mathrm{T}} \mathbf{1}_A(\Theta, \mathbf{z}(t, \boldsymbol{z}[k], \Theta[k])) dt \nu(x_p, x_c, d\Theta) \delta(dx_p, dx_c)}{\int_{\mathbb{E}} t_*(\Theta, x_p, x_c) \nu(x_p, x_c, d\Theta) \delta(dx_p, dx_c)},$$

where $\delta(dx_p, dx_c)$ is equal to one for $x_p = x_c = 0$, and at this point $\mathbf{z}(t, \boldsymbol{z}[k], \Theta[k])) = 0$. This implies that

$$\begin{aligned} \mu(A) &= \frac{\int \int_0^{\mathrm{T}} \mathbf{1}_A(\Theta, 0) dt \nu(0,0,d\Theta)}{\int \mathrm{T}\nu(0,0,d\Theta)} \\ &= \frac{\int \mathrm{T} \cdot \mathbf{1}_A(\Theta, 0) \nu(0,0,d\Theta)}{\int \mathrm{T} \cdot \nu(0,0,d\Theta)} \\ &= \frac{\int \mathbf{1}_A(\Theta, 0) \nu(0,0,d\Theta)}{\int \nu(0,0,d\Theta)} = \delta(0,0,\Theta), \end{aligned}$$

where $A = (\Theta, A_{x_p}, A_{x_c}) \in \beta(\mathbb{E})$. □

The main idea of Theorem 17 is that it provides a testable sufficient condition for convergence to the delta Dirac distribution of the sampled-data PDP.

## IV.6 CHAPTER SUMMARY

One of the main objectives of this chapter was to give a methodology for embedding the sampling instants in a PDP model. This methodology had two main parts. First, the subordinated Markov chain technique was used to obtain a CTMC model from the associated DTMC of the PDP model. Second, the uniformization technique was used to obtain a DTMC at each sampling instant $k\mathrm{T}$. If $\frac{1}{\mathrm{T}} = \lambda_1 + \lambda_2$ then it was shown that the DTMC at each sampling instant has the same transition matrix as the DTMC model associated with the PDP model. Also, this chapter gives some initial results on the representation of sampled-data stochastic systems as piecewise-deterministic Markov processes. Using known PDP properties, relationships between the invariant measures of $(\Theta[k], \boldsymbol{x}_p(t), \boldsymbol{x}_c[k])$ and $(\Theta[k], \boldsymbol{x}_p[k], \boldsymbol{x}_c[k])$ were given. Using the PDP representation of the sampled-data system with stochastic upsets, a sufficient condition for the stability in distribution to the delta Dirac distribution was obtained.

# CHAPTER V

# CONCLUSIONS AND FUTURE RESEARCH

## V.1 CONCLUSIONS

The main contributions of this dissertation are the development of models of sampled-data systems with stochastic upsets; the establishment of the equivalence between these models and their discretized versions; and the development of tools to analyze the stability of these systems that may avoid the numerical issues encountered with other techniques.

The sampled-data systems of interest were introduced in Chapter I. Models to analyze them were presented in Chapters II, III and IV. In Chapter II, in particular, the appropriate framework to analyze these stochastic sampled-data systems was presented. This made it possible to establish the equivalence between the sampled-data systems and their discretized version. In Chapter III, the sampled-data system with stochastic upsets was modeled as a dynamically colored Petri net (DCPN), where the possibility of inducing a failure to the controller is modeled as a continuous time Markov chain (CTMC). This model is intuitively more appealing than a DTMC, because the faults can happen at any time. On the other hand, this model has the following problem. In a sampled-data system, the upset times and the recovery durations are an integer multiple of the sample period.

In Chapter IV, the embedding of the sampling instants in the PDP model was accomplished in two steps. First, using DCPN formalism the recovery duration was modeled as taking values on $\mathbb{R}^+$. It was modeled using a guard transition. The failures were modeled using delay transitions. These transitions give a family of stopping times that model the random process of the occurrence or not of a fault and recovery times. Intuitively, at these stopping times, the state of the sampled-data system is subject to a major change and between the jumps the state evolves in a smooth way. At these stopping times, the DCPN jumps from one state to another from which a reachabilty graph is constructed. Basically, the reachability graph gives a partition of a DCPN state into classes. For analysis with the PDP model, a transition matrix for the switching process is derived. This is called the transition kernel of a PDP. Second, from the first step a family of stopping times and a transition kernel of a PDP were obtained. Using the associated DTMC of a PDP it is possible to obtain a DTMC taking values in the classes of the DCPN state and stopping times as discrete-times. But because the recovery frames take values at integer multiples of the sample period T, it is necessary to take a subsequence of a stopping times that only considers the recovery frames at integer multiples of the sample period. More important is to obtain the Markov chain associated with this subsequence. This was developed using the subordinated process and the uniformization techniques. From these two steps a transition matrix of a DTMC model of the switching between the nominal and recovery mode was obtained. This transition has two main characteristics. First, the dimension of the matrix only depends on the number of classes that result from the partition of the DCPN state space. Second, the number of recovery cycles is related to the elements of the transition matrix and not to its size, showing an advantage of the DCPN model over the SFSM model.

Now, having characterized the switching between nominal and upset mode, the sampled-data PDP was presented where the local characteristics were obtained using stochastic motions, DCPN-PDP models and Feller semigroups. A stability analysis for the sampled-data PDP was presented and more importantly a mean square stability equivalence between the sampled-data PDP and its associated discrete-version was obtained. The discrete-time version was shown to be a Markov jump linear system.

## V.2 FUTURE RESEARCH

From the description of the results of the dissertation, there are two techniques developed for a specific case that can be generalized. First, the abstraction procedure for the state space of a DCPN. Using the reachability graph it is possible to induce a partition of the DCPN state space. Basically the classes are recognized through firing the delay and guard transitions and observing the jumps from one state to another. Using the DCPN-PDP relation a transition kernel for going from one class to another can be obtained. Second, using the associated DTMC of a PDP, it is possible to generate a DTMC of a system modeled as a DCPN. This Markov chain has as a state space the partition of the state space of a DCPN.

From the software point of view there is another possible extension. *Smart* [43] is a tool that given a formal description of a system as a Petri net, so *Smart* can generate the state space and attempts to recognize the system as a DTMC and perform formal verification of a system. It will be interesting to develop the same idea when the system is formally described as a DCPN and to use this DCPN as an input to *Smart*. This could be implemented using the state space abstraction and time discretization proposed in this dissertation. Having a *Smart* with DCPN formalism as an input could help to develop analysis and verification of temporal logic properties of a system with the following characteristics: systems that are subject to stochastic failures and have fault tolerant capabilities. With the DCPN models of systems with stochastic failures and fault tolerant capabilities, it is possible to interconnect these DCPN models and obtain one DCPN model which could be used as an input to *Smart*.

Finally, runway safety monitor (RSM) is a protocol playing a key role in accident avoidance, which detect incidents then alerts the pilot, one application of the abstraction described in this dissertation could be to model the RSM as a DCPN when the RSM is subject to communication failures and have a recovery mechanism. Because *Smart* operates on discrete systems, the abstraction technique and time discretization proposed will help to discretize the continuous variables of RSM with fault tolerant mechanism and under stochastic upsets.

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## V.3 VITA

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  - Investigated stability and control of a class of sampled-data stochastic systems.

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