Spring 2011

On the Radius of Convergence of Interconnected Analytic Nonlinear Systems

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A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

ELECTRICAL AND COMPUTER ENGINEERING

OLD DOMINION UNIVERSITY
May 2011

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ABSTRACT

ON THE RADIUS OF CONVERGENCE OF
INTERCONNECTED ANALYTIC NONLINEAR SYSTEMS

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A complete analysis is presented of the radii of convergence of the parallel, product, cascade and unity feedback interconnections of analytic nonlinear input-output systems represented as Fliess operators. Such operators are described by convergent functional series, indexed by words over a noncommutative alphabet. Their generating series are therefore specified in terms of noncommutative formal power series. Given growth conditions on the coefficients of the generating series for the component systems, the radius of convergence of each interconnected system is computed assuming the component systems are either all locally convergent or all globally convergent. In the process of deriving the radius of convergence for the unity feedback connection, it is shown definitively that local convergence is preserved under unity feedback. This had been an open question in the literature.

This dissertation is dedicated to my mother, Yin Yin Nwé.

ACKNOWLEDGMENTS

First and foremost, I would like to express my gratitude to my dear advisor Dr. Steven Gray, whose guidance, support and inspiration have played a central role in my growth as a person and as a researcher. I am incredibly fortunate to be one of the many individuals to have benefitted from his brilliance, wisdom and scholarship. I also appreciate the assistance of Dr. Richard Noren, Dr. Oscar González, and Dr. Dimitrie Popescu in polishing the ideas in this dissertation to their best possible state.

I would like to express my special gratitude to my father, Khin Maung Maung, for all the sacrifices he made to enable my dreams. During my long academic journey, the love and emotional support from my family and friends has been essential. I will forever be indebted to my parents: Yin Yin Nwé, Khin Maung Maung, and Vickie-Chiodo Maung, my four siblings, my grandparents, and my aunts and uncles for their unconditional love. My friends Ei Ei, Thet, and Lin Lin have been my primary source of good cheer.

Finally, I thank Thien for standing by me all during my lengthy graduate studies. The patience and understanding from his kind heart made my difficult moments less painful. His encouragement and reassurance gave me strength to go forward no matter how great the challenges that I faced.

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CHAPTER 1

INTRODUCTION

This chapter provides the motivation for the research described in this dissertation. Subsequently, the problem statement is presented followed by a chapter-by-chapter outline of the document.

1.1 MOTIVATION

Most complex systems found in applications can be viewed as a collection of interconnected subsystems. Generally, an interconnection is said to be *well-posed* when the output signal and every internal signal is uniquely defined on some interval $[t_0, t_0+T]$, $T > 0$, when the inputs are, for example, Lebesgue measurable functions on the same interval. Sometimes additional properties like causality, continuity and regularity are also included as part of the definition of well-posedness [5, 34]. If one or more subsystems is nonlinear, a variety of sufficient conditions are available to ensure that an interconnected system is well-posed [1, 2, 31]. One example for feedback systems is the incremental small gain theorem, which imposes a bound on the $L_\mathfrak{p}$ loop gain [5].

This dissertation focuses on a class of analytic nonlinear input-output systems known as *Fliess operators* [14–16]. Such operators are described by functional series indexed by the set of words X^* over the noncommutative alphabet $X = \{x_0, x_1, \ldots x_m\}$. Their generating series are, therefore, specified in terms of noncommutative formal power series, the set of which is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. (The set of all formal power series over a commutative alphabet X is denoted by $\mathbb{R}^\ell\,[[X]]$.) A formal power series c is a mapping $c : X^* \mapsto \mathbb{R}^\ell$. The value of c at $\eta \in X^*$ is denoted by (c, η), and is called the *coefficient* of η in c. Specifically, one can formally associate with any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L_\mathfrak{p}^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B_\mathfrak{p}^m(R)[t_0, t_1] := \{u \in L_\mathfrak{p}^m[t_0, t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define recursively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0).$$

If there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \forall \eta \in X^*, \tag{1.1.1}$$

where $|\eta|$ denotes the length of the word η, the series c is said to be *locally convergent*, and the set of all locally convergent formal power series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X \rangle\rangle$. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^\ell$.) In this case, F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0,\, t_0 + T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0,\, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [20]. In particular, when $\mathfrak{p} = 1$, the series defining $y = F_c[u]$ converges if

$$\max\{R, T\} < \frac{1}{M_c(1 + m)} \tag{1.1.2}$$

[6, 8]. Let $\pi : \mathbb{R}^\ell_{LC}\langle\langle X \rangle\rangle \to \mathbb{R}^+ \cup \{0\}$ take each nonzero series c to the *smallest* possible geometric growth constant M_c satisfying (1.1.1). In this case, $\mathbb{R}^\ell_{LC}\langle\langle X \rangle\rangle$ can be partitioned into equivalence classes, and the number $1/(M_c(1 + m))$ will be referred to as the *radius of convergence* for the class $\pi^{-1}(M_c)$. This is in contrast to the usual situation where a radius of convergence is assigned to individual series [25]. In practice, it is not difficult to estimate the minimal M_c for many series, in which case, the radius of convergence for $\pi^{-1}(M_c)$ provides an easily computed *lower bound* for the radius of convergence of c in the usual sense. Finally, given any measurable function u on $[t_0, \infty]$, let $u[t_0, t_1]$ denotes its restriction to the interval $[t_0, t_1]$. Define the extended space $L^m_{\mathfrak{p},e}(t_0)$ as

$$L^m_{\mathfrak{p},e}(t_0) = \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L^m_{\mathfrak{p}}[t_0, t_1], \forall t_1 \in (t_0, \infty)\}.$$

When c satisfies the more stringent growth condition

$$|(c, \eta)| \leq K_c M_c^{|\eta|}, \quad \forall \eta \in X^*, \tag{1.1.3}$$

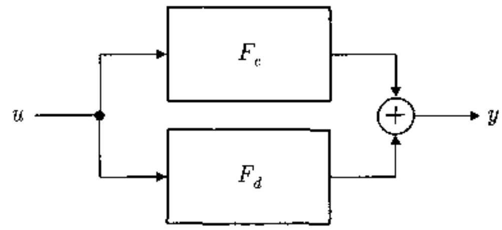

Fig. 1: The parallel connection of two Fliess operators

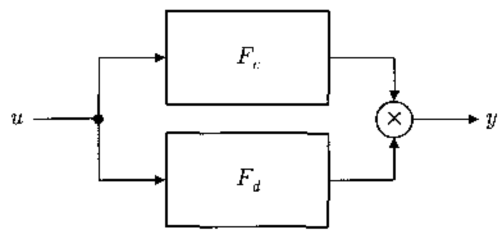

Fig. 2: The product connection of two Fliess operators

the series F_c defines an operator from the extended space $L_{\mathrm{p,e}}^m(t_0)$ into $C[t_0, \infty)$ [20]. Such generating series are called *globally convergent series*, and the set of all such series is denoted by $\mathbb{R}_{GC}^{\ell}\langle\langle X\rangle\rangle$.

Given two input-output systems F_c and F_d, there are four fundamental system interconnections normally encountered in applications : the parallel connection, the product connection, the cascade connection and the feedback connection. For any admissible input, u, the parallel and product connections as shown in Figures 1 and 2 are described, respectively, by

$$y = F_c[u] + F_d[u], \quad y = F_c[u]F_d[u].$$

The cascade connection depicted in Figure 3 is equivalent to

$$y = F_c[F_d[u]].$$

Finally, the feedback connection as shown in Figure 4 is described by the solution y to the feedback equation

$$y = F_c[u + F_d[y]].$$

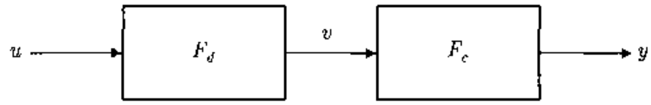

Fig. 3: The cascade connection of two Fliess operators

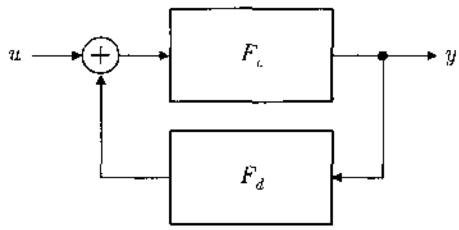

Fig. 4: The feedback connection of two Fliess operators

It is known that the parallel, product and cascade connection of two locally convergent Fliess operators always yields another locally convergent Fliess operator [19]. The feedback connection is known to be well-posed in a certain sense, but it is not known at present whether it has a locally convergent Fliess operator representation. An important exception to this state of affairs is the self-excited case ($u = 0$) [19]. In addition, global convergence is preserved by the parallel and product connections but not in general by the cascade or feedback connection [18]. Little else is known about the subject. In particular, there is no proof that the unity feedback interconnection (that is, when F_d is replaced by the identity map I) preserves local convergence. Furthermore, the radius of convergence is not known for any of the four interconnections. As discussed in later chapters, the parallel connection is straightforward, and lower bounds are available in [32] for the product connection and in [19] for the cascade and self-excited feedback connections. However, these bounds are in general very conservative. Hence, the primary goal of this dissertation is to address these specific gaps in the literature.

1.2 PROBLEM STATEMENT

The specific goals of this dissertation are to:

1. Compute the radii of convergence of the parallel, product, cascade and unity feedback interconnections of input-output systems represented by Fliess operators. The cases where the components are either all locally convergent or all globally convergent will be considered individually.

2. Show that the unity feedback connection preserves local convergence.

3. Provide for each interconnection specific examples under which the radius of convergence is achieved.

1.3 DISSERTATION OUTLINE

The remainder of this dissertation is organized as follows. In Chapter 2, the mathematical tools used to solve the main problems are presented. First, the basic theory of formal power series is introduced in the context of formal language theory. Then the basic interconnection theory for Fliess operators is reviewed. This includes the definitions of the composition and feedback products of formal power series. The goal of Chapter 3 is to calculate the radii of convergence for the parallel, product and cascade connection of two convergent Fliess operators. The case where the operators are locally convergent is considered first, followed by the globally convergent case. In Chapter 4, the radius of convergence is determined for the feedback connection. First, self-excited feedback systems are addressed. Subsequently, the analysis for the unity feedback case is presented. Again, separate analyses are done for closed-loop systems having components with locally convergent generating series and globally convergent generating series. Chapter 5 summarizes the conclusions and describes future work that could be done in this area.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

The generating series of Fliess operators are specified by noncommutative formal power series. Therefore, this chapter presents some basic definitions concerning these objects and describes a set of key operations one can apply to them. Specifically, connecting two Fliess operators in the parallel or product configuration is equivalent to adding or *shuffling* the corresponding generating series, respectively. Connecting them in a cascade or feedback fashion is equivalent to performing the *composition product* or *feedback product*, respectively, on the generating series. But first some notation and terminology from formal language theory is introduced.

2.1 NOTATION AND TERMINOLOGY FOR FORMAL POWER SERIES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η, while $|\eta|_{x_i}$ is the number of times the letter x_i appears in η. The set of all words with length k will be denoted by X^k. Joining two words $\xi, \nu \in X^*$ from end to end to form the new word $\eta = \xi\nu$ is called *catenation*. The power η^i means catenating η with itself i times. Furthermore, the empty word, $\emptyset$, is an identity element for catenation, that is,

$$\emptyset\eta = \eta\emptyset = \eta.$$

The empty word $\emptyset$ has length zero. The set of all words including the empty word will be denoted by X^*. Since catenation is associative, X^* forms a monoid under this product.

Definition 2.1.1. Formal Power Series
Given an alphabet $X = \{x_0, x_1, \ldots, x_m\}$, a *formal power series* c is any mapping of the form

$$c : X^* \to \mathbb{R}^\ell.$$

The image of a word $\eta \in X^*$ under c is denoted by (c, η) and is called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The notation $c \leq d$ means that the component series satisfy $(c_i, \eta) \leq (d_i, \eta)$ for all $\eta \in X^*$ and $i = 1, 2, \ldots, l$. When $(c, \eta) \in \mathbb{R}^\ell$, $|(c, \eta)| := \max_i |(c_i, \eta)|$ The definition of the catenation product can be extended to $\mathbb{R}\langle\langle X\rangle\rangle$ as follows.

Definition 2.1.2. Catenation Product
The *catenation product* of two series $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ is

$$(cd, \eta) = \sum_{\substack{\xi, \nu \in X^* \\ \eta = \xi\nu}} (c, \xi)(d, \nu), \ \forall \eta \in X^*.$$

$\mathbb{R}\langle\langle X\rangle\rangle$ forms an associative $\mathbb{R}$-algebra under the catenation product with identity element 1.

Definition 2.1.3. The Sum and Scalar Product
The *sum* of two series $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is defined as

$$(c + d, \eta) = (c, \eta) + (d, \eta), \ \forall \eta \in X^*,$$

and the *scalar product* is given by

$$(\alpha c, \eta) = \alpha(c, \eta), \ \forall \eta \in X^*, \ \alpha \in \mathbb{R}.$$

With these definitions, $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ admits an $\mathbb{R}$-vector space structure. The following theorem relates the sum of the generating series to the parallel connection of the corresponding Fliess operators.

Theorem 2.1.1. *[14] Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, the parallel connection $F_c + F_d$ has the generating series $c + d$. That is,*

$$F_c + F_d = F_{c+d}.$$

The local convergence is preserved under summation.

The following set of definitions will be used throughout the dissertation.

Definition 2.1.4. Left-Shift Operator
Given any $\xi \in X^*$, the corresponding *left-shift operator* on X^* is defined as

$$\xi^{-1} \quad : \quad X^* \to \mathbb{R}^\ell\langle\langle X\rangle\rangle$$

$$\xi^{-1}(\eta) = \begin{cases} \eta' & : \text{ if } \eta = \xi\eta' \\ 0 & : \text{ otherwise.} \end{cases}$$

This definition can be extended linearly as follows For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$,

$$\xi^{-1}(c) = \sum_{\eta \in X^*} (c, \eta)\xi^{-1}(\eta).$$

In addition, $\xi^{-i}(\cdot)$ denotes the left-shift operator $\xi^{-1}(\cdot)$ applied i times.

Definition 2.1.5. Support of a Formal Power Series

The *support* of a formal power series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is defined as

$$\text{supp}(c) := \{\eta \in X^* : (c, \eta) \neq 0\}.$$

Definition 2.1.6. Order of a Formal Power Series

The *order* of a formal power series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is defined as

$$\text{ord(c)} = \begin{cases} \min\{|\eta| : \eta \in \text{supp}(c)\} & : \ c \neq 0 \\ \infty & : \ c = 0. \end{cases}$$

The following theorem will be essential in computing the radius of convergence for a given interconnection.

Theorem 2.1.2. *[33] Let $f(z) = \sum_{n\geq 0} a_n z^n$ be analytic in some neighborhood of the origin in the complex plane. Suppose $z_0 \neq 0$ is a singularity of $f(z)$ having the smallest modulus. Given any $\epsilon > 0$, there exists an integer $N \geq 0$ such that for all $n > N$,*

$$|a_n| < (1/|z_0| + \epsilon)^n .$$

Furthermore, for infinitely many n,

$$|a_n| > (1/|z_0| - \epsilon)^n .$$

The following definition will be used extensively in the analysis of feedback systems in Chapter 4.

Definition 2.1.7. Realization of a Fliess Operator

A Fliess operator F_c defined on $B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$ is said to be *realized* by a state space realization when there exists a system of n analytic differential equations and ℓ output equations

$$\dot{z} = g_0(z) + \sum_{i=1}^{m} g_i(z)\, u_i, \quad z(t_0) = z_0 \tag{2.1.1}$$

$$y = h(z), \tag{2.1.2}$$

where each g_i is an analytic vector field on some neighborhood $\mathcal{W}$ of z_0, and h is an analytic function on $\mathcal{W}$, such that (2.1.1) has a well defined solution $z(t)$, $t \in [t_0, t_0 + T]$ on $\mathcal{W}$ for any given input $u \in B_{\mathfrak{p}}^m(R)[t_0, t_0 + T]$, and

$$F_c[u](t) = h(z(t)), \quad t \in [t_0, t_0 + T]$$

[15,20,23].

Let $G = \{g_0, g_1, \ldots, g_m\}$. It is well known that when F_c is realizable, the generating series c is related to the realization (G, h, z_0) by

$$(c, \eta) = L_{g_\eta} h(z_0), \quad \forall \eta \in X^*, \tag{2.1.3}$$

where the iterated *Lie derivatives* are defined by

$$L_{g_\eta} h = L_{g_{i_1}} \cdots L_{g_{i_k}} h, \quad \eta = x_{i_k} \cdots x_{i_1} \in X^*$$

with $L_{g_i} : h \mapsto \partial h / \partial z \cdot g_i$ and $L_\emptyset h = h$ [15,16,23]. The analyticity of G and h ensures that c is locally convergent [30].

2.2 SHUFFLE PRODUCT AND THE PRODUCT CONNECTION

The central definition in this section is given below [3,14,28].

Definition 2.2.1. Shuffle Product
The *shuffle product* of two words $\eta, \xi \in X^*$ is defined as the $\mathbb{R}$-bilinear mapping uniquely specified by the recursive definition

$$\begin{aligned} \eta \sqcup\!\sqcup\, \xi &= (x_i \eta') \sqcup\!\sqcup\, (x_j \xi') \\ &= x_i(\eta' \sqcup\!\sqcup\, (x_j \xi')) + x_j((x_i \eta') \sqcup\!\sqcup\, \xi'), \end{aligned}$$

where $\eta = x_i \eta'$, $\xi = x_j \xi'$ and $\nu \sqcup\!\sqcup\, \emptyset = \emptyset \sqcup\!\sqcup\, \nu = \nu$, $\forall \nu \in X^*$. This definition is extended linearly to any two series $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ by letting

$$c \sqcup\!\sqcup\, d = \sum_{\eta, \xi \in X^*} (c, \eta)(d, \xi) \eta \sqcup\!\sqcup\, \xi.$$

Given two series $c, d \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, the shuffle product $c \sqcup\!\sqcup d$ is defined componentwise, i.e., the i-th component of $c \sqcup\!\sqcup d$ is $(c \sqcup\!\sqcup d, \nu)_i = (c_i \sqcup\!\sqcup d_i, \nu)$ for any $\nu \in X^*$ and $i = 1, 2, \ldots, \ell$. $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ forms a commutative and associative $\mathbb{R}$-algebra under the shuffle product. For any $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, the power $c^{\sqcup\!\sqcup k}$ is equivalent to shuffling the series c with itself k times and $c^{\sqcup\!\sqcup 0} = 1$. The following properties and identities of shuffle product will be used extensively in the analysis presented in subsequent chapters.

Lemma 2.2.1. *[32] The following identities hold:*

1. *For any $x \in X$, $x^{\sqcup\!\sqcup k} = k!\, x^k$.*

2. $(c \sqcup\!\sqcup d, \nu) = \displaystyle\sum_{i=0}^{|\nu|} \sum_{\substack{\eta \in X^i \\ \xi \in X^{|\nu|-i}}} (c, \eta)(d, \xi)(\eta \sqcup\!\sqcup \xi, \nu)$.

3. $\displaystyle\sum_{\substack{\eta \in X^i \\ \xi \in X^{|\nu|-i}}} (\eta \sqcup\!\sqcup \xi, \nu) = \binom{|\nu|}{i}$, $i = 0, 1, \ldots, |\nu|$.

Theorem 2.2.1. *[3] The left-shift operator acts as a derivation on the shuffle product, i.e., for all $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ and any $x_k \in X$*

$$x_k^{-1}(c \sqcup\!\sqcup d) = x_k^{-1}(c) \sqcup\!\sqcup d + c \sqcup\!\sqcup x_k^{-1}(d).$$

The following theorem relates the shuffle product of the generating series to the product connection of the corresponding Fliess operators.

Theorem 2.2.2. *[14, 32] Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, the product connection F_cF_d has generating series $c \sqcup\!\sqcup d$. That is,*

$$F_cF_d = F_{c \sqcup\!\sqcup d}.$$

Furthermore, local convergence is preserved under the shuffle product.

2.3 COMPOSITION PRODUCT AND THE CASCADE CONNECTION

The composition product can be traced back to the work of Ferfera in [9, 10]. The interpretation given below first appeared in [17, 18]

Definition 2.3.1. Composition Product
Let $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ and define the family of mappings

$$D_{x_i} : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. Assume $D_\emptyset$ is the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$. Such maps can be composed in an obvious way so that $D_{x_i x_j} := D_{x_i} D_{x_j}$ provides an $\mathbb{R}$-algebra which is isomorphic to the usual $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X \rangle\rangle$ under the catenation product. The *composition product* of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ is defined as

$$\underbrace{(x_{i_k} x_{i_{k-1}} \cdots x_{i_1})}_{\eta} \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ the definition is extended linearly as

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, D_\eta(1).$$

From this definition, it is clear that the composition product is linear in its left argument, i.e., $(\alpha c + \beta d) \circ e = \alpha(c \circ e) + \beta(d \circ e)$, where $\alpha, \beta \in \mathbb{R}$, $c, d \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, and $e \in \mathbb{R}^m\langle\langle X \rangle\rangle$. It is sometimes useful to express the composition product in the following alternative ways:

(i) An arbitrary word $\eta \in X^*$ can be written as

$$\eta = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} \ldots x_0^{n_1} x_{i_1} x_0^{n_0},$$

where $i_j \neq 0$ for $j = 1, \ldots, k$, and $n_0, n_1, \ldots, n_k \geq 0$. Then it follows that

$$\eta \circ d = x_0^{n_k+1} \left[d_{i_k} \sqcup\!\sqcup x_0^{n_{k-1}+1} \left[d_{i_{k-1}} \sqcup\!\sqcup \cdots x_0^{n_1+1} \left[d_{i_1} \sqcup\!\sqcup x_0^{n_0} \right] \cdots \right] \right].$$

(ii) For any word $\eta \in X^*$, one can uniquely associate a set of right factors $\{\eta_0, \eta_1, \ldots, \eta_k\}$ by the iteration

$$\eta_{i+1} = x_0^{n_{j+1}} x_{i_{j+1}} \eta_j, \quad \eta_0 = x_0^{n_0}, \ \ i_{j+1} \neq 0,$$

so that $\eta = \eta_k$ with $k = |\eta| - |\eta|_{x_0}$. Then, $\eta \circ d = \eta_k \circ d$, where

$$\eta_{j+1} \circ d = x_0^{n_{j+1}+1} [d_{i_{j+1}} \sqcup\!\sqcup (\eta_j \circ d)],$$

and $\eta_0 \circ d = x_0^{n_0}$.

The following lemma states some important properties of the composition product.

Lemma 2.3.1. *[9, 19] For $c, d \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and $e \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$ the following identities hold:*

1. $0 \circ c = 0$.
2. $c \circ 0 = \sum_{n\geq 0}(c, x_0^n)x_0^n$.
3. $(c \sqcup\!\sqcup d) \circ e = (c \circ e) \sqcup\!\sqcup (d \circ e)$.

An important observation is that the composition product induces a contraction on $\mathbb{R}^{m}\langle\langle X\rangle\rangle$. To see this precisely, consider first the following definition.

Definition 2.3.2. Ultrametric Space

Given a set S, a function $\delta : S \times S \to \mathbb{R}$ is called an *ultrametric* if it satisfies the following properties for all $s, s', s'' \in S$:

1. $\delta(s, s') \geq 0$
2. $\delta(s, s') = 0$ if and only if $s = s'$
3. $\delta(s, s') = \delta(s', s)$
4. $\delta(s, s') \leq \max\{\delta(s, s''), \delta(s', s'')\}$.

The pair (S, δ) is referred to as an *ultrametric space*. It is easily shown that every ultrametric space is a metric space.

Theorem 2.3.1. *[3] The $\mathbb{R}$-vector space $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ with the mapping*

$$\begin{aligned} dist \quad &: \quad \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \times \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \to \mathbb{R} \\ &: \quad (c, d) \mapsto \sigma^{ord(c-d)} \end{aligned}$$

is an ultrametric space for any real number $0 < \sigma < 1$.

Definition 2.3.3. Contractive Mapping

Let (S, δ) be a metric space. A mapping $\mathcal{T} : S \to S$ is called a *contractive mapping* if there exists a real number $0 < \alpha < 1$ such that

$$\delta(\mathcal{T}(s), \mathcal{T}(s')) \leq \alpha\, \delta(s, s'), \quad s, s' \in S.$$

Given any mapping $\mathcal{T}$, a point $s^* \in S$ is said to be a *fixed point* if $\mathcal{T}(s^*) = s^*$. The following theorem gives a condition under which a fixed point exists and is unique.

Theorem 2.3.2. *[24] Let (S, δ) be a complete nonempty metric space. Then every contractive mapping $\mathcal{T} : S \to S$ has precisely one fixed point in S.*

Theorem 2.3.3. *[19] For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $c \mapsto c \circ d$ is a contractive map on $\mathbb{R}^m\langle\langle X\rangle\rangle$ in the ultrametric sense.*

The following theorem states that local convergence is preserved under composition.

Theorem 2.3.4. *[19] Suppose $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Then $c \circ d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ Specifically,*

$$|(c \circ d, \nu)| \leq K_c((\phi(mK_d) + 1)M)^{|\nu|}(|\nu| + 1)!, \quad \forall \nu \in X^*,$$

where $\phi(x) := x/2 + \sqrt{x^2/4 + x}$ and $M = \max\{M_c, M_d\}$. (Here $\phi(1) = \phi_g := (1 + \sqrt{5})/2$, the golden ratio. See Table 1 for some specific values of $\phi(mK_d) + 1$.)

TABLE 1. Some specific values of $\phi(mK_d) + 1$

mK_d	$\phi(mK_d) + 1$
0	1
$\ll 1$	$\simeq \sqrt{mK_d} + 1$
$1/2$	2
1	$\phi_g + 1 = \phi_g^2$
$\gg 1$	$\approx mK_d$
$+\infty$	$+\infty$

In light of (1.1 2) and the theorem above, a lower bound on the radius of convergence for $c \circ d$ is $1/(\phi(mK_d) + 1)M(1 + m)$. To date no example has been presented for which the radius of convergence corresponds exactly to this bound. Thus, it is believed that this result is conservative. In addition, it can be shown by a simple counterexample that global convergence is *not* always preserved under composition [7,9].

However, if c and d are globally convergent, one would expect this stronger property to produce a correspondingly larger radius of convergence for $c \circ d$. Finally, in much of the work to follow, the subset of $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ described below will be useful.

Definition 2.3.4. [13, 14] **Exchangeable Series**
A series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is said to be *exchangeable* if for arbitrary $\eta, \xi \in X^*$

$$|\eta|_{x_i} = |\xi|_{x_i},\ i = 0, 1, \ldots, m \quad \Rightarrow \quad (c, \eta) = (c, \xi).$$

Theorem 2.3.5. *If $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is an exchangeable series and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$ is arbitrary then the composition product can be written in the form*

$$c \circ d = \sum_{k=0}^{\infty} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} (c, x_0^{r_0}\cdots x_m^{r_m})\, D_{x_0}^{r_0}(1) \sqcup\!\sqcup \cdots \sqcup\!\sqcup D_{x_m}^{r_m}(1).$$

Proof: For fixed $r_i \geq 0$, $i = 0, 1, \ldots, m$ define the polynomial

$$X(r_0, \ldots, r_m) = \sum_{\substack{\eta \in X^* \\ |\eta|_{x_i} = r_i \\ i=0,1,\ldots,m}} \eta.$$

Using the identity

$$X(r_0, r_1, \ldots, r_m) = x_0^{r_0} \sqcup\!\sqcup x_1^{r_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{r_m}$$

[6], observe that

$$\begin{aligned}
c \circ d &= \sum_{k=0}^{\infty} \sum_{\eta \in X^k} (c, \eta)\, \eta \circ d \\
&= \sum_{k=0}^{\infty} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} (c, x_0^{r_0}, \ldots, x_m^{r_m})\, X(r_0, \ldots, r_m) \circ d \\
&= \sum_{k=0}^{\infty} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} (c, x_0^{r_0}, \ldots, x_m^{r_m})\, (x_0^{r_0} \circ d) \sqcup\!\sqcup \cdots \sqcup\!\sqcup (x_m^{r_m} \circ d) \\
&= \sum_{k=0}^{\infty} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} (c, x_0^{r_0}, \ldots, x_m^{r_m})\, D_{x_0}^{r_0}(1) \sqcup\!\sqcup \cdots \sqcup\!\sqcup D_{x_m}^{r_m}(1).
\end{aligned}$$

■

The following theorem relates the composition product of the generating series to the cascade connection of the corresponding Fliess operators.

Theorem 2.3.6. *[9, 10, 19] Given Fliess operators F_c and F_d, where $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$, the cascade connection $F_c \circ F_d$ has generating series $c \circ d$, that is,*

$$F_c \circ F_d = F_{c \circ d}.$$

Furthermore, local convergence is preserved under the composition product.

2.4 FEEDBACK PRODUCT AND THE FEEDBACK CONNECTION

Consider two Fliess operators interconnected to form a feedback system as shown in Figure 4. The output y must satisfy the feedback equation

$$y = F_c[u + F_d[y]]$$

for every admissible input u. It was shown in [19, 21] that there always exists a generating series e so that $y = F_e[u]$. In which case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d \circ e}[u]]. \tag{2.4.1}$$

The *feedback product* of c and d is thus defined as $c@d = e$. F_e is the composition of two operators, namely, F_c and $I + F_{d \circ e}$. The latter is not realizable by a Fliess operator due to the *direct feed* term I. To compensate for the presence of this term the following definition of the *modified composition product* is needed.

Definition 2.4.1. Modified Composition Product
The *modified composition product* of $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$ is defined as

$$c \tilde{\circ} d = \sum_{\eta \in X^*} (c, \eta)\, \tilde{D}_\eta(1),$$

where

$$\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup\, e)$$

with $d_0 := 0$.

Alternatively, the *modified composition product* can be expressed as follows. For any $\eta \in X^*$ and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$

$$\eta \tilde{\circ} d = \begin{cases} \eta & : \eta = x_0^n \\ x_0^n x_i(\eta' \tilde{\circ} d) + x_0^{n-1}(d_i \sqcup\!\sqcup\, (\eta' \tilde{\circ} d)) & : \eta = x_0^n x_i \eta',\ \eta' \in X^* \\ & \quad i \neq 0, \end{cases}$$

where $n \geq 0$. For $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the definition is extended linearly as

$$c\tilde{\circ}d = \sum_{\eta\in X^*} (c,\eta)\eta\tilde{\circ}d.$$

Theorem 2.4.1. *[19] For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, it follows that*

$$F_{c\tilde{\circ}d}[u] = F_c\left[u + F_d[u]\right].$$

The feedback equation (2.4.1) can be written in terms of the modified composition product as

$$F_e[u] = F_{c\tilde{\circ}(d\circ e)}[u].$$

It was shown in [27, Corollary 2.2] that if $F_c = F_d$ on any $B_{\mathrm{p}}^m(R)[t_0, t_0+T]$ then $c = d$. A similar uniqueness result for the formal case is described in [21]. Therefore,

$$e = c\tilde{\circ}(d\circ e).$$

Theorem 2.4.2. *[19, 26] For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $d \mapsto c\tilde{\circ}d$ is a contractive map on $\mathbb{R}^m\langle\langle X\rangle\rangle$.*

Theorem 2.4.3. *[21] For any $c, d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, it follows that :*

1. *e is the unique fixed point of the contractive iterated map*

$$\tilde{S} : e(k) \mapsto e(k+1) = c\tilde{\circ}(d\circ e(k)).$$

2. *$c@d = e$ satisfies the fixed point equation*

$$e = c\tilde{\circ}(d\circ e). \tag{2.4.2}$$

In the case of a unity feedback system, where the operator F_d in the feedback path is replaced by I, equation (2.4.2) reduces to $e = c\tilde{\circ}e$. In the self-excited case, i.e., when u=0, equation (2.4.2) becomes $e = (c\circ d)\circ e$. Thus, when $c\circ d$ is redefined as c, it reduces further to $e = c\circ e$. Moreover, since a self-excited feedback system can be described by $F_{c@d}[0] = F_{(c@d)\circ 0}[u]$, the generating series $e = (c@d)\circ 0$. Thus, $e \in \mathbb{R}^m[[X_0]]$, where $X_0 = \{x_0\}$. The next theorem states that local convergence of a self-excited unity feedback system is preserved.

Theorem 2.4.4. *[19] Let $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c \geq 1$ and $M_c > 0$. If $e \in \mathbb{R}^m[[X_0]]$ satisfies $e = c \circ e$ then*

$$|(e, x_0^n)| \leq K_c\phi_g\left((mK_c(2+\phi_g)+1)M_c\right)^n s_n\, n!, \quad n \geq 0,$$

where $s_0 := 1/\phi_g$ and $s_{n+1} = \mathcal{B}(C_n) := \sum_{k=0}^{n}\binom{n}{k}C_k$, i.e., s_{n+1}, $n \geq 0$ is the binomial transformation of the Catalan sequence.

TABLE 2: Selected sequences from the OEIS concerning the local convergence of the feedback product in the self-excited case

sequence	OEIS number	$n = 0, 1, 2, \ldots$
C_n	A000108	$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$
s_{n+1}	A007317	$1, 2, 5, 15, 51, 188, 731, 2950, \ldots$

The Catalan sequence is a sequence of natural numbers which appears in many counting problems. The n-th Catalan number is described as

$$C_n = \frac{1}{n-1}\binom{2n}{n}.$$

The sequence s_{n+1}, $n \geq 0$ is sequence number A007317 in the Online Encyclopedia of Integer Sequences (OEIS) [29] See Table 2 for the first few entries of both C_n, $n \geq 0$ and s_{n+1}, $n \geq 0$. The asymptotic behavior of s_{n+1}, $n \geq 0$ is known to be

$$s_n \sim \frac{\sqrt{5}}{8\sqrt{\pi n^3}}\, 5^n$$

[22]. Therefore, for the single-input, single-output case

$$|(e, x_0^n)| \leq (\beta(K_c)M_c)^n\, n!, \quad n \geq 0,$$

where $\beta(K_c) := K_c(10 + 5\phi_g) + 5$ for $K_c \geq 1$. For a self-excited unity feedback system, it follows from (1.1.2) with $R = m = 0$ that $F_c[0]$ is guaranteed to converge on at least the interval $[0, 1/\beta(K_c)M_c)$. But again no example has been presented to date for which this interval corresponds exactly to the interval of convergence. Little else is known concerning the local convergence of the closed-loop system, but as in the cascade connection, global convergence is known not to be preserved under feedback [9, 18]. However, a version of Theorem 2.4.4 tailored to the case where $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ should intuitively yield a larger interval of convergence for the closed-loop system. Most importantly, when the input is nonzero, the question of whether or not the unity feedback system preserves local convergence remains open.

CHAPTER 3

THE RADIUS OF CONVERGENCE OF THE NONRECURSIVE CONNECTIONS (PARALLEL, PRODUCT AND CASCADE)

The goal of this chapter is to calculate the radius of convergence of the parallel, product and cascade connections of two convergent Fliess operators. The case where the component operators are locally convergent is considered first, followed by the globally convergent case.

3.1 THE PARALLEL CONNECTION

3.1.1 Local Convergence

The analysis begins with the parallel connection shown in Figure 1, which can be considered as the simplest of all the interconnections. The following theorem is a prerequisite for proving the main theorem of this section.

Theorem 3.1.1. *Let* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $\bar{c}, \bar{d} \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, *where each component of* $(\bar{c}, \eta)$ *and* $(\bar{d}, \eta)$ *is* $K_c M_c^{|\eta|}|\eta|!$, $\eta \in X^*$ *with* $K_c, M_c > 0$ *and* $K_d M_d^{|\eta|}|\eta|!$, $\eta \in X^*$ *with* $K_d, M_d > 0$, *respectively. If* $\bar{b} = \bar{c} + \bar{d}$, *then the sequence* $(\bar{b}_i, x_0^k)$, $k \geq 0$ *has the exponential generating function*

$$
\begin{aligned}
f(x_0) &:= \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k)\frac{x_0^k}{k!} \\
&= \frac{K_c}{1 - M_c x_0} + \frac{K_d}{1 - M_d x_0}
\end{aligned}
$$

for any $i = 1, 2, \ldots, \ell$. *Moreover, the smallest possible geometric growth constant for* $\bar{b}$ *is*

$$M_b = \max\{M_c, M_d\}.$$

Proof: There is no loss of generality in assuming $\ell = 1$. Observe for any $\nu \in X^n$, $n \geq 0$ that

$$
\begin{aligned}
(\bar{b}, \nu) &= (\bar{c}, \nu) + (\bar{d}, \nu) \\
&= (K_c M_c^n + K_d M_d^n)\, n!.
\end{aligned}
$$

Furthermore, $(\bar{b},\nu) = (\bar{b}, x_0^n)$, $n \geq 0$. The key idea is that $f(t)$ is the zero-input response of $F_{\bar{b}}$. Specifically,

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k)\frac{t^k}{k!} = F_{\bar{b}}[0] \\ &= F_{\bar{c}}[0] + F_{\bar{d}}[0] \\ &= \sum_{k=0}^{\infty} K_c M_c^k t^k + \sum_{k=0}^{\infty} K_d M_d^k t^k \\ &= \frac{K_d}{1 - M_c t} + \frac{K_d}{1 - M_d t}. \end{aligned} \tag{3.1.1}$$

Since f is analytic at the origin, by Theorem 2.1.2 the smallest geometric growth constant for the sequence $(\bar{b}, x_0^n)$, $n \geq 0$, and thus for the entire formal power series $\bar{b}$, is determined by the location of any singularity nearest to the origin in the complex plane, say x_0'. Specifically, $M_b = 1/|x_0'|$, where it is easily verified from (3.1.1) that x_0' is the positive real number

$$x_0' = \frac{1}{\max\{M_c, M_d\}}.$$

This proves the theorem. ∎

The following theorem describes the radius of convergence of the parallel connection of two locally convergent Fliess operators.

Theorem 3.1.2. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c + d$ then*

$$|(b,\nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{3.1.2}$$

for some $K_b > 0$, where

$$M_b = \max\{M_c, M_d\}.$$

Furthermore, no smaller geometric growth constant can satisfy (3.1.2), and thus the radius of convergence is

$$\frac{1}{\max\{M_c, M_d\}(1+m)}.$$

Proof: First observe that

$$\begin{aligned} |(c+d,\nu)| &\leq |(c,\nu)| + |(d,\nu)| \\ &\leq (\bar{c}_i, \nu) + (\bar{d}_i, \nu) \\ &= (\bar{b}_i, \nu), \end{aligned}$$

where $\bar{c}, \bar{d}$ and $\bar{b}$ are defined as in Theorem 3.1.1 and $i = 1, 2, \ldots, \ell$. In light of Theorem 3.1.1 and Theorem 2.1.2, $(\bar{b}_i, \nu)$ is *asymptotically* bounded by $M_b^{|\nu|}|\nu|!$. Thus, some $K_b > 0$ can always be introduced such that

$$(\bar{b}_i, \nu) \leq K_b M_b^{|\nu|}|\nu|!, \; \nu \in X^*.$$

Furthermore, $(\bar{b}_i, x_0^n)$, $n \geq 0$ is growing exactly at this rate. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ■

3.1.2 Global Convergence

In this section, the radius of convergence of the parallel connection of two globally convergent Fliess operators is calculated. The following theorem is a prerequisite for proving the main theorem of this section.

Theorem 3.1.3. *Let* $X = \{x_0, x_1, \ldots, m\}$. *Let* $\bar{c}, \bar{d} \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$, *where each component of* $(\bar{c}, \eta)$ *and* $(\bar{d}, \eta)$ *is* $K_c M_c^{|\eta|}$, $\eta \in X^*$ *with* $K_c, M_c > 0$ *and* $K_d M_d^{|\eta|}$, $\eta \in X^*$ *with* $K_d, M_d > 0$, *respectively. If* $\bar{b} = \bar{c} + \bar{d}$, *then* $(\bar{b}_i, \nu) \leq (\bar{b}_i, x_0^{|\nu|})$, $\nu \in X^*$, *and the sequence* $(\bar{b}_i, x_0^k)$, $k \geq 0$ *has the exponential generating function*

$$f(x_0) = K_c \exp(M_c t) + K_d \exp(M_d t)$$

for any $i = 1, 2, \ldots, \ell$.

Proof: There is no loss of generality in assuming $\ell = 1$. Observe for any $\nu \in X^n$, $n \geq 0$ that

$$\begin{aligned}(\bar{b}, \nu) &= (\bar{c}, \nu) + (\bar{d}, \nu) \\ &= K_c M_c^n + K_d M_d^n. \end{aligned} \tag{3.1.3}$$

Thus, $(\bar{b}, \nu) = (\bar{b}, x_0^n)$, $n \geq 0$. As in the local case, $f(t)$ is the zero-input response of $F_{\bar{b}}$. Specifically,

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k)\frac{t^k}{k!} = F_{\bar{b}}[0] \\ &= F_{\bar{c}}[0] + F_{\bar{d}}[0] \\ &= \sum_{k=0}^{\infty} \frac{K_c M_c^k t^k}{k!} + \sum_{k=0}^{\infty} \frac{K_d M_d^k t^k}{k!} \\ &= K_c \exp(M_c t) + K_d \exp(M_d t). \end{aligned}$$

Thus, the theorem is proved. ■

Now the main result of this section is presented.

Theorem 3.1.4. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c, d \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$ respectively. If $b = c + d$ then*

$$|(b, \nu)| \leq (\bar{b}_i, x_0^{|\nu|}), \quad \nu \in X^*, \quad i = 1, 2, \ldots, \ell,$$

where the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function

$$f(x_0) = K_c \exp(M_c x_0) + K_d \exp(M_d x_0).$$

Thus, the radius of convergence is infinity.

Proof: The proof is perfectly analogous to the local case, and hence, is omitted. ■

From equation (3.1.3), it can be seen that global convergence is preserved in general under the parallel connection. In addition, the nearest singularity to the origin of the function f, say x_0', is at infinity. Thus, the smallest geometric growth constant of $\bar{b}$ is

$$M_b = 1/|x_0'| = 0.$$

This implies that the radius of convergence is infinite, and therefore F_b defines an operator from the extended space $L^m_{\mathfrak{p},e}(t_0)$ into $C[t_0, \infty)$.

3.2 THE PRODUCT CONNECTION

3.2.1 Local Convergence

In this section the radius of convergence of the product connection of two locally convergent Fliess operators will be calculated. The following theorem is a prerequisite for proving the main theorem of this section.

Theorem 3.2.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c}, \bar{d} \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, where each component of $(\bar{c}, \eta)$ and $(\bar{d}, \eta)$ is $K_c M_c^{|\eta|}|\eta|!$, $\eta \in X^*$ with $K_c, M_c > 0$ and $K_d M_d^{|\eta|}|\eta|!$, $\eta \in X^*$ with $K_d, M_d > 0$, respectively. If $\bar{b} = \bar{c} \sqcup\!\sqcup \bar{d}$, then the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function*

$$f(x_0) = \frac{K_c K_d}{(1 - M_c x_0)(1 - M_d x_0)}$$

for any $i = 1, 2, \ldots, \ell$*. Moreover, the smallest possible geometric growth constant for* $\bar{b}$ *is*

$$M_b = \max\{M_c, M_d\}.$$

Proof: There is no loss of generality in assuming $\ell = 1$. Observe for any $\nu \in X^n$, $n \geq 0$ that

$$\begin{aligned}
(\bar{b}, \nu) &= \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\bar{c}, \eta)(\bar{d}, \xi)(\eta \sqcup\!\sqcup \xi, \nu) \\
&= \sum_{j=0}^{n} K_c M_c^j \, j! \, K_d M_d^{n-j} (n-j)! \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\eta \sqcup\!\sqcup \xi, \nu) \\
&= \sum_{j=0}^{n} K_c M_c^j \, j! \, K_d M_d^{n-j} (n-j)! \binom{n}{j} \\
&= K_c K_d \sum_{j=0}^{n} M_c^j M_d^{n-j} n!.
\end{aligned}$$

Furthermore, $\bar{b}$ and the sequence $(\bar{b}, x_0^n)$, $n \geq 0$ have the same growth constants. Observe that $f(t)$ is the zero-input response of $F_{\bar{b}}$. Specifically,

$$\begin{aligned}
f(t) &= \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k) \frac{t^k}{k!} = F_{\bar{b}}[0] \\
&= F_{\bar{c}}[0] F_{\bar{d}}[0] \\
&= \sum_{k=0}^{\infty} K_c M_c^k t^k \sum_{k=0}^{\infty} K_d M_d^k t^k \\
&= \frac{K_c K_d}{(1 - M_c t)(1 - M_d t)}.
\end{aligned} \tag{3.2.1}$$

Since f is analytic at the origin, Theorem 2.1.2 is applied to compute the smallest geometric constant. Specifically, $M_b = 1/|x_0'|$, where it is easily verified from (3.2.1) that the singularity nearest to the origin is the positive real number

$$x_0' = \frac{1}{\max\{M_c, M_d\}}.$$

This proves the theorem. ∎

Now the main result of this section is presented.

Theorem 3.2.2. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c \sqcup\!\sqcup\, d$ then*

$$|(b, \nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{3.2.2}$$

for some $K_b > 0$, where

$$M_b = \max\{M_c, M_d\}.$$

Furthermore, no smaller geometric growth constant can satisfy (3.2.2), and thus the radius of convergence is

$$\frac{1}{\max\{M_c, M_d\}(1+m)}.$$

Proof: First observe that

$$\begin{aligned}
|(c \sqcup\!\sqcup\, d, \nu)| &\leq \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} |(c, \eta)||(d, \xi)|(\eta \sqcup\!\sqcup\, \xi, \nu) \\
&\leq \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\bar{c}_i, \eta)(\bar{d}_i, \xi)(\eta \sqcup\!\sqcup\, \xi, \nu) \\
&= (\bar{b}_i, \nu),
\end{aligned}$$

where $\bar{c}, \bar{d}$ and $\bar{b}$ are defined as in Theorem 3.2.1 and $i = 1, 2, \ldots, \ell$. By Theorem 3.2.1, and Theorem 2.1.2, $(\bar{b}_i, \nu)$ is asymptotically bounded by $M_b^{|\nu|}|\nu|!$. Thus, some $K_b > 0$ can always be introduced such that

$$(\bar{b}_i, \nu) \leq K_b M_b^{|\nu|} |\nu|!, \; \nu \in X^*.$$

Furthermore, $(\bar{b}_i, x_0^n)$, $n \geq 0$ is growing exactly at this rate. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ■

One observation is that the exponential generating functions in Theorem 3.1.1 and Theorem 3.2.1 have identical sets of singularities. Therefore, the minimal geometric growth constants for the generating series of the parallel and product connections are the same. As a result, for locally convergent component systems the two interconnections have the same radius of convergence.

3.2.2 Global Convergence

In this section the radius of convergence of the product connection of two globally convergent Fliess operators will be calculated. The following theorem is a prerequisite for proving the main theorem of this section.

Theorem 3.2.3. *Let $X = \{x_0, x_1, \ldots, m\}$. Let $\bar{c}, \bar{d} \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$, where each component of $(\bar{c}, \eta)$ and $(\bar{d}, \eta)$ is $K_c M_c^{|\eta|}$, $\eta \in X^*$ with $K_c, M_c > 0$ and $K_d M_d^{|\eta|}$, $\eta \in X^*$ with $K_d, M_d > 0$, respectively. If $\bar{b} = \bar{c} \sqcup\!\sqcup \bar{d}$, then $(b_i, \nu) \le (b_i, x_0^{|\nu|})$, $\nu \in X^*$, and the sequence $(\bar{b}_i, x_0^k)$, $k \ge 0$ has the exponential generating function*

$$f(x_0) = K_c K_d \exp[(M_c + M_d)x_0]$$

for any $i = 1, 2, \ldots, \ell$.

Proof: There is no loss of generality in assuming $\ell = 1$. Observe for any $\nu \in X^n$, $n \ge 0$ that

$$\begin{aligned}
(\bar{c} \sqcup\!\sqcup \bar{d}, \nu) &= \sum_{j=0}^{n} \sum_{\substack{\eta\in X^j \\ \xi\in X^{n-j}}} (\bar{c}, \eta)(\bar{d}, \xi)(\eta \sqcup\!\sqcup \xi, \nu) \\
&= \sum_{j=0}^{n} K_c M_c^j \; K_d M_d^{n-j} \sum_{\substack{\eta\in X^j \\ \xi\in X^{n-j}}} (\eta \sqcup\!\sqcup \xi, \nu) \\
&= \sum_{j=0}^{n} K_c M_c^j \; K_d M_d^{n-j} \binom{n}{j} \\
&= K_c K_d (M_c + M_d)^n. \qquad (3.2.3)
\end{aligned}$$

Furthermore, $\bar{b}$ and the sequence $(\bar{b}, x_0^n)$, $n \ge 0$ have the same growth constants. Observe that $f(t)$ is the zero-input response of $F_{\bar{b}}$. Specifically,

$$\begin{aligned}
f(t) &= \sum_{k=0}^{\infty} (\bar{b}_i, x_0^k) \frac{t^k}{k!} = F_{\bar{b}}[0] \\
&= F_{\bar{c}}[0] F_{\bar{d}}[0] \\
&= \sum_{k=0}^{\infty} \frac{K_c M_c^k t^k}{k!} \sum_{k=0}^{\infty} \frac{K_d M_d^k t^k}{k!} \\
&= K_c K_d \exp[(M_c + M_d)t].
\end{aligned}$$

This proves the theorem. ∎

Now the main result of this section is presented.

Theorem 3.2.4. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c, d \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c \sqcup\!\sqcup d$ then*

$$|(b, \nu)| \le (\bar{b}_i, x_0^{|\nu|}), \quad \nu \in X^*, \quad i = 1, 2, \ldots, \ell,$$

where the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ *has the exponential generating function* f.

$$f(x_0) = K_c K_d \exp[(M_c + M_d)x_0].$$

Thus, the radius of convergence is infinity.

Proof: First observe that

$$\begin{aligned}
(c \sqcup\!\sqcup d, \nu) &= \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (c, \eta)(d, \xi)(\eta \sqcup\!\sqcup \xi, \nu) \\
|(c \sqcup\!\sqcup d, \nu)| &\leq \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} |(c, \eta)||(d, \xi)|(\eta \sqcup\!\sqcup \xi, \nu) \\
&\leq \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\bar{c}_i, \eta)(\bar{d}_i, \xi)(\eta \sqcup\!\sqcup \xi, \nu) \\
&= (\bar{b}_i, \nu),
\end{aligned}$$

where $\bar{c}, \bar{d}, \bar{b}$ are defined as in Theorem 3.2.3, and $i = 1, 2, \ldots, \ell$. In light of Theorem 3.2.3, $(\bar{b}_i, \nu)$ is bounded by $(\bar{b}_i, x_0^{|\nu|})$, which has the exponential generating function f. Therefore, the theorem is proved. ∎

From equation (3.2.3), it can be seen that global convergence is preserved in general under the product connection. In addition, the nearest singularity to the origin of the function f, say x_0', is at infinity. Thus, by Theorem 2.1.2, the smallest geometric growth constant of $\bar{b}$ is

$$M_b = 1/|x_0'| = 0.$$

Hence, the radius of convergence is infinite, and therefore F_b defines an operator from the extended space $L_{p,e}^m(t_0)$ into $C[t_0, \infty)$.

3.3 THE CASCADE CONNECTION

3.3.1 Local Convergence

The goal of this section is to calculate the radius of convergence of the cascade connection of two locally convergent Fliess operators. The analysis for this interconnection is substantially more complex as compared to that for the parallel and product connections. A preliminary theorem and a lemma will be needed to prove the following main result.

Theorem 3.3.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c \circ d$ then*

$$|(b, \nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{3.3.1}$$

for some $K_b > 0$, where

$$M_b = \frac{M_d}{1 - mK_d W\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)}.$$

Furthermore, no smaller geometric growth constant can satisfy (3.3.1), and thus the radius of convergence is

$$\frac{1}{M_d(1+m)}\left[1 - mK_d W\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)\right].$$

The following theorem and lemma are prerequisites for the proof of the main result above.

Theorem 3.3.2. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c} \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $\bar{d} \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$, where each component of $(\bar{c}, \eta)$ is $K_c M_c^{|\eta|}|\eta|!$, $\eta \in X^*$ with $K_c, M_c > 0$, and likewise, each component of $(\bar{d}, \eta)$ is $K_d M_d^{|\eta|}|\eta|!$, $\eta \in X^*$ with $K_d, M_d > 0$. If $\bar{b} = \bar{c} \circ \bar{d}$, then the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function*

$$f(x_0) = \frac{K_c}{1 - M_c x_0 + (mM_cK_d/M_d)\ln(1 - M_d x_0)}$$

for any $i = 1, 2, \ldots, \ell$. Moreover, the smallest possible geometric growth constant for $\bar{b}$ is

$$M_b = \frac{M_d}{1 - mK_d W\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)},$$

where W denotes the Lambert W-function, namely, the inverse of the function

$$g(W) = W\exp(W) \tag{3.3.2}$$

[4].

Proof: There is no loss of generality in assuming $\ell = 1$ First observe that $\bar{c}$ is exchangeable, and thus, from Theorem 2.3.5 it follows that

$$\begin{aligned}
\bar{b} &= \sum_{k=0}^{\infty} K_c M_c^k \sum_{\substack{r_0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = k}} k! \frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{(x_m \circ \bar{d})^{\sqcup\!\sqcup\, r_m}}{r_m!} \\
&= \sum_{k=0}^{\infty} K_c \left(M_c(x_0 + m x_0 \bar{d}_1)\right)^{\sqcup\!\sqcup\, k}.
\end{aligned}$$

Note that the identity $\bar{d}_i = \bar{d}_j$ for every $i, j = 1, 2, \ldots, m$ has been used above. Shuffling both sides of this equation by $M_c(x_0 + mx_0\bar{d}_1)$ yields

$$\bar{b} ⧢ M_c(x_0 + mx_0\bar{d}_1) = \sum_{k=0}^{\infty} K_c(M_c(x_0 + mx_0\bar{d}_1))^{⧢ k+1}.$$

Adding K_c to both sides gives

$$\bar{b} = K_c + M_c[\bar{b} ⧢ (x_0 + mx_0\bar{d}_1)]. \tag{3.3.3}$$

By inspection, $(\bar{b}, \emptyset) = K_c$, $(\bar{b}, x_0) = K_c M_c(1 + mK_d)$ and $(\bar{b}, x_i) = 0$ for $i = 1, 2, \ldots, m$. Let $(\bar{b}, \nu_n) := \max\{(\bar{b}, \nu) : \nu \in X^n\}$. For any $\nu \in X^n, n \geq 2$ it follows from (3.3.3) that

$$\begin{aligned}
(\bar{b}, \nu) &= M_c \sum_{i=0}^{n} \sum_{\substack{\eta \in X^i \\ \xi \in X^{n-i}}} (\bar{b}, \eta)(x_0 + mx_0\bar{d}_1, \xi)(\eta ⧢ \xi, \nu) \\
&= M_c \sum_{i=0}^{n-1} \sum_{\substack{\eta \in X^i \\ \xi \in X^{n-i}}} (\bar{b}, \eta)(x_0 + mx_0\bar{d}_1, \xi)(\eta ⧢ \xi, \nu) \\
&\leq M_c \sum_{i=0}^{n-1} (\bar{b}, \nu_i) \sum_{\substack{\eta \in X^i \\ x_0\xi' \in X^{n-i}}} (x_0 + mx_0\bar{d}_1, x_0\xi')\,(\eta ⧢ x_0\xi', \nu) \\
&= M_c \sum_{i=0}^{n-2} (\bar{b}, \nu_i) \sum_{\substack{\eta \in X^i \\ \xi' \in X^{n-i-1}}} (1 + m\bar{d}_1, \xi')(\eta ⧢ x_0\xi', \nu) \\
&\quad + M_c(\bar{b}, \nu_{n-1}) \sum_{\eta \in X^{n-1}} (1 + m\bar{d}_1, \emptyset)(\eta ⧢ x_0, \nu).
\end{aligned}$$

In the first summation directly above, note that $|\xi'| \geq 1$, and thus, $(1 + m\bar{d}_1, \xi') = m(\bar{d}_1, \xi')$. Consequently,

$$\begin{aligned}
(\bar{b}, \nu) &\leq M_c \sum_{i=0}^{n-2} (\bar{b}, \nu_i)\, mK_d M_d^{(n-i-1)}(n-i-1)! \sum_{\substack{\eta \in X^i \\ \xi' \in X^{n-i-1}}} (\eta ⧢ x_0\xi', \nu) + \\
&\qquad (\bar{b}, \nu_{n-1}) M_c(1 + mK_d) \sum_{\eta \in X^{n-1}} (\eta ⧢ x_0, \nu) \\
&\leq M_c \sum_{i=0}^{n-2} (\bar{b}, \nu_i)\, mK_d M_d^{(n-i-1)}(n-i-1)! \sum_{\substack{\eta \in X^i \\ \xi \in X^{n-i}}} (\eta ⧢ \xi, \nu) + \\
&\qquad (\bar{b}, \nu_{n-1}) M_c(1 + mK_d) \sum_{\substack{\eta \in X^{n-1} \\ \xi \in X}} (\eta ⧢ \xi, \nu)
\end{aligned}$$

$$= M_c \sum_{i=0}^{n-2} (\bar{b}, \nu_i)\; mK_d M_d^{(n-i-1)} (n-i-1)! \binom{n}{i} + (\bar{b}, \nu_{n-1}) M_c (1 + mK_d) n.$$

Note that the inequality above still holds when the left-hand side is replaced with $(\bar{b}, \nu_n)$. Now let a_n, $n \geq 0$ be the sequence satisfying the recursive formula

$$a_n = M_c \sum_{i=0}^{n-2} a_i m K_d M_d^{(n-i-1)} (n-i-1)! \binom{n}{i} + a_{n-1} M_c (1 + mK_d) n, \; n \geq 2,$$

where $a_0 = K_c$ and $a_1 = K_c M_c (1 + mK_d)$. Since the recursion above involves only positive terms, it follows that $(\bar{b}, \nu_n) \leq a_n$, $\forall n \geq 0$. It is easily verified that the sequence $a_n, n \geq 0$ has the exponential generating function

$$f(x_0) = \frac{K_c}{1 - M_c x_0 + (m M_c K_d / M_d) \ln(1 - M_d x_0)}. \tag{3.3.4}$$

When all the growth constants and m are unity, a_n, $n \geq 0$ is the integer sequence number A052820 in [29]. See the first row of Table 3 for the first few entries.

TABLE 3: Selected sequences from the OEIS for some cascade examples

sequence	OEIS number	$n = 0, 1, 2, \ldots$
a_n (local)	A052820	$1, 2, 9, 62, 572, 6604, 91526, \ldots$
b_n (global)	A000110	$1, 2, 5, 15, 52, 203, 877, 4140, \ldots$

Next it will be shown that $(\bar{c} \circ \bar{d}, x_0^n) = a_n$. It is sufficient to show that the zero-input response of the cascade system represented by the Fliess operator $F_{\bar{c} \circ \bar{d}}$, shown in Figure 3 is equal to f. Clearly,

$$v_1(t) = F_{d_1}[0] = \sum_{k=0}^{\infty} K_d M_d^k t^k = \frac{K_d}{1 - M_d t}.$$

From (3.3.3) observe

$$\bar{c} \circ \bar{d} = K_c + (\bar{c} \circ \bar{d}) \sqcup\!\sqcup M_c (x_0 + m x_0 \bar{d}_1).$$

Note that $x_0\bar{d}_1$ has the exponential generating function $\int_0^t v_1(\tau)\,d\tau$. Therefore,

$$\begin{aligned} y(t) &= F_{\bar{c}}[v](t) = F_{\bar{c}}[F_{\bar{d}}[0]](t) = F_{\bar{c}\circ\bar{d}}[0](t) \\ &= K_c + y(t)M_c\left(t + m\int_0^t v_1(\tau)\,d\tau\right) \\ &= \frac{K_c}{1 - M_c\left(t + m\int_0^t v_1(\tau)\,d\tau\right)} \\ &= \frac{K_c}{1 - M_c t + (mM_cK_d/M_d)\ln(1-M_dt)} \\ &= f(t). \end{aligned}$$

This proves that for every $n \geq 0$

$$(\bar{b},\nu) \leq (\bar{b},\nu_n) \leq a_n = (\bar{b},x_0^n),\ \nu \in X^n.$$

Since f is analytic at the origin, the smallest geometric growth constant is $M_b = 1/|x_0'|$, where it is easily verified from (3.3.4) that x_0' is the positive real number

$$x_0' = \frac{1}{M_d}\left[1 - mK_dW\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)\right]$$

This proves the theorem. ■

It is known that if u is analytic with generating series c_u, then $y = F_c[u]$ is also analytic [32], and its generating series is given by $c_y = c \circ c_u$ [19, 26, 27]. In this situation, the following corollary is useful for estimating a lower bound on the interval of convergence for the output.

Corollary 3.3.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $X_0 = \{x_0\}$. Suppose $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $c_u \in \mathbb{R}^m_{LC}[[X_0]]$ with growth constants K_{c_u}, M_{c_u}, respectively. Then, $c_y = c \circ c_u$ satisfies*

$$|(c_y, x_0^k)| \leq K_{c_y} M_{c_y}^k k!,\ k \geq 0$$

for some $K_{c_y} > 0$ and

$$M_{c_y} = \frac{M_{c_u}}{\left[1 - mK_{c_u}W\left(\frac{1}{mK_{c_u}}\exp\left(\frac{M_c - M_{c_u}}{mM_cK_{c_u}}\right)\right)\right]}.$$

Thus, the interval of convergence for the output $y = F_{c_y}[u]$ is at least as large as $T = 1/M_{c_y}$.

The following lemma is also needed for proving the main result.

Lemma 3.3.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ such that $|c| \leq d$, where $|c| := \sum_{\eta\in X^*} |(c, \eta)|\, \eta$. Then for any fixed $\xi \in X^*$ it follows that $|\xi \circ c| \leq \xi \circ d$.*

Proof: The proof is by induction on $k = |\xi| - |\xi|_{x_0}$. Let $\xi_0 = x_0^{n_0}$ and $\xi_k = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} \cdots x_{i_1} x_0^{n_0}$ for $k > 0$, where $1 \leq i_j \leq m$. For $k = 0$, the claim is trivial since

$$\xi_0 \circ c = x_0^{n_0} \circ c = x_0^{n_0} = x_0^{n_0} \circ d = \xi_0 \circ d.$$

Assume now that $|(\xi_k \circ c, \eta)| \leq (\xi_k \circ d, \eta)$ up to some fixed $k \geq 0$. Observe that

$$\begin{aligned}
\xi_{k+1} \circ c &= x_0^{n_{k+1}+1}(c_{i_{k+1}} \sqcup\!\sqcup (\xi_k \circ c)) \\
(\xi_{k+1} \circ c, \eta) &= (c_{i_{k+1}} \sqcup\!\sqcup (\xi_k \circ c), x_0^{-(n_{k+1}+1)}(\eta)) \\
&= \sum_{j=0}^{n} \sum_{\substack{\alpha\in X^j \\ \beta\in X^{n-j}}} (c_{i_{k+1}}, \alpha)(\xi_k \circ c, \beta)(\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}+1)}(\eta)),
\end{aligned}$$

where $n := |x_0^{-(n_{k+1}+1)}(\eta)| \geq 0$. Therefore,

$$\begin{aligned}
|(\xi_{k+1} \circ c, \eta)| &\leq \sum_{j=0}^{n} \sum_{\substack{\alpha\in X^j \\ \beta\in X^{n-j}}} \left|(c_{i_{k+1}}, \alpha)\right| |(\xi_k \circ c, \beta)|\, (\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}+1)}(\eta)) \\
&\leq \sum_{j=0}^{n} \sum_{\substack{\alpha\in X^j \\ \beta\in X^{n-j}}} (d_{i_{k+1}}, \alpha)(\xi_k \circ d, \beta)(\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}+1)}(\eta)) \\
&= (\xi_{k+1} \circ d, \eta).
\end{aligned}$$

Thus, the inequality holds for all $k \geq 0$, and the lemma is proved. ∎

Proof of Theorem* 3.3.1*:

Since $|d| \leq \bar{d}$, it follows from Lemma 3.3.1 that for any $\nu \in X^*$

$$\begin{aligned}
|(b, \nu)| &\leq \sum_{\eta\in X^*} |(c, \eta)||(\eta \circ d, \nu)| \\
&\leq \sum_{\eta\in X^*} K_c M_c^{|\eta|} |\eta|!\, (\eta \circ \bar{d}, \nu) \\
&= (\bar{b}_i, \nu),
\end{aligned}$$

where $\bar{b} = \bar{c} \circ \bar{d}$ and $i = 1, 2, \ldots, \ell$. In light of Theorem 3.3.2, $(\bar{b}_i, \nu)$ is asymptotically bounded by $M_b^{|\nu|}|\nu|!$. Thus, some $K_b > 0$ can always be introduced such that

$$(\bar{b}_i, \nu) \leq K_b M_b^{|\nu|}|\nu|!, \quad \nu \in X^*.$$

Furthermore, $(\bar{b}_i, x_0^n)$ is growing exactly at this rate. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ■

Example 3.3.1. Let $X = \{x_0, x_1\}$ and $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ such that $M = M_c = M_d$. Then

$$\begin{aligned} M_b &= \frac{M}{1 - K_d W(1/K_d)} \\ &= \left(\frac{3}{2} + K_d + O\left(\frac{1}{K_d}\right)\right) M \\ &\approx K_d M \end{aligned}$$

when $K_d \gg 1$. This is consistent with Theorem 2.3.4 and Table 1. On the other hand, if $K_d = 1$ then $M_b = (1 - W(1))^{-1}M = 2.3102M$, which is less than the estimate $(\phi_g + 1)M = 2.6180M$ given by Theorem 2.3.4. □

Example 3.3.2. Suppose $X = \{x_0, x_1\}$ and $\bar{b} = \bar{c} \circ \bar{d}$ with $\bar{c} = \sum_{\eta \in X^*} K_c M_c^{|\eta|}|\eta|!\, \eta$ and $\bar{d} = \sum_{\eta \in X^*} K_d M_d^{|\eta|}|\eta|!\, \eta$. The output of the cascaded system as shown in Figure 3 is described by the state space system

$$\begin{aligned} \dot{z}_1 &= \frac{M_c}{K_c} z_1^2 (1 + z_2), \quad z_1(0) = K_c \\ \dot{z}_2 &= \frac{M_d}{K_d} z_2^2 (1 + u), \quad z_2(0) = K_d \\ y &= z_1. \end{aligned}$$

A MATLAB generated zero-input response is shown in Figure 5 when $K_c = 1$, $M_c = 2$, $K_d = 3$ and $M_d = 4$. As expected from Theorem 3.3.2, the finite escape time of the output is $t_{esc} = 1/M_b = 0.1028$. The output responses corresponding to the analytic inputs $u_1(t) = 1/1-t$ and $u_2(t) = 1/1-t^2$, each having growth constants $K_{c_u} = M_{c_u} = 1$, are also shown in the figure. Their respective finite escape times are $t_{esc} = 0.08321$ and $t_{esc} = 0.08377$. Here u_1 has the shortest escape time since its generating series

$$c_{u_1} = \sum_{k=0}^{\infty} k!\, x_0^k$$

has all its coefficients growing at the maximum rate. Where as

$$c_{u_2} = \sum_{k=0}^{\infty} (2k)!\, x_0^{2k}$$

has all its odd coefficients equal to zero. By Corollary 3.3.1, any finite escape time for the output corresponding to any analytic input with the given growth constants K_{c_u}, M_{c_u} must be at least as large as $T = 1/M_{c_y} = 0.05073$. □

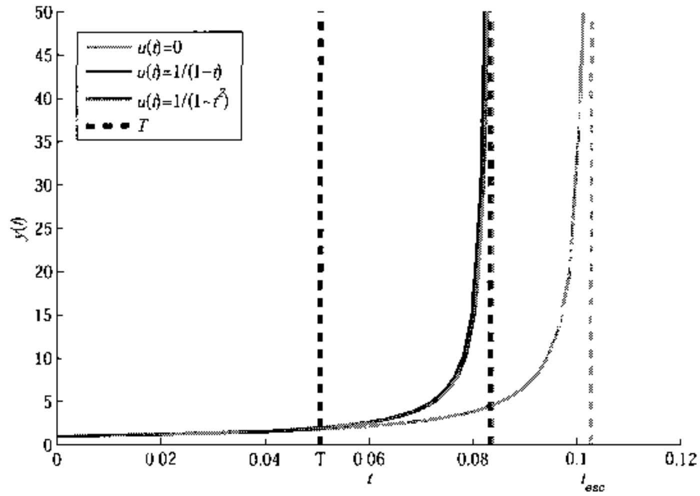

Fig. 5: Output responses of the cascaded system $F_{\bar{c}\circ\bar{d}}$ to various analytic inputs in Example 3.3.2

3.3.2 Global Convergence

A parallel analysis is done in this section to compute the radius of convergence of the cascade connection of two globally convergent Fliess operators. The following theorem contains the main result.

Theorem 3.3.3. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Assume $\bar{c}$ and $\bar{d}$ are defined as in Theorem* 3.3.4. *If $b = c \circ d$ and $\bar{b} = \bar{c} \circ \bar{d}$ then*

$$|(b,\nu)| \leq (\bar{b}_i, x_0^{|\nu|}),\ \nu \in X^*,\ i = 1, 2, \ldots, \ell,$$

where the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ *has the exponential generating function*

$$f(x_0) = K_c \exp\left(\frac{mK_d \exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right).$$

Therefore, the radius of convergence is infinity

An intermediate result is essential in proving the main theorem above.

Theorem 3.3.4. *Let* $X = \{x_0, x_1, \ldots, x_m\}$. *Let* $\bar{c} \in \mathbb{R}^\ell_{GC}\langle\langle X\rangle\rangle$ *and* $\bar{d} \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$, *where each component of* $(\bar{c}, \eta)$ *is* $K_c M_c^{|\eta|}$, $\eta \in X^*$ *with* $K_c, M_c > 0$, *and likewise, each component of* $(\bar{d}, \eta)$ *is* $K_d M_d^{|\eta|}$, $\eta \in X^*$ *with* $K_d, M_d > 0$. *If* $\bar{b} = \bar{c} \circ \bar{d}$, *then* $(\bar{b}_i, \nu) \leq (\bar{b}_i, x_0^{|\nu|})$, $\nu \in X^*$, *and the sequence* $(\bar{b}_i, x_0^k)$, $k \geq 0$ *has the exponential generating function*

$$f(x_0) = K_c \exp\left(\frac{mK_d \exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right)$$

for any $i = 1, 2, \ldots, \ell$.

Proof: As in the local case, there is no loss of generality in assuming $\ell = 1$. Using Theorem 2.3.5, observe that

$$\begin{aligned}
\bar{b} &= \sum_{k=0}^{\infty} \frac{K_c M_c^k}{k!} \sum_{\substack{r_0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = k}} k! \frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{(x_m \circ \bar{d})^{\sqcup\!\sqcup\, r_m}}{r_m!} \\
&= K_c \sum_{k=0}^{\infty} \frac{\left(M_c(x_0 + m x_0 \bar{d}_1)\right)^{\sqcup\!\sqcup\, k}}{k!}.
\end{aligned}$$

Therefore, $(\bar{b}, \emptyset) = K_c$ and

$$\begin{aligned}
x_0^{-1}(\bar{b}) &= K_c \sum_{k=1}^{\infty} \frac{\left(M_c(x_0 + m x_0 \bar{d}_1)\right)^{\sqcup\!\sqcup\, k-1}}{(k-1)!} \sqcup\!\sqcup\, M_c(1 + m\bar{d}_1) \\
&= \bar{b} \sqcup\!\sqcup\, M_c(1 + m\bar{d}_1).
\end{aligned} \tag{3.3.5}$$

By inspection,

$$\begin{aligned}
(x_0^{-1}(\bar{b}), \emptyset) &= K_c M_c(1 + mK_d) \\
(x_0^{-1}(\bar{b}), x_0) &= K_c M_c m K_d M_d + K_c(M_c(1 + mK_d))^2 \\
(x_0^{-1}(\bar{b}), x_i) &= K_c M_c m K_d M_d, \; i = 1, 2, \ldots, m.
\end{aligned}$$

For any $\nu \in X^n$, $n \geq 2$, it follows that

$$
\begin{aligned}
(x_0^{-1}(\bar{b}), \nu) &= M_c \sum_{i=0}^{n} \sum_{\substack{\eta \in X^i \\ \xi \in X^{n-i}}} (\bar{b}, \eta)(1 + m\bar{d}_1, \xi)(\eta \sqcup \xi, \nu) \\
&= M_c \sum_{i=1}^{n-1} \sum_{\substack{x_0\eta' \in X^i \\ \xi \in X^{n-i}}} (\bar{b}, x_0\eta')(1 + m\bar{d}_1, \xi)(x_0\eta' \sqcup \xi, \nu) \\
&\quad + M_c \sum_{x_0\eta' \in X^n} (\bar{b}, x_0\eta')(1 + m\bar{d}_1, \emptyset)(x_0\eta', \nu) \\
&\quad + M_c \sum_{\xi \in X^n} (\bar{b}, \emptyset)(1 + m\bar{d}_1, \xi)(\xi, \nu) \\
&= M_c \sum_{i=1}^{n-1} \sum_{\substack{\eta' \in X^{i-1} \\ \xi \in X^{n-i}}} (x_0^{-1}(\bar{b}), \eta')(1 + m\bar{d}_1, \xi)(x_0\eta' \sqcup \xi, \nu) \\
&\quad + M_c \sum_{\eta' \in X^{n-1}} (x_0^{-1}(\bar{b}), \eta')(1 + m\bar{d}_1, \emptyset)(x_0\eta', \nu) + M_c(\bar{b}, \emptyset)m(\bar{d}_1, \nu).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(x_0^{-1}(\bar{b}), \nu) &\leq M_c \sum_{i=1}^{n-1} (x_0^{-1}(\bar{b}), \eta_{i-1}) m K_d M_d^{n-i} \sum_{\substack{\eta \in X^i \\ \xi \in X^{n-i}}} (\eta \sqcup \xi, \nu) \\
&\quad + (x_0^{-1}(\bar{b}), \eta_{n-1}) M_c(1 + mK_d) + K_c M_c m K_d M_d^n \\
&= M_c \sum_{i=1}^{n-1} (x_0^{-1}(\bar{b}), \eta_{i-1}) m K_d M_d^{n-i} \binom{n}{i} \\
&\quad + (x_0^{-1}(\bar{b}), \eta_{n-1}) M_c(1 + mK_d) + K_c M_c m K_d M_d^n.
\end{aligned}
$$

Similar to the analysis in the previous section, let $a_n, \ n \geq 0$ be the sequence satisfying the recursive formula

$$
a_n = M_c \sum_{i=1}^{n-1} a_{i-1} m K_d M_d^{n-i} \binom{n}{i} + a_{n-1} M_c(1 + mK_d) + K_c M_c m K_d M_d^n, \ n \geq 2,
$$

where $a_0 = K_c M_c(1 + mK_d)$ and $a_1 = K_c M_c m K_d M_d + K_c(M_c(1 + mK_d))^2$. It follows that $(x_0^{-1}(\bar{b}), \nu_n) \leq a_n, \ \forall n \geq 0$, and thus, $(\bar{b}, \nu_n) \leq b_n, \ \forall n \geq 0$, where $b_n = a_{n-1}$ and $b_0 = K_c$. It is easily verified that the sequence $b_n, \ n \geq 0$ has the exponential generating function

$$
f(x_0) = K_c \exp\left(\frac{mK_d \exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c} \right).
$$

When all the growth constants and m are unity, b_n, $n \geq 0$ is the integer sequence number A000110 (shifted one position to the left) in the OEIS. These integers are called the *Bell numbers*. See the second row of Table 3 for the first few entries.

Next it will be shown that $(\bar{c} \circ \bar{d}, x_0^n) = b_n$. It is sufficient to show that the zero-input response of the cascade system represented by the Fliess operator $F_{\bar{c}\circ\bar{d}}$, shown in Figure 3 is equal to f. Clearly,

$$v_1(t) = F_{\bar{d}_1}[0](t) = \sum_{k=0}^{\infty} K_d M_d^k \frac{t^k}{k!} = K_d \exp(M_d t).$$

From (3.3.5) and the fact that $x_i^{-1}(\bar{b}) = 0$, $i = 1, 2, \ldots, m$, it follows that

$$y'(t) = M_c y(t)(1 + mK_d \exp(M_d t)), \quad y(0) = K_c.$$

Solving this differential equation yields

$$y(t) = K_c \exp\left(\frac{mK_d \exp(M_d t) + M_d t - mK_d}{M_d/M_c}\right).$$

Thus, for every $n \geq 0$

$$(\bar{b}, \nu) \leq (\bar{b}, \nu_n) \leq b_n = (\bar{b}, x_0^n), \; \nu \in X^n,$$

and the theorem is proved. ■

***Proof of Theorem* 3.3.3**

Again from Lemma 3.3.1, it follows that for any $\nu \in X^*$

$$\begin{aligned} |(b, \nu)| &\leq \sum_{\eta \in X^*} |(c, \eta)||(\eta \circ d, \nu)| \\ &\leq \sum_{\eta \in X^*} K_c M_c^{|\eta|} \, (\eta \circ \bar{d}, \nu) \\ &= (\bar{b}_i, \nu). \end{aligned}$$

By Theorem 3.3.4, $(\bar{b}_i, \nu)$ is bounded by $(\bar{b}_i, x_0^{|\nu|})$, which has the exponential generating function f. Thus, the theorem is proved. ■

It is worth noting that the Bell numbers (without any left shift), B_n, have the exponential generating function $e^{e^x - 1}$. Their asymptotic behavior is

$$B_n \sim n^{-\frac{1}{2}} (\lambda(n))^{n+\frac{1}{2}} e^{\lambda(n) - n - 1},$$

where $\lambda(n) = n/W(n)$. Thus, the Lambert W-function appears to also play a role in the global problem. It is also known that the Bell numbers play a central role in the analysis of function composition [11]. Most importantly, since the double exponential appearing in Theorem 3.3.4 has no finite singularities, as appeared in the local analysis in Section 3.3.1, the following main result is immediate.

Theorem 3.3.5. *The cascade connection of two globally convergent Fliess operators has a radius of convergence equal to infinity. Therefore, the output of such a system is always well defined over any finite interval of time when* $u \in L_{1,e}^m(t_0)$.

It is important to understand that this theorem is *not* saying that the composite system has a globally convergent generating series in the sense of (1.1.3). If this were the case, then it would be possible to bound $y(t) = F_{c\circ d}[0]$ by a single exponential function rather than a double exponential function (see [20, Theorem 3.1]). Thus, the fastest possible growth rate for the coefficients of a cascade connection involving components with globally convergent generating series falls somewhere strictly *in between* the local growth condition (1.1.1) and the global growth condition (1.1.3).

Example 3.3.3. Suppose $X = \{x_0, x_1\}$ and $\bar{b} = \bar{c} \circ \bar{d}$ with $\bar{c} = \sum_{\eta \in X^*} K_c M_c^{|\eta|}\, \eta$ and $\bar{d} = \sum_{\eta \in X^*} K_d M_d^{|\eta|}\, \eta$. The output of the cascade system is described by the state space realization

$$\begin{aligned} \dot{z}_1 &= M_c z_1(1 + z_2), \quad z_1(0) = K_c \\ \dot{z}_2 &= M_d z_2(1 + u), \quad z_2(0) = K_d \\ y &= z_1. \end{aligned}$$

A MATLAB generated zero-input response of this system is shown on a double logarithmic scale in Figure 6 when $K_c = M_c = K_d = M_d = 1$. As expected from Theorem 3.3.4, this plot asymptotically approaches that of $y(t) = t$ as $t \to \infty$. □

3.4 SUMMARY

A complete analysis of the radius of convergence of the parallel, product and cascade connections of two analytic nonlinear input-output systems represented as Fliess operators has been presented. For the parallel and product connections, if the component systems are both locally convergent, then the radius of convergence

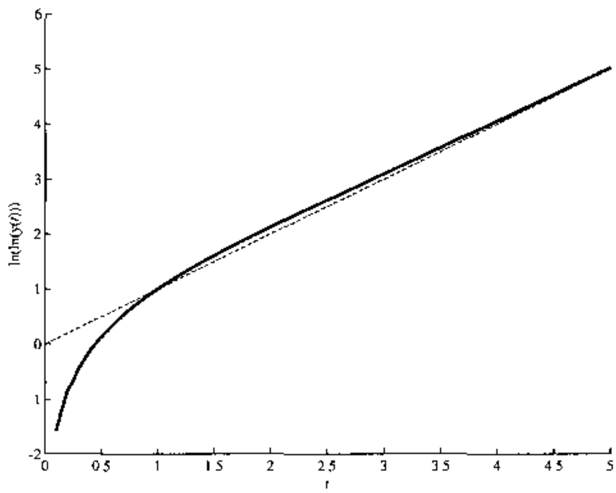

Fig. 6: Zero-input response of the cascade system $F_{\bar{c}\circ\bar{d}}$ in Example 3.3.3 on a double logarithmic scale and the function $y(t) = t$

of the overall system was found to be the minimum of the radii of convergence of the component systems. If they are globally convergent, so is the overall system. Therefore, the radius of convergence of the overall system is infinite. For the cascade connection, if the component systems are both locally convergent, then the radius of convergence is finite and can be computed in terms of the Lambert W-function. A similar method was used in the case of analytic inputs to compute a lower bound on the interval of convergence of the output function. On the other hand, if both systems are globally convergent, then the radius of convergence was shown to be infinite, even though it is known that the global convergence property is not preserved in general. This means in particular that if the input is well defined and absolutely integrable over any finite time interval, then the output of the composite system is also well defined over the same interval. The Lambert W-function played an implicit role in the analysis of the global case.

CHAPTER 4

THE RADIUS OF CONVERGENCE OF THE FEEDBACK CONNECTION

In this chapter, the radius of convergence is determined for the feedback connection. First, self-excited feedback systems are addressed. Subsequently, the analysis for the unity feedback case is presented. In each case, separate analyses are done for closed-loop systems having components with locally convergent generating series and globally convergent generating series.

4.1 THE SELF-EXCITED CASE

As discussed in Chapter 2, the generating series e for the self-excited feedback interconnection of F_c and F_d shown in Figure 4 satisfies the identity $e = (c \circ d) \circ e$. Letting $c \circ d$ be redefined as c, a unity feedback system involving F_c is characterized by $e = c \circ e$. Therefore, there is no loss of generality in assuming unity feedback in the self-excited case.

4.1.1 Local Convergence

The main result of this section is the following theorem.

Theorem 4.1.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$. If $e \in \mathbb{R}^m[[X_0]]$ satisfies $e = c \circ e$ then*

$$|(e, x_0^n)| \leq K_e \left(\alpha(K_c) M_c\right)^n n!, \quad n \geq 0, \tag{4.1.1}$$

for some $K_e > 0$ and

$$\alpha(K_c) = \frac{1}{1 - mK_c \ln\left(1 + 1/mK_c\right)}.$$

Furthermore, no geometric growth constant smaller than $\alpha(K_c)M_c$ can satisfy 4.1.1, and thus the radius of convergence is $1/(\alpha(K_c)M_c)$.

Note that if $m = 1$, the function $\alpha(K_c)$ can be written as the series expansion about $K_c = \infty$

$$\alpha(K_c) = \frac{4}{3} + 2K_c + O\left(\frac{1}{K_c}\right).$$

It is easy to show that $\alpha(K_c) < \beta(K_c)$ for all $K_c \geq 1$ and $\beta(K_c)/\alpha(K_c) \approx 9$ for $K_c \gg 1$, where $\beta(K_c)$ is defined in Theorem 2.4.4. Thus, Theorem 4.1.1, which describes the radius of convergence in this case, constitutes an order of magnitude improvement over the lower bound given in Theorem 2.4.4. Before presenting the proof of this theorem, a variety of intermediate results are required involving exchangeable series. The following theorem characterizes the self-excited feedback connection of a Fliess operator having a particular type of exchangeable generating series.

Theorem 4.1.2. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $\bar{c} \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$, where each component of $(\bar{c}, \eta)$ is $K_c M_c^{|\eta|} |\eta|!$, $\eta \in X^*$ with $K_c, M_c > 0$. Then each component of the solution $\bar{e} \in \mathbb{R}^m_{LC}[[X_0]]$ of the self-excited unity feedback equation $\bar{e} = \bar{c} \circ \bar{e}$ has the exponential generating function*

$$f(x_0) = \frac{-1}{m\left[1 + W\left(-\frac{1+mK_c}{mK_c} \exp\left[\frac{M_c x_0 - (1+mK_c)}{mK_c}\right]\right)\right]}. \tag{4.1.2}$$

In addition, the smallest possible geometric growth constant for $\bar{e}$ is

$$M_e = \alpha(K_c) M_c, \tag{4.1.3}$$

where

$$\alpha(K_c) = \frac{1}{1 - mK_c \ln(1 + 1/mK_c)}.$$

Proof: Since all the component series of $\bar{c}$ are identical, the same is true for $\bar{e}$. Therefore, the focus will be a single component, say $\bar{e}_1$. First it is shown that $\bar{e}_1$ must satisfy the shuffle identity

$$\bar{e}_1 = K_c + M_c[\bar{e}_1 \sqcup\!\sqcup (x_0 + m x_0 \bar{e}_1)].$$

Observe that from Theorem 2.3.5 and the shuffle product version of the binomial theorem it follows that

$$\begin{aligned}
\bar{e}_1 = (\bar{c} \circ \bar{e})_1 &= \sum_{k=0}^{\infty} K_c M_c^k \sum_{\substack{r_0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = k}} k! \, \frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{(x_m \circ \bar{e})^{\sqcup\!\sqcup\, r_m}}{r_m!} \\
&= \sum_{k=0}^{\infty} K_c \left(M_c \left(x_0 + \sum_{i=1}^{m} x_0 \bar{e}_i \right) \right)^{\sqcup\!\sqcup\, k} \\
&= \sum_{k=0}^{\infty} K_c \left(M_c (x_0 + m x_0 \bar{e}_1) \right)^{\sqcup\!\sqcup\, k}.
\end{aligned}$$

Shuffling both sides of this equation by $M_c(x_0 + mx_0\bar{e}_1)$ yields

$$\bar{e}_1 \sqcup\!\sqcup M_c(x_0 + mx_0\bar{e}_1) = \sum_{k=0}^{\infty} K_c(M_c(x_0 + mx_0\bar{e}_1))^{\sqcup\!\sqcup k+1}.$$

Adding K_c to both sides gives

$$\bar{e}_1 = K_c + M_c[\bar{e}_1 \sqcup\!\sqcup (x_0 + mx_0\bar{e}_1)]. \tag{4.1.4}$$

When written in terms of generating functions, (4.1.4) is equivalent to

$$f(x_0) = K_c + M_c\left(x_0 f(x_0) + m\int_0^{x_0} f(\xi)\, d\xi\, f(x_0)\right),\ f(0) = K_c.$$

A simple calculation shows that this equation is equivalent to

$$K_c f'(x_0) = M_c\left(f^2(x_0) + mf^3(x_0)\right),\quad f(0) = K_c. \tag{4.1.5}$$

One can verify by brute force, since M_c is nonzero, that (4.1.2) is the solution of (4.1.5). Since $f(x_0)$ is analytic at $x_0 = 0$, the smallest geometric growth constant is determined by the location of its singularity nearest to the origin, $x_0' \in \mathbb{C}$. In which case, $M_e = 1/|x_0'|$, where x_0' satisfies

$$m\left[1 + W\left(-\frac{1+mK_c}{mK_c}\exp\left[\frac{M_c x_0' - (1+mK_c)}{mK_c}\right]\right)\right] = 0.$$

Equation (4.1.3) and the subsequent identities then follow directly by solving this equation for x_0' via (3.3.2). ∎

One additional technical lemma is needed before the proof of Theorem 4.1.1 can be presented. Given any series $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, it is convenient to define $|c| = \sum_{\eta\in X^*} |(c,\eta)|\,\eta$ in the lemma below.

Lemma 4.1.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $c, \bar{c} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ have growth constants $K_c, M_c > 0$, and specifically each component of $\bar{c}$ is $K_c M_c^{|\eta|}|\eta|!$, $\eta \in X^*$. If $e, \bar{e} \in \mathbb{R}^m[[X_0]]$ satisfy, respectively, $e = c \circ e$ and $\bar{e} = \bar{c} \circ \bar{e}$ then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.*

Proof: Since the mapping $d \mapsto c \circ d$ is a contraction, it follows that if $e_i(k) := (c^{\circ k} \circ 0)_i$, $k \geq 1$ then $e_i = \lim_{k\to\infty} e_i(k)$. Likewise, one can define a sequence $\bar{e}_i(k)$ using $\bar{c}$. It will first be shown by induction that $|e_i(k)| \leq \bar{e}_i(k)$, $k \geq 1$. Observe that $e_i(1) = \sum_{n\geq 0}(c_i, x_0^n)\, x_0^n$ and $\bar{e}_i(1) = \sum_{n\geq 0} K_c M_c^n n!\, x_0^n$. Therefore, $|e_i(1)| \leq \bar{e}_i(1)$.

Now assume the claim holds up to some fixed $k \geq 1$. Then, using Lemma 3.3.1, for any $\xi \in X^*$

$$\begin{aligned} |(e_i(k+1), \xi)| &= |((c \circ e(k))_i, \xi)| = \left| \sum_{\eta \in X^*} (c_i, \eta) \, (\eta \circ e(k), \xi) \right| \\ &\leq \sum_{\eta \in X^*} |(c_i, \eta)| \, |(\eta \circ e(k), \xi)| \\ &\leq \sum_{\eta \in X^*} K_c M_c^{|\eta|} \, |\eta|! \, (\eta \circ \bar{e}(k), \xi) \\ &= (\bar{e}_i(k+1), \xi). \end{aligned}$$

Thus,

$$|e_i(k)| \leq \bar{e}_i(k), \;\; k \geq 1,$$

and the initial claim is established. Next, by a property of the limit supremum,

$$\limsup_{k \to \infty} |(e_i(k), \xi)| \leq \limsup_{k \to \infty} (\bar{e}_i(k), \xi).$$

Since each sequence converges, it follows that $|e_i| \leq \bar{e}_i$. ∎

***Proof of Theorem* 4.1.1:**

If e, $\bar{c}$ and $\bar{e}$ are defined as in Lemma 4.1.1 then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$. From Theorem 4.1.2, $(\bar{e}_i, x_0^n)$ is asymptotically bounded by $(\alpha(K_c)M_c)^n \, n!$. In which case,

$$|(e_i, x_0^n)| \leq (\bar{e}_i, x_0^n) \leq K_e \left(\alpha(K_c) M_c\right)^n \, n!, \;\; n \geq 0,$$

for some constant $K_e > 0$. This proves the theorem. ∎

The following examples illustrate the main results of this section.

Example 4.1.1. Let $X = \{x_0, x_1\}$. Suppose $\bar{e}$ satisfies $\bar{e} = \bar{c} \circ \bar{e}$ with $\bar{c} = \sum_{\eta \in X^*} K_c M_c^{|\eta|} |\eta|! \; \eta$. This series is exchangeable, so by Theorem 4.1.2, $M_e = \alpha(K_c) M_c$. From (4.1.5) it follows that the output of the self-excited unity feedback system is described by the solution of the state space system

$$\begin{aligned} \dot{z} &= \frac{M_c}{K_c}(z^2 + z^3), \;\; z(0) = K_c \\ y &= z. \end{aligned}$$

MATLAB generated solutions of this system are shown in Figure 7 when $K_c = M_c = 1$ and when $K_c = 0.5$, $M_c = 2$. As expected, the respective finite escape times are

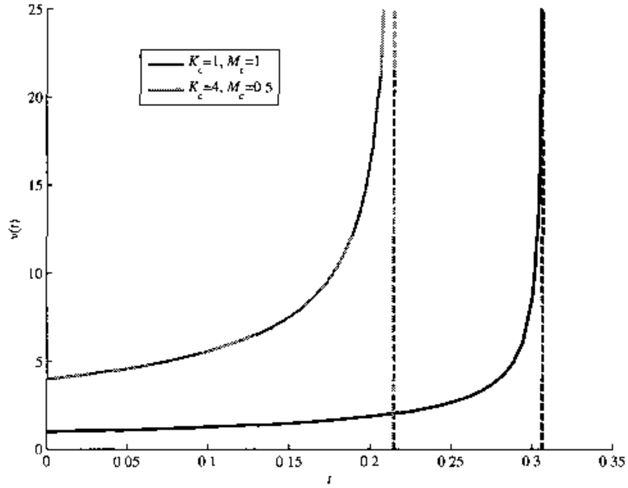

Fig. 7: Outputs of the self-excited loop in Example 4.1.1

$t_{esc} = 1/\alpha(1) = 1 - \ln(2) \approx 0.3069$ and $t_{esc} = 2/\alpha(4) \approx 0.2149$, which in this case are the radii of convergence. Also, from (4.1.2) it follows when $K_c = M_c = 1$ that

$$\begin{aligned} f(x_0) &= \frac{-1}{1 + W(-2\exp(x_0 - 2))} \\ &= 1 + 2\,x_0 + 5\,{x_0}^2 + \frac{41}{3}{x_0}^3 + \frac{469}{12}{x_0}^4 + O\big({x_0}^5\big)\,. \end{aligned}$$

The coefficients (e, x_0^n), $n \geq 0$ correspond in this case to OEIS sequence A112487 as shown in Table 4. □

TABLE 4: Selected sequences from the OEIS for feedback examples

sequence	OEIS number	$n = 0, 1, 2, \ldots$
$(\bar{e}, x_0^n)$ (Example 4.1.1)	A112487	$1, 2, 10, 82, 938, 13778, 247210, \ldots$
$(\bar{e}, x_0^n)$ (Example 4.1.4)	A000629	$1, 2, 6, 26, 150, 1082, 9366, \ldots$

Example 4.1.2. Let $X = \{x_0, x_1\}$ and consider the case where e satisfies $e = c \circ e$ with $c = \sum_{n \geq 0} n!\, x_1^n$. This c is also an exchangeable series except here many of the coefficients have been zeroed out in comparison with the previous example.

Therefore, it is likely that the radius of convergence will be *larger*. In this special case, equation (4.1.5) reduces to

$$f'(x_0) = f^3(x_0), \ \ f(0) = 1,$$

which has the solution

$$f(x_0) = \frac{1}{\sqrt{1-2x_0}}.$$

The singularity at $x_0' = 1/2$ implies that $M_e = 2 < 1/(1-\ln(2)) \approx 3.2589$. So in fact the radius of convergence is 0.5, which is larger than 0.3069 obtained in the previous example. The function f is known to be the exponential generating function for the sequence $(e, x_0^n) = (2n-1)!!$, $n \geq 0$. (The double factorial for a positive odd integer n is defined as $n!! = n(n-2)\cdots 1$ and $-1!! := 1$.) Using an identity for the double factorial, it follows that

$$\begin{aligned}(e, x_0^n) &= \frac{(2n)!}{2^n n!} = \frac{n+1}{2^n}\,\frac{(2n)!}{(n+1)n!n!}\,n! \\ &= \frac{(n+1)}{2^n}\,C_n\,n!.\end{aligned}$$

Thus, the generating series for the feedback system is

$$e = \sum_{n=0}^{\infty} \frac{C_n}{2^n}\,(n+1)!\,x_0^n.$$

The output of the corresponding self-excited unity feedback system is described by the solution of

$$\begin{aligned}\dot{z} &= z^3, \ \ z(0) = 1 \\ y &= z.\end{aligned}$$

A MATLAB generated solution of this system is shown in Figure 8. As expected, it has a finite escape time of $t_{esc} = 1/M_e = 0.5 > 1 - \ln(2) \approx 0.3069$. □

Example 4.1.3. Consider the feedback system shown in Figure 4 with $c = d = \sum_{\eta \in X^*} |\eta|!\,\eta$. Clearly $c \circ d$ is locally convergent, but not exchangeable. Thus, only Theorem 4.1.1 applies. The output y of the feedback system with $u = 0$ is described by the state space system

$$\begin{aligned}\dot{z}_1 &= z_1^2(1+z_2), \ \ z_1(0) = 1 \\ \dot{z}_2 &= z_2^2, \ \ z_2(0) = 1 \\ y &= z_2.\end{aligned}$$

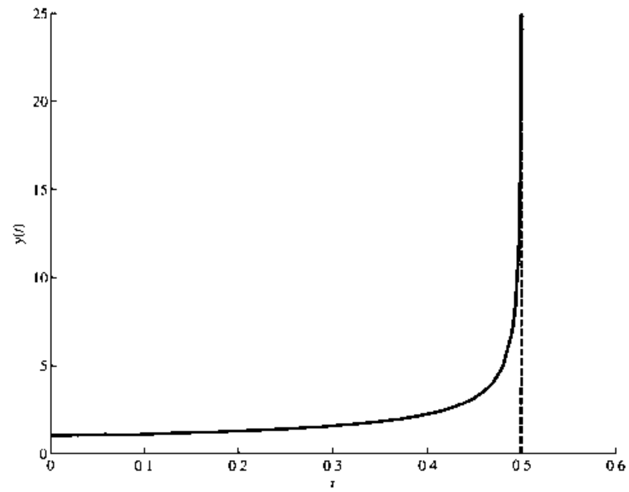

Fig. 8: Output of the self-excited loop in Example 4.1.2

The output y, as computed by MATLAB, is numerically indistinguishable from the $K_c = M_c = 1$ case shown in Figure 7 for Example 4.1.1. This is expected since $e(k+1) = c \circ e(k)$ and $e(k+1) = (c \circ c) \circ e(k)$ have the same fixed point. Hence, $t_{esc} = 1 - \ln(2) \approx 0.3069$. $\square$

4.1.2 Global Convergence

The global analogue of Theorem 4.1.2 regarding self-excited systems is given next.

Theorem 4.1.3. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$. If $e \in \mathbb{R}^m[[X_0]]$ satisfies $e = c \circ e$ then*

$$|(e, x_0^n)| \leq K_e \left(\gamma(K_c) M_c\right)^n n!, \quad n \geq 0, \tag{4.1.6}$$

for some $K_e > 0$ and

$$\gamma(K_c) = \frac{1}{\ln\left(1 + 1/mK_c\right)}.$$

Furthermore, no geometric growth constant smaller than $\gamma(K_c)M_c$ can satisfy (4.1.6), and thus the radius of convergence is $1/(\gamma(K_c)M_c)$.

It is known in general that global convergence is not preserved under feedback [18], but e is always at least locally convergent [19]. When $m = 1$, in light of the expansion

$$\gamma(K_c) = \frac{1}{2} + K_c + O\left(\frac{1}{K_c}\right),$$

the global growth condition on c gives a radius of convergence that is about twice that for the local case. The following theorem is essential for proving the main result.

Theorem 4.1.4. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $\bar{c} \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$, where each component of $(\bar{c}, \eta)$ is $K_c M_c^{|\eta|}$, $\eta \in X^*$ with $K_c, M_c > 0$. Then each component of the solution $\bar{e} \in \mathbb{R}^m_{LC}[[X_0]]$ of the self-excited unity feedback equation $\bar{e} = \bar{c} \circ \bar{e}$ has the exponential generating function*

$$f(x_0) = \frac{K_c \exp(M_c x_0)}{(1 + mK_c) - mK_c \exp(M_c x_0)}. \tag{4.1.7}$$

In addition, the smallest possible geometric growth constant of e is

$$M_e = \gamma(K_c) M_c, \tag{4.1.8}$$

where

$$\gamma(K_c) = \frac{1}{\ln(1 + 1/mK_c)}.$$

Proof: Without loss of generality, the focus is on the single component $\bar{e}_1$. First it is shown that $\bar{e}_1$ must satisfy the shuffle identity

$$x_0^{-1}(\bar{e}_1) = M_c(1 + m\bar{e}_1) ⧢ \bar{e}_1. \tag{4.1.9}$$

Observe that from Theorem 2.3 5 and the shuffle product version of the binomial theorem it follows that

$$\begin{aligned}
\bar{e}_1 &= \sum_{k=0}^{\infty} \frac{K_c M_c^k}{k!} \sum_{\substack{r_0, \ldots, r_m \geq 0 \\ r_0 + \cdots + r_m = k}} k! \, \frac{x_0^{⧢ r_0}}{r_0!} ⧢ \cdots ⧢ \frac{(x_m \circ \bar{e}_1)^{⧢ r_m}}{r_m!} \\
&= K_c \sum_{k=0}^{\infty} \frac{(M_c(x_0 + m x_0 \bar{e}_1))^{⧢ k}}{k!}.
\end{aligned}$$

Therefore, $(\bar{e}_1, \emptyset) = K_c$ and

$$\begin{aligned}
x_0^{-1}(\bar{e}_1) &= K_c \sum_{k=1}^{\infty} \frac{(M_c(x_0 + m x_0 \bar{e}_1))^{⧢ k-1}}{(k-1)!} ⧢ M_c(1 + m\bar{e}_1) \\
&= \bar{e}_1 ⧢ M_c(1 + m\bar{e}_1).
\end{aligned} \tag{4.1.10}$$

Therefore,

$$x_0^{-1}(\bar{e}_1) = M_c(1 + m\bar{e}_1) \sqcup\!\sqcup \bar{e}_1.$$

Since $x_0^{-1}(\bar{e}_1)$ has the exponential generating f', equation (4.1.9) is equivalent to

$$f'(x_0) = M_c(f(x_0) + mf^2(x_0)), \quad f(0) = K_c. \tag{4.1.11}$$

It can be verified directly, since $M_c > 0$, that the solution of this differential equation is

$$f(x_0) = \frac{K_c \exp(M_c x_0)}{(1 + mK_c) - mK_c \exp(M_c x_0)}.$$

Since f is analytic at $x_0 = 0$, the smallest geometric growth constant is again determined from Theorem 2.1.2 by computing the location of the singularity nearest to the origin, x_0'. In this case, $M_e = 1/|x_0'|$, where x_0' is a root of

$$(1 + mK_c) - mK_c \exp(M_c x_0) = 0.$$

Equation (4.1.8) and the subsequent identities then follow from solving this equation for x_0'. ■

The following lemma is a global version of Lemma 4.1.1. Its proof is perfectly analogous.

Lemma 4.1.2. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $c, \bar{c} \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ have growth constants $K_c, M_c > 0$, and specifically each component of $\bar{c}$ is $K_c M_c^{|\eta|}$, $\eta \in X^*$. If $e, \bar{e} \in \mathbb{R}^m[[X_0]]$ satisfy, respectively, $e = c \circ e$ and $\bar{e} = \bar{c} \circ \bar{e}$ then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.*

***Proof of Theorem* 4.1.3:**

If e, $\bar{c}$ and $\bar{e}$ are defined as in Lemma 4.1.2 then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$. The remainder of the proof is exactly analogous to that given for the local case. ■

The following examples illustrate the main results of this subsection.

Example 4.1.4. Let $X = \{x_0, x_1\}$. Suppose $\bar{e}$ satisfies $\bar{e} = \bar{c} \circ \bar{e}$ with $\bar{c} = \sum_{\eta \in X^*} K_c M_c^{|\eta|}\, \eta$. From Theorem 4.1.4 it follows that $M_e = \gamma(K_c)M_c$. From (4.1.11), the output of the self-excited unity feedback system is described by the solution of the state space system

$$\begin{aligned} \dot{z} &= M_c(z + z^2), \quad z(0) = K_c \\ y &= z. \end{aligned}$$

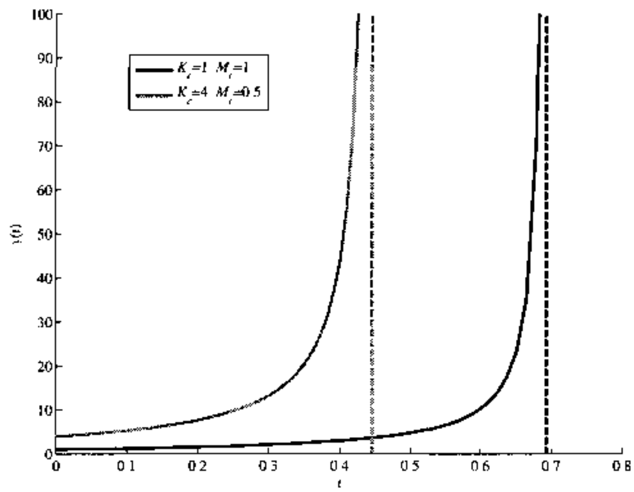

Fig. 9: Outputs of the self-excited loop in Example 4.1.4

MATLAB generated solutions of this system are shown in Figure 9 when $K_c = M_c = 1$ and when $K_c = 4$, $M_c = 0.5$. As expected, the respective finite escape times are $t_{esc} = 1/\gamma(1) = \ln(2) \approx 0.6931$ and $t_{esc} = 2/\gamma(4) \approx 0.4463$. Note that these escape times are in fact about twice that of the respective cases in Example 4.1.1. Also, from (4.1.7) it follows when $K_c = M_c = 1$ that

$$f(x_0) = \frac{\exp(x_0)}{2 - \exp(x_0)}.$$

The sequence (e, x_0^n), $n \geq 0$ corresponds to OEIS sequence A000629 as shown in Table 4. $\square$

Example 4.1.5. Suppose $X = \{x_0, x_1\}$ and consider the case where e satisfies $e = c \circ e$ with $c = \sum_{n\geq 0} x_1^n$. Following the steps in the proof of Theorem 4.1.4 with $r_0 = 0$, the exponential generating function of e is found to satisfy

$$f'(x_0) = f^2(x_0), \quad f(0) = 1.$$

Solving this equation directly yields

$$f(x_0) = \frac{1}{1 - x_0}.$$

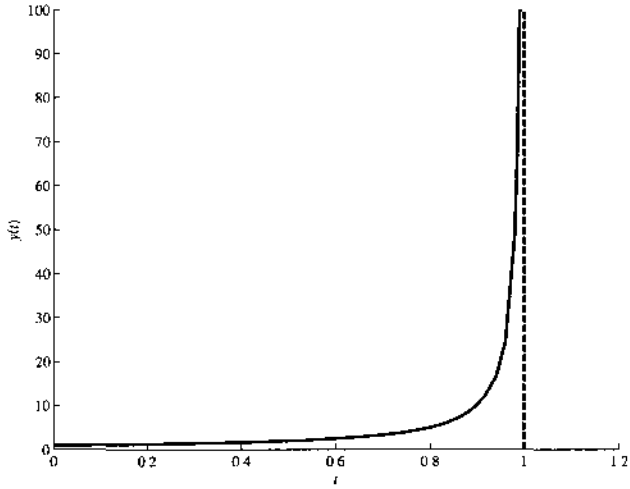

Fig. 10: Output of the self-excited loop in Example 4.1.5

The singularity at $x_0' = 1$ implies that $M_e = 1 < 1/\ln(2) \approx 1.4427$. Thus, the radius of convergence is 1. The coefficients of e correspond to $n!$. The output of the self-excited unity feedback system is described by the solution of

$$\begin{aligned} \dot{z} &= z^2, \quad z(0) = 1 \\ y &= z. \end{aligned}$$

A MATLAB generated solution of this system is shown in Figure 10. It has the finite escape time $t_{esc} = 1/M_e = 1 > \ln(2) \approx 0.6931$. □

Example 4.1.6. Consider the feedback connection involving the globally convergent series $c = x_1$ and $d = \sum_{k\geq 0} x_1^k$ as discussed in [18]. $F_{c@d}$ has the state space realization

$$\begin{aligned} \dot{z}_1 &= z_1 z_2, \quad z_1(0) = 1 \\ \dot{z}_2 &= z_1 + u, \quad z_2(0) = 0 \\ y &= z_2. \end{aligned}$$

Setting $u = 0$, the natural response y satisfies $\ddot{y} - \dot{y}y = 0$, $y(0) = 0$, $\dot{y}(0) = 1$, which has the solution

$$\begin{aligned} y(t) &= \sqrt{2}\tan\left(\frac{t}{\sqrt{2}}\right) \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}2^k(2^{2k-1})\frac{\mathcal{B}_{2k}}{k}\frac{t^{2k-1}}{(2k-1)!} \\ &= t+\frac{t^3}{3!}+4\,\frac{t^5}{5!}+34\,\frac{t^7}{7!}+496\,\frac{t^9}{9!}+\cdots \end{aligned}$$

for $0 \le t < \pi/\sqrt{2} = t_{esc}$, where $\mathcal{B}_k$ denotes the k-th *Bernoulli number*. Observe that $c \circ d = x_0 x_1^*$, and thus, $M_{cod} = 1$. In which case, $t_{esc} \approx 2.2214 > \ln(2)/M_{cod} \approx 0.6931$ as expected by Theorem 4.1.3. The existence of $t_{esc} < \infty$ implies that $c@d$ is not globally convergent. Therefore, this example illustrates the fact that the global convergence is in general not preserved under feedback. □

4.2 THE UNITY FEEDBACK CASE

4.2.1 Local Convergence

Now the convergence analysis proceeds to the unity feedback case, where a nonzero input can be applied to the closed-loop system. The following theorem, which describes the radius of convergence of the unity feedback connection with a locally convergent component system, is the main result of this section.

Theorem 4.2.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$. If $e \in \mathbb{R}^m\langle\langle X\rangle\rangle$ satisfies $e = c\tilde{\circ}e$ then*

$$|(e,\eta)| \le K_e(\alpha(K_c)M_c)^{|\eta|}|\eta|!, \quad \eta \in X^*,$$

for some $K_e > 0$, where

$$\alpha(K_c) = \frac{1}{1 - mK_c\ln(1 + 1/mK_c)}.$$

Furthermore, no geometric growth constant smaller than $\alpha(K_c)M_c$ can satisfy the inequality above, and thus the radius of convergence is

$$\frac{1}{(1+m)\alpha(K_c)M_c}.$$

The following lemmas are needed for the proof.

Lemma 4.2.1. *Let $X = \{x_0, x_1, \ldots, x_m\}$. The Fliess operator $F_e : u \to y$ having the state space representation*

$$\begin{aligned} \dot{z} &= \lambda\left(z^2 + mz^3 + z^2 \sum_{i=1}^{m} u_i\right), \quad z(0) = z_0, \\ y &= z, \end{aligned}$$

where $\lambda, z_0 \in \mathbb{R}^+$, has a generating series $e \in \mathbb{R}\langle\langle X \rangle\rangle$ whose coefficients satisfy the inequality

$$0 < (e, \eta) \leq \left(e, x_0^{|\eta|}\right), \quad \eta \in X^*.$$

Proof: First observe that for the vector fields $g_0(z) = \lambda(z^2 + mz^3)$ and $g_i(z) = \lambda z^2$, $i = 1, 2, \ldots, m$, the Lie derivatives of $h(z) = z$ consist of products of polynomials with non-negative coefficients. Therefore, using (2.1.3),

$$0 < (e, \eta) = L_{g_\eta} h(z_0), \quad \eta \in X^*.$$

For any $k > 0$, let $\eta_k = x_0^{n_0} x_{i_1} x_0^{n_1} \cdots x_{i_k} x_0^{n_k}$, where $1 \leq i_j \leq m$. Then the Lie derivative corresponding to the word $\eta_k x_0^{n_{k+1}+1}$ is

$$\begin{aligned} L_{g_{\eta_k x_0^{n_{k+1}+1}}} h &= L_{g_{x_0^{n_{k+1}+1}}} L_{g_{\eta_k}} h \\ &= L_{g_{x_0^{n_{k+1}}}} \left[\lambda(z^2 + mz^3) \frac{d}{dz} L_{g_{\eta_k}} h\right] \\ &= L_{g_{x_0^{n_{k+1}}}} \left[\lambda z^2 \frac{d}{dz} L_{g_{\eta_k}} h\right] + L_{g_{x_0^{n_{k+1}}}} \left[\lambda m z^3 \frac{d}{dz} L_{g_{\eta_k}} h\right] \\ &= L_{g_{x_0^{n_{k+1}}}} L_{g_{\eta_k x_{i_k}}} h + L_{g_{x_0^{n_{k+1}}}} \left[\lambda m z^3 \frac{d}{dz} L_{g_{\eta_k}} h\right] \\ &= L_{g_{\eta_{k+1}}} h + L_{g_{x_0^{n_{k+1}}}} \left[\lambda m z^3 \frac{d}{dz} L_{g_{\eta_k}} h\right]. \end{aligned}$$

When evaluated at $z(0) = z_0$,

$$L_{g_{\eta_k x_0^{n_{k+1}+1}}} h(z_0) = L_{g_{\eta_{k+1}}} h(z_0) + L_{g_{x_0^{n_{k+1}}}} \left[\lambda m z^3 \frac{d}{dz} L_{g_{\eta_k}} h(z_0)\right].$$

Clearly, the second term on the right-hand side above also consists of products of polynomials with non-negative coefficients. Thus, it is strictly positive, and therefore,

$$L_{g_{\eta_{k+1}}} h(z_0) < L_{g_{\eta_k x_0^{n_{k+1}+1}}} h(z_0), \quad k > 0. \tag{4.2.1}$$

This inequality is used to complete the proof of the lemma. Specifically, it will be shown by induction on k that

$$L_{g_{\eta_k}} h(z_0) \leq L_{g_{x_0^{|\eta_k|}}} h(z_0), \quad k \geq 0.$$

The claim is trivially true when $k = 0$. Now, assume it is true up to some fixed $k \geq 0$. Then using (4.2.1), it follows that

$$\begin{aligned} L_{g_{\eta_{k+1}}} h(z_0) &\leq L_{g_{\eta_k x_0^{n_{k+1}+1}}} h(z_0) \\ &= L_{g_{\eta_{k-1} x_{i_k} x_0^{n_k} x_0^{n_{k+1}+1}}} h(z_0) \\ &\leq L_{g_{x_0^{|\xi_k|}}} h(z_0) \\ &= L_{g_{x_0^{|\eta_{k+1}|}}} h(z_0), \end{aligned}$$

where $\xi_k := \eta_{k-1} x_{i_k} x_0^{n_k} x_0^{n_{k+1}+1}$. Therefore, the claim is verified for all $k \geq 0$, and the lemma is proved. ■

Lemma 4.2.2. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c, d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ such that $|c| \leq d$. Then for any fixed $\xi \in X^*$ it follows that $|\xi \tilde{\circ} c| \leq \xi \tilde{\circ} d$.*

Proof: Let $\xi_0 = x_0^{n_0}$ and $\xi_k = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} \cdots x_{i_1} x_0^{n_0}$ for $k > 0$, where $1 \leq i_j \leq m$. The proof is by induction on k. For $k = 0$ the claim is trivial since

$$\xi_0 \tilde{\circ} c = x_0^{n_0} \tilde{\circ} c = x_0^{n_0} = x_0^{n_0} \tilde{\circ} d = \xi_0 \tilde{\circ} d.$$

Assume now that $|(\xi_k \tilde{\circ} c, \eta)| \leq (\xi_k \tilde{\circ} d, \eta)$ up to some fixed $k \geq 0$, and observe

$$\xi_{k+1} \tilde{\circ} c = x_0^{n_{k+1}} x_{i_{k+1}} (\xi_k \tilde{\circ} c) + x_0^{n_{k+1}+1} \left(c_{i_{k+1}} ⧢ (\xi_k \tilde{\circ} c) \right).$$

Therefore,

$$\begin{aligned} (\xi_{k+1} \tilde{\circ} c, \eta) &= \left(x_0^{n_{k+1}} x_{i_{k+1}} (\xi_k \tilde{\circ} c), \eta \right) + \left(c_{i_{k+1}} ⧢ (\xi_k \tilde{\circ} c), x_0^{-(n_{k+1}+1)}(\eta) \right) \\ &= \left(\xi_k \tilde{\circ} c, x_{i_{k+1}}^{-1} x_0^{-(n_{k+1})}(\eta) \right) + \sum_{i=0}^{n} \sum_{\substack{\alpha \in X^i \\ \beta \in X^{n-i}}} (c_{i_{k+1}}, \alpha)(\xi_k \tilde{\circ} c, \beta) \\ &\quad \left(\alpha ⧢ \beta, x_0^{-(n_{k+1}+1)}(\eta) \right). \end{aligned}$$

In which case,

$$\begin{aligned}
&|(\xi_{k+1}\tilde{\circ}c, \eta)| \\
&\leq \left|\left(\xi_k\tilde{\circ}c, x_{i_{k+1}}^{-1}x_0^{-(n_{k+1})}(\eta)\right)\right| + \sum_{i=0}^{n} \sum_{\substack{\alpha\in X^i \\ \beta\in X^{n-i}}} \left|(c_{i_{k+1}}, \alpha)\right| |(\xi_k\tilde{\circ}c, \beta)| \cdot \\
&\qquad \left(\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}+1)}(\eta)\right) \\
&\leq \left(\xi_k\tilde{\circ}d, x_{i_{k+1}}^{-1}x_0^{-(n_{k+1})}(\eta)\right) + \sum_{i=0}^{n} \sum_{\substack{\alpha\in X^i \\ \beta\in X^{n-i}}} (d_{i_{k+1}}, \alpha)(\xi_k\tilde{\circ}d, \beta)\left(\alpha \sqcup\!\sqcup \beta, x_0^{-(n_{k+1}-1)}(\eta)\right) \\
&= (\xi_{k+1}\tilde{\circ}d, \eta),
\end{aligned}$$

where $n := |x_0^{-(n_{k+1}+1)}(\eta)| \geq 0$. Thus, the inequality holds for all $k \geq 0$, and the lemma is proved. ∎

Lemma 4.2.3. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $c, \bar{c} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ have growth constants $K_c, M_c > 0$, and specifically each component of $\bar{c}$ is $K_c M_c^{|\eta|}|\eta|!$, $\eta \in X^*$. If $e, \bar{e} \in \mathbb{R}^m[[X_0]]$ satisfy, respectively, $e = c\tilde{\circ}e$ and $\bar{e} = \bar{c}\tilde{\circ}\bar{e}$ then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.*

Proof: Since the mapping $d \mapsto c\tilde{\circ}d$ is a contraction, it follows that if $e_i(k) := (c^{\tilde{\circ}k}\tilde{\circ}0)_i$, $k \geq 1$ then $e_i = \lim_{k\to\infty} e_i(k)$. Likewise, one can define a sequence $\bar{e}_i(k)$ using $\bar{c}$. It will first be shown by induction that $|e_i(k)| \leq \bar{e}_i(k)$, $k \geq 1$. Observe that $e_i(1) = \sum_{\eta\in X^*}(c_i, \eta)\eta$ and $\bar{e}_i(1) = \sum_{\eta\in X^*} K_c M_c^{|\eta|}|\eta|!\,\eta$. Therefore, $|e_i(1)| \leq \bar{e}_i(1)$. Now assume the claim holds up to some fixed $k \geq 1$. Then, using Lemma 3.3.1, for any $\xi \in X^*$

$$\begin{aligned}
|(e_i(k+1), \xi)| &= |((c\tilde{\circ}e(k))_i, \xi)| = \left|\sum_{\eta\in X^*}(c_i, \eta)\,(\eta\tilde{\circ}e(k), \xi)\right| \\
&\leq \sum_{\eta\in X^*} |(c_i, \eta)|\; |(\eta\tilde{\circ}e(k), \xi)| \\
&\leq \sum_{\eta\in X^*} K_c M_c^{|\eta|}\,|\eta|!\,(\eta\tilde{\circ}\bar{e}(k), \xi) \\
&= (\bar{e}_i(k+1), \xi).
\end{aligned}$$

Thus,

$$|e_i(k)| \leq \bar{e}_i(k),\; k \geq 1,$$

and the initial claim is established. Next, by a property of the limit supremum,

$$\limsup_{k\to\infty} |(e_i(k),\xi)| \leq \limsup_{k\to\infty} (\bar{e}_i(k),\xi).$$

Since each sequence converges, it follows that $|e_i| \leq \bar{e}_i$. ■

Finally, the following distributive property concerning the modified composition product will be important in the work that follows. It is the counterpart of the distributive property for the (regular) composition product [19].

Lemma 4.2.4. *Let $X = \{x_0, x_1, \ldots, x_m\}$. The modified composition product is distributive to the left over the shuffle product, that is,*

$$(c \sqcup\!\sqcup d)\tilde{\circ}e = (c\tilde{\circ}e) \sqcup\!\sqcup (d\tilde{\circ}e), \quad c, d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle.$$

Proof: Since the shuffle product is defined componentwise, and the modified composition product is linear in its left argument, it is sufficient to assume $m = 1$ and show that

$$(\eta \sqcup\!\sqcup \xi)\tilde{\circ}e = (\eta\tilde{\circ}e) \sqcup\!\sqcup (\xi\tilde{\circ}e), \quad \eta, \xi \in X^*.$$

Let $k = |\eta| + |\xi|$. The claim is trivially true when at least one of the words is empty. Thus, the identity is true for $k = 0$ and $k = 1$. Assume it is true up to some fixed $k \geq 0$. Let $\eta = x_i\eta'$ and $\xi = x_j\xi'$ such that $k + 1 = |\eta| + |\xi|$. First consider the case when $i, j \neq 0$. Then

$$\begin{aligned}
(\eta \sqcup\!\sqcup \xi)\tilde{\circ}e &= [x_i(\eta' \sqcup\!\sqcup \xi) + x_j(\eta \sqcup\!\sqcup \xi')]\tilde{\circ}e \\
&= [x_i(\eta' \sqcup\!\sqcup \xi)]\tilde{\circ}e + [x_j(\eta \sqcup\!\sqcup \xi')]\tilde{\circ}e \\
&= x_i[(\eta' \sqcup\!\sqcup \xi)\tilde{\circ}e] + x_0(e_i \sqcup\!\sqcup [(\eta' \sqcup\!\sqcup \xi)\tilde{\circ}e]) + x_j[(\eta \sqcup\!\sqcup \xi')\tilde{\circ}e] + \\
&\qquad x_0(e_j \sqcup\!\sqcup [(\eta \sqcup\!\sqcup \xi')\tilde{\circ}e]) \\
&= x_i[(\eta'\tilde{\circ}e) \sqcup\!\sqcup (\xi\tilde{\circ}e)] + x_0[e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e) \sqcup\!\sqcup (\xi\tilde{\circ}e)] + \\
&\qquad x_j[(\eta\tilde{\circ}e) \sqcup\!\sqcup (\xi'\tilde{\circ}e)] + x_0[e_j \sqcup\!\sqcup (\eta\tilde{\circ}e) \sqcup\!\sqcup (\xi'\tilde{\circ}e)] \\
&= x_i[(\eta'\tilde{\circ}e) \sqcup\!\sqcup [x_j(\xi'\tilde{\circ}e) + x_0(e_j \sqcup\!\sqcup (\xi'\tilde{\circ}e))]] + \\
&\qquad x_0[e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e) \sqcup\!\sqcup [x_j(\xi'\tilde{\circ}e) + x_0(e_j \sqcup\!\sqcup (\xi'\tilde{\circ}e))]] \\
&\qquad x_j[[x_i(\eta'\tilde{\circ}e) + x_0(e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e))] \sqcup\!\sqcup (\xi'\tilde{\circ}e)] + \\
&\qquad x_0[e_j \sqcup\!\sqcup [x_i(\eta'\tilde{\circ}e) + x_0(e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e))] \sqcup\!\sqcup (\xi'\tilde{\circ}e)] \\
&= x_i[(\eta'\tilde{\circ}e) \sqcup\!\sqcup x_j(\xi'\tilde{\circ}e)] + x_i[(\eta'\tilde{\circ}e) \sqcup\!\sqcup x_0(e_j \sqcup\!\sqcup (\xi'\tilde{\circ}e))] + \\
&\qquad x_0[e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e) \sqcup\!\sqcup x_j(\xi'\tilde{\circ}e)] +
\end{aligned}$$

$$
\begin{aligned}
& x_0[e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e) \sqcup\!\sqcup x_0(e_j \sqcup\!\sqcup (\xi'\tilde{\circ}e))] + x_j[x_i(\eta'\tilde{\circ}e) \sqcup\!\sqcup (\xi'\tilde{\circ}e)] + \\
& x_j[x_0(e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e)) \sqcup\!\sqcup (\xi'\tilde{\circ}e)] + x_0[e_j \sqcup\!\sqcup x_i(\eta'\tilde{\circ}e) \sqcup\!\sqcup (\xi'\tilde{\circ}e)] + \\
& x_0[e_j \sqcup\!\sqcup x_0(e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e)) \sqcup\!\sqcup (\xi'\tilde{\circ}e)] \\
= \; & [x_i(\eta'\tilde{\circ}e)] \sqcup\!\sqcup [x_j(\xi'\tilde{\circ}e)] + [x_i(\eta'\tilde{\circ}e)] \sqcup\!\sqcup [x_0(e_j \sqcup\!\sqcup (\xi'\tilde{\circ}e))] + \\
& [x_0(e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e))] \sqcup\!\sqcup [x_j(\xi'\tilde{\circ}e)] + \\
& [x_0(e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e))] \sqcup\!\sqcup [x_0(e_j \sqcup\!\sqcup (\xi'\tilde{\circ}e))] \\
= \; & [x_i(\eta'\tilde{\circ}e) + x_0(e_i \sqcup\!\sqcup (\eta'\tilde{\circ}e))] \sqcup\!\sqcup [x_j(\xi'\tilde{\circ}e) + x_0(e_j \sqcup\!\sqcup (\xi'\tilde{\circ}e))] \\
= \; & (\eta\tilde{\circ}e) \sqcup\!\sqcup (\xi\tilde{\circ}e).
\end{aligned}
$$

Thus, the identity holds for all $\eta, \xi \in X^*$. The cases when $i \neq 0, j = 0$ and $i = j = 0$, can be proved in a similar manner using the identity $(x_0\eta)\tilde{\circ}e = x_0(\eta\tilde{\circ}e)$. Therefore, the lemma is proved. ∎

***Proof of Theorem* 4.2.1:**
Assume $\bar{e}$ is the solution of $\bar{e} = \bar{c}\tilde{\circ}\bar{e}$. Since all the components of $\bar{c}$ are identical, the focus will be on $\bar{e}_1$. Observe

$$
\begin{aligned}
\bar{e}_1 = (\bar{c}\tilde{\circ}\bar{e})_1 &= \sum_{k=0}^{\infty} K_c M_c^k \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} k!\, \frac{x_0^{\sqcup\!\sqcup r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{(x_m\tilde{\circ}\bar{e})^{\sqcup\!\sqcup r_m}}{r_m!} \\
&= \sum_{k=0}^{\infty} K_c \left[M_c \left(x_0 + \sum_{i=1}^{m} x_0\bar{e}_i + \sum_{i=1}^{m} x_i \right) \right]^{\sqcup\!\sqcup k} \\
&= \sum_{k=0}^{\infty} K_c \left[M_c \left(x_0 + m x_0\bar{e}_1 + \sum_{i=1}^{m} x_i \right) \right]^{\sqcup\!\sqcup k}.
\end{aligned}
$$

Shuffling both sides of this equation by $M_c\left(x_0 + m x_0\bar{e}_1 + \sum_{i=1}^{m} x_i\right)$ yields

$$
\bar{e}_1 \sqcup\!\sqcup M_c \left(x_0 + m x_0\bar{e}_1 + \sum_{i=1}^{m} x_i \right) = \sum_{k=0}^{\infty} K_c \left[M_c \left(x_0 + m x_0\bar{e}_1 + \sum_{i=1}^{m} x_i \right) \right]^{\sqcup\!\sqcup k-1}.
$$

Adding K_c to both sides gives

$$
\bar{e}_1 = K_c + \bar{e}_1 \sqcup\!\sqcup M_c \left(x_0 + m x_0\bar{e}_1 + \sum_{i=1}^{m} x_i \right).
$$

Therefore,

$$
F_{\bar{e}_1}[u] = K_c + M_c F_{\bar{e}_1}[u] \left(E_{x_0}[u] + m F_{x_0\bar{e}_1}[u] + \sum_{i=1}^{m} E_{x_i}[u] \right).
$$

Set $y_1 = F_{\bar{e}_1}[u]$ and note that $F_{\bar{e}_1}[u] \neq 0$ since $y_1(0) = K_c \neq 0$. Then it follows that

$$y_1(t) = K_c + M_c y_1(t) \left(t + m \int_0^t y_1(\tau)\, d\tau + \sum_{i=1}^{m} \int_0^t u_i(\tau)\, d\tau \right),$$

or equivalently,

$$\dot{z} = \frac{M_c}{K_c} \left(z^2 + mz^3 + z^2 \sum_{i=1}^{m} u_i \right), \quad z(0) = K_c \tag{4.2.2}$$

$$y_1 = z. \tag{4.2.3}$$

Therefore, by Lemma 4.2.1, $(\bar{e}_1, \eta) \leq (\bar{e}_1, x_0^{|\eta|})$, $\eta \in X^*$. But $(\bar{e}_1, x_0^n) \leq K_e(\alpha(K_c)M_c)^n n!$ by Theorem 4.1.1. Using Lemma 4.2.3, $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$. Hence, $|(e_i, \eta)| \leq K_e(\alpha(K_c)M_c)^{|\eta|}|\eta|!$, $\eta \in X^*$. From Theorem 4.1.2 and Example 4.1.1, $\bar{c}$ is the series for which each component of the corresponding feedback generating series $\bar{e}$ achieves exactly the growth rate $K_e(\alpha(K_c)M_c)^{|\eta|}|\eta|!$. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ∎

The following corollary addresses a question that was left unresolved in [19].

Corollary 4.2.1. *Let $c \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$. Then the generating series for the unity feedback connection, namely the series e satisfying $e = c\tilde{\circ}e$, is locally convergent.*

The final theorem in this section is useful for convergence analysis of feedback systems having analytic inputs.

Theorem 4.2.2. *Let $c \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$, and assume e satisfies $e = c\tilde{\circ}e$. If $c_u \in \mathbb{R}^m_{LC}[[X_0]]$ with growth constants $K_{c_u}, M_{c_u} > 0$ then $c_y = e \circ c_u$ satisfies*

$$|(c_y, x_0^k)| \leq K_{c_y} M_{c_y}^k k!, \ k \geq 0$$

for some $K_{c_y} > 0$ and

$$M_{c_y} = \frac{M_{c_u}}{\left[1 - mK_{c_u} W\left(\frac{1}{mK_{c_u}} \exp\left(\frac{\alpha(K_c)M_c - M_{c_u}}{m\alpha(K_c)M_c K_{c_u}}\right)\right)\right]}.$$

Thus, the interval of convergence for the output $y = F_{c_y}[u]$ is at least as large as $T = 1/M_{c_y}$.

Proof: The theorem is an immediate consequence of Theorem 4.2.1 and Corollary 3.3.1. ∎

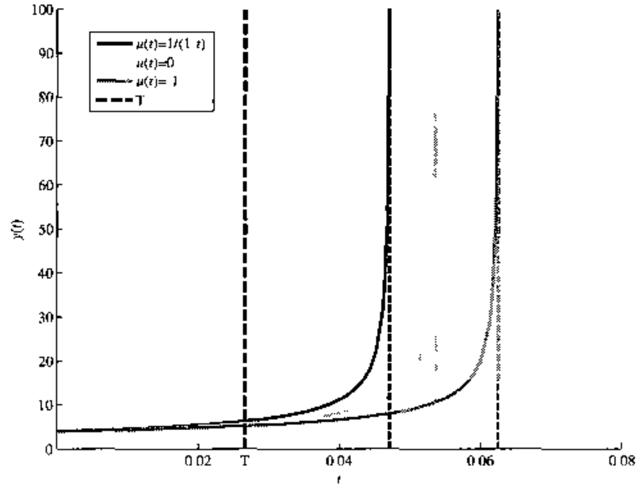

Fig. 11: Output responses of the unity feedback system to analytic inputs in Example 4.2.1

Example 4.2.1. Let $\bar{c} = \sum_{\eta \in X^*} K_c M_c^{|\eta|} |\eta|!\, \eta$ and $\bar{e} = \bar{c} \tilde{\circ} \bar{e}$. The corresponding feedback system has the state space realization (4.2.2)-(4.2.3). By Theorem 4.2.1, the finite escape time of the zero-input response is $t_{esc} = \frac{1}{\alpha(K_c) M_c}$. By Theorem 4.2.2, any finite escape time for an output corresponding to an analytic input with growth constants K_{c_u}, M_{c_u} must be at least as large as $T = 1/M_{c_y}$. A MATLAB generated solution of this system is shown in Figure 11 when $K_c = 4$ and $M_c = 2$. As predicted, $t_{esc} = \frac{1}{\alpha(K_c) M_c} = 0.0537$ when $u = 0$. When $K_{c_u} = M_{c_u} = 1$ it follows that $T = 0.0267$ as also shown in the figure. The output corresponding to the input $u = 1/(1-t)$ has $t_{esc} = 0.0472 > T$ as expected. For comparison, the $u = -1$ response is also shown.

$\square$

4.2.2 Global Convergence

A parallel analysis is done next for the unity feedback case, where the component system has a globally convergent generating series. The main theorem below describes the radius of convergence.

Theorem 4.2.3. *Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{GC}\langle\langle X \rangle\rangle$ with growth constants*

$K_c, M_c > 0$. *If* $e \in \mathbb{R}^m\langle\langle X\rangle\rangle$ *satisfies* $e = c\tilde{\circ}c$ *then*

$$|(e,\eta)| \le K_e(\gamma(K_c)M_c)^{|\eta|}|\eta|!, \quad \eta \in X^*,$$

for some $K_e > 0$, *where*

$$\gamma(K_c) = \frac{1}{\ln(1 + 1/mK_c)}.$$

Furthermore, no geometric growth constant smaller than $\gamma(K_c)M_c$ *can satisfy the inequality above, and thus the radius of convergence is*

$$\frac{1}{(1+m)\gamma(K_c)M_c}.$$

The following lemmas are essential in proving the main theorem above.

Lemma 4.2.5. *Let* $X = \{x_0, x_1, \ldots, x_m\}$. *The Fliess operator* $F_e : u \mapsto y$ *having the state space representation*

$$\begin{aligned} \dot{z} &= \lambda\left(z + mz^2 + z\sum_{i=1}^m u_i\right), \quad z(0) = z_0, \\ y &= z, \end{aligned}$$

where $\lambda, z_0 \in \mathbb{R}^+$, *has a generating series* $e \in \mathbb{R}\langle\langle X\rangle\rangle$ *whose coefficients satisfy the inequality*

$$0 < (e,\eta) \le \left(e, x_0^{|\eta|}\right), \quad \eta \in X^*.$$

Proof: First observe that for the vector fields $g_0(z) = \lambda(z + mz^2)$ and $g_i(z) = \lambda z$, the Lie derivatives of $h(z) = z$ consist of products of polynomials with non-negative coefficients. Therefore,

$$0 < (e,\eta) = L_{g_\eta}h(z_0), \ \eta \in X^*.$$

For any $k > 0$, let $\eta_k = x_0^{n_0}x_{i_1}x_0^{n_1}\cdots x_{i_k}x_0^{n_k}$. Then the Lie derivative corresponding to the word $\eta_k x_0^{n_{k+1}+1}$ is

$$\begin{aligned} L_{g_{\eta_k x_0^{n_{k+1}+1}}}h &= L_{g_{x_0^{n_{k+1}+1}}}L_{g_{\eta_k}}h \\ &= L_{g_{x_0^{n_{k+1}}}}\left[\lambda(z+mz^2)\frac{d}{dz}L_{g_{\eta_k}}h\right] \\ &= L_{g_{x_0^{n_{k+1}}}}\left[\lambda z\frac{d}{dz}L_{g_{\eta_k}}h\right] + L_{g_{x_0^{n_{k+1}}}}\left[\lambda mz^2\frac{d}{dz}L_{g_{\eta_k}}h\right] \\ &= L_{g_{x_0^{n_{k+1}}}}L_{g_{\eta_k x_{i_k}}}h + L_{g_{x_0^{n_{k+1}}}}\left[\lambda mz^2\frac{d}{dz}L_{g_{\eta_k}}h\right] \\ &= L_{g_{\eta_{k-1}}}h + L_{g_{x_0^{n_{k+1}}}}\left[\lambda mz^2\frac{d}{dz}L_{g_{\eta_k}}h\right]. \end{aligned}$$

When evaluated at $z(0) = z_0$,

$$L_{g_{\eta_k x_0^{n_{k+1}+1}}} h(z_0) \;=\; L_{g_{\eta_{k+1}}} h(z_0) + L_{g_{x_0^{n_{k+1}}}} \left[\lambda m z^2 \frac{d}{dz} L_{g_{\eta_k}} h(z_0)\right].$$

Clearly, the second term on the right-hand side above also consists of the products of polynomials with non-negative coefficients. Thus, it is strictly positive, and therefore,

$$L_{g_{\eta_{k+1}}} h(z_0) < L_{g_{\eta_k x_0^{n_{k+1}+1}}} h(z_0), \quad k > 0. \tag{4.2.4}$$

This inequality is used to complete the proof of the lemma. Specifically, it will be shown by induction on k that

$$L_{g_{\eta_k}} h(z_0) \leq L_{g_{x_0^{|\eta_k|}}} h(z_0), \quad k \geq 0.$$

The claim is trivially true when $k = 0$. Now, assume it is true up to some fixed $k \geq 0$. Then using (4.2.4), it follows that

$$\begin{aligned} L_{g_{\eta_{k+1}}} h(z_0) &\leq L_{g_{\eta_k x_0^{n_{k+1}+1}}} h(z_0) \\ &= L_{g_{\eta_{k-1} x_{i_k} x_0^{n_k} x_0^{n_{k+1}+1}}} h(z_0) \\ &\leq L_{g_{x_0^{|\xi_k|}}} h(z_0) \\ &= L_{g_{x_0^{|\eta_{k+1}|}}} h(z_0), \end{aligned}$$

where $\xi_k := \eta_{k-1} x_{i_k} x_0^{n_k} x_0^{n_{k+1}+1}$. Therefore, the claim is verified for all $k \geq 0$, and the lemma is proved. ∎

Lemma 4.2.6. *Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $c, \bar{c} \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ have growth constants $K_c, M_c > 0$, and specifically each component of $\bar{c}$ is $K_c M_c^{|\eta|}$, $\eta \in X^*$. If $e, \bar{e} \in \mathbb{R}^m[[X_0]]$ satisfy, respectively, $e = c\tilde{\circ}e$ and $\bar{e} = \bar{c}\tilde{\circ}\bar{e}$ then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.*

Proof: The proof is perfectly analogous to its local counterpart, Lemma 4.2.3. ∎

***Proof of Theorem* 4.2.3:**

Assume $\bar{e}$ is the solution of $\bar{e} = \bar{c}\tilde{\circ}\bar{e}$. As in the local case, there is no loss of generality in considering the single component e_1. Observe

$$\begin{aligned}
\bar{e}_1 &= \sum_{k=0}^{\infty} \frac{K_c M_c^k}{k!} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} k! \, \frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{(x_m \tilde{\circ} \bar{e})^{\sqcup\!\sqcup\, r_m}}{r_m!} \\
&= K_c \sum_{k=0}^{\infty} \frac{\left(M_c \left(x_0 + \sum_{i=1}^{m} x_0 \bar{e}_1 + \sum_{i=1}^{m} x_i\right)\right)^{\sqcup\!\sqcup\, k}}{k!} \\
&= K_c \sum_{k=0}^{\infty} \frac{\left(M_c \left(x_0 + m x_0 \bar{e}_1 + \sum_{i=1}^{m} x_i\right)\right)^{\sqcup\!\sqcup\, k}}{k!}.
\end{aligned}$$

Therefore, $(\bar{e}_1, \emptyset) = K_c$ and

$$\begin{aligned}
x_0^{-1}(\bar{e}_1) &= K_c \sum_{k=1}^{\infty} \frac{\left(M_c \left(x_0 + m x_0 \bar{e}_1 + \sum_{i=1}^{m} x_i\right)\right)^{\sqcup\!\sqcup\, k-1}}{(k-1)!} \sqcup\!\sqcup M_c(1 + m\bar{e}_1) \\
&= \bar{e}_1 \sqcup\!\sqcup M_c(1 + m\bar{e}_1).
\end{aligned}$$

In which case,

$$x_0^{-1}(\bar{e}_1) = M_c(1 + m\bar{e}_1) \sqcup\!\sqcup \bar{e}_1. \tag{4.2.5}$$

After applying the left-shift operation with respect to x_i on $\bar{e}_1$, where $i = 1, 2, \ldots, m$, it follows that

$$\begin{aligned}
x_i^{-1}(\bar{e}_1) &= \sum_{k=0}^{\infty} \frac{K_c k (M_c(x_0 + x_i \tilde{\circ} \bar{e}))^{\sqcup\!\sqcup\, k-1}}{k!} \sqcup\!\sqcup M_c x_i^{-1} \left(x_0 + m x_0 \bar{e}_1 + \sum_{i=1}^{m} x_i \right) \\
&= \sum_{k=0}^{\infty} \frac{K_c (M_c(x_0 + x_i \tilde{\circ} \bar{e}))^{\sqcup\!\sqcup\, k}}{k!} \sqcup\!\sqcup M_c \\
&= M_c \bar{e}_1. \qquad (4.2.6)
\end{aligned}$$

If $z = F_{\bar{e}_1}[u]$ then (4.2.5) and (4.2.6) yield

$$F_{x_0^{-1}(\bar{e}_1)}[u] = M_c z (1 + mz) \tag{4.2.7}$$

$$F_{x_i^{-1}(\bar{e}_1)}[u] = M_c z. \tag{4.2.8}$$

Therefore,

$$\frac{d}{dt} F_{\bar{e}_1}[u] = F_{x_0^{-1}(\bar{e}_1)}[u] + \sum_{i=1}^{m} u_i F_{x_i^{-1}(\bar{e}_1)}[u].$$

From (4.2.7) and (4.2.8), the following state space realization is obtained.

$$\begin{aligned}
\dot{z} &= M_c \left(z + m z^2 + z \sum_{i=1}^{m} u_i \right), \quad z(0) = K_c \\
y &= z.
\end{aligned}$$

Lemma 4.2.5 gives $(\bar{e}_1, \eta) \leq (\bar{e}_1, x_0^{|\eta|})$, $\eta \in X^*$. But $(\bar{e}_1, x_0^n) \leq K_e(\gamma(K_c)M_c)^n n!$ by Theorem 4.1.3. Using Lemma 4.2.6, one has $|e| \leq \bar{e}$. Hence, $|(e,\eta)| \leq K_e(\gamma(K_c)M_c)^{|\eta|}|\eta|!$, $\eta \in X^*$. From Theorem 4.1.4 and Example 4.1.4, $\bar{c}$ is the series for which the corresponding coefficients, $(\bar{e}, x_0^k)$, achieve exactly the growth rate $K_e(\gamma(K_c)M_c)^{|\eta|}|\eta|!$. Thus, no smaller geometric growth constant is possible, and the theorem is proved. ■

4.3 SUMMARY

A complete analysis of the radius of convergence of the unity feedback connection of an analytic nonlinear input-output system represented as a Fliess operator has been presented. First, the self-excited case was considered. If the component system is locally convergent, then the radius of convergence is finite and can be computed in terms of the Lambert W-function. Unlike the cascade connection, even if the component system is globally convergent, the radius of convergence of the overall feedback system is still finite. An explicit formula was derived for it. Surprisingly, the radius of convergence of the unity feedback systems with a non-zero input was found to be identical to that of the self-excited connection in both the local and global cases. In the process of computing the radii of convergence, it is shown definitively that local convergence is preserved under unity feedback.

CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH

This dissertation described the radius of convergence for the four fundamental interconnections of two convergent Fliess operators, specifically, the parallel, product, cascade and unity feedback connections. For either locally convergent or globally convergent subsystems, the radius of convergence for the composite system was computed explicitly. The results are summarized in Table 5. In the process, it was also shown that the unity feedback connection preserves local convergence, which was an open problem. A number of specific examples for which the radius of convergence is achieved were provided. It was found that the Lambert-W function plays a central role in computing the radii of convergence for the composition and feedback connections. This suggests a direct connection to the combinatorics of rooted nonplanar labeled trees [4, 12]. That aspect of the problem was not pursued in this dissertation. However, future research could focus on a more fundamental combinatoric interpretation of the composition and feedback products of formal power series. This may give deeper insight into the analysis presented here and perhaps simplify some of the arguments used. In addition, one could continue to investigate the radius of convergence for other types of system interconnections. For example, the non-unity feedback system and interconnections involving component systems which have a mixture of locally convergent and globally convergent generating series. Many of the basic methods presented in the dissertation should apply to such problems. Finally, there are many practical engineering applications to which the analysis used here will be helpful.

TABLE 5: Radii of convergence for the four elementary system connections

connection	$c, d \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$	$c, d \in \mathbb{R}^m_{GC}\langle\langle X \rangle\rangle$
parallel	$\frac{1}{\max\{M_c, M_d\}(1-m)}$	∞
product	$\frac{1}{\max\{M_c, M_d\}(1+m)}$	∞
cascade	$\frac{1}{M_d(1+m)}\left[1 - mK_d W\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)\right]$	∞
unity feedback	$\frac{1}{M_c(1+m)}\left[1 - mK_c \ln\left(1 + \frac{1}{mK_c}\right)\right]$	$\frac{1}{M_c(1-m)} \ln\left(1 + \frac{1}{mK_c}\right)$

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Recent Publications

1. M. Thitsa and W. S. Gray, 'On the radius of convergence of cascaded analytic nonlinear systems,' *Proc. 50th IEEE Conf. on Decision and Control and European Control Conference*, Orlando, Florida, 2011 (invited), under review.
2. W S. Gray and M. Thitsa, 'A unified approach to generating series for cascaded nonlinear input-output systems,' *International Journal of Mathematics and Mathematical Sciences*, under review.
3. M. Thitsa and W. S. Gray, 'On the radius of convergence of cascaded analytic nonlinear systems: the SISO case,' *Proc. 43rd IEEE Southeastern Symp. on System Theory*, Auburn, Alabama, 2011, pp. 32-38.
4. W. S. Gray and M. Thitsa, 'On the radius of convergence of self-excited feedback connected analytic nonlinear systems,' *Proc. 49th IEEE Conf. on Decision and Control*, Atlanta, Georgia, 2010, pp. 7092-7098.
5. W. S. Gray and M. Thitsa, 'On the radius of convergence of self-excited feedback connected analytic nonlinear systems: the local case,' *Proc. 42nd IEEE Southeastern Symp. on System Theory*, Tyler, Texas, 2010, pp. 116-121.