Spring 2014

Ray- and Wave-Theoretic Approach to Electromagnetic Scattering from Radially Inhomogeneous Spheres and Cylinders

Michael A. Pohrivchak
Old Dominion University

RAY- AND WAVE-THEORETIC APPROACH TO ELECTROMAGNETIC SCATTERING FROM RADIALLY INHOMOGENEOUS SPHERES AND CYLINDERS

by

Michael A. Pohrivchak
B.S. 2007, Virginia Polytechnic Institute and State University
M.S. 2011, Old Dominion University

A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

COMPUTATIONAL AND APPLIED MATHEMATICS

OLD DOMINION UNIVERSITY
May 2014

Approved by:

John A. Adam (Director)

Gordon Melrose (Member)

Daniel Gunlycke (Member)

Tony Slaba (Member)

Linda Vahala (Member)

ABSTRACT

RAY- AND WAVE-THEORETIC APPROACH TO ELECTROMAGNETIC SCATTERING FROM RADIALLY INHOMOGENEOUS SPHERES AND CYLINDERS

Michael A. Pohrivchak
Old Dominion University, 2014
Director: Dr. John A. Adam

With applications in the areas of chemistry, physics, microbiology, meteorology, radar, astronomy, and many other fields, electromagnetic scattering is an important area of research. Many everyday phenomena that we experience are a result of the scattering of electromagnetic and acoustic waves. In this dissertation, the scattering of plane electromagnetic waves from radially inhomogeneous spheres and cylinders using both ray- and wave-theoretic principles is considered. Chapters 2 and 3 examine the use of the ray approach. The deviation undergone by an incident ray from its original direction is related to the angle through which the radius vector turns from the point at which the ray enters the sphere to its point of exit. This angle can be expressed in terms of a complicated improper integral. The resulting deviation for several different refractive index profiles (some being singular) is examined to investigate properties of the refractive index profiles that allow for direct transmission bows to exist. In Chapter 4, the complementary approach of wave-theoretic analysis leads to the construction of exact electromagnetic solutions for the asymptotic backscattered field produced by an incident plane wave. This has direct relevance to radar applications in particular. The radial eigenfunctions can be evaluated exactly (and also asymptotically) for the transverse electric and transverse magnetic modes. This allows a determination of the high-frequency backscattered field by means of a modified Watson transformation.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my advisor Dr. John Adam for his abundant guidance, support, and humor throughout my graduate studies. Without his valuable guidance and persistent help, this dissertation would not have been possible. I would also like to extend my sincere appreciation to my committee members, Dr. Gordon Melrose, Dr. Daniel Gunlycke, Dr. Tony Slaba, and Dr. Linda Vahala, for their help and support. I would like to thank Dr. Hideaki Kaneko, Dr. Richard Noren, and Dr. Raymond Cheng who served as Graduate Program Director during my graduate studies. Without their knowledge of the program requirements, my graduation from Old Dominion University would be impossible. I would also like to thank the entire Department of Mathematics and Statistics of Old Dominion University for the great opportunity they have given me over the years. Without their funding and support, I would have been unable to attend graduate school. In addition, I would like to thank the Department of Modeling and Simulation of Old Dominion University for the scholarship they awarded me for two of the years of my graduate studies.

I would like to thank my family for their love, support, and encouragement. I would like to give very special thanks to my father, Ronald, my mother, Alfreda, and my brothers, Nikolai and Stephen, for always supporting me and helping me succeed. I would also like to thank my friends Vincent and Dustin for giving me reasons to take needed breaks from my research.

TABLE OF CONTENTS

Page

LIST OF FIGURES viii

Chapter

I. INTRODUCTION 1

II. RAY APPROACH - NON-SINGULAR PROFILES 4
- II.1 INTRODUCTION 4
- II.2 PROFILE 1 6
- II.3 PROFILE 2 13
- II.4 PROFILE 3 17
- II.5 PROFILE 4 23

III. RAY APPROACH - SINGULAR PROFILES 36
- III.1 INTRODUCTION 36
- III.2 PROFILE 1 37
- III.3 PROFILE 2 41
- III.4 PROFILE 3 47
- III.5 PROFILE 4 51
- III.6 PROFILE 5 54
- III.7 PROFILE 6 56

IV. WAVE APPROACH 62
- IV.1 INTRODUCTION 62
- IV.2 PROFILE 1 64
- IV.3 PROFILE 2 96
- IV.4 PROFILE 3 102

V. CONCLUSIONS 107

REFERENCES 109

APPENDICES

A. DERIVATION OF TWO INTEGRALS 112

B. SOLUTION OF A SYSTEM OF NON-LINEAR EQUATIONS 114

C. VERIFICATION OF SOLUTIONS FROM WESTCOTT (1968) 117
C.1 PROFILE BASED ON BESSEL'S EQUATION 117
C.2 PROFILES BASED ON WHITTAKER'S EQUATION 119
C.3 PROFILES BASED ON THE HYPERGEOMETRIC EQUATION 121

D. RELATING WESTCOTT (1968) TO USLENGHI AND WESTON (1970) 128

E. COUPLING TWO EQUATIONS FROM SAMADDAR (1970) 131

VITA 140

LIST OF FIGURES

Figure Page

1. The ray path for direct transmission through a radially inhomogeneous unit sphere for $n'(r) < 0$ (reproduced from [1] and [2]). 5

2. Plot of equation (5) with $n_1 = \frac{4}{3}$ and $c = \frac{1}{3}$. 7

3. Case 1: Plot of equation (21) with $n_0 = \frac{4}{3}$ and $n_1 = 1.01$. 18

4. Case 1: Plot of equation (37) with $n_0 = \frac{4}{3}$ and $n_1 = 1.01$. 18

5. Case 2: Plot of equation (21) with $n_0 = \frac{4}{3}$ and $n_1 = 2$. 19

6. Case 2: Plot of equation (37) with $n_0 = \frac{4}{3}$ and $n_1 = 2$. 19

7. Plot of equation (40) with $a = 2$ and $b = -5$. 24

8. Plot of equation (55) with $a = 2$ and $b = -5$. 24

9. Plot of equation (56) with $a = 2$ and $b = 7$. 35

10. Plot of equation (71) with $a = 2$ and $b = 7$. 35

11. Plot of equation (72) with $n_1 = \frac{4}{3}$ and $b = 4$. 42

12. Plot of equation (81) with $n_1 = \frac{4}{3}$ and $b = 4$. 42

13. Plot of equation (85) with $a = -205$ and $b = 204.5$. 48

14. Plot of equation (93) with $a = -205$ and $b = 204.5$. 48

15. Plot of equation (96) with $a = \frac{1}{10}$ and $b = \frac{3}{2}$. 49

16. Plot of equation (103) with $a = 5$ and $b = 11$. 51

17. Plot of equation (111) with $a = -\frac{1}{1000}$, $b = 2$, and $c = 5$. 57

18. Plot of equation (117) with $a = -\frac{1}{1000}$, $b = 2$, and $c = 5$. 57

19. Plot of equation (118) with $\alpha = \frac{5}{4}$ and $\eta = -\frac{1}{3}$. 61

20. Plot of equation (123) with $\alpha = \frac{5}{4}$ and $\eta = -\frac{1}{3}$. 61

21. Contours of integration in the complex ν plane, where a description of C and C_1 can be found on the previous page and a description of C_2 can be found on page 93. 66

CHAPTER I

INTRODUCTION

The following quote from a paper by Adam [1] explains in part why the ray- and wave-theoretic approach to electromagnetic scattering is such a fundamental topic:

> "Geometrical optics and wave (or physical) optics are two very different but complementary approaches to describing many optical phenomena such as the rainbow. However, there is a broad 'middle ground', the '*semiclassical*' régime. Thus there are essentially three domains within which scattering phenomena may be described: the scattering of waves by objects which in size are (i) small, (ii) comparable with, and (iii) large, compared to the wavelength of the incident (plane wave) radiation. There may be considerable overlap of region (ii) with the others, depending on the problem of interest, but basically, the wave-theoretic principles in region (i) tell us why the sky is blue (amongst many other things). At the other extreme, the 'classical' domain (iii) enables us in particular to be able to describe the basic features of the rainbow in terms of ray optics. The wave-particle duality so fundamental in quantum mechanics is relevant to region (ii) because the more subtle features exhibited by such phenomena involve both these aspects of description and explanation."

Solutions of spherical scattering problems have practical applications in chemistry, physics, microbiology, meteorology, radar, astronomy, and other fields [10]. Many phenomena that we experience every day are related to the scattering of plane electromagnetic waves. Sound and light waves are scattered around objects that enable us to hear the sound and be illuminated by the light. The scattering of plane electromagnetic waves provides an explanation of why the sky is blue and how a rainbow is formed. There are several other reasons why it is important to have a deep understanding of electromagnetic scattering by radially inhomogeneous media. Methods that are employed in this area can be very useful in exploring the combustion of liquid hydrocarbons, the injection of sprays in high pressure environments, as well as the spraying and drying techniques utilized in the food, agricultural, and pharmaceutical industries. Another example where the scattering of electromagnetic waves by a radially inhomogeneous sphere is used is in biological studies to detect blood and bacteria cells. Medical imaging uses the scattering of plane electromagnetic waves to identify and diagnose a range of health-related issues. Electromagnetic scattering is also utilized in geophysical exploration to identify a new deposit of a certain resource. Another example of the importance of electromagnetic scattering is in the area of nondestructive testing of artifacts without causing damage to the environment or other objects. The scattering of electromagnetic plane waves by a radially inhomogeneous sphere is a vast field with many practical and research applications.

This dissertation is organized as follows. In Chapter 2, the ray approach to electromagnetic scattering from a radially inhomogeneous sphere for non-singular refractive index profiles is discussed. The refractive index varies as a function of the distance from the center of the sphere, r, where the refractive index profile is defined for all values of r. This discussion leads to the definition of a ray path integral which is related to the deviation of a ray incident from infinity from its original direction upon the sphere at angle of incidence i. The level of difficulty in evaluating this integral increases greatly even for simple profiles such as the linear profile. Chapter 3 presents some singular refractive index profiles and evaluates the ray path integral for those profiles. The refractive index profiles in Chapter 2 and Chapter 3 were restricted to be within the range of 1 to 4. These bounds were chosen because values less than 1 and large positive values are physically impractical for optical wavelengths. The theoretical considerations in this work do not depend on the upper bound of 4 or the lower bound of 1 and the graphs of the refractive index profiles could be easily modified to accommodate larger and smaller values. It should be noted that for nanomaterials, microwave wavelengths, plasmas, and other materials, there is no limit to the value that the refractive index can take, as large positive values have been observed as well as values less than 1. For each profile in Chapter 2 and Chapter 3, certain conditions are discussed that allow for a zero-order (or direct transmission) bow to exist. At the time of this paper being published, there is no known theorem stating necessary and sufficient conditions on the refractive index profile for a zero-order (or higher order) bow to exist. It is hoped that the research found in Chapter 2 and Chapter 3 will serve as valuable background for such conditions to be obtained in due course. Some progress in this direction has been made by Adam [2]. In Chapter 4, a wave-theoretic approach is used to study electromagnetic scattering for a specific refractive index profile. On the basis of this analysis, several other important profiles can be investigated through (in principle) a judicious choice of variable transformations. At short wavelengths, the leading term of the backscattered field of a plane electromagnetic wave cannot be determined fully by simple geometrical optics considerations (illustrating the comment made about régime (ii) mentioned in the first paragraph). Rather, it is obtained by utilizing a modified Watson transformation of the exact solution. This transformation was developed for accelerating the convergence rate of infinite series. This technique previously has been utilized in the field of radar technology. The wave-theoretic analysis leads to the construction of exact electromagnetic solutions for the asymptotic backscattered field produced by an incident plane wave. The radial eigenfunctions can be evaluated exactly (and also asymptotically) for the transverse electric (TE) and transverse magnetic (TM) modes. Subsequently, a determination of the high-frequency backscattered field can be made. For two other profiles based on the hypergeometric equation, the corresponding radial eigenfunctions are derived exactly so that the methods employed in Chapter 4 may be applied to these profiles in future work.

CHAPTER II

RAY APPROACH - NON-SINGULAR PROFILES

II.1 INTRODUCTION

In the following work, i will refer to the angle of incidence for the incoming ray, r is the radial distance within a sphere of radius $\tilde{a}$, and *D(i)* is the total angle of deviation undergone by the ray from its original direction. The subscripts 0 and 1 will be used to distinguish between the deviations of the exiting ray for the direct transmission and the primary bow, respectively. The angle of incidence, i, is measured with respect to the surface normal, as shown in Figure 1, and is therefore restricted between 0 and $\frac{\pi}{2}$. However, it will be assumed that $i \in [0.005, \frac{\pi}{2}]$. This lower bound is placed on the angle of incidence in order to avoid any potential numerical singularities of the deviation angle that may arise at an angle of incidence of 0. Rescaling by the radius $\tilde{a}$, $r \in [0, 1]$. A well-known result is that the curvature of the ray path is towards regions of higher refractive index n which is a consequence of Snel's law of refraction generalized to continuously varying media. This tells us that within a sphere, if $\frac{dn(r)}{dr} \equiv n'(r) < 0$ an incoming ray bends towards the origin; if $n'(r) > 0$, it bends away from it. From Figure 1 it can be seen that in the case of $n'(r) < 0$, the relationship between the angle of incidence and $D_0(i)$ is

$$i + 2\Theta(i) + (i - |D_0(i)|) = \pi.$$

Thus it follows that

$$|D_0(i)| = 2i - \pi + 2\Theta(i). \tag{1}$$

In equation (1), $2\Theta(i)$ is the angle through which the radius vector turns from the point at which the ray enters the sphere to its point of exit. For one internal reflection, which corresponds to a primary bow, the ray undergoes an extra $2\Theta(i)$ deviation so that

$$|D_1(i)| = 2i - \pi + 4\Theta(i).$$

We will follow the common approach and drop the absolute value notation. The deviation formulae can be extended to higher order bows by adding an additional

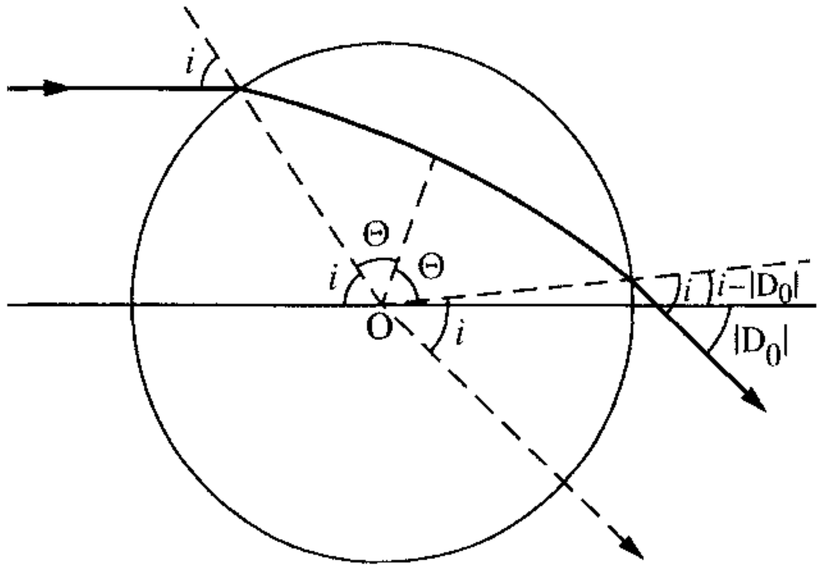

Fig. 1. The ray path for direct transmission through a radially inhomogeneous unit sphere for $n'(r) < 0$ (reproduced from [1] and [2]).

$2\Theta(i)$ for each subsequent bow. The quantity $\Theta(i)$ is an improper definite integral to be defined in this section. Except for a few specific $n(r)$ profiles, analytic expressions for $\Theta(i)$ are difficult to obtain. In this chapter, we evaluate $\Theta(i)$ for four profiles and will see how for even simple profiles (such as the linear profile) the evaluation of the integral becomes quite challenging.

In a spherically symmetric medium with refractive index $n(r)$ each ray path satisfies the following equation [4]

$$rn(r)\sin\hat{\phi} = \text{constant},$$

where $\hat{\phi}$ is the angle between the radius vector $\mathbf{r}$ and the tangent to the ray at that point, where we note that $r = |\mathbf{r}|$. The above expression may be thought of as the optical analogue of the conservation of angular momentum for a particle moving under the action of a central force. The result, known as *Bouguer's formula* (for Pierre Bouguer, 1698-1758), implies that all the ray paths $r(\theta)$ are curves lying in planes through the origin where θ is the polar angle. By using elementary differential geometry, we can establish that

$$\sin\hat{\phi} = \frac{r(\theta)}{\sqrt{r^2(\theta) + \left(\frac{dr}{d\theta}\right)^2}}.$$

From the equation for $\sin\hat{\phi}$, we can determine a formula for the angular deviation of a ray, $\Theta(i)$, within the sphere and, as a result, the total angle of deviation $D(i)$ through which an incoming ray at an angle of incidence i is rotated may be calculated. The formula for $\Theta(i)$ has been found to be [2], [17]

$$\Theta(i) = \sin i \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2 n^2(r) - \sin^2 i}}, \tag{2}$$

where the lower limit $r_c(i)$ is the point at which the integrand is singular and is therefore the solution of

$$r_c(i) n(r_c(i)) = \sin i. \tag{3}$$

The quantity $r_c(i)$ is the radial point of closest approach to the center of the sphere which is sometimes called the *turning point*. For a zero-order bow to exist for some critical angle of incidence $i_c \in \left[0.005, \frac{\pi}{2}\right]$, it is necessary and sufficient that

$$D_0'(i_c) = 0. \tag{4}$$

In the next several sections in this chapter, we will evaluate equation (2) for various profiles to determine equation (1). We will then employ that result in equation (4), and utilize the resulting equation to impose conditions on the refractive index that allow for a zero-order bow to exist. This technique will be applied only to those refractive index profiles whose derivative of the deviation angle is readily obtainable algebraically. In the plots of the refractive index profiles in this chapter, we indicate the portion of the profile that is naturally practical for optical wavelengths by a solid line and the naturally impractical values by a dashed line.

II.2 PROFILE 1

Consider (as previously discussed in [23])

$$n(r) = \frac{2n_1 r^{\frac{1}{c}-1}}{1 + r^{\frac{2}{c}}}, \qquad n_1 = n(1), \tag{5}$$

where c is a positive real constant. The refractive index profile in equation (5) is singular at $r = 0$ for $c > 1$. The refractive index profile is also singular at $r = 0$ for $c < 0$. In order to avoid this singularity, it will be assumed that $0 < c \leq 1$. We give the plot of equation (5) in Figure 2 with $n_1 = \frac{4}{3}$ and $c = \frac{1}{3}$.

When $c = 1$, the refractive index profile in equation (5) results in the well-known

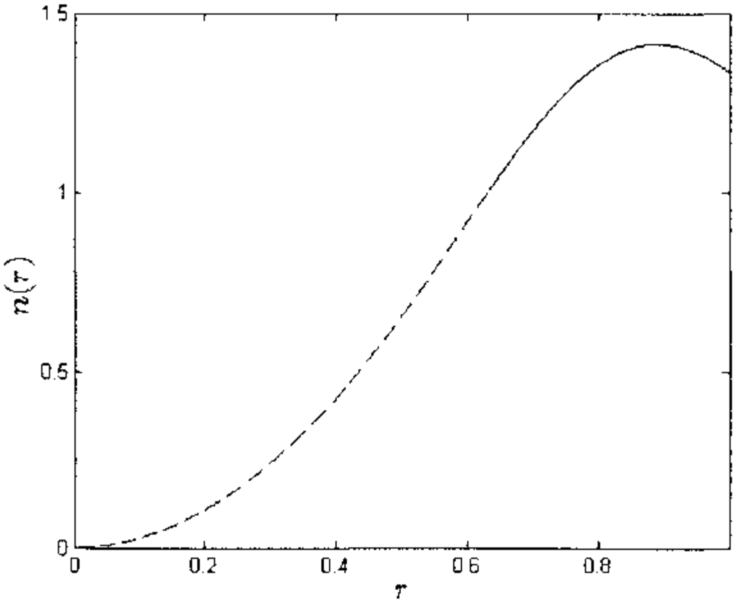

Fig. 2. Plot of equation (5) with $n_1 = \frac{4}{3}$ and $c = \frac{1}{3}$.

Maxwell's fish-eye profile. Maxwell's fish-eye profile was studied by James Clerk Maxwell in 1854, but without the spherical mirror. The reason why it is called the fish-eye profile is because Maxwell thought of this profile by considering the crystalline lens in fish. The fish-eye mirror makes a perfect lens, but it is a rather peculiar lens that contains both the object and the image inside the optical medium. The fish-eye mirror could transfer embedded images with details significantly smaller than the wavelength of light over distances much larger than the wavelength, a useful feature for nanolithography [3].

We note that

$$r^2 n^2(r) = \frac{4n_1^2 r^{\frac{2}{c}}}{(1 + r^{\frac{2}{c}})^2}.$$

Substituting this into equation (2) gives

$$\Theta(i) = \sin i \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{\frac{4n_1^2 r^{\frac{2}{c}}}{(1+r^{\frac{2}{c}})^2} - \sin^2 i}} = \frac{\sin i}{2n_1} \int_{r_c(i)}^{1} \frac{(1 + r^{\frac{2}{c}})}{r\sqrt{r^{\frac{2}{c}} - \frac{\sin^2 i}{4n_1^2}(1 + r^{\frac{2}{c}})^2}} dr.$$

Then, letting

$$\hat{a} \equiv \frac{\sin i}{2n_1},$$

the integral for $\Theta(i)$ becomes

$$\Theta(i) = \hat{a}\left\{\int\limits_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^{\frac{2}{c}} - \hat{a}^2(1+r^{\frac{2}{c}})^2}} + \int\limits_{r_c(i)}^{1} \frac{r^{\frac{2}{c}-1}}{\sqrt{r^{\frac{2}{c}} - \hat{a}^2(1+r^{\frac{2}{c}})^2}}dr\right\}$$

$$= \hat{a}\left\{\int\limits_{r_c(i)}^{1} \frac{dr}{r\sqrt{-\hat{a}^2(r^{\frac{2}{c}})^2 + (1-2\hat{a}^2)r^{\frac{2}{c}} - \hat{a}^2}} + \int\limits_{r_c(i)}^{1} \frac{r^{\frac{2}{c}-1}}{\sqrt{r^{\frac{2}{c}} - \hat{a}^2(1+r^{\frac{2}{c}})^2}}dr\right\}$$

$$\equiv \hat{a}[I_1(r) + I_2(r)]. \tag{6}$$

Next, we calculate $r_c(i)$, which is determined from equation (3). Then we will evaluate the integrals $I_1(r)$ and $I_2(r)$.

We have from equation (5) that

$$n(r_c(i)) = \frac{2n_1 r_c^{\frac{1}{c}-1}}{1+r_c^{\frac{2}{c}}}$$

and therefore by multiplying both sides of this equation by $r_c(i)$, we obtain that

$$r_c(i)n(r_c(i)) = \frac{2n_1 r_c^{\frac{1}{c}}}{1+r_c^{\frac{2}{c}}}.$$

In order to determine $r_c(i)$, we must solve the equation

$$\frac{2n_1 r_c^{\frac{1}{c}}}{1+r_c^{\frac{2}{c}}} = \sin i. \tag{7}$$

Accordingly, we find that

$$\frac{r_c^{\frac{1}{c}}}{1+r_c^{\frac{2}{c}}} = \frac{\sin i}{2n_1} = \hat{a}.$$

$$\hat{a}(r_c^{\frac{1}{c}})^2 - r_c^{\frac{1}{c}} + \hat{a} = 0.$$

This is simply a quadratic equation in terms of $r^{\frac{1}{c}}$ which has the solutions

$$r_c^{\frac{1}{c}} = \frac{1 \pm \sqrt{1-4\hat{a}^2}}{2\hat{a}}.$$

Since $r_c(i)$ is the radial point of closest approach to the center of the sphere, we take the smaller of the two quadratic solutions

$$r_c^{\frac{1}{c}} = \frac{1 - \sqrt{1-4\hat{a}^2}}{2\hat{a}}.$$

Hence it follows that

$$r_c(i) = \left[\frac{1 - \sqrt{1-4\hat{a}^2}}{2\hat{a}}\right]^c. \tag{8}$$

Now we will utilize equation (8) in the expressions for $I_1(r)$ and $I_2(r)$ and solve both integrals. First

$$I_1(r) = \int_{\left[\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}}\right]^c}^{1} \frac{dr}{r\sqrt{-\hat{a}^2(r^{\frac{2}{c}})^2 + (1-2\hat{a}^2)r^{\frac{2}{c}} - \hat{a}^2}}. \tag{9}$$

In equation (9), make the change of variables

$$v_1 = r^{\frac{2}{c}}.$$

Then we obtain

$$dv_1 = \frac{2}{c} r^{\frac{2}{c}-1} dr$$

and

$$\frac{dr}{r} = \frac{c}{2}(r^{\frac{2}{c}})^{-1} dv_1 = \frac{c}{2}\frac{dv_1}{v_1}.$$

We note that

$$\begin{aligned} v_1(r_c(i)) = v_1\left(\left[\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}}\right]^c\right) &= \left(\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}}\right)^2 \\ &= \frac{1 - 2\sqrt{1-4\hat{a}^2} + 1 - 4\hat{a}^2}{4\hat{a}^2} \\ &= \frac{1 - 2\hat{a}^2 - \sqrt{1-4\hat{a}^2}}{2\hat{a}^2} \end{aligned}$$

and $v_1(1) = 1$. Using this and the relationships from the previous page, equation (9) becomes

$$I_1(v_1) = \frac{c}{2} \int_{\frac{1-2\hat{a}^2-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}}^{1} \frac{dv_1}{v_1\sqrt{-\hat{a}^2 v_1^2 + (1-2\hat{a}^2)v_1 - \hat{a}^2}}. \tag{10}$$

Furthermore, using equation (A.5), where $r = v_1$, $A = -\hat{a}^2$, $B = 1 - 2\hat{a}^2$, and $C = -\hat{a}^2$, equation (10) can be evaluated as follows

$$\begin{aligned}
I_1(v_1) &= \frac{c}{2\hat{a}} \arcsin\left[\frac{-2\hat{a}^2 + (1-2\hat{a}^2)v_1}{v_1\sqrt{(1-2\hat{a}^2)^2 - 4\hat{a}^4}}\right]\Bigg|_{\frac{1-2\hat{a}^2-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}}^{1} \\
&= \frac{c}{2\hat{a}} \arcsin\left[\frac{(1-2\hat{a}^2)v_1 - 2\hat{a}^2}{v_1\sqrt{1-4\hat{a}^2}}\right]\Bigg|_{\frac{1-2\hat{a}^2-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}}^{1} \\
&= \frac{c}{2\hat{a}} \left\{\arcsin\left[\frac{1-4\hat{a}^2}{\sqrt{1-4\hat{a}^2}}\right] - \arcsin\left[\frac{(1-2\hat{a}^2)\left[\frac{1-2\hat{a}^2-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}\right] - 2\hat{a}^2}{\left[\frac{1-2\hat{a}^2-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}\right]\sqrt{1-4\hat{a}^2}}\right]\right\} \\
&= \frac{c}{2\hat{a}} \left\{\arcsin\left[\sqrt{1-4\hat{a}^2}\right] - \arcsin\left[\frac{(1-2\hat{a}^2)^2 - (1-2\hat{a}^2)\sqrt{1-4\hat{a}^2} - 4\hat{a}^4}{(1-2\hat{a}^2)\sqrt{1-4\hat{a}^2} - (1-4\hat{a}^2)}\right]\right\} \\
&= \frac{c}{2\hat{a}} \left\{\arcsin\left[\sqrt{1-4\hat{a}^2}\right] - \arcsin\left[\frac{1-4\hat{a}^2 - (1-2\hat{a}^2)\sqrt{1-4\hat{a}^2}}{(1-2\hat{a}^2)\sqrt{1-4\hat{a}^2} - (1-4\hat{a}^2)}\right]\right\} \\
&= \frac{c}{2\hat{a}} \left\{\arcsin\left[\sqrt{1-4\hat{a}^2}\right] - \arcsin(-1)\right\} \\
&= \frac{c}{2\hat{a}} \left\{\arcsin\left[\sqrt{1-4\hat{a}^2}\right] + \frac{\pi}{2}\right\}.
\end{aligned} \tag{11}$$

Let

$$\hat{\alpha} = \arcsin\left[\sqrt{1-4\hat{a}^2}\right].$$

Then it follows that

$$\cos\hat{\alpha} = 2\hat{a},$$

where we take the positive root since $\hat{\alpha} \in [0, \frac{\pi}{2}]$. Therefore we discover that

$$\hat{\alpha} = \frac{\pi}{2} - \arcsin(2\hat{a}),$$

and hence we obtain the result

$$I_1(r) = \frac{c}{2\hat{a}}\left\{\pi - \arcsin(2\hat{a})\right\}. \tag{12}$$

Next we evaluate $I_2(r)$, where

$$I_2(r) = \int_{\left[\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}}\right]^c}^{1} \frac{r^{\frac{2}{c}-1}}{\sqrt{r^{\frac{2}{c}} - \hat{a}^2\left(1+r^{\frac{2}{c}}\right)^2}} dr. \tag{13}$$

In equation (13), make the change of variables

$$v_2 = 1 + r^{\frac{2}{c}}.$$

Then we find that

$$dv_2 = \frac{2}{c} r^{\frac{2}{c}-1} dr$$

and

$$\frac{c}{2} dv_2 = r^{\frac{2}{c}-1} dr.$$

We note that

$$\begin{aligned} v_2\left(r_c(i)\right) &= 1 + \left(\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}}\right)^2 \\ &= 1 + \frac{1-2\hat{a}^2-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2} = \frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2} \end{aligned}$$

and $v_2(1) = 2$. Using this and the above relationships, equation (13) becomes (noting that $v_2 - 1 = r^{\frac{2}{c}}$)

$$I_2\left(v_2\right) = \frac{c}{2} \int_{\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}}^{2} \frac{dv_2}{\sqrt{v_2 - 1 - \hat{a}^2 v_2^2}}. \tag{14}$$

Using equation (A.9), where $r = v_2$, $D = -\hat{a}^2$, $E = 1$, and $F = -1$, equation (14) can be evaluated as follows (where $j = \sqrt{-1}$)

$$\begin{aligned} I_2\left(v_2\right) &= -\frac{cj}{2\hat{a}} \log\left[2\sqrt{\hat{a}^2\left(\hat{a}^2 v_2^2 - v_2 + 1\right)} - 2\hat{a}^2 v_2 + 1\right] \Bigg|_{\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}}^{2} \\ &= -\frac{cj}{2\hat{a}} \Bigg\{ \log\left[2\hat{a}\sqrt{4\hat{a}^2-1} - 4\hat{a}^2 + 1\right] \end{aligned}$$

$$
\begin{aligned}
&-\log\left[2\hat{a}\sqrt{\hat{a}^2\left(\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}\right)^2-\left(\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}\right)+1}\right.\\
&\qquad\left.\left.-2\hat{a}^2\left(\frac{1-\sqrt{1-4\hat{a}^2}}{2\hat{a}^2}\right)+1\right]\right\}\\
&=-\frac{cj}{2\hat{a}}\left\{\log\left[2j\hat{a}\sqrt{1-4\hat{a}^2}+1-4\hat{a}^2\right]\right.\\
&\quad-\log\left[\sqrt{1-2\sqrt{1-4\hat{a}^2}+1-4\hat{a}^2-2(1-\sqrt{1-4\hat{a}^2})+4\hat{a}^2}\right.\\
&\qquad\left.\left.+\sqrt{1-4\hat{a}^2}\right]\right\}\\
&=-\frac{cj}{2\hat{a}}\left\{\log\left[\frac{2j\hat{a}\sqrt{1-4\hat{a}^2}+1-4\hat{a}^2}{\sqrt{1-4\hat{a}^2}}\right]\right\}\\
&=-\frac{cj}{2\hat{a}}\log\left[2j\hat{a}+\sqrt{1-4\hat{a}^2}\right].
\end{aligned}
\tag{15}
$$

Using the formula

$$\arcsin z = -j\log\left[jz+\sqrt{1-z^2}\right], \tag{16}$$

we have that

$$\arcsin(2\hat{a}) = -j\log\left[2j\hat{a}+\sqrt{1-4\hat{a}^2}\right]. \tag{17}$$

Applying this to equation (15), we find that

$$I_2(r)=\frac{c}{2\hat{a}}\arcsin(2\hat{a}). \tag{18}$$

Therefore, upon using equations (12) and (18) in equation (6) yields

$$\Theta(i)=\frac{c}{2}\pi. \tag{19}$$

Hence, for the refractive index in equation (5) we have the result

$$
\begin{aligned}
D_0(i) &= 2i-\pi+2\Theta(i)\\
&= 2i-\pi+c\pi\\
&= \pi(c-1)+2i.
\end{aligned}
\tag{20}
$$

We note that $D_0'(i)=2\neq 0$ for any value of i. Thus, no zero-order bow is possible for the profile given by equation (5).

II.3 PROFILE 2

Consider next the refractive index profile

$$n(r) = n(0)\sqrt{1 - \frac{r^2}{L^2}} \equiv n_0\sqrt{1 - \frac{r^2}{L^2}}, \tag{21}$$

which was considered in [19]. The refractive index profile in equation (21) is known as the parabolic refractive index profile. An application of the parabolic refractive index profile is round optical fibers where the refractive index of the core of the fiber is a quadratic function that varies with the distance from the optical fiber axis. We note that L^2 is a constant that will be determined. Let $n(1) \equiv n_1 > 1$. Then we find that

$$L^2 = \frac{n_0^2}{n_0^2 - n_1^2}. \tag{22}$$

Let $K = \sin i$. We first calculate $r_c(i)$. Using equation (3), we want to solve

$$r_c^4(n_0^2 - n_1^2) - r_c^2 n_0^2 + K^2 = 0. \tag{23}$$

Consequently, we determine that

$$r_c^2 = \frac{n_0^2 \pm \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)}. \tag{24}$$

For a sphere of radius 1, the radial point of closest approach is bounded by

$$0 \leq r_c < 1, \quad r_c \in \mathbb{R}. \tag{25}$$

Further restrictions are imposed on $r_c(i)$ that are dependent on the values of n_0 and n_1. We will analyze two cases here.

<u>Case 1</u>: $\quad 1 < n_1 < n_0$.

This implies that $1 < n_1^2 < n_0^2$ and $n_0^2 - n_1^2 > 0$. If the expression for r_c^2 in equation (24) is substituted into equation (25), we have that

$$\begin{aligned} 0 &\leq \frac{n_0^2 \pm \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)} < 1, \\ 0 &\leq n_0^2 \pm \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2} < 2(n_0^2 - n_1^2), \\ -n_0^2 &\leq \pm\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2} < n_0^2 - 2n_1^2, \\ n_0^2 &\geq \mp\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2} > 2n_1^2 - n_0^2. \end{aligned} \tag{26}$$

We are interested in determining whether we choose the positive root or negative root in equation (24). Suppose $n_0^2 - 2n_1^2 < 0$. If this is the case, then we must take the minus root in equation (24). On the other hand, if we suppose $n_0^2 - 2n_1^2 > 0$, then $n_0^2 > 2n_1^2$. Then squaring both sides of equation (26) yields

$$0 \le K^2 < n_1^2. \tag{27}$$

Since $n_1^2 > 1$ and $K^2 = \sin^2 i < 1$, equation (27) is always satisfied. As a result, we may take the plus or minus root if $n_0^2 - 2n_1^2 > 0$. Since $r_c(i)$ is the radial point of closest approach, we will take the minus root in equation (26). Thus, we take the minus root in equation (24). As a result of the restriction in the second equation of equation (25), we must have the constraint

$$n_0^4 > 4(n_0^2 - n_1^2)K^2. \tag{28}$$

The largest number that K^2 can attain is 1 so that the strongest requirement we can have is that

$$n_0^4 > 4(n_0^2 - n_1^2). \tag{29}$$

Summarizing for Case 1, we find that

$$r_c = \sqrt{\frac{n_0^2 - \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)}}, \tag{30}$$

where equation (29) must be satisfied.

Case 2: $\quad 1 < n_0 < n_1$.

This implies that $1 < n_0^2 < n_1^2$ and $n_0^2 - n_1^2 < 0$. If the expression for r_c^2 in equation (24) is substituted into equation (25), we have that

$$\begin{gathered} 0 \le \frac{n_0^2 \pm \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)} < 1, \\ 0 \ge n_0^2 \pm \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2} > 2(n_0^2 - n_1^2). \end{gathered} \tag{31}$$

Since we need the middle term in equation (31) to be negative, we must take the negative root in equation (24). Squaring both sides of equation (31) yields

$$0 \le K^2 < n_1^2. \tag{32}$$

Since $n_1^2 > 1$, equation (32) is always satisfied. We note that $n_0^4 - 4(n_0^2 - n_1^2)K^2 > 0$ since $n_0^2 - n_1^2 < 0$. As a result, the second equation of equation (25) will always be satisfied for Case 2. Hence for Case 2 we determine that

$$r_c = \sqrt{\frac{n_0^2 - \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)}}. \tag{33}$$

We have looked at Cases 1 and 2 and have determined that the expression for $r_c(i)$ is the same for these two cases. Now we evaluate $\Theta(i)$.

Substituting equation (21) into equation (2) gives us

$$\begin{aligned}\Theta(i) = K \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2 n^2(r) - K^2}} &= K \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2\left[n_0^2\left(1 - \frac{r^2}{L^2}\right)\right] - K^2}} \\ &= K \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{n_0^2 r^2 - \frac{n_0^2}{L^2} r^4 - K^2}} \\ &= K \int_{r_c(i)}^{1} \frac{r dr}{r^2\sqrt{n_0^2 r^2 - \frac{n_0^2}{L^2} r^4 - K^2}}.\end{aligned} \tag{34}$$

Let $x = r^2$. Then $dx = 2rdr$ and $rdr = \frac{1}{2}dx$. We note that

$$x\left(r_c(i)\right) = r_c^2(i) = \frac{n_0^2 - \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)}$$

and $x(1) = 1$. Using these relationships, we obtain that

$$\Theta(i) = \frac{K}{2} \int_{r_c^2(i)}^{1} \frac{dx}{x\sqrt{n_0^2 x - \frac{n_0^2}{L^2} x^2 - K^2}}. \tag{35}$$

Upon using equation (A.5), where $A = -\frac{n_0^2}{L^2}$, $B = n_0^2$, and $C = -K^2$, we find that

$$\Theta(i) = \frac{K}{2} \frac{1}{\sqrt{K^2}} \arcsin\left[\frac{-2K^2 + n_0^2 x}{x\sqrt{n_0^4 - 4\frac{n_0^2 K^2}{L^2}}}\right]\Bigg|_{r_c^2(i)}^{1}$$

$$
\begin{aligned}
&= \frac{1}{2}\arcsin\left[\frac{n_0^2 x - 2K^2}{x\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}\right]\Bigg|_{r_c^2(i)}^{1} \\
&= \frac{1}{2}\Bigg\{\arcsin\left[\frac{n_0^2 - 2K^2}{\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}\right] \\
&\qquad - \arcsin\left[\frac{\frac{n_0^4 - n_0^2\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)} - 2K^2}{\frac{n_0^2 - \sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{2(n_0^2 - n_1^2)}\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}\right]\Bigg\} \\
&= \frac{1}{2}\Bigg\{\arcsin\left[\frac{n_0^2 - 2K^2}{\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}\right] \\
&\qquad - \arcsin\left[\frac{n_0^4 - 4(n_0^2 - n_1^2)K^2 - n_0^2\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}{n_0^2\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2} - \{n_0^4 - 4(n_0^2 - n_1^2)K^2\}}\right]\Bigg\} \\
&= \frac{1}{2}\left\{\arcsin\left[\frac{n_0^2 - 2K^2}{\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}\right] + \frac{\pi}{2}\right\}. \qquad (36)
\end{aligned}
$$

Therefore, for the refractive index in equation (21), we obtain the result

$$
\begin{aligned}
D_0(i) &= 2i - \pi + 2\Theta(i) \\
&= 2i - \frac{\pi}{2} + \arcsin\left[\frac{n_0^2 - 2K^2}{\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2}}\right]. \qquad (37)
\end{aligned}
$$

Differentiating equation (37) with respect to i yields

$$
\begin{aligned}
&D_0'(i) \\
&= 2 - \frac{4\sin i\cos i}{\sqrt{n_0^4 - 4(n_0^2 - n_1^2)K^2 - (n_0^2 - 2K^2)^2}}\left\{1 - \frac{(n_0^2 - n_1^2)(n_0^2 - 2K^2)}{n_0^4 - 4(n_0^2 - n_1^2)K^2}\right\} \\
&= 2 - 2\frac{\cos i}{\sqrt{n_1^2 - K^2}}\left\{\frac{n_1^2 n_0^2 - 2K^2(n_0^2 - n_1^2)}{n_0^4 - 4(n_0^2 - n_1^2)K^2}\right\}. \qquad (38)
\end{aligned}
$$

Recall that a zero-order bow exists if $D_0'(i_c) = 0$, where $i_c \in [0.005, \frac{\pi}{2}]$. This is satisfied in equation (38) if

$$
\begin{aligned}
\frac{\cos(i_c)}{\sqrt{n_1^2 - \sin^2 i_c}}\left[\frac{n_1^2 n_0^2 - 2(n_0^2 - n_1^2)\sin^2 i_c}{n_0^4 - 4(n_0^2 - n_1^2)\sin^2 i_c}\right] &= 1, \\
\frac{\cos(i_c)}{\sqrt{n_1^2 - \sin^2 i_c}}\left[\frac{n_1^2 L^2 - 2\sin^2 i_c}{n_0^2 L^2 - 4\sin^2 i_c}\right] &= 1. \qquad (39)
\end{aligned}
$$

Let

$$
h(i_c) \equiv \frac{\cos i_c}{\sqrt{n_1^2 - \sin^2 i_c}}\frac{n_1^2 L^2 - 2\sin^2 i_c}{n_0^2 L^2 - 4\sin^2 i_c}.
$$

For a zero-order bow to exist, we must have that $h(i_c) = 1$. Since $n_1^2 > 1$, then $n_1^2 - \sin^2 i_c > 1 - \sin^2 i_c = \cos^2 i_c$. For this reason, we must have that

$$\frac{1}{\sqrt{n_1^2 - \sin^2 i_c}} < \frac{1}{\sqrt{\cos^2 i_c}} = \frac{1}{\cos i_c}.$$

As a consequence, we obtain that

$$h(i_c) < \frac{n_1^2 L^2 - 2\sin^2 i_c}{n_0^2 L^2 - 4\sin^2 i_c} \equiv g(i_c).$$

We note that $g(i_c) < 1$, if we have the condition

$$\frac{n_1^2 L^2 - 2\sin^2 i_c}{n_0^2 L^2 - 4\sin^2 i_c} < 1.$$

Consider Case 1 where $n_0^2 - n_1^2 > 0$. Since $n_0^2 L^2 - 4\sin^2 i_c > 0$ for Case 1, we have that $g(i_c) < 1$ if

$$\begin{aligned} n_1^2 L^2 - 2\sin^2 i_c &< n_0^2 L^2 - 4\sin^2 i_c, \\ 2\sin^2 i_c &< (n_0^2 - n_1^2) L^2 = n_0^2. \end{aligned}$$

Accordingly, $h(i_c) < 1$ if $\sin^2 i_c < \frac{n_0^2}{2}$ for Case 1. As a result, a zero-order bow cannot exist for Case 1 if $\sin^2 i_c < \frac{n_0^2}{2}$. Consequently, if $\frac{n_0^2}{2} > 1$, a zero-order bow cannot exist for Case 1. Therefore, in order to guarantee the existence of a zero-order bow for Case 1, we must have that $\frac{n_0^2}{2} < 1$. In other words, we must have that $n_0 < \sqrt{2}$ in order for a zero-order bow to exist for Case 1.

In Figures 3-6, we give the plots of equations (21) and (37) for both Case 1 and Case 2. For Case 1, $D_0'(i) = 0$ at approximately $i = 1.28$ (or 73.34°) and $i = 1.49$ (or 85.52°), and for Case 2, $D_0'(i) = 0$ at approximately $i = 0.22$ (or 12.78°).

II.4 PROFILE 3

Consider the profile (as mentioned in [8])

$$n(r) = a - br^2, \tag{40}$$

where a and b are constants. The refractive index profile in equation (40) has been used as an example profile to discuss an extension of the rainbow Airy theory to nonuniform spheres [8]. We are interested in evaluating, where $K = \sin i$,

$$\Theta(i) = K \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2 n^2(r) - K^2}}. \tag{41}$$

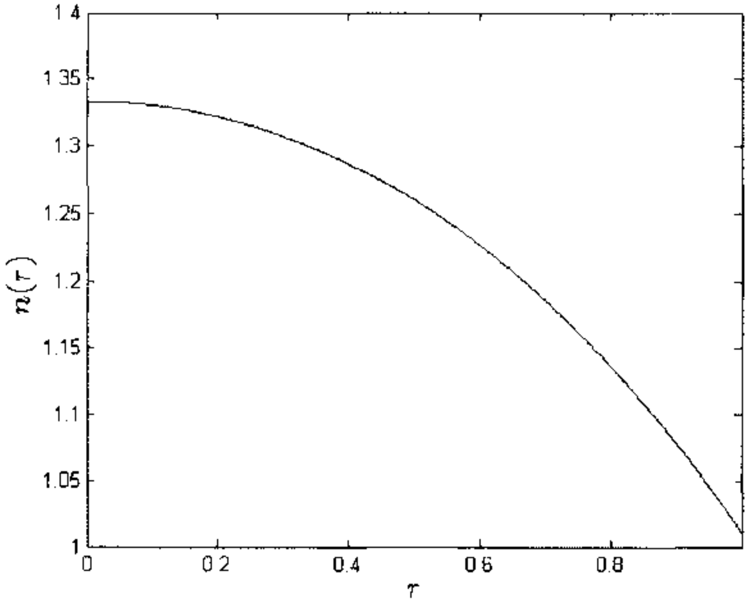

Fig. 3. Case 1: Plot of equation (21) with $n_0 = \frac{4}{3}$ and $n_1 = 1.01$.

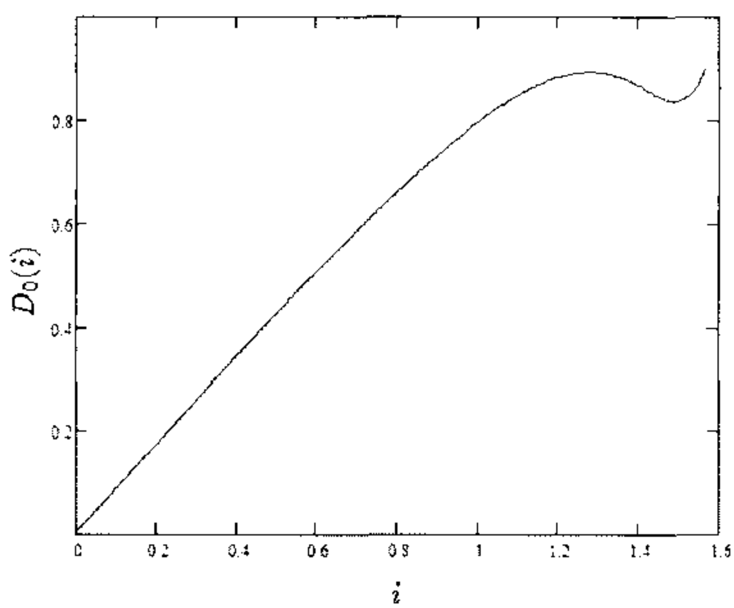

Fig. 4. Case 1: Plot of equation (37) with $n_0 = \frac{4}{3}$ and $n_1 = 1.01$.

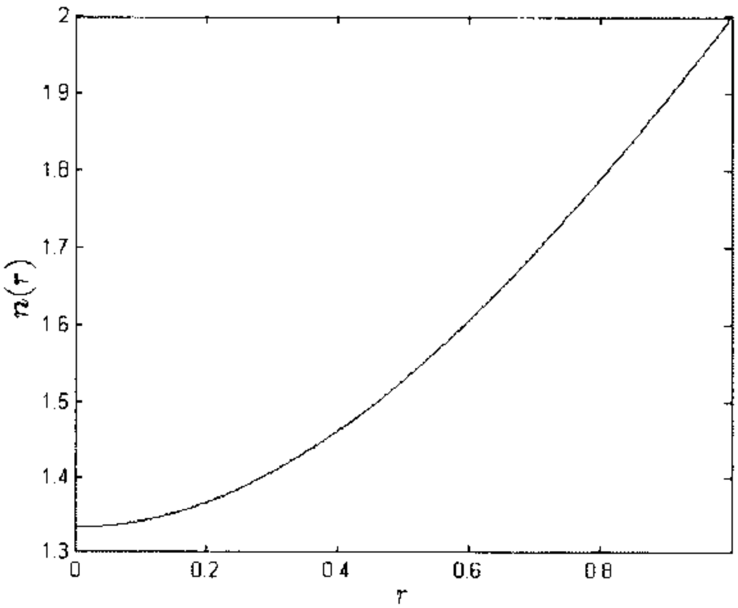

Fig. 5. Case 2: Plot of equation (21) with $n_0 = \frac{4}{3}$ and $n_1 = 2$.

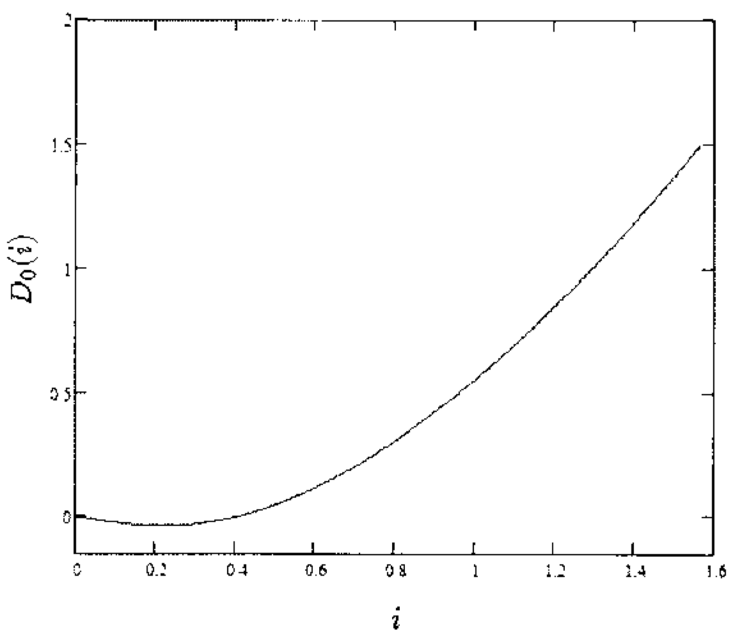

Fig. 6. Case 2: Plot of equation (37) with $n_0 = \frac{4}{3}$ and $n_1 = 2$.

First, as a consequence of equation (3), we study the equation

$$br^3 - ar + K = 0. \tag{42}$$

Equation (42) has three solutions, which we call r_1, r_2, and r_c where r_c is the minimal distance of approach of the light ray from the center of the sphere. Using the standard results between solutions of a cubic equation, we have the following relations among the solutions of equation (42):

$$r_1 + r_2 + r_c = 0, \tag{43a}$$

$$r_1 r_2 + r_2 r_c + r_1 r_c = -\frac{a}{b}, \tag{43b}$$

$$r_1 r_2 r_c = -\frac{K}{b}. \tag{43c}$$

We will use equations (43a)-(43c) to derive three more relations between the solutions of equation (42). From equation (43a), we obtain that

$$(r_1 + r_2 + r_c)(r_1 + r_2 + r_c) = 0.$$

Correspondingly, we discover that

$$r_1^2 + 2r_1 r_2 + 2r_1 r_c + r_2^2 + 2r_2 r_c + r_c^2 = 0.$$

Then we have that

$$\begin{aligned} r_1^2 + r_2^2 + r_c^2 &= -2(r_1 r_2 + r_1 r_c + r_2 r_c) \\ &= -2\left(-\frac{a}{b}\right) \\ &= 2\frac{a}{b}, \end{aligned} \tag{44}$$

where equation (43b) has been used in the second line. From equation (43b), we determine that

$$(r_1 r_2 + r_2 r_c + r_1 r_c)(r_1 r_2 + r_2 r_c + r_1 r_c) = \frac{a^2}{b^2}.$$

As a result, we find that

$$\begin{aligned} r_1^2 r_2^2 + 2r_1 r_2^2 r_c + 2r_1^2 r_2 r_c + r_2^2 r_c^2 + 2r_1 r_2 r_c^2 + r_1^2 r_c^2 &= \frac{a^2}{b^2}, \\ r_1^2 r_2^2 + r_2^2 r_c^2 + r_1^2 r_c^2 + 2r_1 r_2 r_c(r_1 + r_2 + r_c) &= \frac{a^2}{b^2}, \\ r_1^2 r_2^2 + r_c^2(r_1^2 + r_2^2) &= \frac{a^2}{b^2}, \end{aligned} \tag{45}$$

where we used equation (43a) in the third line. Squaring both sides of (43c) yields that

$$r_1^2 r_2^2 r_c^2 = \frac{K^2}{b^2}. \tag{46}$$

Now we rewrite the integrand in equation (41). We have that

$$\begin{aligned}
\frac{1}{r\sqrt{r^2n^2(r)-K^2}} &= \frac{1}{r\sqrt{r^2(a-br^2)^2-K^2}} = \frac{1}{r\sqrt{r^2(a^2-2abr^2+b^2r^4)-K^2}} \\
&= \frac{1}{r\sqrt{a^2r^2-2abr^4+b^2r^6-K^2}} = \frac{1}{jr\sqrt{K^2-a^2r^2+2abr^4-b^2r^6}} \\
&= \frac{1}{j|b|r\sqrt{\frac{K^2}{b^2}-\frac{a^2}{b^2}r^2+2\frac{a}{b}r^4-r^6}} \\
&= \frac{1}{j|b|r\sqrt{r_1^2r_2^2r_c^2-\left[r_1^2r_2^2+r_c^2\left(r_1^2+r_2^2\right)\right]r^2+\left[r_1^2+r_2^2+r_c^2\right]r^4-r^6}},
\end{aligned} \tag{47}$$

where we have utilized equations (44)-(46) (and recall $j = \sqrt{-1}$). Focusing on the polynomial under the radical, we find that

$$\begin{aligned}
&r_1^2r_2^2r_c^2-\left[r_1^2r_2^2+r_c^2\left(r_1^2+r_2^2\right)\right]r^2+\left[r_1^2+r_2^2+r_c^2\right]r^4-r^6 \\
&\quad= r_c^2r^4-r_c^2(r_1^2+r_2^2)r^2+r_1^2r_2^2r_c^2-r^6+(r_1^2+r_2^2)r^4-r_1^2r_2^2r^2 \\
&\quad= r_c^2\left[r^4-(r_1^2+r_2^2)r^2+r_1^2r_2^2\right]-r^2\left[r^4-(r_1^2+r_2^2)r^2+r_1^2r_2^2\right] \\
&\quad= \left(r_c^2-r^2\right)\left[r^4-(r_1^2+r_2^2)r^2+r_1^2r_2^2\right] \\
&\quad= \left(r_c^2-r^2\right)\left(r^2-r_1^2\right)\left(r^2-r_2^2\right).
\end{aligned} \tag{48}$$

Utilizing equation (48) in equation (47) yields

$$\frac{1}{r\sqrt{r^2n^2(r)-K^2}} = \frac{1}{j|b|r\sqrt{\left(r_c^2-r^2\right)\left(r^2-r_1^2\right)\left(r^2-r_2^2\right)}}. \tag{49}$$

Equation (49) may be further rewritten as

$$\begin{aligned}
&\frac{1}{j|b|r\sqrt{\left(r_c^2-r^2\right)\left(r^2-r_1^2\right)\left(r^2-r_2^2\right)}} \\
&= \frac{r}{j|b|r_c^2\sqrt{r_c^2-r_1^2}\sqrt{r_c^2-r_2^2}\sqrt{r_c^2-r^2}}\left[\frac{1}{\frac{r^2}{r_c^2}\sqrt{\frac{r_1^2-r^2}{r_1^2-r_c^2}}\sqrt{\frac{r^2-r_2^2}{r_c^2-r_2^2}}}\right] \\
&= \frac{j}{|b|r_c^2\sqrt{r_c^2-r_1^2}}\left[\frac{1}{\frac{r^2}{r_c^2}\sqrt{\frac{r_1^2-r^2}{r_1^2-r_c^2}}\sqrt{\frac{r^2-r_2^2}{r_c^2-r_2^2}}}\right]\left[-\frac{r}{\sqrt{r_c^2-r_2^2}\sqrt{r_c^2-r^2}}\right].
\end{aligned} \tag{50}$$

Let

$$\sin\phi = \sqrt{\frac{r_c^2 - r^2}{r_c^2 - r_2^2}}, \tag{51a}$$

$$p = \sqrt{\frac{r_2^2 - r_c^2}{r_1^2 - r_c^2}}, \tag{51b}$$

and

$$\alpha^2 = 1 - \frac{r_2^2}{r_c^2} = \frac{r_c^2 - r_2^2}{r_c^2}. \tag{51c}$$

We have the following relations

$$1 - \alpha^2 \sin^2\phi = \frac{r^2}{r_c^2}, \tag{52a}$$

$$1 - p^2 \sin^2\phi = \frac{r_1^2 - r^2}{r_1^2 - r_c^2}, \tag{52b}$$

$$1 - \sin^2\phi = \frac{r^2 - r_2^2}{r_c^2 - r_2^2}, \tag{52c}$$

and

$$\frac{d}{dr}(\sin\phi) = -\frac{r}{\sqrt{r_c^2 - r_2^2}\sqrt{r_c^2 - r^2}}. \tag{52d}$$

Upon using equations (52a)-(52d) in equation (50) gives us

$$\begin{aligned}
\frac{1}{r\sqrt{r^2 n^2(r) - K^2}} &= \frac{j}{|b| r_c^2 \sqrt{r_c^2 - r_1^2}} \frac{\frac{d}{dr}(\sin\phi)}{\left(1 - \alpha^2 \sin^2\phi\right)\sqrt{\left(1 - p^2 \sin^2\phi\right)\left(1 - \sin^2\phi\right)}} \\
&= \frac{j}{|b| r_c^2 \sqrt{r_c^2 - r_1^2}} \frac{d}{dr}\left\{ \int_0^{\sin\phi} \frac{dt}{\left(1 - \alpha^2 t^2\right)\sqrt{\left(1 - p^2 t^2\right)\left(1 - t^2\right)}} \right\} \\
&= \frac{j}{|b| r_c^2 \sqrt{r_c^2 - r_1^2}} \frac{d}{dr} \Pi\left(\phi, \alpha^2, p\right),
\end{aligned} \tag{53}$$

where $\Pi\left(\phi, \alpha^2, p\right)$ is the incomplete elliptic integral of the third kind, ϕ is given by equation (51a), α^2 is given by equation (51c), and p is given by equation (51b).

Integrating (53) in the interval $r \in [r_c(i), 1]$ we obtain that

$$\Theta(i) = \frac{Kj}{|b| r_c^2 \sqrt{r_c^2 - r_1^2}} \Pi\left(\phi, \alpha^2, p\right). \tag{54}$$

Therefore we determine that

$$D_0(i) = 2i - \pi + 2\frac{Kj}{|b| r_c^2 \sqrt{r_c^2 - r_1^2}} \Pi\left(\phi, \alpha^2, p\right). \tag{55}$$

As can be seen from equation (55), the calculation of $D_0'(i)$ is quite difficult. In order to avoid these highly involved calculations, we will only provide the graph of $D_0(i)$ from equation (55).

In Figures 7 and 8, we plot equations (40) and (55) for the case $n'(r) > 0$, where $a = 2$ and $b = -5$. When $n'(r) > 0$, a zero-order bow exists at an incidence angle of approximately 0.26 (or 14.82°).

II.5 PROFILE 4

Consider the linear profile (previously mentioned in [1])

$$n(r) = a + br. \tag{56}$$

where a and b are constants. The linear refractive index profile has been utilized with respect to absorption measurements of nonlinear optical liquids in the visible and near-infrared spectral region [9].

Using equation (56) in equation (2) gives us

$$\begin{aligned} \Theta(i) &= \sin i \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2 (a+br)^2 - \sin^2 i}} \\ &= \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2 (A+Br)^2 - 1}}, \end{aligned} \tag{57}$$

where

$$A = \frac{a}{q}, \qquad B = \frac{b}{q},$$

and

$$q = \sin i.$$

We note that

$$\frac{1}{r\sqrt{r^2 (A+Br)^2 - 1}} = \frac{1}{r\sqrt{(Br^2 + Ar + 1)(Br^2 + Ar - 1)}} \equiv g(r)$$

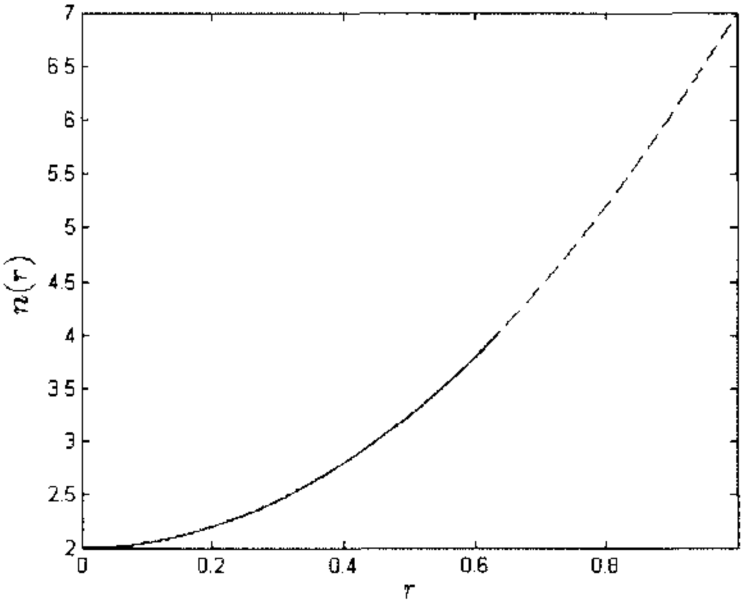

Fig. 7. Plot of equation (40) with $a = 2$ and $b = -5$.

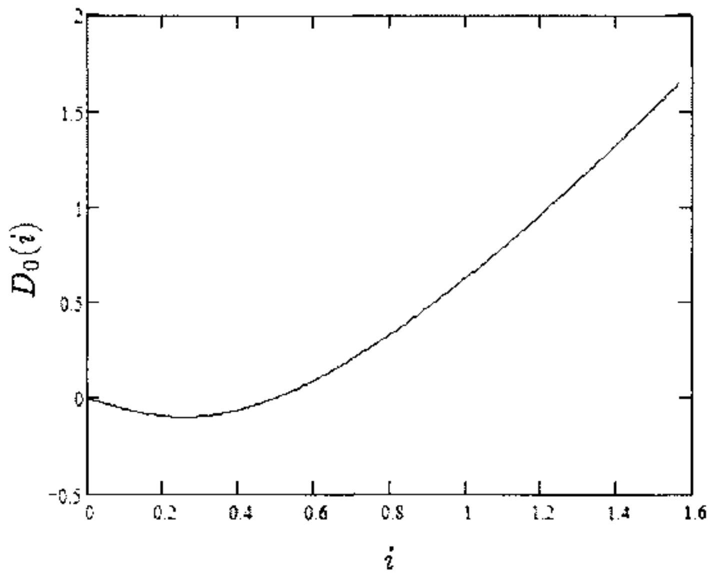

Fig. 8. Plot of equation (55) with $a = 2$ and $b = -5$.

and

$$Br^2 + Ar - 1 = \left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right]\left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right].$$

Let

$$k = -\frac{1}{4B}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right], \tag{58}$$

where we take the positive root if $4B - A^2 > 0$ and we take the negative root if $4B - A^2 < 0$. We now give two results that relate to k that we use later. We have that

$$\begin{aligned} k - 1 &= -\frac{1}{4B}\left[A^2 + 4B \pm j\sqrt{(A^2+4B)(4B-A^2)}\right] \\ &= -\frac{\sqrt{A^2+4B}}{4B}\left[\sqrt{A^2+4B} \pm j\sqrt{4B-A^2}\right]. \end{aligned}$$

Accordingly, we discover that

$$\begin{aligned} (k-1)^2 &= \frac{A^2+4B}{16B^2}\left[A^2 + 4B - 4B + A^2 \pm 2j\sqrt{(A^2+4B)(4B-A^2)}\right] \\ &= \frac{A^2+4B}{8B^2}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]. \end{aligned} \tag{59}$$

Also we have that

$$(k+1)^2 = (k-1)^2 + 4k = \frac{A^2-4B}{8B^2}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]. \tag{60}$$

We will now take several steps to simplify the integrand in equation (57) in order to allow us to perform the integration. We find that

$$\begin{aligned} g(r) &= \frac{\frac{\sqrt{A^2+4B}}{2\sqrt{2B}}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]^{\frac{1}{2}}}{\frac{\sqrt{A^2+4B}}{2\sqrt{2B}}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]^{\frac{1}{2}}} g(r) \\ &= \frac{\frac{\sqrt{A^2+4B}}{2\sqrt{2B}}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]^{\frac{1}{2}}}{k-1} g(r) \\ &= -\frac{j\sqrt{A^2+4B}}{\sqrt{2B}}\frac{\sqrt{k}}{k-1} g(r) = -\frac{\sqrt{2Bk}}{B(k-1)}\left[\frac{\sqrt{A^2+4B}}{2} j g(r)\right]. \end{aligned}$$

We also find that

$$g(r) = -\frac{\sqrt{2Bk}}{B(k-1)}\left[\frac{\sqrt{A^2+4B}}{2} j \frac{1}{r\sqrt{(Br^2+Ar+1)(Br^2+Ar-1)}}\right]$$

$$= -\frac{\sqrt{2Bk}}{B(k-1)}\left\{\frac{\left[\frac{A+\sqrt{A^2+4B}}{2}\right]\frac{\sqrt{A^2+4B}}{2}jr}{r\sqrt{(Br^2+Ar+1)(Br^2+Ar-1)}}\right.$$

$$\left. - \frac{\left(\frac{A+\sqrt{A^2+4B}}{2}r-1\right)\frac{\sqrt{A^2+4B}}{2}j}{r\sqrt{(Br^2+Ar+1)(Br^2+Ar-1)}}\right\}$$

$$= -\frac{\sqrt{2Bk}}{B}\left\{\frac{\left[\frac{A+\sqrt{A^2+4B}}{2}\right]\frac{\sqrt{A^2+4B}}{2}j}{(k-1)\sqrt{(Br^2+Ar+1)(Br^2+Ar-1)}}\right.$$

$$\left. - \frac{\left(\frac{A+\sqrt{A^2+4B}}{2}r-1\right)\frac{\sqrt{A^2+4B}}{2}j}{r(k-1)\sqrt{(Br^2+Ar+1)(Br^2+Ar-1)}}\right\}$$

$$= -\frac{\sqrt{2Bk}}{B}\left[\left(\frac{A+\sqrt{A^2+4B}}{2}\right)H(r) - \sqrt{A^2+4B}P(r)\right],$$

where

$$H(r) \equiv \frac{\frac{\sqrt{A^2+4B}}{2}j}{(k-1)\sqrt{(Br^2+Ar+1)}} \times \frac{1}{\sqrt{\left[-\frac{1}{2}\left(A-\sqrt{A^2+4B}\right)r+1\right]\left[\frac{1}{2}\left(A+\sqrt{A^2+4B}\right)r-1\right]}}$$

and

$$P(r) \equiv \frac{\left(\frac{A+\sqrt{A^2+4B}}{2}r-1\right)}{r\sqrt{A^2+4B}}H(r).$$

Once we determine $H(r)$ and $P(r)$, we will be able to evaluate $\Theta(i)$. We now give some results that will be used to evaluate $H(r)$ and $P(r)$.

$$\left(A+\sqrt{A^2+4B}\right)\left(A-\sqrt{A^2+4B}\right) = -4B. \tag{61a}$$

$$\frac{A+\sqrt{A^2+4B}}{A-\sqrt{A^2+4B}} = -\frac{4B}{\left(A-\sqrt{A^2+4B}\right)^2}, \tag{61b}$$

and

$$4B(k-1)^2 = \frac{A^2+4B}{2B}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]$$

$$= -\frac{(A^2-4B)}{2B}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]$$

$$+ \frac{A^2}{B}\left[A^2 \pm j\sqrt{(A^2+4B)(4B-A^2)}\right]$$

$$= -4B\left(k+1\right)^2 - 4A^2k = -4B\left(k^2+2k+1\right) - 4A^2k$$
$$= -4B\left(k^2+1\right) - 2k\left(4B+2A^2\right). \tag{61c}$$

Utilizing equation (61c), we are able to derive another useful result that will be used in the calculation of $H(r)$.

$$\begin{aligned}
&4B^2(k-1)^2r^2 + 4AB(k-1)^2r + 4B(k-1)^2 \\
&= 4B^2(k-1)^2r^2 + 4AB(k-1)^2r - 4B(k^2+1) - 2k(4B+2A^2) \\
&= r^2\left[4B^2 + 4B^2k^2 - 2k(4B^2)\right] + r\left[4AB + 4ABk^2 - 2k(4AB)\right] \\
&\qquad - \left[4B + 4Bk^2 + 2k(4B+2A^2)\right] \\
&= -2k\left[4B^2r^2 + 4ABr + 4B + 2A^2\right] + 4B(1+k^2)(Br^2 + Ar - 1) \\
&= -k\left(A - \sqrt{A^2+4B}\right)^2\left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right]^2 \\
&\qquad + 4B(1+k^2)\left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right]\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right] \\
&\qquad - k\left(A + \sqrt{A^2+4B}\right)^2\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right]^2 \\
&= -k\left(A - \sqrt{A^2+4B}\right)^2\left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right]^2 \\
&\qquad - \left(A + \sqrt{A^2+4B}\right)\left(A - \sqrt{A^2+4B}\right) \\
&\qquad\quad \times \left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right]\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right] \\
&\qquad - k^2\left(A + \sqrt{A^2+4B}\right)\left(A - \sqrt{A^2+4B}\right) \\
&\qquad\quad \times \left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right]\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right] \\
&\qquad - k\left(A + \sqrt{A^2+4B}\right)^2\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right]^2.
\end{aligned}$$

Thus we obtain the result

$$
\begin{aligned}
&4B^2(k-1)^2r^2 + 4AB(k-1)^2r + 4B(k-1)^2 \\
&= \Bigg\{ -k\left(A - \sqrt{A^2+4B}\right)\left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right] \\
&\qquad - \left(A + \sqrt{A^2+4B}\right)\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right]\Bigg\} \\
&\times \Bigg\{ \left(A - \sqrt{A^2+4B}\right)\left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right] \\
&\qquad + k\left(A + \sqrt{A^2+4B}\right)\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right]\Bigg\}.
\end{aligned}
\tag{61d}
$$

We provide two more results that will be useful in the evaluation of $H(r)$ and $P(r)$.

$$
\begin{aligned}
&\left(A - \sqrt{A^2+4B}\right)\left[\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1\right] \\
&\qquad + \left(A + \sqrt{A^2+4B}\right)\left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right] \\
&= -A + \sqrt{A^2+4B} + A + \sqrt{A^2+4B} = 2\sqrt{A^2+4B}
\end{aligned}
\tag{61e}
$$

and

$$
\begin{aligned}
r\sqrt{A^2+4B} &= r\frac{\sqrt{A^2+4B}}{2} + r\frac{\sqrt{A^2+4B}}{2} \\
&= \frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1 - \frac{Ar}{2} + 1 + r\frac{\sqrt{A^2+4B}}{2} \\
&= \frac{1}{2}\left(A + \sqrt{A^2+4B}\right)r - 1 + \left[-\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)r + 1\right].
\end{aligned}
\tag{61f}
$$

Now we calculate $H(r)$. To this end, we see that

$$
\begin{aligned}
&H(r) \\
&= \frac{\frac{\sqrt{A^2+4B}}{2}j}{\sqrt{B(k-1)^2r^2 + A(k-1)^2r + (k-1)^2}} \\
&\qquad \times \frac{1}{\sqrt{\left[-\frac{1}{2}(A - \sqrt{A^2+4B})r + 1\right]\left[\frac{1}{2}(A + \sqrt{A^2+4B})r - 1\right]}} \\
&= \frac{1}{\sqrt{-B(k-1)^2r^2 - A(k-1)^2r - (k-1)^2}}
\end{aligned}
$$

$$
\times \left[- \frac{\frac{\sqrt{A^2+4B}}{2}}{\sqrt{\left[-\frac{1}{2}(A - \sqrt{A^2+4B})r + 1 \right] \left[\frac{1}{2}(A + \sqrt{A^2+4B})r - 1 \right]}} \right]
$$

$$
= \frac{1}{\sqrt{\left[\frac{A+\sqrt{A^2+4B}}{A-\sqrt{A^2+4B}} \right] \left[-B(k-1)^2 r^2 - A(k-1)^2 r - (k-1)^2 \right]}}
$$

$$
\times \left[- \frac{\frac{\sqrt{A^2+4B}}{2} \sqrt{\frac{A+\sqrt{A^2+4B}}{A-\sqrt{A^2+4B}}}}{\sqrt{\left[-\frac{1}{2}(A - \sqrt{A^2+4B})r + 1 \right] \left[\frac{1}{2}(A + \sqrt{A^2+4B})r - 1 \right]}} \right].
$$

Using equations (61a) and (61b), we find that

$H(r)$

$$
= \frac{1}{\sqrt{\left[\frac{1}{(A-\sqrt{A^2+4B})^2} \right] \left[4B^2(k-1)^2 r^2 + 4AB(k-1)^2 r + 4B(k-1)^2 \right]}}
$$

$$
\times \left[- \frac{\frac{\sqrt{A^2+4B}}{2} \sqrt{\frac{A+\sqrt{A^2+4B}}{A-\sqrt{A^2+4B}}}}{\sqrt{\left[-\frac{1}{2}(A - \sqrt{A^2+4B})r + 1 \right] \left[\frac{1}{2}(A + \sqrt{A^2+4B})r - 1 \right]}} \right].
$$

Upon using equations (61c) and (61d), we obtain that

$H(r)$

$$
= \frac{1}{\sqrt{\frac{\left\{ -k(A-\sqrt{A^2+4B}) \left[\frac{1}{2}(A+\sqrt{A^2+4B})r - 1 \right] - (A+\sqrt{A^2+4B}) \left[-\frac{1}{2}(A-\sqrt{A^2+4B})r + 1 \right] \right\}}{(A-\sqrt{A^2+4B})}}}
$$

$$
\times \frac{1}{\sqrt{\frac{\left\{ (A-\sqrt{A^2+4B}) \left[\frac{1}{2}(A+\sqrt{A^2+4B})r - 1 \right] + k(A+\sqrt{A^2+4B}) \left[-\frac{1}{2}(A-\sqrt{A^2+4B})r + 1 \right] \right\}}{(A-\sqrt{A^2+4B})}}}
$$

$$
\times \left[- \frac{\frac{\sqrt{A^2+4B}}{2} \sqrt{\frac{A+\sqrt{A^2+4B}}{A-\sqrt{A^2+4B}}}}{\sqrt{\left[-\frac{1}{2}(A - \sqrt{A^2+4B})r + 1 \right] \left[\frac{1}{2}(A + \sqrt{A^2+4B})r - 1 \right]}} \right].
$$

As a result, we determine that

$$H(r) = \frac{1}{\sqrt{\frac{\left\{-k(A-\sqrt{A^2+4B})\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]-(A+\sqrt{A^2+4B})\left[-\frac{1}{2}(A-\sqrt{A^2+4B})r+1\right]\right\}}{-k(A-\sqrt{A^2+4B})\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]}}}$$

$$\times \frac{1}{\sqrt{\frac{\left\{(A-\sqrt{A^2+4B})\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]-k(A+\sqrt{A^2+4B})\left[-\frac{1}{2}(A-\sqrt{A^2+4B})r+1\right]\right\}}{(A-\sqrt{A^2+4B})\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]}}}$$

$$\times \left\{-\frac{\frac{\sqrt{A^2+4B}}{2}}{\sqrt{-k}\sqrt{-\frac{1}{2}\left(A-\sqrt{A^2+4B}\right)r+1}} \times \frac{\sqrt{\frac{A+\sqrt{A^2+4B}}{A-\sqrt{A^2+4B}}}}{\left[\frac{1}{2}\left(A+\sqrt{A^2+4B}\right)r-1\right]^{\frac{3}{2}}}\right\}.$$

Hence, we discover that

$$H(r) = \frac{1}{\sqrt{\left\{1+\left[\frac{A+\sqrt{A^2+4B}}{k(A-\sqrt{A^2+4B})}\right]\left[\frac{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}{\frac{1}{2}(A+\sqrt{A^2+4B})r-1}\right]\right\}}}$$

$$\times \frac{1}{\sqrt{\left\{1+\left[\frac{k(A+\sqrt{A^2+4B})}{A-\sqrt{A^2+4B}}\right]\left[\frac{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}{\frac{1}{2}(A+\sqrt{A^2+4B})r-1}\right]\right\}}}$$

$$\times \left\{-\frac{\frac{\sqrt{A^2+4B}}{2}}{\sqrt{-\frac{1}{2}\left(A-\sqrt{A^2+4B}\right)r+1}} \times \frac{\sqrt{-\frac{(A+\sqrt{A^2+4B})}{k(A-\sqrt{A^2+4B})}}}{\left[\frac{1}{2}\left(A+\sqrt{A^2+4B}\right)r-1\right]^{\frac{3}{2}}}\right\}. \quad (62)$$

Let

$$\sin\phi = \sqrt{\left[-\frac{(A+\sqrt{A^2+4B})}{k(A-\sqrt{A^2+4B})}\right]\left[\frac{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}{\frac{1}{2}(A+\sqrt{A^2+4B})r-1}\right]} \tag{63}$$

and

$$\beta = -\frac{(A+\sqrt{A^2+4B})}{k(A-\sqrt{A^2+4B})}.$$

Using the definition of β, equation (63) may be written as

$$\sin\phi = \sqrt{\beta}\sqrt{\frac{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}{\frac{1}{2}(A+\sqrt{A^2+4B})r-1}}. \tag{64}$$

Differentiating equation (64) with respect to r yields that

$$\frac{d}{dr}\left(\sin\phi\right)$$

$$= -\frac{\sqrt{\beta}}{4}\left[\frac{A-\sqrt{A^2+4B}}{\sqrt{\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]\left[-\frac{1}{2}(A-\sqrt{A^2+4B})r+1\right]}} + \frac{\left(A+\sqrt{A^2+4B}\right)\sqrt{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}}{\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]^{\frac{3}{2}}}\right]$$

$$= -\frac{\sqrt{\beta}}{4}\left[\frac{(A-\sqrt{A^2+4B})\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]}{\sqrt{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]^{\frac{3}{2}}} + \frac{(A+\sqrt{A^2+4B})\left[-\frac{1}{2}(A-\sqrt{A^2+4B})r+1\right]}{\sqrt{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]^{\frac{3}{2}}}\right].$$

Using equation (61e), we have that

$$\frac{d}{dr}\left(\sin\phi\right) = -\frac{\frac{\sqrt{A^2+4B}}{2}\sqrt{\beta}}{\sqrt{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}\left[\frac{1}{2}(A+\sqrt{A^2+4B})r-1\right]^{\frac{3}{2}}}$$

$$= -\frac{\frac{\sqrt{A^2+4B}}{2}\sqrt{-\frac{(A+\sqrt{A^2+4B})}{k(A-\sqrt{A^2+4B})}}}{\sqrt{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}\left[\frac{1}{2}\left(A+\sqrt{A^2+4B}\right)r-1\right]^{\frac{3}{2}}}. \tag{65}$$

Using equation (63) and equation (65) in equation (62) gives us that

$$
\begin{aligned}
H(r) &= \frac{1}{\sqrt{\left(1-\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)}}\frac{d}{dr}\left(\sin\phi\right) \\
&= \frac{d}{dr}\left\{\int_0^{\sin\phi}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\right\} \\
&= \frac{d}{dr}F(\phi,k). \qquad (66)
\end{aligned}
$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind, ϕ is given by equation (63), and k is given by equation (58). We now turn our attention to evaluating $P(r)$. To that end, we have that

$$
\begin{aligned}
P(r) &\equiv \frac{\left(\frac{A+\sqrt{A^2+4B}}{2}r-1\right)}{r\sqrt{A^2+4B}}H(r) \\
&= \frac{\left(\frac{A+\sqrt{A^2+4B}}{2}r-1\right)}{r\sqrt{A^2+4B}}\frac{1}{\sqrt{\left(1-\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)}}\frac{d}{dr}\left(\sin\phi\right).
\end{aligned}
$$

Utilizing equation (61f), we find that

$$
\begin{aligned}
&P(r) \\
&= \frac{\left[\frac{1}{2}\left(A+\sqrt{A^2+4B}\right)r-1\right]}{\frac{1}{2}\left(A+\sqrt{A^2+4B}\right)r-1+\left[-\frac{1}{2}\left(A-\sqrt{A^2+4B}\right)r+1\right]} \\
&\qquad \times\frac{1}{\sqrt{\left(1-\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)}}\frac{d}{dr}\left(\sin\phi\right) \\
&= \frac{1}{\frac{\frac{1}{2}(A+\sqrt{A^2+4B})r-1+\left[-\frac{1}{2}(A-\sqrt{A^2+4B})r+1\right]}{\frac{1}{2}(A+\sqrt{A^2+4B})r-1}} \\
&\qquad \times\frac{1}{\sqrt{\left(1-\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)}}\frac{d}{dr}\left(\sin\phi\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1 + \left[\frac{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}{\frac{1}{2}(A+\sqrt{A^2+4B})r-1}\right]} \\
&\qquad \times \frac{1}{\sqrt{\left(1-\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)}}\frac{d}{dr}\left(\sin\phi\right) \\
&- \frac{1}{1 - \left[\frac{k(A-\sqrt{A^2+4B})}{A+\sqrt{A^2+4B}}\right]\left[\frac{-(A+\sqrt{A^2+4B})}{k(A-\sqrt{A^2+4B})}\right]\left[\frac{-\frac{1}{2}(A-\sqrt{A^2+4B})r+1}{\frac{1}{2}(A+\sqrt{A^2+4B})r-1}\right]} \\
&\qquad \times \frac{1}{\sqrt{\left(1-\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)}}\frac{d}{dr}\left(\sin\phi\right). \qquad (67)
\end{aligned}
$$

Let

$$\alpha^2 = \frac{k(A-\sqrt{A^2+4B})}{A+\sqrt{A^2+4B}}. \qquad (68)$$

Applying this and equation (63) in equation (67) yields

$$
\begin{aligned}
P(r) &= \frac{1}{1-\alpha^2\sin^2\phi}\frac{1}{\sqrt{\left(1-\sin^2\phi\right)\left(1-k^2\sin^2\phi\right)}}\frac{d}{dr}\left(\sin\phi\right) \\
&= \frac{d}{dr}\left\{\int_0^{\sin\phi}\frac{dt}{(1-\alpha^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}}\right\} \\
&= \frac{d}{dr}\Pi(\phi,\alpha^2,k), \qquad (69)
\end{aligned}
$$

where $\Pi(\phi,\alpha^2,k)$ is the incomplete elliptic integral of the third kind, ϕ is given by equation (63), α^2 is given by equation (68), and k is given by equation (58). As a consequence of equations (66) and (69), we obtain the result

$$g(r) = -\frac{\sqrt{2Bk}}{B}\left\{\left(\frac{A+\sqrt{A^2+4B}}{2}\right)\frac{d}{dr}F(\phi,k) - \sqrt{A^2+4B}\frac{d}{dr}\Pi(\phi,\alpha^2,k)\right\}.$$

Therefore, integrating the above in the interval $r \in [r_c(i), 1]$ gives us that

$$
\begin{aligned}
\Theta(i) &= \int_{r_c(i)}^{1} g(r)dr \\
&= -\frac{\sqrt{2Bk}}{B}\left\{\left(\frac{A+\sqrt{A^2+4B}}{2}\right)F(\phi,k) - \sqrt{A^2+4B}\,\Pi(\phi,\alpha^2,k)\right\}. \qquad (70)
\end{aligned}
$$

As a result, we discover that

$$D_0(i) = 2i - \pi - 2\frac{\sqrt{2Bk}}{B}\left\{ \left(\frac{A + \sqrt{A^2 + 4B}}{2}\right) F(\phi, k) - \sqrt{A^2 + 4B}\,\Pi(\phi, \alpha^2, k) \right\}. \tag{71}$$

As can be seen from equation (71), the calculation of $D_0'(i)$ is extremely difficult. The calculation of the derivative of the deviation angle would involve differentiating the incomplete elliptic integrals of the first and third kind since their arguments ϕ, α^2, and k are functions of the angle of incidence i. In order to avoid these highly involved calculations, we will only provide the graph of $D_0(i)$ from equation (71).

In Figures 9 and 10, we plot equations (56) and (71) for the case $n'(r) > 0$. A zero-order bow exists at an angle of incidence of approximately 0.15 (or 8.41°).

In this chapter, we have shown that the evaluation of the ray path integral $\Theta(i)$ is algebraically possible for a few refractive index profiles using the geometrical optics approach. As a consequence, the deviation angle $D_0(i)$ is readily determined. For two of the refractive index profiles in this chapter, we were able to discuss certain constraints that needed to be satisfied in order for a zero-order bow to exist. As we have seen with the refractive index profiles in equations (40) and (56), the formula for the deviation angle can become quite complicated. As a result, the calculation of the derivative of the deviation angle is not readily obtainable algebraically.

In the next chapter, we will extend the geometrical optics approach to several singular refractive index profiles. We will follow the same procedure as we have done in this chapter by first determining the ray path integral $\Theta(i)$ and then the deviation angle $D_0(i)$. For the refractive index profiles in the next chapter whose derivative of the deviation angle is readily obtainable algebraically, we will then determine necessary conditions on the parameters of the refractive index profile that allow for the existence of a zero-order bow.

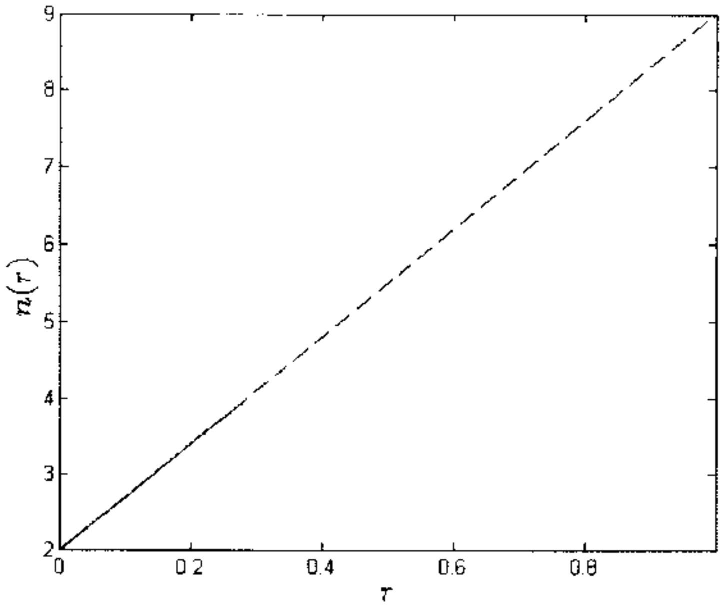

Fig. 9. Plot of equation (56) with $a = 2$ and $b = 7$.

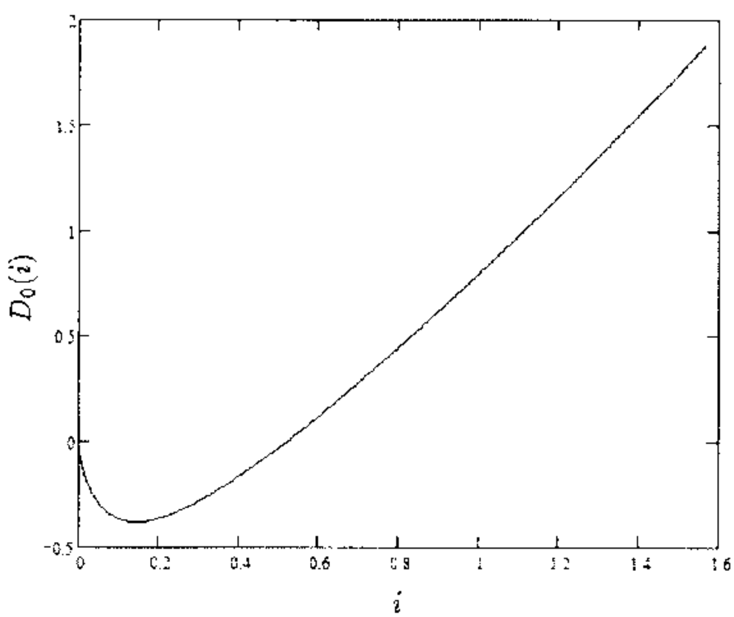

Fig. 10. Plot of equation (71) with $a = 2$ and $b = 7$.

CHAPTER III

RAY APPROACH - SINGULAR PROFILES

III.1 INTRODUCTION

In this chapter, we study refractive index profiles that have at least one singular point in the domain $r \in [0, 1]$. For each profile, we evaluate the integral (2) and examine the derivative of (1) for those refractive index profiles whose derivative of the deviation angle is readily obtainable. With respect to the refractive index profiles discussed in this chapter, the asymptotes and the less practical portions of the profile (for optical wavelengths) are indicated by a dashed line. Geometrical optics describes light propagation by considering how individual rays will be modified as they pass through a particular medium. This branch of optics provides guidelines, which may rely on the wavelength of the ray, that allow for the propagation of rays through a medium. A drawback of the geometrical approach is its inability to account for optical phenomena such as diffraction and interference. For example, as stated in [14], it has been shown that, in the optical limit (as the wavelength approaches 0), geometrical optics incorrectly predicts the amplitude and phase of the backscattered signal for spherically symmetric lenses with refractive indices that include a point singularity at the origin. Consequently, in this case, correction factors to the geometrical optics approach need to be obtained by an asymptotic analysis of the exact solution that is valid for short wavelengths. When rays propagate in inhomogeneous media, a condition that is sometimes placed on the applicability of geometrical optics is that the refractive index profile in the medium must be slowly varying [13]. Plasmas are an excellent example of media, which in limiting cases, may exhibit poles, zeros or both in the refractive index. For example, a cylindrically confined laboratory plasma may possess a resonance, cutoff, or both at some finite radii. If the frequency of the incident wave is much greater than the collision frequency of the plasma, as it occurs when the plasma is probed by a laser beam, then the squared refractive index is essentially a real quantity, and it is infinite at a resonance and zero at a cutoff (this is equivalent to saying that collisions are neglected) [13], [15]. The current and future applications of plasmas provides reasons why the study of singular refractive index profiles is of considerable value.

III.2 PROFILE 1

Consider the refractive index profile (as previously discussed in [18] and [23])

$$n(r) = n_1 r^{\frac{1}{b}-1}(2 - r^{\frac{2}{b}})^{\frac{1}{2}}, \qquad n_1 = n(1) > 1. \tag{72}$$

The profile given by equation (72) is singular at $r = 0$ if one of the following conditions is satisfied:

$$b < 0,$$

$$b > 1.$$

We will show that in order for a zero-order bow to exist, we must have $b > 2$. Consequently, the second singularity condition $b > 1$ will be met in order to guarantee the existence of a zero-order bow.

When $b = 1$, the refractive index profile in equation (72) corresponds to a Luneburg lens. A Luneburg lens is a dielectric sphere whose refractive index decreases from a large value at the center of the sphere to the index of the surrounding medium at its surface [5] in such a way that, in the context of ray theory, an electromagnetic plane wave incident on the lens focuses at the axial point on the shadowed side of the lens surface [18]. The Luneburg lens focuses light rays or other electromagnetic radiation on points on the surface of its sphere that lie in the direction of incidence. Luneburg lenses are useful in microwave technology, for example for satellite tracking. A single Luneburg lens can focus signals from several satellites on its surface. Movable detectors placed at the radio images can follow the signal by feedback-they track the satellites on the Luneburg sphere [3].

We note that

$$r^2 n^2(r) = {n_1}^2 r^{\frac{2}{b}}(2 - r^{\frac{2}{b}}).$$

Using this in equation (2) gives us

$$\Theta(i) = \sin i \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{{n_1}^2 r^{\frac{2}{b}}(2 - r^{\frac{2}{b}}) - \sin^2 i}} = \frac{\sin i}{n_1} \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{-\frac{\sin^2 i}{{n_1}^2} + 2r^{\frac{2}{b}} - (r^{\frac{2}{b}})^2}}.$$

Letting

$$a \equiv \frac{\sin i}{n_1},$$

the above equation for $\Theta(i)$ becomes

$$\Theta(i) = a \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{-a^2 + 2r^{\frac{2}{b}} - (r^{\frac{2}{b}})^2}}. \tag{73}$$

We also have that

$$n(r_c(i)) = n_1 r_c^{\frac{1}{b}-1}(2 - r_c^{\frac{2}{b}})^{\frac{1}{2}}$$

and

$$r_c(i)n(r_c(i)) = n_1 r_c^{\frac{1}{b}}(2 - r_c^{\frac{2}{b}})^{\frac{1}{2}}.$$

In order to determine $r_c(i)$, we must solve the equation

$$n_1 r_c^{\frac{1}{b}}(2 - r_c^{\frac{2}{b}})^{\frac{1}{2}} = \sin i. \tag{74}$$

Then we obtain the result

$$r_c^{\frac{2}{b}}(2 - r_c^{\frac{2}{b}}) = \left(\frac{\sin i}{n_1}\right)^2 = a^2,$$

$$(r_c^{\frac{2}{b}})^2 - 2r_c^{\frac{2}{b}} + a^2 = 0,$$

$$r_c^{\frac{2}{b}} = \frac{2 \pm \sqrt{4 - 4a^2}}{2} = 1 \pm \sqrt{1 - a^2}.$$

As a consequence of the definition of $r_c(i)$ as a minimum value, we must have that

$$r_c^{\frac{2}{b}} = 1 - \sqrt{1 - a^2},$$

where we note that $a^2 < 1$ so that $r_c \in \mathbb{R}$. As a result of the definitions of a and n_1, we see that $a^2 < 1$ is always satisfied. Thus it follows that

$$r_c(i) = \left[1 - \sqrt{1 - a^2}\right]^{\frac{b}{2}}. \tag{75}$$

Substituting equation (75) into equation (73) yields the expression for $\Theta(i)$ as

$$\Theta(i) = a \int_{\left(1-\sqrt{1-a^2}\right)^{\frac{b}{2}}}^{1} \frac{dr}{r\sqrt{-a^2 + 2r^{\frac{2}{b}} - (r^{\frac{2}{b}})^2}}. \tag{76}$$

In equation (76), make the change of variables

$$u = r^{\frac{2}{b}}.$$

Then we see that

$$du = \frac{2}{b} r^{\frac{2}{b}-1} dr$$

and

$$\frac{dr}{r} = \frac{b}{2}(r^{\frac{2}{b}})^{-1} du = \frac{b}{2}\frac{du}{u}.$$

We note that

$$u(r_c(i)) = u\left\{ \left[1 - \sqrt{1-a^2}\right]^{\frac{b}{2}} \right\} = 1 - \sqrt{1-a^2}$$

and $u(1) = 1$. Using this and the above relationships, equation (76) becomes

$$\Theta(i) = \frac{ab}{2} \int_{1-\sqrt{1-a^2}}^{1} \frac{du}{u\sqrt{-a^2 + 2u - u^2}}. \tag{77}$$

Utilizing equation (A.5), where $r = u$, $A = -1$, $B = 2$, and $C = -a^2$, equation (77) can be evaluated as follows

$$\begin{aligned}
\Theta(i) &= \frac{ab}{2}\frac{1}{a} \arcsin\left[\frac{-2a^2 + 2u}{u\sqrt{4 - 4a^2}}\right]\Bigg|_{1-\sqrt{1-a^2}}^{1} \\
&= \frac{b}{2} \arcsin\left[\frac{u - a^2}{u\sqrt{1-a^2}}\right]\Bigg|_{1-\sqrt{1-a^2}}^{1} \\
&= \frac{b}{2}\left\{ \arcsin\left[\frac{1-a^2}{\sqrt{1-a^2}}\right] - \arcsin\left[\frac{1-\sqrt{1-a^2}-a^2}{(1-\sqrt{1-a^2})\sqrt{1-a^2}}\right]\right\} \\
&= \frac{b}{2}\left\{ \arcsin\left[\sqrt{1-a^2}\right] - \arcsin\left[\frac{1-a^2-\sqrt{1-a^2}}{\sqrt{1-a^2}-(1-a^2)}\right]\right\} \\
&= \frac{b}{2}\left\{ \arcsin\left[\sqrt{1-a^2}\right] - \arcsin(-1)\right\}
\end{aligned}$$

$$= \frac{b}{2}\left\{ \arcsin\left[\sqrt{1-a^2}\right] + \frac{\pi}{2}\right\}. \tag{78}$$

Let

$$\alpha = \arcsin\left[\sqrt{1-a^2}\right].$$

Then we have that

$$\cos\alpha = a,$$

where we have taken the positive root since $\alpha \in [0, \frac{\pi}{2}]$. Thus we find that

$$\alpha = \arcsin\left[\sqrt{1-a^2}\right] = \frac{\pi}{2} - \arcsin a. \tag{79}$$

Using equation (79), equation (78) becomes

$$\Theta(i) = \frac{b}{2}\left\{\frac{\pi}{2} - \arcsin a + \frac{\pi}{2}\right\} = \frac{b}{2}\left\{\pi - \arcsin a\right\}. \tag{80}$$

As a consequence, we obtain the result

$$D_0(i) = 2i - \pi + 2\Theta(i) = 2i - \pi + b\pi - b\arcsin\left(\frac{\sin i}{n_1}\right).$$

Therefore we determine that

$$D_0(i) = \pi(b-1) + 2i - b\arcsin\left(\frac{\sin i}{n_1}\right). \tag{81}$$

For a zero-order bow to exist for some critical angle of incidence $i_c \in \left[0.005, \frac{\pi}{2}\right]$, it is necessary and sufficient that

$$D_0'(i_c) = 0. \tag{82}$$

Differentiating equation (81) with respect to i and setting the result equal to zero yields that

$$2 = b\frac{\cos i_c}{\sqrt{n_1^2 - \sin^2 i_c}}.$$

Then we discover that

$$4 = b^2\frac{\cos^2 i_c}{n_1^2 - \sin^2 i_c},$$

$$4(n_1^2 - 1 + \cos^2 i_c) = b^2\cos^2 i_c,$$

$$4(n_1^2 - 1) = \cos^2 i_c(b^2 - 4),$$
$$\cos^2 i_c = \frac{4(n_1^2 - 1)}{b^2 - 4}.$$

Accordingly, we see that

$$\cos i_c = 2\left(\frac{n_1^2 - 1}{b^2 - 4}\right)^{\frac{1}{2}}. \quad (83)$$

If we restrict ourselves to the case $b > 0$, then we must have that

$$2\left(\frac{n_1^2 - 1}{b^2 - 4}\right)^{\frac{1}{2}} \leq 1,$$
$$\frac{n_1^2 - 1}{b^2 - 4} \leq \frac{1}{4},$$
$$4n_1^2 - 4 \leq b^2 - 4,$$
$$b^2 \geq 4n_1^2,$$
$$b \geq 2n_1 > 2. \quad (84)$$

where we drop the minus sign since $n_1 > 1$ and $b > 0$. Therefore, we have established that a zero-order bow can exist for $n(r)$ in equation (72) if $b \geq 2n_1$. We note that a zero-order bow cannot exist if $n_1 = 1$. If $n_1 = 1$, then $D_0'(i) = 2 - b$ which has a root at $b = 2$. However, the right hand side of equation (83) becomes singular at $b = 2$, and consequently, $D_0'(i) \neq 0$.

In Figures 11 and 12, we plot equations (72) and (81) with $n_1 = \frac{4}{3}$ and $b = 4$. In Figure 11, we notice the singularity at $r = 0$ since $b > 1$. We note that an extremum of $D_0(i)$ occurs at an angle of incidence of approximately 1.04 (or 59.39°).

III.3 PROFILE 2

Consider the inversely linear refractive index profile, previously considered by Gould and Burman [16] and Adam and Laven [17], which is given by

$$n(r) = \frac{1}{ar + b}, \quad (85)$$

where a and b are constants. The refractive index profile given by equation (85) is singular when $r = -\frac{b}{a}$. The inversely linear profile given by equation (85) has applications in atmospheric and terrestrial physics [16].

We note that

$$r^2 n^2(r) = \frac{r^2}{(ar + b)^2}.$$

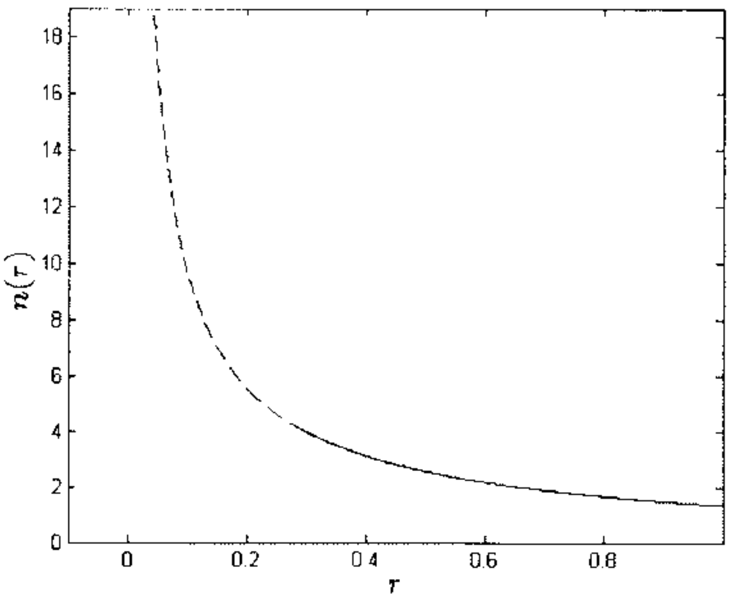

Fig. 11. Plot of equation (72) with $n_1 = \frac{4}{3}$ and $b = 4$.

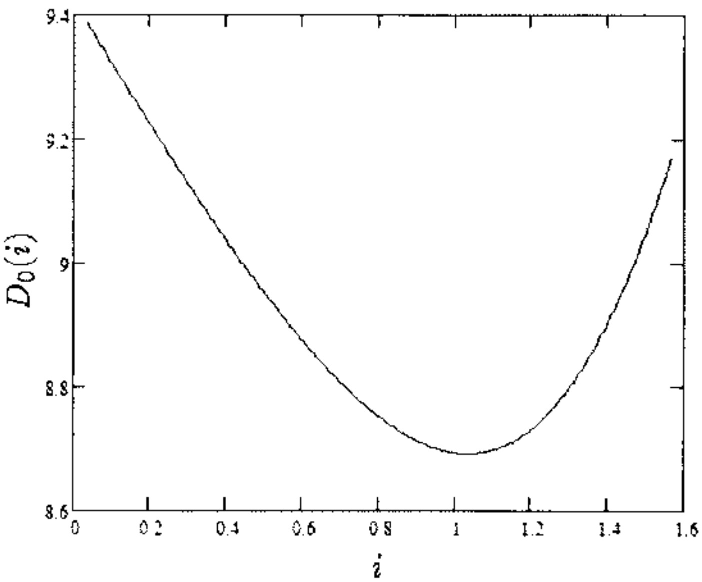

Fig. 12. Plot of equation (81) with $n_1 = \frac{4}{3}$ and $b = 4$.

Substituting this into equation (2) yields

$$\Theta(i) = \sin i \int_{r_c(i)}^{1} \frac{(ar+b)}{r\sqrt{r^2 - (ar+b)^2 \sin^2 i}} dr$$

$$= a \sin i \int_{r_c(i)}^{1} \frac{dr}{\sqrt{\left(1 - a^2 \sin^2 i\right) r^2 - 2abr \sin^2 i - b^2 \sin^2 i}}$$

$$+ b \sin i \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{\left(1 - a^2 \sin^2 i\right) r^2 - 2abr \sin^2 i - b^2 \sin^2 i}}.$$

Let $\alpha = a \sin i$ and $\beta = b \sin i$. Then we determine that

$$\Theta(i) = \alpha \int_{r_c(i)}^{1} \frac{dr}{\sqrt{(1-\alpha^2)\, r^2 - 2\alpha\beta r - \beta^2}} + \beta \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{(1-\alpha^2)\, r^2 - 2\alpha\beta r - \beta^2}}$$

$$\equiv \alpha \hat{I}_1 + \beta \hat{I}_2. \tag{86}$$

Next we calculate $r_c(i)$. Using equation (3), we want to solve the following equation for $r_c(i)$:

$$\frac{r_c(i)}{ar_c(i) + b} = \sin i.$$

This equation may be rearranged to obtain

$$r_c(i) = r_c(i) a \sin i + b \sin i$$

$$= \alpha r_c(i) + \beta.$$

Then we have the result

$$r_c(i) = \frac{\beta}{1-\alpha}. \tag{87}$$

We now turn our attention back to the integrals $\hat{I}_1$ and $\hat{I}_2$, appearing in equation (86). The integral for $\hat{I}_1$ is given as

$$\hat{I}_1 = \int\limits_{\frac{\beta}{1-\alpha}}^{1} \frac{dr}{\sqrt{(1-\alpha^2)\,r^2 - 2\alpha\beta r - \beta^2}}.$$

Utilizing equation (A.9), with $D = 1-\alpha^2, E = -2\alpha\beta$, and $F = -\beta^2$, we find that

$$\begin{aligned}
\hat{I}_1 &= \frac{1}{\sqrt{1-\alpha^2}} \log\left[2\sqrt{1-\alpha^2}\sqrt{(1-\alpha^2)\,r^2 - 2\alpha\beta r - \beta^2} + 2\left(1-\alpha^2\right)r - 2\alpha\beta\right]\Bigg|_{\frac{\beta}{1-\alpha}}^{1} \\
&= \frac{1}{\sqrt{1-\alpha^2}}\Bigg\{\log 2 + \log\left[\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-2\alpha\beta-\beta^2} + 1 - \alpha^2 - \alpha\beta\right] \\
&\qquad - \log 2 - \log\Bigg[\sqrt{1-\alpha^2}\sqrt{(1-\alpha^2)\,\frac{\beta^2}{(1-\alpha)^2} - \frac{2\alpha\beta^2}{1-\alpha} - \beta^2} \\
&\qquad\qquad + \left(1-\alpha^2\right)\frac{\beta}{1-\alpha} - \alpha\beta\Bigg]\Bigg\} \\
&= \frac{1}{\sqrt{1-\alpha^2}}\Bigg\{\log\left[\sqrt{1-\alpha^2}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)\right] \\
&\qquad - \log\left[\sqrt{1-\alpha^2}\sqrt{\frac{\beta^2(1+\alpha)}{1-\alpha} - \frac{2\alpha\beta^2}{1-\alpha} - \beta^2} + \beta(1+\alpha) - \alpha\beta\right]\Bigg\} \\
&= \frac{1}{\sqrt{1-\alpha^2}}\Bigg\{\log\left[\sqrt{1-\alpha^2}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)\right] \\
&\qquad - \log\left[\sqrt{1-\alpha^2}\sqrt{\frac{\beta^2+\alpha\beta^2-2\alpha\beta^2-\beta^2+\alpha\beta^2}{1-\alpha}} + \beta\right]\Bigg\} \\
&= \frac{1}{\sqrt{1-\alpha^2}}\Bigg\{\log\left[\sqrt{1-\alpha^2}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)\right] - \log\beta\Bigg\} \\
&= \frac{1}{\sqrt{1-\alpha^2}}\Bigg\{\log\left[\frac{\sqrt{1-\alpha^2}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)}{\beta}\right]\Bigg\}. \qquad (88)
\end{aligned}$$

Next we evaluate $\hat{I}_2$, where

$$\hat{I}_2 = \int_{\frac{\beta}{1-\alpha}}^{1} \frac{dr}{r\sqrt{(1-\alpha^2)\,r^2 - 2\alpha\beta r - \beta^2}}.$$

Using $A = 1-\alpha^2$, $B = -2\alpha\beta$, and $C = -\beta^2$ in equation (A.5) gives us

$$\begin{aligned}
\hat{I}_2 &= \frac{1}{\beta}\arcsin\left[\frac{-2\beta^2 - 2\alpha\beta r}{r\sqrt{4\alpha^2\beta^2 + 4\beta^2(1-\alpha^2)}}\right]\Bigg|_{\frac{\beta}{1-\alpha}}^{1} \\
&= \frac{1}{\beta}\arcsin\left[\frac{-2\beta^2 - 2\alpha\beta r}{2\beta r}\right]\Bigg|_{\frac{\beta}{1-\alpha}}^{1} \\
&= \frac{1}{\beta}\arcsin\left[-\frac{(\beta+\alpha r)}{r}\right]\Bigg|_{\frac{\beta}{1-\alpha}}^{1} \\
&= \frac{1}{\beta}\left\{\arcsin\left[-(\beta+\alpha)\right] - \arcsin\left[-\frac{(\beta+\frac{\alpha\beta}{1-\alpha})}{\frac{\beta}{1-\alpha}}\right]\right\} \\
&= \frac{1}{\beta}\left\{\arcsin\left[-(\beta+\alpha)\right] - \arcsin\left[-(1-\alpha+\alpha)\right]\right\} \\
&= \frac{1}{\beta}\left\{\arcsin\left[-(\beta+\alpha)\right] - \arcsin(-1)\right\} \\
&= \frac{1}{\beta}\left\{\arcsin\left[-(\beta+\alpha)\right] + \frac{\pi}{2}\right\} \\
&= \frac{1}{\beta}\left\{\frac{\pi}{2} - \arcsin(\alpha+\beta)\right\}.
\end{aligned} \tag{89}$$

Utilizing equations (88) and (89) in equation (86), we find that

$$\Theta(i) = \frac{\alpha}{\sqrt{1-\alpha^2}}\log\left[\frac{\sqrt{1-\alpha^2}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)}{\beta}\right] + \frac{\pi}{2} - \arcsin(\alpha+\beta). \tag{90}$$

We will rewrite the logarithmic expression in equation (90) in terms of the inverse cosine function. We first note that (where $j = \sqrt{-1}$)

$$\begin{aligned}
&\frac{\alpha}{\sqrt{1-\alpha^2}}\log\left[\frac{\sqrt{1-\alpha^2}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)}{\beta}\right] \\
&\qquad = -\frac{j\alpha}{\sqrt{\alpha^2-1}}\log\left[\frac{j\sqrt{\alpha^2-1}\sqrt{1-(\alpha+\beta)^2} + 1 - \alpha(\alpha+\beta)}{\beta}\right].
\end{aligned}$$

Let

$$z = \frac{1 - \alpha(\alpha + \beta)}{\beta}.$$

Then we see that

$$z^2 = \frac{1 - 2\alpha(\alpha + \beta) + \alpha^2(\alpha + \beta)^2}{\beta^2}$$

and

$$\begin{aligned} 1 - z^2 &= \frac{\beta^2 - 1 + 2\alpha(\alpha + \beta) - \alpha^2(\alpha + \beta)^2}{\beta^2} \\ &= \frac{(\alpha + \beta)^2 + \alpha^2 - 1 - \alpha^2(\alpha + \beta)^2}{\beta^2} \\ &= \frac{\alpha^2 - 1 - (\alpha + \beta)^2[\alpha^2 - 1]}{\beta^2} \\ &= \frac{[\alpha^2 - 1][1 - (\alpha + \beta)^2]}{\beta^2}. \end{aligned}$$

Thus, it is discovered that

$$\sqrt{1 - z^2} = \frac{\sqrt{\alpha^2 - 1}\sqrt{1 - (\alpha + \beta)^2}}{\beta}.$$

Using the formula

$$\arccos z = -j \log[z + j\sqrt{1 - z^2}] \tag{91}$$

and the relationships on this page, $\Theta(i)$ may be expressed as

$$\begin{aligned} \Theta(i) &= \frac{\alpha}{\sqrt{\alpha^2 - 1}} \arccos\left[\frac{1 - \alpha(\alpha + \beta)}{\beta}\right] + \frac{\pi}{2} - \arcsin(\alpha + \beta) \\ &= \frac{\alpha}{\sqrt{\alpha^2 - 1}} \arccos\left[\frac{1 - \alpha(\alpha + \beta)}{\beta}\right] + \arccos(\alpha + \beta) \\ &= \frac{a \sin i}{\sqrt{a^2 \sin^2 i - 1}} \arccos\left[\frac{1 - a(a + b)\sin^2 i}{b \sin i}\right] + \arccos[(a + b)\sin i]. \end{aligned} \tag{92}$$

where we utilized our definitions for α and β.

Using equation (92) in equation (1), it is obtained that

$$D_0(i) = 2i - \pi + 2\left\{\frac{a \sin i}{\sqrt{a^2 \sin^2 i - 1}} \arccos\left[\frac{1 - a(a + b)\sin^2 i}{b \sin i}\right] + \arccos[(a + b)\sin i]\right\}. \tag{93}$$

As can be seen from equation (93), the calculation of $D_0'(i)$ is quite difficult. Once the calculation of the derivative of equation (93) is made, the requirement for the existence of a zero-order bow results in a sixth-order linear equation that is not readily solvable algebraically. In order to avoid these highly involved calculations, we will only study the behavior of $D_0(i)$ from equation (93). Since we want $D_0(i) \in \mathbb{R}$, we require that

$$a^2 \sin^2 i > 1. \tag{94}$$

As a result of the domain of the inverse cosine function, we also require that

$$-1 \leq \frac{1 - a(a+b)\sin^2 i}{b \sin i} \leq 1, \tag{95a}$$

$$-1 \leq (a+b)\sin i \leq 1. \tag{95b}$$

We assume $i \geq 0.005$. The cosecant function is decreasing on the interval $[0.005, \frac{\pi}{2}]$, so that in order to avoid imaginary numbers, we must have that $|a| > \csc 0.005$.

In Figures 13 and 14, we plot equations (85) and (93) with $a = -205$ and $b = 204.5$. We have chosen these values of a and b in order to satisfy the constraints in equations (94), (95a), and (95b). In Figure 13, we notice the singularity at $r = -\frac{b}{a} \approx 0.9976$. In Figure 13, we have focused the view in the vicinity of the singularity of equation (85). We note that to the left of the view, as r approaches zero from the right, the refractive index approaches $\frac{1}{b}$. A zero-order bow exists at an angle of incidence of approximately 0.07 (or 3.78°).

III.4 PROFILE 3

Consider the refractive index profile (as mentioned in [21])

$$n(r) = \frac{a}{r \ln(br)}, \tag{96}$$

where a and b are constants. The refractive index profile in equation (96) is singular at $r = 0$ and $r = \frac{1}{b}$. This profile is undefined if $b \leq 0$. Equation (96) is plotted in Figure 15 with $a = \frac{1}{10}$ and $b = \frac{3}{2}$. A singularity is observed in the figure at $r = 0$ and $r = \frac{1}{b} = \frac{2}{3}$. The refractive index profile in equation (96) has applications in radio wave propagation [21].

We note that

$$r^2 n^2(r) = \frac{a^2}{(\ln(br))^2}.$$

Next we find $r_c(i)$. Using equation (3), it is found that

$$\ln(br_c) = \frac{a}{\sin i}.$$

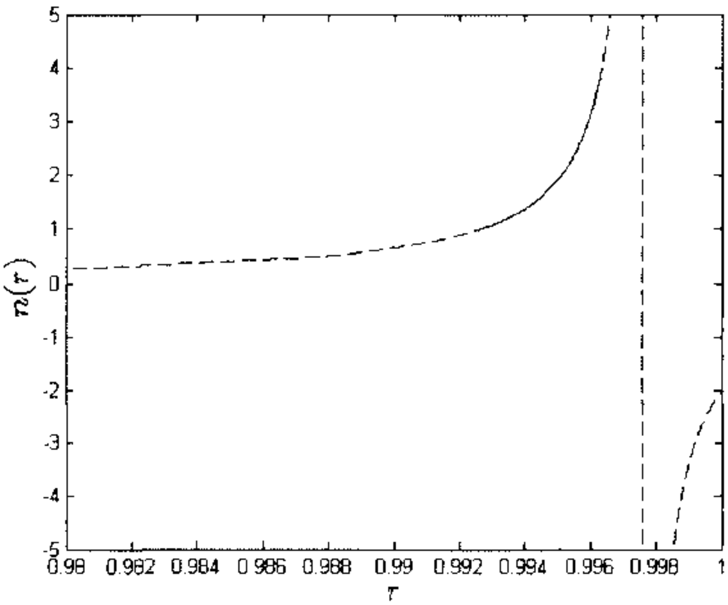

Fig. 13. Plot of equation (85) with $a = -205$ and $b = 204.5$.

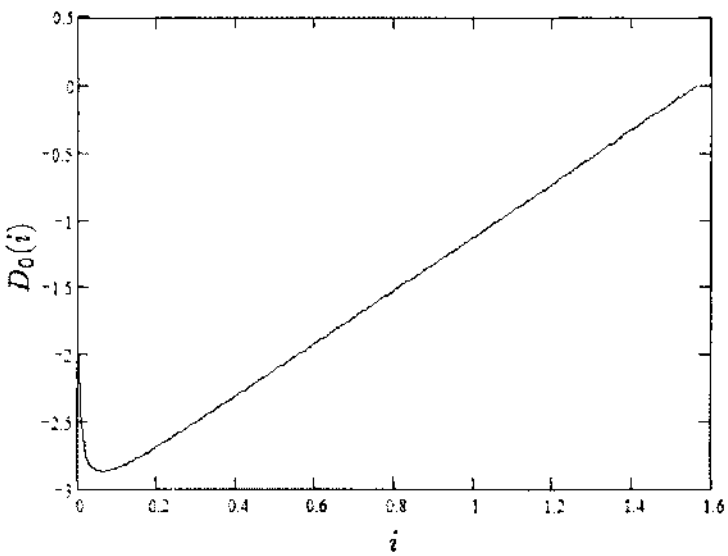

Fig. 14. Plot of equation (93) with $a = -205$ and $b = 204.5$.

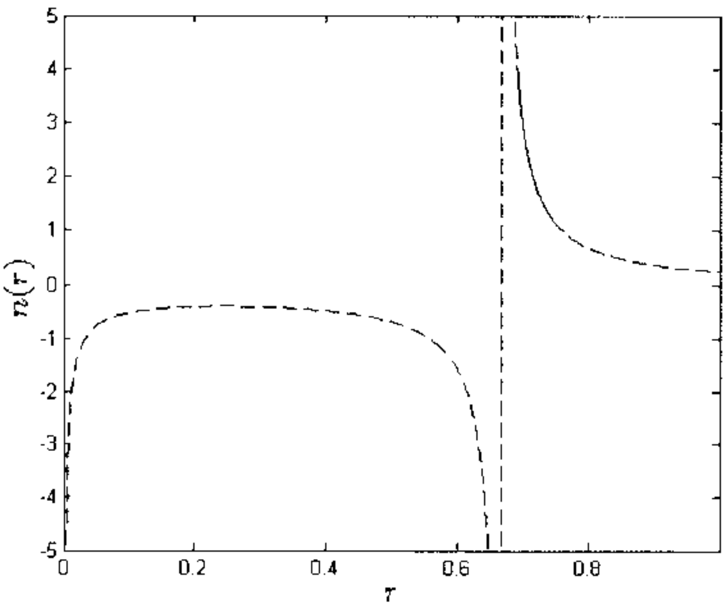

Fig. 15. Plot of equation (96) with $a = \frac{1}{10}$ and $b = \frac{3}{2}$.

Letting $m \equiv \frac{a}{\sin i}$, $r_c(i)$ is given by

$$r_c(i) = \frac{1}{b} e^m. \tag{97}$$

Thus, we obtain the result

$$\Theta(i) = \sin i \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2 n^2(r) - \sin^2 i}} = \sin i \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{\frac{a^2}{(\ln(br))^2} - \sin^2 i}}$$

$$= \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{\frac{a^2}{\sin^2 i (\ln(br))^2} - 1}} = \int_{r_c(i)}^{1} \frac{\ln(br)}{r\sqrt{m^2 - (\ln(br))^2}} dr$$

$$= \int_{\frac{1}{b}e^m}^{1} \frac{\ln(br)}{r\sqrt{m^2 - (\ln(br))^2}} dr. \tag{98}$$

Let $u = \ln(br)$. Then $du = \frac{dr}{r}$. We note that $u(r_c) = m$ and $u(1) = \ln b$. As a result, the integral for $\Theta(i)$ becomes

$$\Theta(i) = \int_{m}^{\ln b} \frac{u}{\sqrt{m^2 - u^2}} du. \tag{99}$$

Let $v = m^2 - u^2$. Then $dv = -2udu$ and $udu = -\frac{1}{2}dv$. We have that $v(m) = 0$ and $v(\ln b) = m^2 - (\ln b)^2$. Now equation (99) becomes

$$\begin{aligned}\Theta(i) &= \int_{0}^{m^2-(\ln b)^2} \left\{ -\frac{1}{2}v^{-\frac{1}{2}} \right\} dv \\ &= -v^{\frac{1}{2}} \Big|_{0}^{m^2-(\ln b)^2} \\ &= -\sqrt{m^2 - (\ln b)^2}.\end{aligned}$$

Hence, we determine that

$$\Theta(i) = -\sqrt{a^2 \csc^2 i - (\ln b)^2}, \tag{100}$$

and the derivative is given by

$$\Theta'(i) = \frac{a^2 \csc^2 i \cot i}{\sqrt{a^2 \csc^2 i - (\ln b)^2}}. \tag{101}$$

Therefore, for the refractive index profile in equation (96), a zero-order bow exists if

$$1 + \frac{a^2 \csc^2 i_c \cot i_c}{\sqrt{a^2 \csc^2 i_c - (\ln b)^2}} = 0. \tag{102}$$

Since i_c is in the first quadrant, and the trigonometric functions are all positive in the first quadrant, equation (102) cannot be satisfied. This tells us that a zero-order bow is not possible for the refractive index profile given by equation (96).

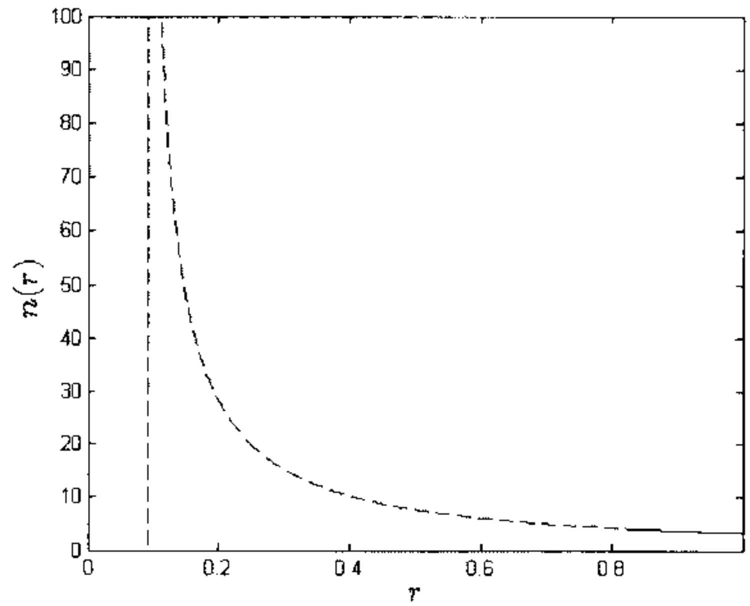

Fig. 16. Plot of equation (103) with $a = 5$ and $b = 11$.

III.5 PROFILE 4

Consider the refractive index profile (which was considered in [21])

$$n(r) = \frac{a}{r\sqrt{\ln(br)}}, \tag{103}$$

where a and b are constants. The refractive index profile in equation (103) is singular at $r = 0$, $r = \frac{1}{b}$, and undefined for $b \leq 0$, as was the case for the previous profile in equation (96). In addition, the refractive index profile in equation (103) is a purely imaginary number in the domain $(0, \frac{1}{b})$. In Figure 16, equation (103) is plotted with $a = 5$ and $b = 11$; the singularity at $r = \frac{1}{b} = \frac{1}{11}$ is clearly visible. The refractive index profile in equation (103) has applications in radio wave propagation [21].

We note that

$$r^2 n^2(r) = \frac{a^2}{\ln(br)}.$$

Next we find $r_c(i)$. Upon using equation (3), it is found that

$$r_c(i) = \frac{1}{b}e^{m^2}. \tag{104}$$

Hence, we determine that

$$\Theta(i) = \sin i \int\limits_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2n^2(r) - \sin^2 i}} = \sin i \int\limits_{r_c(i)}^{1} \frac{dr}{r\sqrt{\frac{a^2}{\ln(br)} - \sin^2 i}}$$

$$= \int\limits_{r_c(i)}^{1} \frac{dr}{r\sqrt{\frac{m^2}{\ln(br)} - 1}} = \int\limits_{\frac{1}{b}e^{m^2}}^{1} \frac{\ln(br)}{r\sqrt{m^2\ln(br) - (\ln(br))^2}}dr. \tag{105}$$

In order to evaluate the integral in (105), let $x = \ln(br)$. Then $dx = \frac{dr}{r}$. We have that $x(r_c) = m^2$ and $x(1) = \ln b$. Consequently, equation (105) becomes

$$\begin{aligned}
\Theta(i) &= \int\limits_{m^2}^{\ln b} \frac{x}{\sqrt{m^2x - x^2}}dx \\
&= \left[-\sqrt{m^2x - x^2} + \frac{m^2}{2}\arccos\left(\frac{\frac{m^2}{2} - x}{\frac{m^2}{2}}\right)\right]\Bigg|_{m^2}^{\ln b} \\
&= -\sqrt{m^2\ln b - (\ln b)^2} + \frac{m^2}{2}\arccos\left(\frac{\frac{m^2}{2} - \ln b}{\frac{m^2}{2}}\right) \\
&\qquad - \frac{m^2}{2}\arccos\left(\frac{\frac{m^2}{2} - m^2}{\frac{m^2}{2}}\right) \\
&= -\sqrt{m^2\ln b - (\ln b)^2} + \frac{m^2}{2}\arccos\left(\frac{\frac{m^2}{2} - \ln b}{\frac{m^2}{2}}\right) \\
&\qquad - \frac{m^2}{2}\arccos(-1) \\
&= -\sqrt{m^2\ln b - (\ln b)^2} + \frac{m^2}{2}\left[\arccos\left(1 - \frac{2\ln b}{m^2}\right) - \pi\right].
\end{aligned} \tag{106}$$

The formula

$$\arccos u = \pi - \arccos(-u)$$

can be rearranged so that

$$\arccos u - \pi = -\arccos(-u).$$

The last term appearing in brackets in equation (106) may therefore be written as

$$\arccos\left(1 - \frac{2\ln b}{m^2}\right) - \pi = -\arccos\left(\frac{2\ln b}{m^2} - 1\right).$$

Thus, we discover that

$$\begin{aligned}\Theta(i) &= \int_{m^2}^{\ln b} \frac{x}{\sqrt{m^2 x - x^2}} dx \\ &= -\left[\sqrt{m^2 \ln b - (\ln b)^2} + \frac{m^2}{2} \arccos\left(\frac{2\ln b}{m^2} - 1\right)\right],\end{aligned}$$

so that the final equation for $\Theta(i)$ is given by

$$\begin{aligned}\Theta(i) &= -\left[\sqrt{a^2 \csc^2 i \ln b - (\ln b)^2} + \frac{a^2}{2}\csc^2 i \arccos\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)\right] \\ &= -\left[\sqrt{\ln b}\sqrt{a^2\csc^2 i - \ln b} + \frac{a^2}{2}\csc^2 i \arccos\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)\right]. \qquad (107)\end{aligned}$$

Differentiation of (107) yields

$$\begin{aligned}\Theta'(i) &= -\Bigg[\sqrt{\ln b}\left(a^2\csc^2 i - \ln b\right)^{-\frac{1}{2}}\left(-a^2\csc^2 i \cot i\right) \\ &\quad + \frac{a^2}{2}\Bigg\{\csc^2 i\left(-\frac{1}{\sqrt{1-\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)^2}}\right)\left(\frac{4\ln b}{a^2}\sin i\cos i\right) \\ &\quad - 2\csc^2 i\cot i \arccos\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)\Bigg\}\Bigg] \\ &= \frac{a^2\csc^2 i\cot i\sqrt{\ln b}}{\sqrt{a^2\csc^2 i - \ln b}} + \frac{2\csc i\cos i\ln b}{\sqrt{1-\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)^2}} \\ &\quad + a^2\csc^2 i\cot i\arccos\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right). \qquad (108)\end{aligned}$$

We note that

$$\begin{aligned}1 - \left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)^2 &= 1 - \left(\frac{4(\ln b)^2}{a^4}\sin^4 i - \frac{4\ln b}{a^2}\sin^2 i + 1\right) \\ &= -\frac{4(\ln b)^2}{a^4}\sin^4 i + \frac{4\ln b}{a^2}\sin^2 i \\ &= \frac{4\ln b}{a^2}\sin^2 i\left[1 - \frac{\ln b\sin^2 i}{a^2}\right] \\ &= \frac{4\ln b}{a^4}\sin^4 i\left[a^2\csc^2 i - \ln b\right].\end{aligned}$$

Then it is found that

$$\Theta'(i) = \frac{a^2\csc^2 i\cot i\sqrt{\ln b}}{\sqrt{a^2\csc^2 i - \ln b}} + \frac{a^2\csc^2 i\cot i\sqrt{\ln b}}{\sqrt{a^2\csc^2 i - \ln b}}$$

$$+ a^2 \csc^2 i \cot i \arccos\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)$$

$$= a^2 \csc^2 i \cot i \left[\frac{2\sqrt{\ln b}}{\sqrt{a^2\csc^2 i - \ln b}} + \arccos\left(\frac{2\ln b}{a^2}\sin^2 i - 1\right)\right]. \tag{109}$$

Therefore, for the refractive index profile in equation (103), a zero-order bow exists if

$$1 + a^2 \csc^2 i_c \cot i_c \left[\frac{2\sqrt{\ln b}}{\sqrt{a^2\csc^2 i_c - \ln b}} + \arccos\left(\frac{2\ln b}{a^2}\sin^2 i_c - 1\right)\right] = 0. \tag{110}$$

The second term in equation (110) is positive due to the definition of i_c. This tells us that in order for equation (110) to be satisfied, the last term must be negative. In other words, we must have that the inverse cosine function is negative. However, for the set of real numbers, the inverse cosine function is always positive. Therefore, a zero-order bow cannot exist for the profile given by equation (103).

III.6 PROFILE 5

Consider the refractive index profile (as previously discussed in [16])

$$n(r) = \sqrt{\frac{a}{r^2} + \frac{b}{r} + c}, \tag{111}$$

where a, b, and c are constants. The refractive index profile in equation (111) is singular at $r = 0$. The refractive index profile given by equation (111) has applications in atmospheric and terrestrial physics [16]. Let $K = \sin i$. Determining $r_c(i)$ from the equation $r_c n(r_c) = K$ yields that

$$a + br_c + cr_c^2 = K^2,$$
$$cr_c^2 + br_c + a - K^2 = 0.$$

Accordingly, we find that

$$r_c(i) = \frac{-b \pm \sqrt{b^2 - 4c(a - K^2)}}{2c}. \tag{112}$$

We must have that $0 \leq r_c(i) < 1$ and $r_c(i) \in \mathbb{R}$. In order to guarantee that the radial point of closest approach to the center of the sphere, $r_c(i)$, is a real quantity, it is required that

$$b^2 > 4c(a - K^2).$$

There are numerous cases for this profile where we can study the behavior of the constants that would determine whether we choose the positive or negative sign in equation (112). Since we require that $n(r) \in \mathbb{R}$, we will not consider the case where all three constants are negative. Regardless of whether $c > 0$ or $c < 0$, the first condition above would result in the following inequality

$$a \leq K^2 < a + b + c.$$

In other words, it is required that $a + b + c > 1$. From the fact that $0 < K^2 \leq 1$, it is found that

$$a > a - K^2 \geq a - 1.$$

Since $a - K^2 < 0$, (we will see from the equation for $\Theta(i)$ why we cannot have $K^2 = a$) this inequality is satisfied only if $a < 1$. Using $a - K^2 \geq a - 1$ in the condition $b^2 > 4c(a - K^2)$ tells us that

$$b^2 > 4c(a - 1).$$

Without loss of generality, we assume that $a < 0, b > 0$, and $c > 0$. As a result, we must take the positive root in equation (112). We note that a and b must be small and large enough, respectively, so that equation (111) is real for all values of r in our domain. Thus we obtain the result

$$\Theta(i) = K \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{r^2 n^2(r) - K^2}} = K \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{(a - K^2) + br + cr^2}}. \tag{113}$$

Utilizing equation (A.5) with $A = c$, $B = b$, and $C = a - K^2$, the integral for $\Theta(i)$ may be evaluated as

$$\begin{aligned}\Theta(i) &= \frac{K}{\sqrt{K^2 - a}} \arcsin\left[\frac{2(a - K^2) + br}{r\sqrt{b^2 - 4c(a - K^2)}}\right]\Bigg|_{r_c(i)}^{1} \\ &= \frac{K}{\sqrt{K^2 - a}}\left\{\arcsin\left[\frac{2(a - K^2) + b}{\sqrt{b^2 - 4c(a - K^2)}}\right] - \arcsin\left[\frac{2(a - K^2) + br_c}{r_c\sqrt{b^2 - 4c(a - K^2)}}\right]\right\}.\end{aligned} \tag{114}$$

We note that

$$\frac{2(a - K^2) + br_c}{r_c\sqrt{b^2 - 4c(a - K^2)}} = \frac{2(a - K^2) + \left(\frac{-b^2 + b\sqrt{b^2 - 4c(a - K^2)}}{2c}\right)}{\left(\frac{-b + \sqrt{b^2 - 4c(a - K^2)}}{2c}\right)\sqrt{b^2 - 4c(a - K^2)}}$$

$$= \frac{4c(a-K^2) + [-b^2 + b\sqrt{b^2 - 4c(a-K^2)}]}{-b\sqrt{b^2 - 4c(a-K^2)} + b^2 - 4c(a-K^2)}$$
$$= -\left[\frac{b^2 - 4c(a-K^2) - b\sqrt{b^2 - 4c(a-K^2)}}{b^2 - 4c(a-K^2) - b\sqrt{b^2 - 4c(a-K^2)}}\right] = -1.$$

Hence, it follows that

$$\Theta(i) = \frac{K}{\sqrt{K^2 - a}}\left\{\arcsin\left[\frac{2(a-K^2)+b}{\sqrt{b^2 - 4c(a-K^2)}}\right] + \frac{\pi}{2}\right\}. \tag{115}$$

From the formula,

$$\arcsin z + \arccos z = \frac{\pi}{2},$$

it is determined that

$$\arcsin z + \frac{\pi}{2} + \arccos z = \pi.$$

Thus we see that

$$\arcsin z + \frac{\pi}{2} = \pi - \arccos z.$$

Using the relation

$$\arccos z = \pi - \arccos(-z),$$

it is found that

$$\arcsin z + \frac{\pi}{2} = \arccos(-z).$$

Using this result in equation (115) yields

$$\Theta(i) = \frac{K}{\sqrt{K^2 - a}}\arccos\left[-\frac{2(a-K^2)+b}{\sqrt{b^2 - 4c(a-K^2)}}\right]. \tag{116}$$

Because of this, we determine that

$$D_0(i) = 2i - \pi + 2\left\{\frac{K}{\sqrt{K^2 - a}}\arccos\left[-\frac{2(a-K^2)+b}{\sqrt{b^2 - 4c(a-K^2)}}\right]\right\}. \tag{117}$$

Due to the complicated nature of $D_0(i)$, we will only provide the graph of $D_0(i)$ from equation (117).

In Figures 17 and 18, we give the plots of equations (111) and (117), respectively, with $a = -\frac{1}{1000}, b = 2$, and $c = 5$. We notice from Figure 17 that equation (111) is singular at $r = 0$. For the values of a, b, and c given above, $D_0(i)$ has two extrema on our interval $[0.005, \frac{\pi}{2}]$ at angles of incidence of approximately 0.13 and 0.62 (or 7.17° and 35.73°, respectively). As a result, two zero-order bows occur on our interval for the refractive index profile given by equation (111).

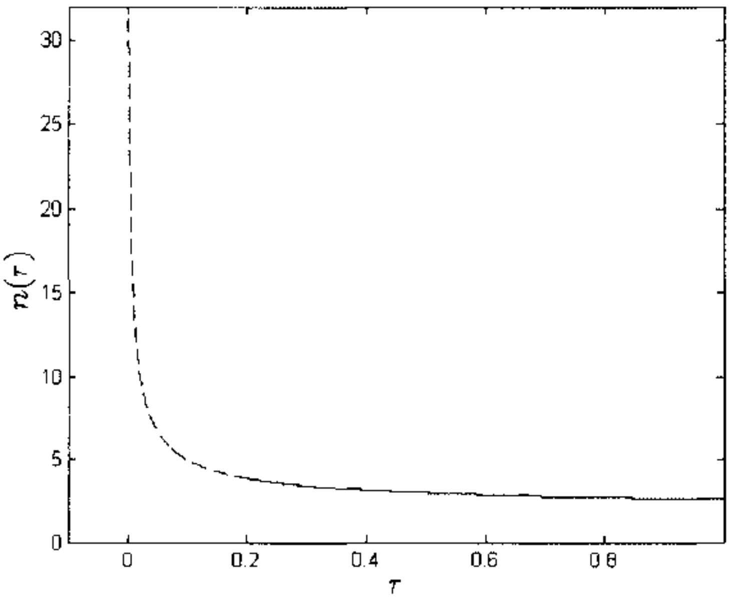

Fig. 17. Plot of equation (111) with $a = -\frac{1}{1000}$, $b = 2$, and $c = 5$.

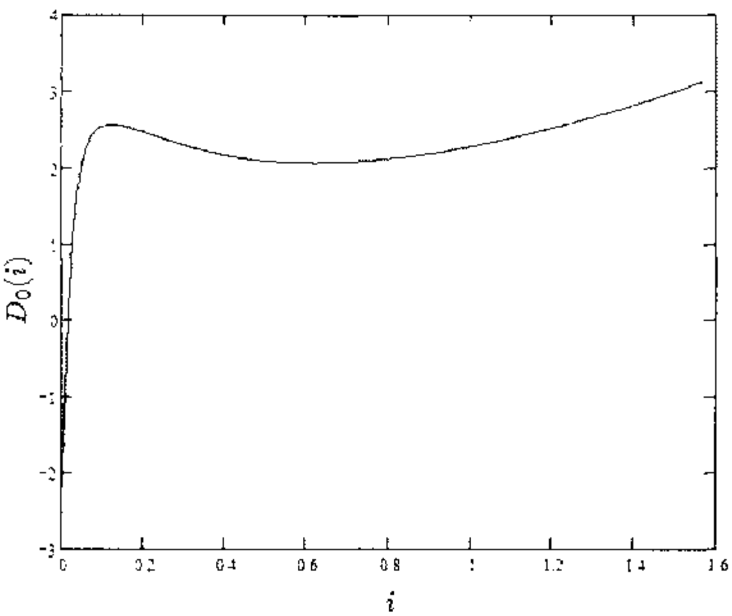

Fig. 18. Plot of equation (117) with $a = -\frac{1}{1000}$, $b = 2$, and $c = 5$.

III.7 PROFILE 6

Consider the refractive index profile, mentioned in [10] and [11], which are given by

$$n(r) = \alpha r^{\eta}, \tag{118}$$

where η is of either sign, $0 < b \leq r \leq a$, and $\alpha > 0$. The profile in (118) has a singularity at $r = 0$ if $\eta < 0$. In order for a zero-order bow to exist, it will be shown that $\eta < 0$. The refractive index profile in equation (118) has been used to show that melting ice crystals may be strong contributors to the glory ray. Also, this refractive index profile was studied to provide a more general explanation of both the rainbow ray and the glory ray phenomena by analyzing the scattering processes of inhomogeneous particles [11]. Using equation (118) in equation (2) where $K = \sin i$, we find that

$$\Theta(i) = K \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{\alpha^2 r^{2(\eta+1)} - K^2}} = \int_{r_c(i)}^{1} \frac{dr}{r\sqrt{Ar^p - 1}}, \tag{119}$$

where $p = 2(\eta + 1)$ and $A = \frac{\alpha^2}{K^2} > 0$. Next we calculate $r_c(i)$. The equation $r_c n(r_c) = K$ implies that

$$\alpha r_c^{\eta+1} = K.$$

Consequently, we determine that

$$r_c(i) = \left(\frac{K}{\alpha}\right)^{\frac{2}{p}}.$$

In order to evaluate the integral in equation (119), make the change of variables

$$v^2 = Ar^p.$$

Thus, it is obtained that

$$\frac{dr}{r} = \frac{2dv}{pv}.$$

We note that

$$v(r_c(i)) = \sqrt{A}(r_c(i))^{\frac{p}{2}} = \sqrt{A}\frac{K}{\alpha} = 1,$$

with $v(1) = \sqrt{A}$. Using these substitutions, equation (119) becomes

$$\Theta(i) = \frac{2}{p} \int_{1}^{\sqrt{A}} \frac{dv}{v\sqrt{v^2 - 1}} = \frac{2}{p} \operatorname{arcsec}(v)\Big|_1^{\sqrt{A}} = \frac{1}{\eta + 1} \operatorname{arcsec}\left(\frac{\alpha}{K} r^{\eta+1}\right)\Bigg|_{r_c(i)}^{1}, \tag{120}$$

where we note that $\eta \neq -1$. Therefore, it is discovered that

$$\Theta(i) = \frac{1}{\eta+1}\left[\operatorname{arcsec}\left(\frac{\alpha}{K}\right) - \operatorname{arcsec}\left(\frac{\alpha}{K} r_c^{\eta+1}\right)\right] = \frac{1}{\eta+1}\left[\operatorname{arcsec}\left(\frac{\alpha}{K}\right) - \operatorname{arcsec}(1)\right]$$
$$= \frac{1}{\eta+1}\operatorname{arcsec}\left(\frac{\alpha}{K}\right). \tag{121}$$

If we have that

$$n(r) \equiv n_b, \qquad 0 \leq r < b;$$
$$= \alpha r^\eta, \qquad b \leq r \leq a,$$

by continuity at $r = b$,

$$\alpha = n_b b^{-\eta}.$$

As a result, it is found that

$$\Theta(i) = \frac{1}{\eta+1}\operatorname{arcsec}\left(\frac{n_b}{K b^\eta}\right). \tag{122}$$

Therefore, we determine that

$$D_0(i) = 2i - \pi + \frac{2}{\eta+1}\operatorname{arcsec}\left(\frac{n_b}{K b^\eta}\right). \tag{123}$$

Differentiating equation (121) with respect to i, we see that

$$\Theta'(i) = \frac{1}{\eta+1}\frac{1}{\frac{\alpha}{K}\sqrt{(\frac{\alpha}{K})^2 - 1}}\left[-\alpha \cot i \csc i\right] = -\frac{1}{\eta+1}\frac{\cos i}{\sqrt{\alpha^2 - K^2}}.$$

A zero-order bow exists if $\Theta'(i) = -1$. Hence, for the refractive index profile in equation (118), a zero-order bow exists if

$$\frac{\cos i_c}{\sqrt{\alpha^2 - \sin^2 i_c}} = (\eta + 1),$$
$$\cos^2 i_c = (\eta+1)^2(\alpha^2 - \sin^2 i_c),$$
$$1 - \sin^2 i_c = (\eta+1)^2\alpha^2 - (\eta+1)^2 \sin^2 i_c,$$
$$1 - (\eta+1)^2\alpha^2 = \sin^2 i_c(1 - (\eta+1)^2),$$
$$\sin^2 i_c = \frac{1 - (\eta+1)^2\alpha^2}{1 - (\eta+1)^2},$$
$$\sin i_c = \sqrt{\frac{1 - (\eta+1)^2\alpha^2}{1 - (\eta+1)^2}}. \tag{124}$$

where $\eta \neq 0$. From the calculation of $\Theta'(i)$, we require that $\alpha^2 - K^2 > 0$ so that $\Theta'(i) \in \mathbb{R}$. Thus $\alpha^2 > K^2$ which implies that $\alpha^2 > 1$. Because of this, we must have that $\alpha > 1$ or $\alpha < -1$. However, due to the equation of $r_c(i)$ and the assumption that $\alpha > 0$, we must have that $\alpha > 0$. From the first line of equation (124), since $i_c \in [0.005, \frac{\pi}{2}]$, we must have that $\eta + 1 > 0$. So $\eta > -1$. There are now two cases to consider:

Case 1: $\quad -1 < \eta < 0$.

Then $(\eta + 1)^2 < 1$ and so $1 - (\eta + 1)^2 > 0$. From equation (124), we must have that $1 - (\eta + 1)^2\alpha^2 > 0$. So we have that $\alpha^2 < \frac{1}{(\eta+1)^2}$. Hence $-\frac{1}{\eta+1} < \alpha < \frac{1}{\eta+1}$. Also from equation (124), we require that

$$\frac{1 - (\eta + 1)^2\alpha^2}{1 - (\eta + 1)^2} \leq 1. \tag{125}$$

Since $1 - (\eta + 1)^2 > 0$, equation (125) tells us that $1 - (\eta + 1)^2\alpha^2 \leq 1 - (\eta + 1)^2$. As a result, $(\eta + 1)^2 - (\eta + 1)^2\alpha^2 \leq 0$. Thus $(\eta + 1)^2[1 - \alpha^2] \leq 0$. Hence $\alpha^2 > 1$, which results in either $\alpha > 1$ or $\alpha < -1$. We already know that we cannot have $\alpha < -1$. As a result, only $\alpha > 1$ is allowed. Summarizing, if $-1 < \eta < 0$, we must have that

$$-\frac{1}{\eta + 1} < \alpha < \frac{1}{\eta + 1}, \tag{126}$$

$$\alpha > 1. \tag{127}$$

Case 2: $\quad \eta > 0$.

Then $(\eta + 1)^2 > 1$ and so $1 - (\eta + 1)^2 < 0$. Utilizing the restriction in equation (125), we must have that $1 - (\eta + 1)^2\alpha^2 \geq 1 - (\eta + 1)^2$. Hence we require that $(\eta + 1)^2[1 - \alpha^2] \geq 0$. Thus

$$\alpha^2 \leq 1,$$

which results in the inequality

$$-1 \leq \alpha \leq 1. \tag{128}$$

However, in order to avoid imaginary numbers in the derivative of $\Theta(i)$ and the calculation of $r_c(i)$, we require that $\alpha > 1$. Since equation (128) contradicts this requirement, a zero-order bow cannot exist when $\eta > 0$. As a result, a zero-order bow is possible only for Case 1 where we have that $-1 < \eta < 0$.

We give the plots of equations (118) and (123) in Figures 19 and 20, respectively, with $\alpha = \frac{5}{4}$ and $\eta = -\frac{1}{3}$. The singularity at $r = 0$ of equation (118) in Figure 19 is clearly visible. For the given values of α and η, a zero-order bow exists at an angle of incidence of approximately 0.84 (or 47.87°).

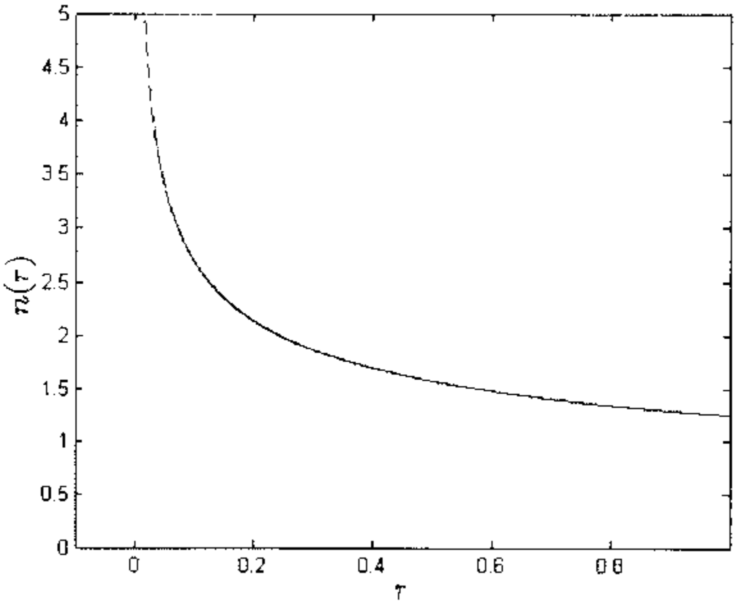

Fig. 19. Plot of equation (118) with $\alpha = \frac{5}{4}$ and $\eta = -\frac{1}{3}$.

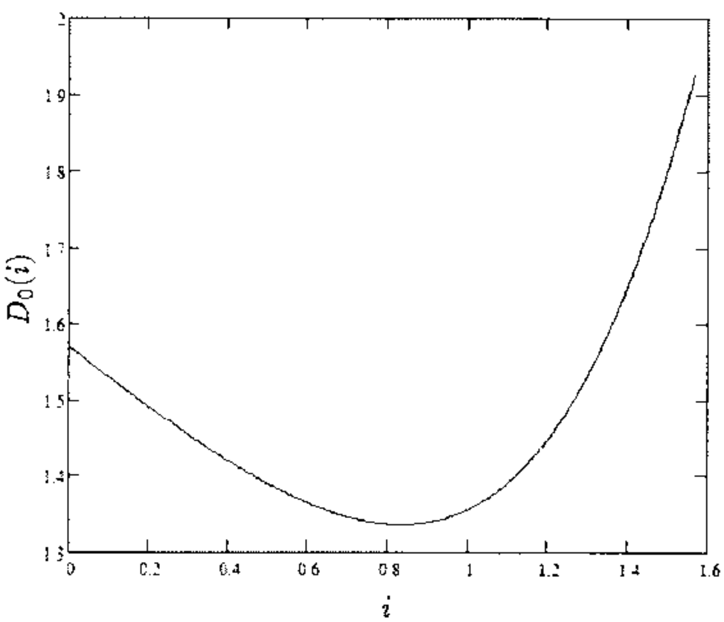

Fig. 20. Plot of equation (123) with $\alpha = \frac{5}{4}$ and $\eta = -\frac{1}{3}$.

CHAPTER IV

WAVE APPROACH

IV.1 INTRODUCTION

In this chapter, the wave approach to electromagnetic scattering will be considered. In the previous two chapters, we considered the ray approach where light is described using ray considerations. With regard to the wave approach, light is described by assuming that it takes on a wave form. Our main goal in this chapter will be to determine the leading order estimate of the far backscattered field at short wavelengths for an electromagnetic wave for a particular refractive index profile. The far backscattered field at short wavelengths is also known as the high-frequency backscattered field. The far backscattered field is given by an infinite series which converges slowly at short wavelengths. The Watson transformation will be employed to speed up the convergence of this series by converting the series to a contour integral. Once this is done, the radial eigenfunctions will be derived for fields of magnetic- and electric-type. These eigenfunctions are necessary in order to calculate the asymptotic expansions for the transverse electric (TE) and transverse magnetic (TM) modes. Once these expansions are obtained, the Mie solutions [6], [7] will be calculated which will allow for the determination of the high-frequency backscattered field. Two cases will be considered where in one case the leading order estimate of the high-frequency backscattered field may be determined.

Consider an incident plane electromagnetic wave propagating in the positive z direction with the free space wavenumber k, whose electric vector

$$\mathbf{E}^{inc} = \hat{e}e^{ikz} \tag{129}$$

has unit amplitude and is polarized in the direction of the constant unit vector $\hat{e}$. We note that $k = \frac{2\pi}{\lambda}$, where λ is the wavelength.

The incident plane wave given by equation (129) produces the far backscattered field (which corresponds to a linear combination of outgoing spherical waves) [22]

$$\mathbf{E}^{b.s.} = \hat{e}\frac{e^{ikr}}{ikr}\sum_{n=1}^{\infty}(-1)^n\left(n+\frac{1}{2}\right)(a_n - b_n), \tag{130}$$

where [22]

$$a_n = -\frac{\psi_n'(k\hat{a}) - M_n\psi_n(k\hat{a})}{\zeta_n'(k\hat{a}) - M_n\zeta_n(k\hat{a})}, \tag{131}$$

$$b_n = -\frac{\psi_n'(k\hat{a}) - \tilde{M}_n\psi_n(k\hat{a})}{\zeta_n'(k\hat{a}) - \tilde{M}_n\zeta_n(k\hat{a})}, \tag{132}$$

$$\psi_n(k\hat{a}) = \sqrt{\frac{\pi k\hat{a}}{2}}J_{n+\frac{1}{2}}(k\hat{a}), \qquad \zeta_n(k\hat{a}) = \sqrt{\frac{\pi k\hat{a}}{2}}H^{(1)}_{n-\frac{1}{2}}(k\hat{a}), \tag{133}$$

$$M_n = \frac{1}{k\hat{a}}\left[\frac{S_n^{(1)'}(x)}{S_n^{(1)}(x)}\right]\Bigg|_{x=1}, \qquad \tilde{M}_n = \frac{1}{k\hat{a}}\left[\frac{T_n^{(1)'}(x)}{T_n^{(1)}(x)}\right]\Bigg|_{x=1}, \tag{134}$$

$i = \sqrt{-1}$, and the prime indicates differentiation with respect to the argument. The functions $\psi_n(k\hat{a})$ and $\zeta_n(k\hat{a})$ are the Riccati-Bessel functions. It should be noted that the n that appears in equations (130)-(134) is not the refractive index profile. In this chapter, n will represent the separation constant. The refractive index profile in this chapter will be denoted by the function $R(x)$. The functions M_n and $\tilde{M}_n$ are known as the transverse electric (TE) and transverse magnetic (TM) modes, respectively. With respect to the TE modes, there is no electric field in the direction of wave propagation. In other words, there is only a magnetic field in the direction of wave propagation. As a result, the functions $S_n(x)$ appearing in the first equation of equation (134) are known as the radial eigenfunctions for fields of magnetic-type. On the other hand, with regard to TM modes, there is no magnetic field in the direction of wave propagation. This means that only an electric field is present in the direction of wave propagation. Because of this, the functions $T_n(x)$ appearing in the second equation of equation (134) are known as the radial eigenfunctions for fields of electric-type. In order for the leading order estimate of the far backscattered field to be calculated for short wavelengths, the Mie solutions which are given by equations (131) and (132) must be determined. Before this can be accomplished, the asymptotic expansions for the TE and TM modes must be calculated using equation (134) which requires the determination of the radial eigenfunctions for fields of magnetic- and electric-type, respectively. We note that $x = \frac{r}{\hat{a}}$ is the radial distance from the center $r = 0$ of the sphere, normalized to the radius $r = \hat{a}$ of the boundary of the sphere. The radial eigenfunctions $S_n^{(1)}(x)$ and $T_n^{(1)}(x)$ are those particular solutions of the radial differential equations

$$S_n''(x) + \left\{[k\hat{a}R(x)]^2 - \frac{n(n+1)}{x^2}\right\}S_n(x) = 0 \tag{135}$$

and

$$T_n''(x) - 2\frac{R'(x)}{R(x)}T_n'(x) + \left\{[k\hat{a}R(x)]^2 - \frac{n(n+1)}{x^2}\right\}T_n(x) = 0. \qquad (136)$$

where $R(x)$ is the refractive index profile of the sphere. We note that the radial eigenfunctions are finite over the interval $0 \leq x \leq 1$ for the refractive index profiles we will consider.

IV.2 PROFILE 1

Consider the wavenumber (from [21])

$$m(r) = \frac{a_0 r^{\frac{\alpha}{2}-1}}{1 + b_0 r^{\alpha}}, \qquad (137)$$

where

$$m(r) = k\tilde{n}(r),$$

k is the free space wavenumber, and $\tilde{n}(r)$ is the refractive index in terms of r. Let

$$x = \frac{r}{\hat{a}} \qquad (138)$$

and

$$b_0 = \hat{a}^{-\alpha} \qquad (139)$$

so that we can write equation (137) in terms of x:

$$m(x) = \frac{a_0[x\hat{a}]^{\frac{\alpha}{2}-1}}{1 + x^{\alpha}} = \frac{a_0\hat{a}^{\frac{\alpha}{2}-1}x^{\frac{\alpha}{2}-1}}{1 + x^{\alpha}}.$$

The differential equations from [21] and [22] must coincide in order to connect the analysis from these two papers. In order for the differential equations from Westcott [21] (which are in terms of r) and Uslenghi and Weston [22] (which are in terms of x) to coincide, we must have that

$$a_0 = c_0 k\hat{a}^{1-\frac{\alpha}{2}}, \qquad (140)$$

where c_0 is a constant. We note that in [22] $c_0 = 2$. Using equation (140), it is determined that

$$kR(x) = \frac{c_0 k x^{\frac{\alpha}{2}-1}}{1 + x^{\alpha}} = m(x), \qquad (141)$$

where it is noted that in [22]

$$R(x) = \frac{c_0 x^{\frac{\alpha}{2}-1}}{1 + x^{\alpha}} \qquad \text{and} \qquad \alpha = \frac{2}{d_0},$$

where $c_0 = 2$ in [22] and d_0 is any positive real constant. In Appendix D, we show how to relate the differential equations from [21], which are in terms of r, to the ones in equations (135) and (136), which are in terms of x. Consider the refractive index profile given by

$$R(x) = \frac{c_0 x^{\frac{\alpha}{2}-1}}{1+x^{\alpha}}. \tag{142}$$

IV.2.1 Converting the Sum to a Contour Integral

The high-frequency backscattered field that is given by equation (130) converges extremely slowly in the limit $k\hat{a} \to \infty$, in other words, for short wavelengths. As a result, we will utilize the Watson transformation which replaces a slowly converging series with a contour integral. This integral converges at a much faster rate than the series. Let

$$d_0 = \frac{2}{\alpha}. \tag{143}$$

If we consider $\nu = n + \frac{1}{2}$ as a complex number, then, using equations (143) and (151), γ_- has branch points at $\nu = \pm\frac{1}{d_0}\sqrt{d_0 - 1}$ and γ_+ at $\nu = \pm i\frac{\sqrt{d_0+1}}{d_0}$. As a result of the location of the poles of the integrand, we choose the branch cuts in the complex ν plane along the real axis between $-\frac{1}{2}$ and $+\frac{1}{2}$. Along the imaginary axis, we choose the branch cuts between $-i\frac{\sqrt{d_0+1}}{d_0}$ and $+i\frac{\sqrt{d_0+1}}{d_0}$. Now we will replace the summation in equation (130) with a line integral taken along the clockwise contour C of Figure 21, which encloses those poles of the integrand that are located at $\nu = p + \frac{1}{2}$, where p is a positive integer.

By following a transformation of the type of Watson's, the line integral along C is replaced by the sum of:

- A line integral whose contour consists of a path C_1 extending from the fourth through the first to the second quadrant, plus the arc of a circle of large radius with center at $\nu = 0$ extending from the second through the first to the fourth quadrant, and
- A residue series due to the poles of the integrand which lie in the first quadrant.

The contour C_1 crosses the real ν axis between $\frac{1}{2}$ and $\frac{3}{2}$ and the imaginary ν axis above $+i\frac{\sqrt{d_0+1}}{d_0}$, avoiding the branch cuts. The result obtained thus far is still

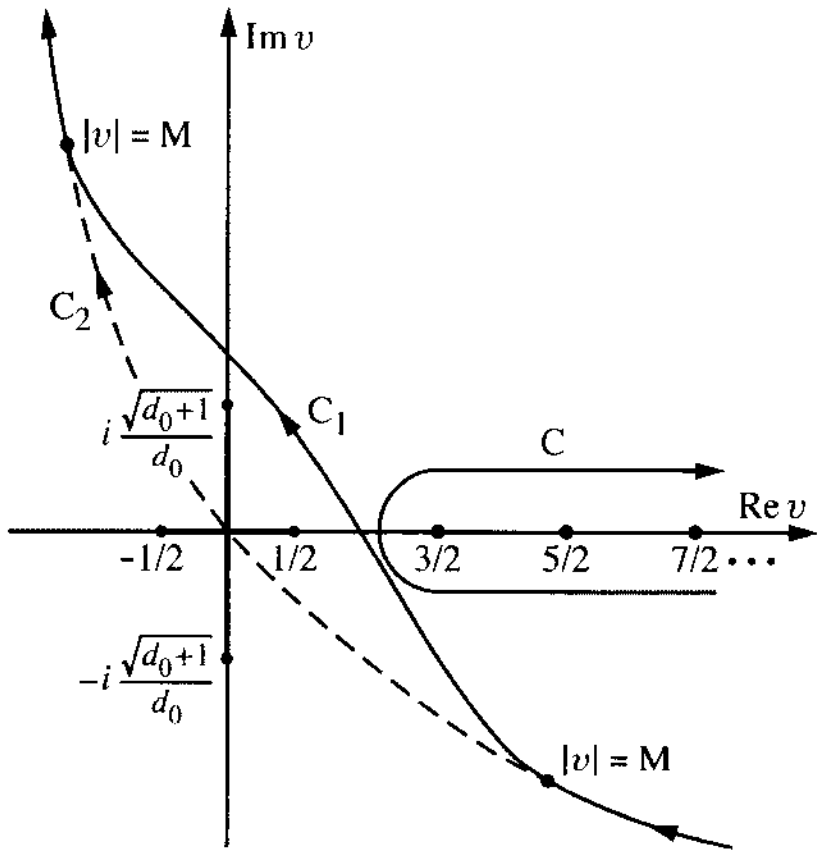

Fig. 21. Contours of integration in the complex ν plane, where a description of C and C_1 can be found on the previous page and a description of C_2 can be found on page 93.

exact. Hence, upon using the Watson transformation, we determine that the high-frequency backscattered field is given by

$$\mathbf{E}^{b.s.} \sim -\hat{e}\frac{e^{ikr}}{2kr}\int_{C_1} \frac{\nu}{\cos \pi\nu}\left(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}}\right)d\nu, \qquad (d_0 k\hat{a} \gg 1). \tag{144}$$

In Figure 21, we show the branch cuts and the contours that we will use to evaluate equation (144). The quantity $(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}})$ in equation (144) must now be evaluated for $|\nu| = O\left[(k\hat{a})^{\frac{1}{2}+\epsilon}\right]$ where ϵ is an arbitrarily small positive number. Before this can be accomplished, the radial eigenfunctions for fields of magnetic- and electric-type must be calculated. Then, the asymptotic expansions for the TE and TM modes must be determined. Once this is done, the Mie solutions can be evaluated and, as a result, the high-frequency backscattered field can be calculated.

IV.2.2 Radial Eigenfunctions for Fields of Magnetic-Type

First consider fields of magnetic-type. From [21], the radial eigenfunctions for the refractive index profile $R(x)$ in equation (142) are given by

$$u(r) = [b_0^{\frac{1}{\alpha}} r]^{\frac{1+(c-1)\alpha}{2}} (1 + b_0 r^\alpha)^{\frac{A}{2}} {}_2F_1(a. b; c; -b_0 r^\alpha),$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function. Writing this in terms of x and renaming the function $S_n^{(1)}(x)$, it is found that

$$S_n^{(1)}(x) = x^{\frac{1\pm(c-1)\alpha}{2}} (1 + x^\alpha)^{\frac{A}{2}} {}_2F_1(a, b; c; -x^\alpha),$$

where [21]

$$L = \frac{1}{2}c - \frac{1}{4}c^2, M = \frac{1}{2}A - \frac{1}{4}A^2, N = \frac{1}{2}cA - ab, A = a + b - c + 1,$$

and [21]

$$\begin{aligned} L &= \frac{1}{4} - \frac{1}{4}(2n+1)^2\alpha^{-2}, \\ M &= -a_0^2 b_0^{-1} \alpha^{-2}, \\ N &= -a_0^2 b_0^{-1} \alpha^{-2}. \end{aligned}$$

These two sets of equations will be used in order to determine the constants a, b, and c in the hypergeometric function in $S_n^{(1)}(x)$. Utilizing our values for a_0 and b_0 in the second set of equations for L, M, and N, we see that

$$\begin{aligned} L &= \frac{1}{4} - \frac{1}{4}(2n+1)^2\alpha^{-2}, \\ M &= -c_0^2 k^2 \hat{a}^{2-\alpha} \hat{a}^\alpha \alpha^{-2} = -(c_0 k \hat{a})^2 \alpha^{-2}, \end{aligned}$$

and

$$N = -(c_0 k \hat{a})^2 \alpha^{-2}.$$

We now determine the constants a, b, and c of the hypergeometric function. Solving the equation that relates L to c,

$$c^2 - 2c + 4L = 0,$$

yields that

$$c = \frac{2 \pm 2\sqrt{1-4L}}{2} = 1 \pm \sqrt{1-4L} = 1 \pm \sqrt{1 - 1 + (2n+1)^2\alpha^{-2}}$$

$$= 1 \pm \frac{2}{\alpha}\sqrt{\left(n+\frac{1}{2}\right)^2} = 1 + \frac{2}{\alpha}\nu, \tag{145}$$

where $\nu = n + \frac{1}{2}$. We are now interested in solving for A since A relates the three constants of the hypergeometric function by the equation $A = a + b - c + 1$. To that end, solving the equation that relates A to M,

$$A^2 - 2A + 4M = 0,$$

implies that

$$A = \frac{2 \pm 2\sqrt{1-4M}}{2} = 1 \pm \sqrt{1 + 4(c_0 k\hat{a})^2\alpha^{-2}} = 1 - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}, \tag{146}$$

where we have taken the minus sign in (146). From the equation $A = a + b - c + 1$ and equation (145), it is found that

$$b = A - a + c - 1 = A - a + \frac{2}{\alpha}\nu.$$

Multiplying both sides of the equation $N = \frac{1}{2}cA - ab$ by 2, using the above equation for b, and rearranging terms yields

$$\begin{aligned} 2N &= cA - 2ab = cA - 2a(A - a + \frac{2}{\alpha}\nu) \\ &= 2a^2 - 2a(A + \frac{2}{\alpha}\nu) + cA \\ &= 2a^2 - 2a[c - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}] + cA. \end{aligned}$$

Then it follows that

$$2a^2 - 2a[c - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}] + cA + 2(c_0 k\hat{a}\alpha^{-1})^2 = 0.$$

This equation is a quadratic equation in the constant a which is readily solvable. As a result,

$$\begin{aligned} a &= \frac{c - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}}{2} \\ &\qquad \pm \frac{\sqrt{[c - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}]^2 - 2[cA + 2(c_0 k\hat{a}\alpha^{-1})^2]}}{2} \\ &= \frac{1}{2}\left\{c - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2} \pm \sqrt{c^2 - 2c + 1}\right\} \\ &= \frac{1}{2}\left\{c - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2} \pm \sqrt{(c-1)^2}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{c - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2} + c - 1\right\} \\
&= c - \frac{1}{2} - \frac{1}{2}\sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2} \\
&= 1 + \frac{2}{\alpha}\nu - \frac{1}{2} - \frac{1}{2}\sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2} \\
&= \frac{1}{2}\left[1 - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}\right] + \frac{2}{\alpha}\nu.
\end{aligned}
$$

Let

$$\beta = \frac{1}{2}\left[1 - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}\right]. \tag{147}$$

Then the constant a is found to be

$$a = \beta + \frac{2}{\alpha}\nu. \tag{148}$$

We note that $\beta = \frac{A}{2}$. Thus, we determine that the constant b is given by

$$b = A - a + \frac{2}{\alpha}\nu = A - \frac{A}{2} - \frac{2}{\alpha}\nu + \frac{2}{\alpha}\nu = \frac{A}{2} = \beta. \tag{149}$$

It is noted that

$$\frac{1}{2}\left[1 + (c-1)\alpha\right] = \frac{1}{2}\left(1 + \frac{2}{\alpha}\nu\alpha\right) = \frac{1}{2}(1 + 2\nu) = \nu + \frac{1}{2}.$$

Therefore, the radial eigenfunctions for fields of magnetic-type are given by

$$S_n^{(1)}(x) = x^{\nu+\frac{1}{2}}(1 + x^\alpha)^\beta {}_2F_1(\beta + \frac{2}{\alpha}\nu, \beta; 1 + \frac{2}{\alpha}\nu; -x^\alpha). \tag{150}$$

IV.2.3 Radial Eigenfunctions for Fields of Electric-Type

For fields of electric-type, the radial eigenfunctions for the refractive index profile $R(x)$ in equation (142) are given by [21]

$$u(r) = [b_0^{\frac{1}{\alpha}} r]^{\frac{1+(c-1)\alpha}{2}}(1 + b_0 r^\alpha)^{\frac{a-b+1}{2}} {}_2F_1(a, c-b; c; -b_0 r^\alpha).$$

Writing in terms of $x = \frac{r}{\hat{a}}$ (and recalling that $b_0 = \hat{a}^{-\alpha}$), we obtain that

$$
\begin{aligned}
u(x) &= x^{\frac{1+(c-1)\alpha}{2}}(1 + x^\alpha)^{\frac{a-b+1}{2}} {}_2F_1(a, c-b; c; -x^\alpha) \\
&= x^{\frac{(c-1)\alpha+\alpha-1}{2}} x^{\frac{2-\alpha}{2}}(1 + x^\alpha)^{\frac{a-b-1}{2}}(1 + x^\alpha) {}_2F_1(a, c-b; c; -x^\alpha) \\
&= x^{\frac{2-\alpha}{2}}(1 + x^\alpha) x^{\frac{(c-1)\alpha+\alpha-1}{2}}(1 + x^\alpha)^{\frac{a-b-1}{2}} {}_2F_1(a, c-b; c; -x^\alpha)
\end{aligned}
$$

$$= c_0[R(x)]^{-1}T_n^{(1)}(x),$$

where

$$T_n^{(1)}(x) = x^{\frac{c\alpha-1}{2}}(1+x^\alpha)^{\frac{a-b-1}{2}}\,{}_2F_1(a, c-b; c; -x^\alpha)$$

and the equations for the constants L, M, and N which determine the constants a, b and c in the hypergeometric function are now given by [21]

$$\begin{aligned} L &= \frac{1}{2}\alpha^{-1} - \frac{1}{4}(2n+1)^2\alpha^{-2} \\ &= \frac{1}{2}\alpha^{-1} - \nu^2\alpha^{-2}, \\ M &= -\frac{1}{2}\alpha^{-1} - \frac{1}{4}(2n+1)^2\alpha^{-2} \\ &= -\frac{1}{2}\alpha^{-1} - \nu^2\alpha^{-2}, \end{aligned}$$

and

$$\begin{aligned} N &= \left[a_0^2 b_0^{-1} - \frac{1}{2}(2n+1)^2\right]\alpha^{-2} \\ &= \left[a_0^2 b_0^{-1} - 2\nu^2\right]\alpha^{-2} \\ &= \left[(c_0 k\dot{a})^2 - 2\nu^2\right]\alpha^{-2}. \end{aligned}$$

Let

$$\gamma_\pm = \frac{1}{2}\sqrt{1 \pm \frac{2}{\alpha} + \frac{4}{\alpha^2}\nu^2}. \tag{151}$$

Then the equation that relates L to c,

$$c^2 - 2c + 4L = 0,$$

implies that

$$\begin{aligned} c &= 1 \pm \sqrt{1-4L} = 1 + \sqrt{1 + 4\frac{\nu^2}{\alpha^2} - \frac{2}{\alpha}} = 1 + \sqrt{1 - \frac{2}{\alpha} + \frac{4}{\alpha^2}\nu^2} \\ &= 1 + 2\gamma_-. \end{aligned} \tag{152}$$

Also, the equation that relates A to M,

$$A^2 - 2A + 4M = 0,$$

implies that

$$A = 1 \pm \sqrt{1-4M} = 1 + \sqrt{1 + \frac{2}{\alpha} + \frac{4}{\alpha^2}\nu^2} = 1 + 2\gamma_+. \tag{153}$$

We have that

$$b = A - a + c - 1 = 1 + 2\gamma_+ - a + 2\gamma_- = 1 - a + 2(\gamma_+ + \gamma_-).$$

Then it is discovered that

$$\begin{aligned} 2N &= cA - 2ab = cA - 2a[1 - a + 2(\gamma_+ + \gamma_-)] \\ &= 2a^2 - 2a[1 + 2(\gamma_+ + \gamma_-)] + cA. \end{aligned}$$

Thus, we find that

$$2a^2 - 2a[1 + 2(\gamma_+ + \gamma_-)] + cA - 2N = 0.$$

Hence, it follows that

$$\begin{aligned} a &= \frac{2[1 + 2(\gamma_+ + \gamma_-)] \pm 2\sqrt{[1 + 2(\gamma_+ + \gamma_-)]^2 - 2(cA - 2N)}}{4} \\ &= \frac{1}{2}\left\{1 + 2(\gamma_+ + \gamma_-) \pm \sqrt{-1 + 4(\gamma_+^2 + \gamma_-^2) - 4\alpha^{-2}[2\nu^2 - (c_0 k\hat{a})^2]}\right\} \\ &= \frac{1}{2}\left\{1 + 2(\gamma_+ + \gamma_-) \pm \sqrt{-1 + 2 + \frac{8}{\alpha^2}\nu^2 - \frac{8}{\alpha^2}\nu^2 + \frac{4}{\alpha^2}(c_0 k\hat{a})^2}\right\} \\ &= \gamma_+ + \gamma_- + \frac{1}{2}\left[1 - \sqrt{1 + \frac{4}{\alpha^2}(c_0 k\hat{a})^2}\right] \\ &= \gamma_+ + \gamma_- + \frac{1}{2}\left[1 - \sqrt{1 + (2c_0 k\hat{a}\alpha^{-1})^2}\right] \\ &= \beta + \gamma_- + \gamma_+, \end{aligned} \tag{154}$$

where we have used equation (147). As a result, it is obtained that

$$b = 1 - \beta - (\gamma_- + \gamma_+) + 2(\gamma_- + \gamma_+) = 1 - \beta + \gamma_- + \gamma_+. \tag{155}$$

We note that

$$\begin{aligned} \frac{1}{2}(c\alpha - 1) &= \frac{1}{2}\Big[(1 + 2\gamma_-)\alpha - 1\Big] = \alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}, \\ \frac{1}{2}(a - b - 1) &= \frac{1}{2}\Big[\beta + \gamma_- + \gamma_+ - 1 + \beta - \gamma_- - \gamma_+ - 1\Big] = \beta - 1, \end{aligned}$$

and

$$c - b = 1 + 2\gamma_- - 1 + \beta - \gamma_- - \gamma_+ = \beta + \gamma_- - \gamma_+.$$

Therefore, the radial eigenfunctions for fields of electric-type are given by

$$T_n^{(1)}(x) = x^{\alpha\gamma\ +\frac{\alpha}{2}-\frac{1}{2}}(1+x^{\alpha})^{\beta-1}{}_2F_1(\beta+\gamma_- +\gamma_+, \beta+\gamma_- -\gamma_+; 1+2\gamma_-; -x^{\alpha}). \quad (156)$$

IV.2.4 Asymptotic Expansions for the TE Modes

We will now use the expression for the radial eigenfunctions for fields of magnetic-type given by equation (150) to calculate the asymptotic expansions for the TE modes. To that end, we must determine M_n which is given by the first equation of equation (134). First, we must obtain an expression for the derivative of the radial eigenfunctions for fields of magnetic-type. Differentiating equation (150) with respect to x yields that

$$\begin{aligned}
S_n^{(1)\prime}(x) &= x^{\nu+\frac{1}{2}}(1+x^{\alpha})^{\beta}\frac{d\,{}_2F_1(\beta+\frac{2}{\alpha}\nu, \beta; 1+\frac{2}{\alpha}\nu; -x^{\alpha})}{d(-x^{\alpha})}(-\alpha x^{\alpha-1}) \\
&\quad + \left[\alpha\beta x^{\alpha-1}x^{\nu+\frac{1}{2}}(1+x^{\alpha})^{\beta-1} + \left(\nu+\frac{1}{2}\right)x^{\nu-\frac{1}{2}}(1+x^{\alpha})^{\beta}\right] \\
&\qquad \times {}_2F_1(\beta+\frac{2}{\alpha}\nu, \beta; 1+\frac{2}{\alpha}\nu; -x^{\alpha}) \\
&= -x^{\alpha-1}x^{\nu+\frac{1}{2}}(1+x^{\alpha})^{\beta}\frac{\alpha\beta(\beta+\frac{2}{\alpha}\nu)}{1+\frac{2}{\alpha}\nu}{}_2F_1(\beta+\frac{2}{\alpha}\nu+1, \beta+1; 2+\frac{2}{\alpha}\nu; -x^{\alpha}) \\
&\quad + x^{\nu+\frac{1}{2}}(1+x^{\alpha})^{\beta}{}_2F_1(\beta+\frac{2}{\alpha}\nu, \beta; 1+\frac{2}{\alpha}\nu; -x^{\alpha})\left[\frac{\alpha\beta x^{\alpha-1}}{1+x^{\alpha}}+\frac{\nu+\frac{1}{2}}{x}\right].
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\frac{S_n^{(1)\prime}(x)}{S_n^{(1)}(x)} \\
&= \frac{\alpha\beta x^{\alpha}+(1+x^{\alpha})(\nu+\frac{1}{2})}{x(1+x^{\alpha})} \\
&\qquad - \frac{\alpha\beta(\beta+\frac{2}{\alpha}\nu)}{1+\frac{2}{\alpha}\nu}x^{\alpha-1}\frac{{}_2F_1(\beta+\frac{2}{\alpha}\nu+1, \beta+1; 2+\frac{2}{\alpha}\nu; -x^{\alpha})}{{}_2F_1(\beta+\frac{2}{\alpha}\nu, \beta; 1+\frac{2}{\alpha}\nu; -x^{\alpha})}.
\end{aligned}$$

Utilizing this in the first equation of equation (134), we find that the TE modes are given by

$$\begin{aligned}
M_n &= \frac{1}{k\hat{a}}\left[\frac{\alpha\beta+2(\nu+\frac{1}{2})}{2} - \frac{\alpha\beta(\beta+\frac{2}{\alpha}\nu)}{1+\frac{2}{\alpha}\nu}\frac{{}_2F_1(\beta+\frac{2}{\alpha}\nu+1, \beta+1; 2+\frac{2}{\alpha}\nu; -1)}{{}_2F_1(\beta+\frac{2}{\alpha}\nu, \beta; 1+\frac{2}{\alpha}\nu; -1)}\right] \\
&= \frac{1+\alpha\beta+2\nu}{2k\hat{a}} - \frac{\alpha\beta(\beta+\frac{2}{\alpha}\nu)}{k\hat{a}(1+\frac{2}{\alpha}\nu)}\frac{{}_2F_1(\beta+\frac{2}{\alpha}\nu+1, \beta+1; 2+\frac{2}{\alpha}\nu; -1)}{{}_2F_1(\beta+\frac{2}{\alpha}\nu, \beta; 1+\frac{2}{\alpha}\nu; -1)}. \quad (157)
\end{aligned}$$

The hypergeometric functions in equation (157) can be expressed in terms of gamma functions using the following formulae from the Digital Library of Mathematical Functions:

$$ {}_2F_1(a,b;a-b+1;-1)=\frac{\Gamma(a-b+1)\Gamma(\frac{1}{2}a+1)}{\Gamma(a+1)\Gamma(\frac{1}{2}a-b+1)}, \tag{158}$$

$$ {}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1}), \tag{159}$$

and

$$ {}_2F_1\left(a,b;\frac{1}{2}a+\frac{1}{2}b+1;\frac{1}{2}\right)=\frac{2\sqrt{\pi}}{a-b}\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+1\right)\left[\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})}-\frac{1}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b)}\right]. \tag{160}$$

For $2z\neq 0,-1,-2,\ldots,$

$$\Gamma(2z)=\frac{1}{\sqrt{\pi}}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{161}$$

Using equation (161), we obtain that

$$\Gamma(z)=\frac{\sqrt{\pi}2^{1-2z}\Gamma(2z)}{\Gamma(z+\frac{1}{2})}. \tag{162}$$

First consider

$$ {}_2F_1(\beta+\frac{2}{\alpha}\nu,\beta;1+\frac{2}{\alpha}\nu;-1).$$

Since

$$\beta+\frac{2}{\alpha}\nu-\beta+1=1+\frac{2}{\alpha}\nu,$$

from equation (158), it is found that

$$\begin{aligned} {}_2F_1(\beta+\frac{2}{\alpha}\nu,\beta;1+\frac{2}{\alpha}\nu;-1) &= \frac{\Gamma(1-\frac{2}{\alpha}\nu)\Gamma(\frac{\beta+\frac{2}{\alpha}\nu}{2}+1)}{\Gamma(\beta+\frac{2}{\alpha}\nu+1)\Gamma(\frac{\beta+\frac{2}{\alpha}\nu}{2}-\beta+1)} \\ &= \frac{\Gamma(1-\frac{2}{\alpha}\nu)}{\Gamma(1+\frac{1}{\alpha}\nu-\frac{\beta}{2})}\frac{\Gamma(\frac{\beta}{2}+1+\frac{1}{\alpha}\nu)}{\Gamma(\beta+1+\frac{2}{\alpha}\nu)}. \end{aligned} \tag{163}$$

Let

$$z=\frac{1+\frac{2}{\alpha}\nu+\beta}{2}.$$

Then it is discovered that

$$2z=1+\frac{2}{\alpha}\nu+\beta$$

and

$$z+\frac{1}{2}=\frac{\beta}{2}+1+\frac{1}{\alpha}\nu.$$

Thus for $1+\dfrac{2}{\alpha}\nu+\beta\neq 0,-1,-2,\ldots$ (i.e. $\dfrac{2}{\alpha}\nu\neq -1,-2,-3,\ldots$, using (162), we find that

$$\frac{\Gamma(\frac{\beta}{2}+1+\frac{1}{\alpha}\nu)}{\Gamma(\beta+1+\frac{2}{\alpha}\nu)}=\frac{\sqrt{\pi}2^{1-1-\frac{2}{\alpha}\nu-\beta}}{\Gamma\left(\frac{1+\frac{2}{\alpha}\nu+\beta}{2}\right)}=\sqrt{\pi}2^{-\beta-\frac{2}{\alpha}\nu}\frac{1}{\Gamma\left(\frac{1+\frac{2}{\alpha}\nu+\beta}{2}\right)}. \tag{164}$$

Therefore it is obtained that

$$ {}_2F_1(\beta+\frac{2}{\alpha}\nu,\beta;1+\frac{2}{\alpha}\nu;-1)=\frac{\sqrt{\pi}2^{-\beta-\frac{2}{\alpha}\nu}\Gamma(1+\frac{2}{\alpha}\nu)}{\Gamma(1+\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu+\beta}{2}\right)}. \tag{165}$$

Now consider

$$ {}_2F_1(\beta+\frac{2}{\alpha}\nu+1,\beta+1;2+\frac{2}{\alpha}\nu;-1).$$

Using equation (159), we see that

$$\begin{aligned} {}_2F_1(\beta+\frac{2}{\alpha}\nu+1,\beta+1;2+\frac{2}{\alpha}\nu;-1)& \\ &=2^{-\beta-\frac{2}{\alpha}\nu-1}{}_2F_1(\beta+\frac{2}{\alpha}\nu+1,2+\frac{2}{\alpha}\nu-\beta-1;2+\frac{2}{\alpha}\nu;\frac{1}{2}) \\ &=2^{-\beta-\frac{2}{\alpha}\nu-1}{}_2F_1(\beta+\frac{2}{\alpha}\nu+1,1+\frac{2}{\alpha}\nu-\beta;2+\frac{2}{\alpha}\nu;\frac{1}{2}). \end{aligned} \tag{166}$$

Since

$$\frac{1}{2}\left(\beta+\frac{2}{\alpha}\nu+1\right)+\frac{1}{2}\left(1+\frac{2}{\alpha}\nu-\beta\right)+1=2+\frac{2}{\alpha}\nu,$$

upon using equation (160), equation (166) becomes

$$\begin{aligned} {}_2F_1(\beta+\frac{2}{\alpha}\nu+1,\beta+1;2+\frac{2}{\alpha}\nu;-1)& \\ =2^{-\beta-\frac{2}{\alpha}\nu-1}&\frac{2\sqrt{\pi}}{\beta+\frac{2}{\alpha}\nu+1-1-\frac{2}{\alpha}\nu+\beta}\Gamma\left(2+\frac{2}{\alpha}\nu\right) \\ &\times\left[\frac{1}{\Gamma\left[\frac{1}{2}(\beta+\frac{2}{\alpha}\nu+1)\right]\Gamma\left[\frac{1}{2}(1+\frac{2}{\alpha}\nu-\beta)+\frac{1}{2}\right]}\right. \\ &\qquad\left.-\frac{1}{\Gamma\left[\frac{1}{2}(\beta+\frac{2}{\alpha}\nu+1)+\frac{1}{2}\right]\Gamma\left[\frac{1}{2}(1+\frac{2}{\alpha}\nu-\beta)\right]}\right] \\ =2^{-\beta-\frac{2}{\alpha}\nu-1}&\frac{\sqrt{\pi}}{\beta}\Gamma\left(2+\frac{2}{\alpha}\nu\right) \\ &\times\left[\frac{1}{\Gamma\left(\frac{1+\frac{2}{\alpha}\nu+\beta}{2}\right)\Gamma(1+\frac{1}{\alpha}\nu-\frac{\beta}{2})}-\frac{1}{\Gamma(1+\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu-\beta}{2}\right)}\right]. \end{aligned} \tag{167}$$

Hence, it follows that

$$\frac{{}_2F_1(\beta+\frac{2}{\alpha}\nu+1,\beta+1;2+\frac{2}{\alpha}\nu;-1)}{{}_2F_1(\beta+\frac{2}{\alpha}\nu,\beta;1+\frac{2}{\alpha}\nu;-1)} = \frac{1}{2\beta}\frac{\Gamma(2+\frac{2}{\alpha}\nu)}{\Gamma(1+\frac{2}{\alpha}\nu)}\left[1-\frac{\Gamma(1+\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu+\beta}{2}\right)}{\Gamma(1+\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu-\beta}{2}\right)}\right]. \tag{168}$$

From the relation

$$\Gamma(z+1)=z\Gamma(z),$$

it is found that

$$\Gamma\left(2+\frac{2}{\alpha}\nu\right)=\left(1+\frac{2}{\alpha}\nu\right)\Gamma\left(1+\frac{2}{\alpha}\nu\right), \tag{169}$$

$$\Gamma\left(1+\frac{1}{\alpha}\nu-\frac{\beta}{2}\right)=\left(\frac{1}{\alpha}\nu-\frac{\beta}{2}\right)\Gamma\left(\frac{1}{\alpha}\nu-\frac{\beta}{2}\right), \tag{170}$$

and

$$\Gamma\left(1+\frac{1}{\alpha}\nu+\frac{\beta}{2}\right)=\left(\frac{1}{\alpha}\nu+\frac{\beta}{2}\right)\Gamma\left(\frac{1}{\alpha}\nu+\frac{\beta}{2}\right). \tag{171}$$

Using equations (169)-(171) in equation (168) yields that

$$\frac{{}_2F_1(\beta+\frac{2}{\alpha}\nu+1,\beta+1;2+\frac{2}{\alpha}\nu;-1)}{{}_2F_1(\beta+\frac{2}{\alpha}\nu,\beta;1+\frac{2}{\alpha}\nu;-1)} = \frac{1+\frac{2}{\alpha}\nu}{2\beta}\left[1-\frac{(\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu+\beta}{2}\right)}{(\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu-\beta}{2}\right)}\right]. \tag{172}$$

Upon utilizing equation (172) in equation (157) gives us that

$$\begin{aligned}
M_{\nu-\frac{1}{2}} &= \frac{1}{2k\hat{a}}+\frac{\alpha\beta+2\nu}{2k\hat{a}}-\frac{\alpha(\beta+\frac{2}{\alpha}\nu)}{2k\hat{a}}\left\{1-\frac{(\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu+\beta}{2}\right)}{(\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma\left(\frac{1+\frac{2}{\alpha}\nu-\beta}{2}\right)}\right\} \\
&= \frac{1}{2k\hat{a}}+\frac{\alpha(\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu+\frac{1+\beta}{2})}{k\hat{a}\Gamma(\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu+\frac{1-\beta}{2})} \\
&= \frac{1}{2k\hat{a}}\left\{1+\frac{\alpha(\frac{2}{\alpha}\nu-\beta)\Gamma(\frac{1}{\alpha}\nu-\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu+\frac{1+\beta}{2})}{\Gamma(\frac{1}{\alpha}\nu+\frac{\beta}{2})\Gamma(\frac{1}{\alpha}\nu+\frac{1-\beta}{2})}\right\}.
\end{aligned} \tag{173}$$

Let

$$A\equiv\frac{2}{\alpha}\nu-\beta=d_0\nu-\beta$$

and

$$B\equiv\frac{2}{\alpha}\nu+\beta=d_0\nu+\beta,$$

where equation (143) has been utilized. Then we see that

$$M_{\nu-\frac{1}{2}} = \frac{1}{2k\hat{a}}\left\{1 + \frac{\frac{2}{d_0}A\Gamma(\frac{A}{2})\Gamma(\frac{B+1}{2})}{\Gamma(\frac{B}{2})\Gamma(\frac{A+1}{2})}\right\}.$$

We have the following formula:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}\sum_{k=0}^{\infty}\frac{G_k(a,b)}{z^k} \sim z^{a-b}\left[G_0(a,b) + \frac{G_1(a,b)}{z} + \frac{G_2(a,b)}{z^2} + O(z^{-3})\right], \quad (174)$$

where

$$\begin{aligned} G_0(a,b) &= 1 \\ G_1(a,b) &= \frac{1}{2}(a-b)(a+b-1) \\ G_2(a,b) &= \frac{1}{12}\begin{pmatrix} a-b \\ 2 \end{pmatrix}\left[3(a+b-1)^2 - (a-b+1)\right]. \end{aligned}$$

It is noted that

$$\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix} = \begin{pmatrix} 2k \\ k \end{pmatrix}\frac{(-1)^{k+1}}{2^{2k}(2k-1)}.$$

We also have the following formulae:

$$\begin{pmatrix} n \\ k \end{pmatrix} = (-1)^k\begin{pmatrix} -n+k-1 \\ k \end{pmatrix}, \quad n < 0 \quad \text{and} \quad k \geq 0$$

and

$$\begin{pmatrix} \frac{1}{p} \\ k \end{pmatrix} = \frac{(-1)^k}{k!}\prod_{j=0}^{k-1}\left(j - \frac{1}{p}\right).$$

Then it follows that

$$\begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}\frac{(-1)^3}{2^4(3)} = -\frac{4!}{2!2!}\frac{1}{16(3)} = -\frac{1}{8}$$

and

$$\begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix} = \frac{1}{2}\left(0 - \frac{3}{2}\right)\left(1 - \frac{3}{2}\right) = \frac{3}{8}.$$

First, consider

$$\frac{\Gamma(\frac{B+1}{2})}{\Gamma(\frac{B}{2})}.$$

For this quotient of Gamma functions, $a = \dfrac{1}{2}$ and $b = 0$. Then, using equation (174), we obtain that

$$\frac{\Gamma(\frac{B+1}{2})}{\Gamma(\frac{B}{2})} \sim \left(\frac{B}{2}\right)^{\frac{1}{2}} \left\{ 1 + \frac{\frac{1}{2}(\frac{1}{2})(-\frac{1}{2})}{\frac{B}{2}} + \frac{\frac{1}{12}(-\frac{1}{8})\left(3(-\frac{1}{2})^2 - \frac{3}{2}\right)}{(\frac{B}{2})^2} + O(B^{-3}) \right\}$$
$$= \left(\frac{B}{2}\right)^{\frac{1}{2}} \left\{ 1 - \frac{1}{4B} + \frac{\frac{1}{3}(-\frac{1}{8})(-\frac{3}{4})}{B^2} + O(B^{-3}) \right\}$$
$$= \left(\frac{B}{2}\right)^{\frac{1}{2}} \left\{ 1 - \frac{1}{4B} + \frac{1}{32B^2} + O(B^{-3}) \right\}.$$

Thus it is discovered that

$$\frac{\Gamma(\frac{B+1}{2})}{\Gamma(\frac{B}{2})} \sim \left(\frac{B}{2}\right)^{\frac{1}{2}} \left\{ 1 - \frac{1}{4B} + \frac{1}{32B^2} + O(B^{-3}) \right\}. \tag{175}$$

In a similar fashion, we find that

$$\frac{\Gamma(\frac{A}{2})}{\Gamma(\frac{A+1}{2})} \sim \left(\frac{A}{2}\right)^{-\frac{1}{2}} \left\{ 1 + \frac{1}{4A} + \frac{1}{32A^2} + O(A^{-3}) \right\}. \tag{176}$$

Hence, combining equations (175) and (176), it is found that

$$\frac{A\Gamma(\frac{A}{2})\Gamma(\frac{B+1}{2})}{\Gamma(\frac{B}{2})\Gamma(\frac{A+1}{2})} \sim \sqrt{AB} \left\{ 1 + \frac{1}{4A} + \frac{1}{32A^2} + O(A^{-3}) \right\} \left\{ 1 - \frac{1}{4B} + \frac{1}{32B^2} + O(B^{-3}) \right\}. \tag{177}$$

Therefore we determine that

$$M_{\nu-\frac{1}{2}} \sim \frac{1}{2k\hat{a}} \left\{ 1 + \frac{2}{d_0}\sqrt{AB} \left[1 + \frac{1}{4A} + \frac{1}{32A^2} + O(A^{-3}) \right] \left[1 - \frac{1}{4B} + \frac{1}{32B^2} + O(B^{-3}) \right] \right\}, \tag{178}$$

where

$$A \equiv \frac{2}{\alpha}\nu - \beta = d_0\nu - \beta, \qquad B \equiv \frac{2}{\alpha}\nu + \beta = d_0\nu + \beta. \tag{179}$$

We note that (178) is valid provided that

$$|A| \gg 1, \qquad |B| \gg 1. \tag{180}$$

As a consequence, the asymptotic expansions for the TE modes are given by

$$M_{\nu-\frac{1}{2}} \sim \left(1 - \frac{d_0}{2} - i\right) + \left(\frac{3}{4} - \frac{1}{2d_0}\right)(k\hat{a})^{-1}$$
$$+ \frac{i}{2}\left(\frac{\nu}{k\hat{a}}\right)^2 + \frac{2 - d_0 - 4i}{16d_0^2}(k\hat{a})^{-2}$$

$$+\frac{i}{8}\left(\frac{\nu}{k\hat{a}}\right)^4 + O\left[\frac{\nu}{(k\hat{a})^3}\right] + O\left[\frac{\nu^3}{(k\hat{a})^4}\right] + O\left[(k\hat{a})^{-3}\right]. \qquad (181)$$

IV.2.5 Asymptotic Expansions for the TM Modes

We will now use the expression for the radial eigenfunctions for fields of electric-type given by equation (156) to calculate the asymptotic expansions for the TM modes. To that end, we must determine $\tilde{M}_n$ which is given by the second equation of equation (134). First, we must obtain an expression for the derivative of the radial eigenfunctions for fields of electric-type. Differentiating equation (156) with respect to x yields that

$$\begin{aligned}
T_n^{(1)\prime}(x) &= x^{\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}}(1+x^\alpha)^{\beta-1} \\
&\qquad \times \frac{d\,{}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_- -\gamma_+; 1+2\gamma_-; -x^\alpha)}{d(-x^\alpha)}\Big(-\alpha x^{\alpha-1}\Big) \\
&\quad + \Bigg[\alpha(\beta-1)x^{\alpha-1}(1+x^\alpha)^{\beta-2}x^{\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}} \\
&\qquad + \left[\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}\right]x^{\alpha\gamma_- + \frac{\alpha}{2} - \frac{3}{2}}(1+x^\alpha)^{\beta-1}\Bigg] \\
&\qquad \times {}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_- -\gamma_+; 1+2\gamma_-; -x^\alpha) \\
&= x^{\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}}(1+x^\alpha)^{\beta-1}\Bigg\{-\frac{\alpha(\beta+\gamma_-+\gamma_+)(\beta+\gamma_- -\gamma_+)x^{\alpha-1}}{1+2\gamma_-} \\
&\qquad \times {}_2F_1(\beta+\gamma_-+\gamma_+ + 1, \beta+\gamma_- -\gamma_+ + 1; 2+2\gamma_-; -x^\alpha) \\
&\qquad + \left[\frac{\alpha(\beta-1)x^{\alpha-1}}{1+x^\alpha} + \frac{\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}}{x}\right] \\
&\qquad \times {}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_- -\gamma_+; 1+2\gamma_-; -x^\alpha)\Bigg\} \\
&= x^{\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}}(1+x^\alpha)^{\beta-1}\Bigg\{-\frac{\alpha(\beta+\gamma_-+\gamma_+)(\beta+\gamma_- -\gamma_+)x^{\alpha-1}}{1+2\gamma_-} \\
&\qquad \times {}_2F_1(\beta+\gamma_-+\gamma_+ + 1, \beta+\gamma_- -\gamma_+ + 1; 2+2\gamma_-; -x^\alpha) \\
&\qquad + \left[\frac{\alpha(\beta-1)x^\alpha + [\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}](1+x^\alpha)}{x(1+x^\alpha)}\right] \\
&\qquad \times {}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_- -\gamma_+; 1+2\gamma_-; -x^\alpha)\Bigg\}.
\end{aligned}$$

Then we see that

$$\frac{T_n^{(1)'}(x)}{T_n^{(1)}(x)} = \frac{\alpha(\beta-1)x^\alpha + [\alpha\gamma_- + \frac{\alpha}{2} - \frac{1}{2}](1+x^\alpha)}{x(1+x^\alpha)} - \frac{\alpha(\beta+\gamma_-+\gamma_+)(\beta+\gamma_--\gamma_+)x^{\alpha-1}}{1+2\gamma_-} \times \frac{{}_2F_1(\beta+\gamma_-+\gamma_++1, \beta+\gamma_--\gamma_++1; 2+2\gamma_-; -x^\alpha)}{{}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_--\gamma_+; 1+2\gamma_-; -x^\alpha)}.$$

Using this in the second equation of equation (134), it is found that the TM modes are given by

$$\hat{M}_n = \alpha\left\{\frac{\beta-1+2\gamma_-+1-\frac{1}{\alpha}}{2k\hat{a}} - \frac{[(\beta+\gamma_-)^2-\gamma_-^2]}{k\hat{a}(1+2\gamma_-)} \times \frac{{}_2F_1(\beta+\gamma_-+\gamma_++1, \beta+\gamma_--\gamma_++1; 2+2\gamma_-; -1)}{{}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_--\gamma_+; 1+2\gamma_-; -1)}\right\}.$$

Let

$$P = \frac{{}_2F_1(\beta+\gamma_-+\gamma_++1, \beta+\gamma_--\gamma_++1; 2+2\gamma_-; -1)}{{}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_--\gamma_-; 1+2\gamma_-; -1)}.$$

Then it is found that

$$\hat{M}_n = \alpha\left\{\frac{\beta-1+2\gamma_-}{2k\hat{a}} + \frac{1-\frac{1}{\alpha}}{2k\hat{a}} - \frac{[(\beta+\gamma_-)^2-\gamma_+^2]}{k\hat{a}(1+2\gamma_-)}P\right\}. \tag{182}$$

From the relation

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1(a, c-b; c; \frac{z}{z-1}),$$

it is determined that

$${}_2F_1(a,b,;c;-1) - 2^{-a}{}_2F_1(a, c-b; c; \frac{1}{2}).$$

Then we discover that

$$\begin{aligned}&{}_2F_1(\beta+\gamma_-+\gamma_++1, \beta+\gamma_--\gamma_++1; 2+2\gamma_-; -1)\\ &= 2^{-(\beta+\gamma_--\gamma_+-1)}{}_2F_1(\beta+\gamma_-+\gamma_++1, 1-\beta+\gamma_-+\gamma_+; 2+2\gamma_-; \frac{1}{2})\end{aligned}$$

and

$${}_2F_1(\beta+\gamma_-+\gamma_+, \beta+\gamma_--\gamma_+; 1+2\gamma_-; -1)$$

$$= 2^{-(\beta+\gamma_-+\gamma_+)} {}_2F_1(\beta + \gamma_- + \gamma_+, 1 - \beta + \gamma_- + \gamma_+; 1 + 2\gamma_-; \frac{1}{2}).$$

Thus we determine that

$$P = \frac{{}_2F_1(\beta + \gamma_- + \gamma_+ + 1, 1 - \beta + \gamma_- + \gamma_-; 2 + 2\gamma_-; \frac{1}{2})}{2 \cdot {}_2F_1(\beta + \gamma_- + \gamma_+, 1 - \beta + \gamma_- + \gamma_+; 1 + 2\gamma_-; \frac{1}{2})}. \tag{183}$$

We have the integral representation [20]

$${}_2F_1(b, \lambda; C; \frac{h}{z}) = \frac{z^\lambda}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt} {}_1F_1(b; C; ht) dt, \tag{184}$$

which is valid for

$$\operatorname{Re} z > \operatorname{Re} h > 0, \qquad \operatorname{Re} \lambda > 0. \tag{185}$$

We will apply equation (184) to equation (183). Letting $b = \beta + \gamma_- + \gamma_+ + 1, \lambda = 1 - \beta + \gamma_- + \gamma_+, C = 2 + 2\gamma_-, h = 1$, and $z = 2$ in the numerator of equation (183) gives us that

$$\begin{aligned} &{}_2F_1(\beta + \gamma_- + \gamma_+ + 1, 1 - \beta + \gamma_- + \gamma_+; 2 + 2\gamma_-; \frac{1}{2}) \\ &= \frac{2^{1-\beta+\gamma_-+\gamma_+}}{\Gamma(1 - \beta + \gamma_- + \gamma_+)} \int_0^\infty t^{\gamma_-+\gamma_+-\beta} e^{-2t} {}_1F_1(\beta + \gamma_- + \gamma_+ + 1; 2 + 2\gamma_-; t) dt. \end{aligned} \tag{186}$$

Similarly, letting $b = \beta + \gamma_- + \gamma_+, \lambda = 1 - \beta + \gamma_- + \gamma_+, C = 1 + 2\gamma_-, h = 1$, and $z = 2$ in the denominator of equation (183) gives us the result

$$\begin{aligned} &{}_2F_1(\beta + \gamma_- + \gamma_+, 1 - \beta + \gamma_- + \gamma_+; 1 + 2\gamma_-; \frac{1}{2}) \\ &= \frac{2^{1-\beta+\gamma_-+\gamma_+}}{\Gamma(1 - \beta + \gamma_- + \gamma_+)} \int_0^\infty t^{\gamma_-+\gamma_+-\beta} e^{-2t} {}_1F_1(\beta + \gamma_- + \gamma_+; 1 + 2\gamma_-; t) dt. \end{aligned} \tag{187}$$

Utilizing equations (186) and (187) in equation (183), it is obtained that

$$P = \frac{\int_0^\infty t^{\gamma_-+\gamma_+-\beta} e^{-2t} {}_1F_1(\beta + \gamma_- + \gamma_+ + 1; 2 + 2\gamma_-; t) dt}{2 \cdot \int_0^\infty t^{\gamma_-+\gamma_+-\beta} e^{-2t} {}_1F_1(\beta + \gamma_- + \gamma_+; 1 + 2\gamma_-; t) dt}. \tag{188}$$

Using the notations from the Digital Library of Mathematical Functions (DLMF),

$${}_1F_1(a; b; z) = M(a, b, z).$$

which is Kummer's function. We will also use Olver's function which is denoted by $\mathbf{M}(a, b, z)$.

We have the following relations from the DLMF:

$$M(a, b, z) = \Gamma(b)\mathbf{M}(a, b, z), \tag{189}$$

$$M(a, b, z) = e^z M(b - a, b, -z), \tag{190}$$

and

$$\mathbf{M}(a, b, -z) = \frac{z^{\frac{1}{2}(1-b)}}{\Gamma(a)} \int_0^\infty e^{-\tau}\tau^{a-\frac{1}{2}b-\frac{1}{2}} J_{b-1}(2\sqrt{z\tau})d\tau. \tag{191}$$

Combining equations (189)-(191), it is obtained that

$$\begin{aligned} M(a, b, z) &= e^z M(b - a, b, -z) = e^z\Gamma(b)\mathbf{M}(b - a, b, -z) \\ &= \frac{e^z z^{\frac{1}{2}(1-b)}\Gamma(b)}{\Gamma(b-a)} \int_0^\infty e^{-\tau}\tau^{b-a-\frac{1}{2}b-\frac{1}{2}} J_{b-1}(2\sqrt{z\tau})d\tau \\ &= \frac{e^z z^{\frac{1}{2}(1-b)}\Gamma(b)}{\Gamma(b-a)} \int_0^\infty e^{-\tau}\tau^{\frac{1}{2}(b-1)-a} J_{b-1}(2\sqrt{z\tau})d\tau. \end{aligned} \tag{192}$$

Consider

$${}_1F_1(\beta + \gamma_- + \gamma_+ + 1; 2 + 2\gamma_-; t) \equiv M(\beta + \gamma_- + \gamma_+ + 1, 2 + 2\gamma_-, t). \tag{193}$$

In equation (193), $a = \beta + \gamma_- + \gamma_+ + 1$ and $b = 2 + 2\gamma_-$. We have that $b - 1 = 1 + 2\gamma_-$. We note that

$$\begin{aligned} \frac{1}{2}(1-b) &= -\frac{1}{2}(1 + 2\gamma_-), \\ b - a &= 2 + 2\gamma_- - \beta - \gamma_- - \gamma_+ - 1 = 1 - \beta + \gamma_- - \gamma_+, \\ \frac{1}{2}(b-1) - a &= \frac{1}{2}(1 + 2\gamma_-) - \beta - \gamma_- - \gamma_+ - 1 = -\beta - \gamma_+ - \frac{1}{2}. \end{aligned}$$

Consequently it follows that

$${}_1F_1(\beta+\gamma_-+\gamma_++1; 2+2\gamma_-; t) = \frac{e^t t^{-\frac{1}{2}-\gamma_-}\Gamma(2+2\gamma_-)}{\Gamma(1-\beta+\gamma_--\gamma_+)} \int_0^\infty e^{-\tau}\tau^{-\beta-\gamma_+-\frac{1}{2}} J_{1+2\gamma_-}(2\sqrt{t\tau})d\tau. \tag{194}$$

Now consider

$${}_1F_1(\beta + \gamma_- + \gamma_+; 1 + 2\gamma_-; t) \equiv M(\beta + \gamma_- + \gamma_+, 1 + 2\gamma_-, t).$$

For this Kummer function, $a = \beta + \gamma_- + \gamma_+$ and $b = 1 + 2\gamma_-$. We have that $b - 1 = 2\gamma_-$. We note that

$$\frac{1}{2}(1 - b) = -\gamma_-,$$

$$b - a = 1 + 2\gamma_- - \beta - \gamma_- - \gamma_+ = 1 - \beta + \gamma_- - \gamma_+,$$

$$\frac{1}{2}(b - 1) - a = \frac{1}{2}(2\gamma_-) - \beta - \gamma_- - \gamma_+ = -\beta - \gamma_+.$$

Accordingly, it is determined that

$$ {}_1F_1(\beta + \gamma_- + \gamma_+; 1 + 2\gamma_-; t) = \frac{e^t t^{-\gamma_-}\Gamma(1 + 2\gamma_-)}{\Gamma(1 - \beta + \gamma_- - \gamma_+)} \int_0^\infty e^{-\tau} \tau^{-\beta - \gamma_+} J_{2\gamma_-}(2\sqrt{t\tau}) d\tau. \quad (195)$$

The integral representations in equations (194) and (195) are valid if

$$k\hat{a} - |\nu| \gg 1. \quad (196)$$

Therefore we discover that

$$P = \frac{\Gamma(2 + 2\gamma_-)}{2\Gamma(1 + 2\gamma_-)} \left\{ \frac{\int_0^\infty t^{\gamma_- + \gamma_+ - \beta} e^{-2t} e^t t^{-\frac{1}{2} - \gamma_-} dt \int_0^\infty e^{-\tau} \tau^{-\beta - \gamma_+ - \frac{1}{2}} J_{1+2\gamma_-}(2\sqrt{t\tau}) d\tau}{\int_0^\infty t^{\gamma_- + \gamma_+ - \beta} e^{-2t} e^t t^{-\gamma_-} dt \int_0^\infty e^{-\tau} \tau^{-\beta - \gamma_+} J_{2\gamma_-}(2\sqrt{t\tau}) d\tau} \right\}.$$

Using that $\Gamma(2 + 2\gamma_-) = (1 + 2\gamma_-)\Gamma(1 + 2\gamma_-)$, the above equation for P becomes

$$P = \left(\frac{1}{2} + \gamma_-\right) \left\{ \frac{\int_0^\infty t^{\gamma_+ - \beta - \frac{1}{2}} e^{-t} dt \int_0^\infty e^{-\tau} \tau^{-\beta - \gamma_+ - \frac{1}{2}} J_{1+2\gamma_-}(2\sqrt{t\tau}) d\tau}{\int_0^\infty t^{\gamma_+ - \beta} e^{-t} dt \int_0^\infty e^{-\tau} \tau^{-\beta - \gamma_+} J_{2\gamma_-}(2\sqrt{t\tau}) d\tau} \right\}. \quad (197)$$

In equation (197), make the change of variables

$$t = u^2, \qquad \tau = w^2, \quad (198)$$

where it is noted that $dt = 2u du$ and $d\tau = 2w dw$. Using these substitutions, equation (197) becomes

$$P = \left(\frac{1}{2} + \gamma_-\right) \left\{ \frac{\int_0^\infty du e^{-u^2} u^{2\gamma_+ - 2\beta} \int_0^\infty dw e^{-w^2} w^{-2\gamma_+ - 2\beta} J_{1+2\gamma_-}(2uw)}{\int_0^\infty du e^{-u^2} u^{2\gamma_+ - 2\beta + 1} \int_0^\infty dw e^{-w^2} w^{-2\gamma_+ - 2\beta + 1} J_{2\gamma_-}(2uw)} \right\}. \quad (199)$$

Now we apply Sommerfeld's integral representation, which is given by

$$J_\mu(2uw) = \frac{1}{2\pi} \int\limits_{\Sigma} d\tau e^{i\mu\tau - 2iuw\sin\tau} \tag{200}$$

to equation (199) where the contour Σ begins at $\tau = -\pi + i\infty$ and ends at $\tau = \pi + i\infty$. This gives us that

$$P = \left(\frac{1}{2} + \gamma_-\right) \frac{\int\limits_0^\infty du e^{-u^2} u^{2\gamma_+ - 2\beta} \int\limits_0^\infty dw e^{-w^2} w^{-2\gamma_+ - 2\beta} \int\limits_\Sigma d\tau e^{i(1+2\gamma_-)\tau - 2iuw\sin\tau}}{\int\limits_0^\infty du e^{-u^2} u^{2\gamma_+ - 2\beta + 1} \int\limits_0^\infty dw e^{-w^2} w^{-2\gamma_+ - 2\beta + 1} \int\limits_\Sigma d\tau e^{2i\gamma_-\tau - 2iuw\sin\tau}}. \tag{201}$$

Consider the following change of variables

$$u = \sqrt{k\hat{a}}\xi, \qquad w = \sqrt{k\hat{a}}\eta, \tag{202}$$

where it is seen that $du = \sqrt{k\hat{a}}d\xi$ and $dw = \sqrt{k\hat{a}}d\eta$. Using these relationships, the numerator of equation (201) may be written as

$$\begin{aligned}
&\int\limits_0^\infty du e^{-u^2} u^{2\gamma_+ - 2\beta} \int\limits_0^\infty dw e^{-w^2} w^{-2\gamma_+ - 2\beta} \int\limits_\Sigma d\tau e^{i(1+2\gamma_-)\tau - 2iuw\sin\tau} \\
&\quad = k\hat{a} \int\limits_0^\infty d\xi e^{-k\hat{a}\xi^2} \xi^{2\gamma_+ - 2\beta} (k\hat{a})^{\gamma_+ - \beta} \int\limits_0^\infty d\eta e^{-k\hat{a}\eta^2} \eta^{-2\gamma_+ - 2\beta} (k\hat{a})^{-\gamma_+ - \beta} \\
&\qquad\qquad \times \int\limits_\Sigma d\tau e^{i(1+2\gamma_-)\tau - 2ik\hat{a}\xi\eta\sin\tau} \\
&\quad = (k\hat{a})^{1-2\beta} \int\limits_0^\infty d\xi e^{-k\hat{a}\xi^2 + (2\gamma_+ - 2\beta)\ln\xi} \int\limits_0^\infty d\eta e^{-k\hat{a}\eta^2 - (2\gamma_+ + 2\beta)\ln\eta} \\
&\qquad\qquad \times \int\limits_\Sigma d\tau e^{i(1+2\gamma_-)\tau - 2ik\hat{a}\xi\eta\sin\tau} \\
&\quad = (k\hat{a})^{1-2\beta} \int\limits_\Sigma d\tau e^{i(1+2\gamma_-)\tau} \int\limits_0^\infty d\eta e^{-k\hat{a}\eta^2 - (2\gamma_+ + 2\beta)\ln\eta} \\
&\qquad\qquad \times \int\limits_0^\infty d\xi e^{-k\hat{a}\xi^2 - 2ik\hat{a}\xi\eta\sin\tau + (2\gamma_+ - 2\beta)\ln\xi}
\end{aligned} \tag{203}$$

and the denominator of equation (201) may be written as

$$\int\limits_0^\infty du e^{-u^2} u^{2\gamma_+ - 2\beta + 1} \int\limits_0^\infty dw e^{-w^2} w^{-2\gamma_+ - 2\beta + 1} \int\limits_\Sigma d\tau e^{2i\gamma_-\tau - 2iuw\sin\tau}$$

$$
\begin{aligned}
&= k\hat{a} \int_0^\infty d\xi e^{-k\hat{a}\xi^2} \xi^{2\gamma_+ - 2\beta + 1} (k\hat{a})^{\gamma_+ - \beta - \frac{1}{2}} \int_0^\infty d\eta e^{-k\hat{a}\eta^2} \eta^{-2\gamma_+ - 2\beta + 1} (k\hat{a})^{-\gamma_+ - \beta + \frac{1}{2}} \\
&\qquad \times \int_\Sigma d\tau e^{2i\gamma_- \tau - 2ik\hat{a}\xi\eta \sin\tau} \\
&= (k\hat{a})^{2-2\beta} \int_\Sigma d\tau e^{2i\gamma_- \tau} \int_0^\infty d\eta e^{-k\hat{a}\eta^2 - (2\gamma_+ + 2\beta - 1)\ln\eta} \\
&\qquad \times \int_0^\infty d\xi e^{-k\hat{a}\xi^2 - 2ik\hat{a}\xi\eta\sin\tau + (2\gamma_+ - 2\beta + 1)\ln\xi}.
\end{aligned}
\tag{204}
$$

Upon using equations (203) and (204) in equation (201), it is obtained that

$$
P = \frac{\frac{1}{2} + \gamma_-}{k\hat{a}} \frac{\int_\Sigma d\tau e^{i(1+2\gamma_-)\tau} \int_0^\infty d\eta e^{-k\hat{a}\eta^2 - (2\gamma_+ + 2\beta)\ln\eta} \int_0^\infty d\xi e^{-k\hat{a}\xi^2 - 2ik\hat{a}\xi\eta\sin\tau - (2\gamma_+ - 2\beta)\ln\xi}}{\int_\Sigma d\tau e^{2i\gamma_- \tau} \int_0^\infty d\eta e^{-k\hat{a}\eta^2 - (2\gamma_+ - 2\beta - 1)\ln\eta} \int_0^\infty d\xi e^{-k\hat{a}\xi^2 - 2ik\hat{a}\xi\eta\sin\tau + (2\gamma_+ - 2\beta + 1)\ln\xi}}.
\tag{205}
$$

The result in equation (205) is exact and valid for $|\nu| = O\left[(k\hat{a})^{\frac{1}{2}+\epsilon}\right]$. We can asymptotically evaluate the integrals in equation (205) by using the method of steepest descents. It is found that (where $c_0 = 2$)

$$
P \sim -\frac{\frac{1}{2} + \gamma_-}{d_0 k\hat{a}} \left[1 + \tan f(\nu)\right] \left\{1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left[\frac{\nu^3}{(k\hat{a})^2}\right] + O\left(\frac{1}{k\hat{a}}\right)\right\}, \tag{206}
$$

where

$$
f(\nu) = \frac{\pi}{4} - \frac{\pi}{2} d_0 k\hat{a} + \pi\gamma_- - \frac{1}{2}\arctan\frac{1}{2}.
$$

Hence, the asymptotic expansions for the TM modes are given by

$$
\tilde{M}_{\nu-\frac{1}{2}} \sim \left[1 - \frac{d_0}{2} + \tan f(\nu)\right] \left\{1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left[\frac{\nu^3}{(k\hat{a})^2}\right] + O\left(\frac{1}{k\hat{a}}\right)\right\}. \tag{207}
$$

(Note: there is a discrepancy between the present derivation of results (181) and (206) and those in [22]. I have made multiple attempts to resolve this, including getting in contact with both authors, each of whom is now aware of the discrepancy and hopes to help me resolve it. For the present, however, the analysis will proceed on the basis of their results being the correct ones.)

IV.2.6 The High-Frequency Backscattered Field

We have calculated the asymptotic expansions for the TE and TM modes in the previous two sections. We are now in a position to determine the difference of the Mie solutions, given by $a_n - b_n$, which appears in the expression of the high-frequency backscattered field given by equation (144). Once this is accomplished, we will be able to achieve the main objective of this chapter and determine the leading order estimate of the high-frequency backscattered field for a specific value of the positive real constant d_0. The Debye asymptotic expansions are used for the Bessel functions appearing in

$$(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}}).$$

In particular, for $|\nu| = O\left[(k\hat{a})^{\frac{1}{2}+\epsilon}\right]$, we have that

$$H_\nu^{(1)}(k\hat{a}) \sim \sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\} \tag{208}$$

and

$$H_\nu^{(1)'}(k\hat{a}) \sim \sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{i + O\left[\left(\frac{\nu}{k\hat{a}}\right)^2\right] + O\left(\frac{1}{k\hat{a}}\right)\right\}, \tag{209}$$

where

$$\rho = \exp\left[i\left(\pi\nu + \frac{\pi}{2} - 2k\hat{a} - \frac{\nu^2}{k\hat{a}}\right)\right]\left\{1 + O\left[\left(\frac{\nu}{k\hat{a}}\right)^2\right] + O\left[\frac{\nu^4}{(k\hat{a})^3}\right]\right\}. \tag{210}$$

We now calculate the integrand in equation (144). First, we note that

$$\cos\pi\nu = \frac{e^{i\pi\nu} + e^{-i\pi\nu}}{2} = \frac{e^{i\pi\nu}}{2}(1 + e^{-2i\pi\nu}).$$

The contour integral in the high-frequency backscattered field is now given by

$$\int_{C_1} \frac{\nu}{\cos\pi\nu}\left(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}}\right)d\nu = 2\int_{C_1} \frac{\nu e^{-i\pi\nu}}{1 + e^{-2i\pi\nu}}\left(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}}\right)d\nu, \qquad (d_0 k\hat{a} \gg 1). \tag{211}$$

We have from equations (131), (132), and using that $\nu = n + \frac{1}{2}$

$$\begin{aligned} a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}} = a_n - b_n &= \frac{-(\psi_n'(k\hat{a}) - M_n\psi_n(k\hat{a}))}{\zeta_n'(k\hat{a}) - M_n\zeta_n(k\hat{a})} + \frac{\psi_n'(k\hat{a}) - \tilde{M}_n\psi_n(k\hat{a})}{\zeta_n'(k\hat{a}) - \tilde{M}_n\zeta_n(k\hat{a})} \\ &= \frac{[M_n\psi_n - \psi_n'][\zeta_n' - \tilde{M}_n\zeta_n] + [\psi_n' - \tilde{M}_n\psi_n][\zeta_n' - M_n\zeta_n]}{[\zeta_n' - M_n\zeta_n][\zeta_n' - \tilde{M}_n\zeta_n]} \end{aligned}$$

$$= \frac{(M_n - \tilde{M}_n)(\psi_n\zeta_n' - \psi_n'\zeta_n)}{(\zeta_n' - M_n\zeta_n)(\zeta_n' - \tilde{M}_n\zeta_n)}. \tag{212}$$

From equation (133), we have that

$$\psi_n(k\hat{a}) = \sqrt{\frac{\pi k\hat{a}}{2}} J_{n+\frac{1}{2}}(k\hat{a}), \qquad \zeta_n(k\hat{a}) = \sqrt{\frac{\pi k\hat{a}}{2}} H^{(1)}_{n+\frac{1}{2}}(k\hat{a}).$$

Differentiation of these functions yields the results

$$\psi_n' = \sqrt{\frac{\pi k\hat{a}}{2}} J'_{n+\frac{1}{2}} + \frac{1}{2}\sqrt{\frac{\pi}{2}}(k\hat{a})^{-\frac{1}{2}} J_{n+\frac{1}{2}}$$

and

$$\zeta_n' = \sqrt{\frac{\pi k\hat{a}}{2}} H^{(1)\prime}_{n+\frac{1}{2}} + \frac{1}{2}\sqrt{\frac{\pi}{2}}(k\hat{a})^{-\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}}.$$

Thus, it is determined that

$$\begin{aligned}
\psi_n\zeta_n' - \psi_n'\zeta_n &= \frac{\pi k\hat{a}}{2} J_{n+\frac{1}{2}} H^{(1)\prime}_{n+\frac{1}{2}} + \frac{\pi}{4} J_{n+\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}} - \frac{\pi k\hat{a}}{2} J'_{n+\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}} - \frac{\pi}{4} J_{n+\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}} \\
&= \frac{\pi k\hat{a}}{2}\left[J_{n+\frac{1}{2}} H^{(1)\prime}_{n+\frac{1}{2}} - J'_{n+\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}} \right] \\
&= \frac{\pi k\hat{a}}{2} \mathcal{W}\{ J_{n+\frac{1}{2}}(k\hat{a}), H^{(1)}_{n+\frac{1}{2}}(k\hat{a}) \} \\
&= \frac{\pi k\hat{a}}{2}\left[\frac{2i}{\pi k\hat{a}} \right] = i,
\end{aligned}$$

where $\mathcal{W}$ is the Wronskian. As a result, we find that

$$a_n - b_n = \frac{i(M_n - \tilde{M}_n)}{(\zeta_n' - M_n\zeta_n)(\zeta_n' - \tilde{M}_n\zeta_n)}. \tag{213}$$

From equations (181) and (207), it is discovered that

$$\begin{aligned}
M_n - \tilde{M}_n \sim{}& \left(1 - \frac{d_0}{2} - i\right) + \left(\frac{3}{4} - \frac{1}{2d_0}\right)(k\hat{a})^{-1} + \frac{i}{2}\left(\frac{\nu}{k\hat{a}}\right)^2 \\
&+ \frac{2 - d_0 - 4i}{16d_0^2}(k\hat{a})^{-2} + \frac{i}{8}\left(\frac{\nu}{k\hat{a}}\right)^4 \\
&+ O\left[\frac{\nu}{(k\hat{a})^3}\right] + O\left[\frac{\nu^3}{(k\hat{a})^4}\right] + O\left[(k\hat{a})^{-3}\right] \\
&- \left[1 - \frac{d_0}{2} + \tan f(\nu)\right]\left\{1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left[\frac{\nu^3}{(k\hat{a})^2}\right] + O\left(\frac{1}{k\hat{a}}\right)\right\} \\
\sim{}& -i - \tan f(\nu)\left\{1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left(\frac{1}{k\hat{a}}\right)\right\} \\
={}& \left\{\frac{-i}{1 + O(\frac{\nu}{k\hat{a}}) + O(\frac{1}{k\hat{a}})} - \tan f(\nu)\right\}\left[1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left(\frac{1}{k\hat{a}}\right)\right]
\end{aligned}$$

$$\sim [-i - \tan f(\nu)]\left\{1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left(\frac{1}{k\hat{a}}\right)\right\}. \tag{214}$$

Utilizing equation (214) in equation (213) yields that

$$a_n - b_n \sim \frac{1 - i\tan f(\nu)}{(\zeta_n' - M_n\zeta_n)(\zeta_n' - \tilde{M}_n\zeta_n)}\left[1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left(\frac{1}{k\hat{a}}\right)\right]. \tag{215}$$

We are now interested in calculating the denominator in equation (215). To that end, it is found that

$$\begin{aligned}
\zeta_n' - M_n\zeta_n &= \sqrt{\frac{\pi k\hat{a}}{2}}H^{(1)\prime}_{n+\frac{1}{2}} + \frac{1}{2}\sqrt{\frac{\pi}{2}}(k\hat{a})^{-\frac{1}{2}}H^{(1)}_{n-\frac{1}{2}} - \sqrt{\frac{\pi k\hat{a}}{2}}M_n H^{(1)}_{n+\frac{1}{2}} \\
&\sim \sqrt{\frac{\pi k\hat{a}}{2}}\sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{i + O\left[\left(\frac{\nu}{k\hat{a}}\right)^2\right] + O\left(\frac{1}{k\hat{a}}\right)\right\} \\
&\qquad + \frac{1}{2}\sqrt{\frac{\pi}{2k\hat{a}}}\sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\} \\
&\qquad - \sqrt{\frac{\pi k\hat{a}}{2}}\sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\}M_n \\
&\sim \frac{1}{\sqrt{\rho}}\left\{i + O\left[\left(\frac{\nu}{k\hat{a}}\right)^2\right] + O\left(\frac{1}{k\hat{a}}\right) + \frac{1}{2k\hat{a}} + O\left[(k\hat{a})^{-2}\right]\right. \\
&\qquad + O\left[\frac{\nu^2}{(k\hat{a})^4}\right] - \left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\} \\
&\qquad\quad \times \left\{1 - \frac{d_0}{2} - i + \left(\frac{3}{4} - \frac{1}{2d_0}\right)(k\hat{a})^{-1}\right. \\
&\qquad\qquad + \frac{i}{2}\left(\frac{\nu}{k\hat{a}}\right)^2 + \frac{2 - d_0 - 4i}{16d_0^2}(k\hat{a})^{-2} + \frac{i}{8}\left(\frac{\nu}{k\hat{a}}\right)^4 \\
&\qquad\qquad \left.\left. + O\left[\frac{\nu}{(k\hat{a})^3}\right] + O\left[\frac{\nu^3}{(k\hat{a})^4}\right] + O\left[(k\hat{a})^{-3}\right]\right\}\right\} \\
&\sim \frac{1}{\sqrt{\rho}}\left\{i + O\left[\left(\frac{\nu}{k\hat{a}}\right)^2\right] + O\left(\frac{1}{k\hat{a}}\right) + \frac{1}{2k\hat{a}}\right. \\
&\qquad - \left\{1 + O\left(\frac{1}{k\hat{a}}\right)\right\} \\
&\qquad\quad \left. \times \left\{1 - \frac{d_0}{2} - i + \left(\frac{3}{4} - \frac{1}{2d_0}\right)(k\hat{a})^{-1} + \frac{i}{2}\left(\frac{\nu}{k\hat{a}}\right)^2\right\}\right\} \\
&\sim \frac{1}{\sqrt{\rho}}\left\{i - \left(1 - \frac{d_0}{2} - i\right)\right\} \\
&= \frac{2}{\sqrt{\rho}}\left[\frac{d_0}{4} - \frac{1}{2} + i\right]
\end{aligned} \tag{216}$$

and

$$\zeta_n' - \tilde{M}_n\zeta_n = \sqrt{\frac{\pi k\hat{a}}{2}} H^{(1)'}_{n+\frac{1}{2}} + \frac{1}{2}\sqrt{\frac{\pi}{2}}(k\hat{a})^{-\frac{1}{2}} H^{(1)}_{n+\frac{1}{2}} - \sqrt{\frac{\pi k\hat{a}}{2}}\tilde{M}_n H^{(1)}_{n+\frac{1}{2}}$$

$$\sim \sqrt{\frac{\pi k\hat{a}}{2}}\sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{i + O\left[\left(\frac{\nu}{k\hat{a}}\right)^2\right] + O\left(\frac{1}{k\hat{a}}\right)\right\}$$

$$+ \frac{1}{2}\sqrt{\frac{\pi}{2k\hat{a}}}\sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\}$$

$$- \sqrt{\frac{\pi k\hat{a}}{2}}\sqrt{\frac{2}{\pi\rho k\hat{a}}}\left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\}\tilde{M}_n$$

$$\sim \frac{1}{\sqrt{\rho}}\left\{i + O\left[\left(\frac{\nu}{k\hat{a}}\right)^2\right] + O\left(\frac{1}{k\hat{a}}\right) + \frac{1}{2k\hat{a}}\left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\}\right.$$

$$- \left\{1 + O\left(\frac{1}{k\hat{a}}\right) + O\left[\frac{\nu^2}{(k\hat{a})^3}\right]\right\}\left\{1 - \frac{d_0}{2} + \tan f(\nu)\right\}$$

$$\left.\times\left\{1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left[\frac{\nu^3}{(k\hat{a})^2}\right] + O\left(\frac{1}{k\hat{a}}\right)\right\}\right\}$$

$$\sim \frac{1}{\sqrt{\rho}}\left[i - \left(1 - \frac{d_0}{2} + \tan f(\nu)\right)\right]$$

$$= \frac{i}{\sqrt{\rho}}\left[1 + i\left(1 - \frac{d_0}{2}\right) + i\tan f(\nu)\right]. \tag{217}$$

Using equations (216) and (217) in equation (215) gives us that

$$a_n - b_n \sim \frac{1 - i\tan f(\nu)}{\left\{\frac{2}{\sqrt{\rho}}\left[\frac{d_0}{4} - \frac{1}{2} + i\right]\right\}\left\{\frac{i}{\sqrt{\rho}}\left[1 + i\left(1 - \frac{d_0}{2}\right) + i\tan f(\nu)\right]\right\}}$$

$$\times\left[1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left(\frac{1}{k\hat{a}}\right)\right]$$

$$= \frac{(1 - i\tan f(\nu))e^{i\pi\nu}e^{i\frac{\pi}{2}}e^{-2ik\hat{a}}e^{-i\frac{\nu^2}{k\hat{a}}}\left[1 + O\left[(\frac{\nu}{k\hat{a}})^2\right] + O\left[\frac{\nu^4}{(k\hat{a})^3}\right]\right]}{2i\left[\frac{d_0}{4} - \frac{1}{2} + i\right]\left[1 + i\left(1 - \frac{d_0}{2}\right) + i\tan f(\nu)\right]}$$

$$\times\left[1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left(\frac{1}{k\hat{a}}\right)\right]$$

$$\sim \frac{(1 - i\tan f(\nu))e^{i\pi\nu}e^{-2ik\hat{a}}e^{-i\frac{\nu^2}{k\hat{a}}}\left[1 + O(\frac{\nu}{k\hat{a}}) + O(\frac{1}{k\hat{a}})\right]}{2\left[\frac{d_0}{4} - \frac{1}{2} + i\right]\left[1 + i\left(1 - \frac{d_0}{2}\right) + i\tan f(\nu)\right]}. \tag{218}$$

Hence utilizing equation (218) in equation (211) it follows that

$$\int_{C_1} \frac{\nu}{\cos\pi\nu}\left(a_{\nu-\frac{1}{2}} - b_{\nu-\frac{1}{2}}\right)d\nu \sim \frac{e^{-2ik\hat{a}}}{\frac{d_0}{4} - \frac{1}{2} + i}$$

$$\times \int_{C_1} d\nu \left\{ \frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1 + e^{-2i\pi\nu}} \frac{1 - i\tan f(\nu)}{1 + i\left(1 - \frac{d_0}{2}\right) + i\tan f(\nu)} \left[1 + O\left(\frac{\nu}{k\hat{a}}\right) + O\left(\frac{1}{k\hat{a}}\right)\right]\right\}.$$

(219)

We will only consider the case where we have that

- the optical rays do not make more than one turn about the center of the lens and
- at least one ray emerges in the backscattering direction.

These considerations yield the following bounds on d_0:

$$1 \le d_0 \le 2.$$

If $d_0 < 2$, the integrand in equation (219) has poles at those points of the complex ν plane for which

$$1 + i\left(1 - \frac{d_0}{2}\right) + i\tan f(\nu) = 0$$

$$i\left[1 - \frac{d_0}{2} + \tan f(\nu)\right] = -1$$

$$1 - \frac{d_0}{2} + \tan f(\nu) = i$$

$$\tan f(\nu) = -\left(\frac{2 - d_0}{2}\right) + i.$$

Thus, it is found that

$$f(\nu) = \arctan\left[-\left(\frac{2 - d_0}{2}\right) + i\right]. \quad (220)$$

Using the formula

$$\arctan z = \frac{i}{2}\log\left(\frac{i + z}{i - z}\right),$$

it is determined that

$$\arctan\left[-\left(\frac{2-d_0}{2}\right)+i\right] = \frac{i}{2}\log\left(\frac{i-(\frac{2-d_0}{2})+i}{i+(\frac{2-d_0}{2})-i}\right)$$
$$= \frac{i}{2}\log\left\{\frac{-(\frac{2-d_0}{2})+2i}{\frac{2-d_0}{2}}\right\}$$
$$= \frac{i}{2}\log\left[-1+\left(\frac{4}{2-d_0}\right)i\right]. \tag{221}$$

Utilizing equation (221) in equation (220) yields that

$$f(\nu) = \frac{i}{2}\log\left[-1+\left(\frac{4}{2-d_0}\right)i\right]. \tag{222}$$

Let

$$z_1 = -1+\left(\frac{4}{2-d_0}\right)i \equiv x_1 + iy_1.$$

We have the formula

$$\log z_1 = \ln r + i\Theta,$$

where

$$r = \sqrt{x_1^2+y_1^2}, \qquad \Theta = Arg z_1.$$

It is seen that

$$r = \sqrt{1+\frac{16}{(2-d_0)^2}}, \qquad \ln r = \frac{1}{2}\ln\left[1+\frac{16}{(2-d_0)^2}\right].$$

We also have the formula $\Theta = Arg z_1 = arg z_1 + 2\pi r$, where $r \in \mathbb{Z}$. Since z_1 is in quadrant II, $\arctan(-p) = -\arctan p$, and $2-d_0>0$, it is discovered that

$$arg z_1 = \arctan\left[-\left(\frac{4}{2-d_0}\right)\right]+\pi = -\arctan\left(\frac{4}{2-d_0}\right)+\pi.$$

Thus, it is obtained that

$$\Theta = -\arctan\left(\frac{4}{2-d_0}\right)+\pi+2\pi r.$$

As a result, we determine that

$$\log z_1 = \frac{1}{2}\ln\left[1+\frac{16}{(2-d_0)^2}\right]+i\left[-\arctan\left(\frac{4}{2-d_0}\right)+\pi+2\pi r\right].$$

Therefore, it follows that

$$f(\nu) = \frac{i}{4}\ln\left[1+\frac{16}{(2-d_0)^2}\right]+\frac{1}{2}\arctan\left(\frac{4}{2-d_0}\right)-\frac{\pi}{2}-\pi r.$$

Since $-r \in \mathbb{Z}$ and letting $-r \equiv m \in \mathbb{Z}$, we see that

$$f(\nu) = \frac{i}{4}\ln\left[1 + \frac{16}{(2-d_0)^2}\right] + \frac{1}{2}\arctan\left(\frac{4}{2-d_0}\right) - \frac{\pi}{2} + m\pi. \tag{223}$$

Substituting the function $f(\nu)$ into equation (223) gives us that

$$\frac{\pi}{4} - \frac{\pi}{2}d_0k\hat{a} + \pi\gamma_- - \frac{1}{2}\arctan\frac{1}{2} = \frac{i}{4}\ln\left[1+\frac{16}{(2-d_0)^2}\right] + \frac{1}{2}\arctan\left(\frac{4}{2-d_0}\right) - \frac{\pi}{2} + m\pi$$

and

$$\pi\gamma_- = m\pi - \frac{3\pi}{4} + \frac{\pi}{2}d_0k\hat{a} + \frac{1}{2}\left[\arctan\frac{1}{2} + \arctan\left(\frac{4}{2-d_0}\right)\right] + \frac{i}{4}\ln\left[1 + \frac{16}{(2-d_0)^2}\right]$$

Letting $\gamma_- = \operatorname{Re}\gamma_- + i\operatorname{Im}\gamma_-$ and equating real and imaginary parts, it is found that the integrand in equation (219) has poles at those points of the complex ν plane for which

$$\operatorname{Re}\gamma_- = m - \frac{3}{4} + \frac{d_0}{2}k\hat{a} + \frac{1}{2\pi}\left[\arctan\frac{1}{2} + \arctan\left(\frac{4}{2-d_0}\right)\right] \tag{224}$$

and

$$\operatorname{Im}\gamma_- = \frac{1}{4\pi}\ln\left[1 + \frac{16}{(2-d_0)^2}\right], \tag{225}$$

where m is any integer.

The contributions to the backscattered field that arise from the poles enclosed by C_1 and by the semicircle at infinity cannot be neglected when compared with the contour integral contribution when $d_0 < 2$. This tells us that the dominant term in the high-frequency backscattered field does not arise from specular reflection as in the case of the lens $d_0 = 2$. Therefore the dominant term in the high-frequency backscattered field is not obtainable by evaluating the contour integral by the saddle point method if $d_0 < 2$. Because of this, we will no longer consider the case when $d_0 < 2$.

We will now turn our attention to the evaluation of the integral in equation (219) when $d_0 = 2$. This will allow for the determination of the leading order estimate of the high-frequency backscattered field by using equation (144). Substituting $d_0 = 2$ into equation (219) and using the result in equation (144) yields that

$$\mathbf{E}^{b.s} \sim -\hat{e}\frac{e^{ikr}}{2ikr}e^{-2ik\hat{a}}$$
$$\times \int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}}e^{-i\frac{\nu^2}{k\hat{a}}}\frac{1-i\tan f(\nu)}{1+i\tan f(\nu)}\left[1+O\left(\frac{\nu}{k\hat{a}}\right)+O\left(\frac{1}{k\hat{a}}\right)\right]. \tag{226}$$

We have that

$$\begin{aligned}
\frac{1-i\tan f(\nu)}{1+i\tan f(\nu)} &= \frac{(1-i\tan f(\nu))(1-i\tan f(\nu))}{1+\tan^2 f(\nu)} \\
&= \frac{1-\tan^2 f(\nu)}{\sec^2 f(\nu)} - 2i\frac{\tan f(\nu)}{\sec^2 f(\nu)} \\
&= \cos^2 f(\nu) - \sin^2 f(\nu) - 2i\sin f(\nu)\cos f(\nu) \\
&= \cos 2f(\nu) - i\sin 2f(\nu) \\
&= e^{-2if(\nu)} \\
&= e^{-i\frac{\pi}{2}+2i\pi k\hat{a}-2i\pi\gamma_- + i\arctan\frac{1}{2}} \\
&= -ie^{2i\pi k\hat{a}-2i\pi\gamma_- + i\arctan\frac{1}{2}}.
\end{aligned}$$

Using this result in equation (226) gives us the high-frequency backscattered field when $d_0 = 2$ as

$$\mathbf{E}^{b.s} \sim \hat{e}\frac{e^{ikr}}{2kr}e^{i[2k\hat{a}(\pi-1)+\arctan\frac{1}{2}]}$$
$$\times \int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}}e^{-i\frac{\nu^2}{k\hat{a}}-2i\pi\gamma_-}\left[1+O\left(\frac{\nu}{k\hat{a}}\right)+O\left(\frac{1}{k\hat{a}}\right)\right]. \tag{227}$$

Let M be a positive number, large compared with unity but independent of $k\hat{a}$. In other words, M can be described as follows:

$$M \gg 1, \qquad \lim_{k\hat{a}\to\infty}\frac{M}{k\hat{a}} = 0.$$

When $d_0 = 2$, it can be shown that the line integral along the arc of the circle vanishes as the radius tends to infinity. Also, when $d_0 = 2$, the contributions to the backscattered field due to the poles in the first quadrant may be neglected because we only want the dominant term of the high-frequency backscattered field, and this arises from an asymptotic estimate of the line integral along the contour C_1. Now, we will split the contour C_1 into three parts, by singling out the portion near $\nu = 0$ along which $|\nu| < M$ (see Figure 21). Along this central portion, we have that

$$e^{-i\frac{\nu^2}{k\hat{a}}} \sim 1, \qquad (|\nu| < M)$$

so that the corresponding integral is $O(1)$, whereas the integral along the entire contour C_1 is $O(k\hat{a})$. Since we only want the leading term in the asymptotic estimate, we may neglect the central portion of C_1. Along the remaining part of the contour, it is determined that

$$\gamma_- \sim \nu, \qquad (|\nu| > M).$$

Then it follows that

$$\int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}} e^{-i\frac{\nu^2}{k\hat{a}}-2i\pi\gamma_-} \sim \int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}} e^{-i\frac{\nu^2}{k\hat{a}}-2i\pi\nu}.$$

We note that

$$\int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}} \left[1+e^{-2i\pi\nu}\right] e^{-i\frac{\nu^2}{k\hat{a}}} = \int_{C_1} \nu e^{-i\frac{\nu^2}{k\hat{a}}} d\nu$$

$$= \frac{ik\hat{a}}{2} e^{-i\frac{\nu^2}{k\hat{a}}}\bigg|_{C_1} \sim O(1).$$

As a consequence, it is discovered that

$$\int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}} \left[1+e^{-2i\pi\nu}\right] e^{-i\frac{\nu^2}{k\hat{a}}} \sim O(1).$$

Thus we obtain that

$$\int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}} e^{-i\frac{\nu^2}{k\hat{a}}-2i\pi\nu} \sim -\int_{C_1} d\nu \frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}} + O(1) \qquad (|\nu| > M). \tag{228}$$

Consider

$$\int_{C_1} d\nu \frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}.$$

Since the integrand has no branch cuts, we may connect the two portions of the contour with a line through the origin $\nu = 0$. The result of this is an added $O(1)$ term. That being so, it is found that

$$\int_{C_1} d\nu \frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}} \sim \int_{C_2} d\nu \frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}} + O(1),$$

where the contour C_2 in the uncut ν plane consists of that portion of C_1 along which $|\nu| > M$, plus the dashed line of Figure 21. Hence, utilizing this in equation (228), it is determined that

$$\int_{C_1} d\nu \frac{\nu}{1+e^{-2i\pi\nu}} e^{-i\frac{\nu^2}{k\hat{a}}-2i\pi\gamma_-} \sim -\left[\int_{C_2} d\nu \frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}} + O(1)\right].$$

Substituting this into equation (227), we find that

$$\mathbf{E}^{b.s.} \sim -\hat{e}\frac{e^{ikr}}{2kr}e^{i[2k\hat{a}(\pi-1)+\arctan\frac{1}{2}]}$$
$$\times\left\{\int\limits_{C_2} d\nu\frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}\left[1+O\left(\frac{\nu}{k\hat{a}}\right)+O\left(\frac{1}{k\hat{a}}\right)\right]+O(1)\right\}. \quad (229)$$

We now focus on the terms containing integrals in equation (229):

$$\int\limits_{C_2} d\nu\frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}\left[1+O\left(\frac{\nu}{k\hat{a}}\right)+O\left(\frac{1}{k\hat{a}}\right)\right]$$
$$\sim\int\limits_{C_2}\frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}d\nu+\int\limits_{C_2}\frac{\frac{\nu^2}{k\hat{a}}e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}d\nu+\frac{1}{k\hat{a}}\int\limits_{C_2}\frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}d\nu$$
$$\equiv I_1+I_2+\frac{1}{k\hat{a}}I_1. \quad (230)$$

Consider

$$I_1=\int\limits_{C_2}\frac{\nu e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}d\nu. \quad (231)$$

Let $x=2i\pi\nu$. Then $x^2=-4\pi^2\nu^2$. We also note that

$$\nu=\frac{1}{2i\pi}x \quad \text{and} \quad d\nu=\frac{1}{2i\pi}dx.$$

Upon using these substitutions, equation (231) becomes

$$I_1=-\frac{1}{4\pi^2}\int\limits_{C_2}\frac{xe^{i\frac{1}{4\pi^2k\hat{a}}x^2}}{1+e^{-x}}dx. \quad (232)$$

Letting

$$\delta=-\frac{i}{4\pi^2k\hat{a}},$$

and using the result found in [24], we obtain that

$$I_1=-\frac{1}{4\pi^2}\int\limits_{C_2}\frac{xe^{-\delta x^2}}{1+e^{-x}}dx=-\frac{1}{4\pi^2}\left[-\frac{1}{2\delta}-\frac{\pi^2}{6}+\frac{7\pi^4}{60}\delta+\dots\right].$$

Since $\delta\sim O((k\hat{a})^{-1})$ and $k\hat{a}\gg 1$, it is determined that

$$I_1\sim\frac{1}{4\pi^2}\left(\frac{1}{2\delta}\right)=\frac{1}{4\pi^2}\left(\frac{4\pi^2k\hat{a}}{-2i}\right)=\frac{ik\hat{a}}{2}=-\frac{k\hat{a}}{2}(-i)=-\frac{k\hat{a}}{2}e^{-i\frac{\pi}{2}}.$$

Thus, we see that

$$I_1 \sim -\frac{k\hat{a}}{2}e^{-i\frac{\pi}{2}}. \tag{233}$$

Now consider

$$I_2 = \frac{1}{k\hat{a}}\int\limits_{C_2} \frac{\nu^2 e^{-i\frac{\nu^2}{k\hat{a}}}}{1+e^{-2i\pi\nu}}d\nu. \tag{234}$$

Using the same relationship for x that was previously used, equation (234) becomes

$$\frac{1}{k\hat{a}}\left(-\frac{1}{4\pi^2}\right)\left(\frac{1}{2i\pi}\right)\int\limits_{C_2}\frac{x^2e^{i\frac{1}{4\pi^2k\hat{a}}x^2}}{1+e^{-x}}dx$$

$$= \left(\frac{i}{4\pi^2k\hat{a}}\right)\frac{1}{2\pi}\int\limits_{C_2}\frac{x^2e^{i\frac{1}{4\pi^2k\hat{a}}x^2}}{1+e^{-x}}dx. \tag{235}$$

Utilizing the same definition for δ as before, it is found that

$$I_2 = -\frac{\delta}{2\pi}\int\limits_{C_2}\frac{x^2e^{-\delta x^2}}{1+e^{-x}}dx. \tag{236}$$

Since $|1+e^{-x}| > 1$, we obtain that

$$\left|\int\limits_{-\infty}^{\infty}\frac{x^2e^{-\delta x^2}}{1+e^{-x}}dx\right| < \int\limits_{-\infty}^{\infty}x^2e^{-\delta x^2}dx = 2\int\limits_{0}^{\infty}x^2e^{-\delta x^2}dx,$$

where we have noted that the integral is symmetric in x. We know that for $a > 0$ and $p \in \mathbb{R}^+$

$$2\int\limits_{0}^{\infty}u^{2p-1}e^{-au^2}du = \frac{\Gamma(p)}{\delta^p}.$$

Letting $u = x, a = \delta$ and $p = \frac{3}{2}$, it is determined that

$$\int\limits_{-\infty}^{\infty}x^2e^{-\delta x^2}dx = \frac{\Gamma(\frac{3}{2})}{\delta^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2}\delta^{-\frac{3}{2}}. \tag{237}$$

Using equation (237) in equation (236), it is discovered that

$$I_2 \sim -\frac{1}{4\sqrt{\pi}}\delta^{-\frac{1}{2}} = -\frac{1}{4\sqrt{\pi}}\left(\frac{4\pi^2k\hat{a}}{-i}\right)^{\frac{1}{2}} = -\frac{\sqrt{\pi}}{2}(k\hat{a})^{\frac{1}{2}}e^{i\frac{\pi}{4}} = -\frac{k\hat{a}}{2}e^{-i\frac{\pi}{2}}\left[\sqrt{\pi}(k\hat{a})^{-\frac{1}{2}}e^{\frac{3i\pi}{4}}\right]$$

$$\sim -\frac{k\hat{a}}{2}e^{-i\frac{\pi}{2}}O((k\hat{a})^{-\frac{1}{2}}). \tag{238}$$

Substituting equations (233) and (238) into equation (230), the terms containing integrals in the backscattered field reduce to

$$-\frac{k\hat{a}}{2}e^{-i\frac{\pi}{2}}\left[1+O((k\hat{a})^{-\frac{1}{2}})+O((k\hat{a})^{-1})\right] \sim -\frac{k\hat{a}}{2}e^{-i\frac{\pi}{2}}\left[1+O((k\hat{a})^{-\frac{1}{2}})\right]. \tag{239}$$

Utilizing equation (239) in equation (229), when $d_0 = 2$, the leading order estimate of the high-frequency backscattered field is found to be

$$\mathbf{E}^{b.s} \sim \hat{e}\frac{\hat{a}}{4r}e^{i\left[kr-2k\hat{a}(\pi-1)-\frac{\pi}{2}+\arctan\frac{1}{2}\right]}\left\{1+O[(k\hat{a})^{-\frac{1}{2}}]\right\}. \tag{240}$$

IV.3 PROFILE 2

We will now consider another refractive index profile that is based on the hypergeometric equation and derive the radial eigenfunctions for fields of magnetic- and electric-type. The hope is that the analysis that was done in the previous section for the refractive index profile $R(x)$ in equation (142) may be extended to this refractive index profile in future work. First, consider the wavenumber

$$m(r) = \frac{a_0}{r(1+b_0 r^{\alpha})}. \tag{241}$$

Then, writing (241) in terms of $x = \frac{r}{\hat{a}}$, we have that

$$m(x) = \frac{a_0}{x\hat{a}(1+x^{\alpha})},$$

where we have taken $b_0 = \hat{a}^{-\alpha}$. Taking the refractive index profile to be, which we call Profile 2,

$$R_2(x) = \frac{c_0}{x(1+x^{\alpha})},$$

we must have that $a_0 = c_0 k\hat{a}$. Then it is seen that

$$m(x) = \frac{c_0 k}{x(1+x^{\alpha})}.$$

Let

$$\nu = n + \frac{1}{2},$$

$$g_0 \equiv c_0 k\hat{a},$$

$$d_0 \equiv \frac{2}{\alpha}.$$

IV.3.1 Radial Eigenfunctions for Fields of Magnetic-Type

We are now interested in calculating the radial eigenfunctions for fields of magnetic-type for the refractive index profile

$$R_2(x) = \frac{c_0}{x(1+x^\alpha)}.$$

From [21], the radial eigenfunctions for fields of magnetic-type, in terms of r, are given by

$$u(r) = \left[b_0^{\frac{1}{\alpha}} r\right]^{\frac{1+(c-1)\alpha}{2}} (1+b_0 r^\alpha)^{\frac{A}{2}} {}_2F_1(a,b;c;-b_0 r^\alpha).$$

Writing this in terms of x, it is determined that

$$S_n^{(1)}(x) = x^{\frac{1+(c-1)\alpha}{2}} (1+x^\alpha)^{\frac{A}{2}} {}_2F_1(a,b;c;-x^\alpha). \tag{242}$$

where [21]

$$\begin{aligned} L &= \frac{1}{2}c - \frac{1}{4}c^2, \\ M &= \frac{1}{2}A - \frac{1}{4}A^2, \\ N &= \frac{1}{2}cA - ab, \\ A &= a + b - c + 1, \end{aligned}$$

and [21]

$$\begin{aligned} L &= \frac{1}{4} + \left\{a_0^2 - \frac{1}{4}(2n+1)^2\right\}\alpha^{-2} \\ &= \frac{1}{4} + \left\{(c_0 k \hat{a})^2 - \nu^2\right\}\alpha^{-2} \\ &= \frac{1}{4} + \{g_0^2 - \nu^2\}\alpha^{-2}, \\ M &= a_0^2\alpha^{-2} = (c_0 k \hat{a} \alpha^{-1})^2 = g_0^2\alpha^{-2}, \\ N &= 2a_0^2\alpha^{-2} = 2g_0^2\alpha^{-2}. \end{aligned}$$

From the formula $c^2 - 2c + 4L = 0$, it is found that

$$\begin{aligned} c &= \frac{2 \pm 2\sqrt{1-4L}}{2} = 1 \pm \sqrt{1-4L} = 1 + \sqrt{1 - 1 + 4\alpha^{-2}[\nu^2 - g_0^2]} = 1 + \frac{2}{\alpha}\sqrt{\nu^2 - g_0^2} \\ &= 1 + d_0\sqrt{\nu^2 - g_0^2}. \end{aligned}$$

Similarly, we obtain that

$$A = 1 \pm \sqrt{1-4M} = 1 - \sqrt{1-4g_0^2\alpha^{-2}} = 1 - \sqrt{1-d_0^2 g_0^2}$$
$$\equiv 2\beta,$$

where β in this equation is different from the one used with Profile 1 in the previous section. We have that

$$b = A - a + c - 1 = A - a + d_0\sqrt{\nu^2 - g_0^2} = 2\beta - a + d_0\sqrt{\nu^2 - g_0^2}.$$

Consequently, it follows that

$$2N = cA - 2ab = 2c\beta - 2a(2\beta - a + d_0\sqrt{\nu^2 - g_0^2}$$
$$= 2c\beta + 2a^2 - 2a(2\beta + d_0\sqrt{\nu^2 - g_0}).$$

This gives us that

$$2a^2 - 2a(2\beta + d_0\sqrt{\nu^2 - g_0^2}) + 2c\beta - 2N = 0.$$

Accordingly, it is discovered that

$$a = \frac{2(2\beta + d_0\sqrt{\nu^2 - g_0^2}) \pm 2\sqrt{4\beta^2 + 4\beta d_0\sqrt{\nu^2 - g_0^2} + d_0^2(\nu^2 - g_0^2) - 2(2c\beta - 2N)}}{4}$$
$$= \frac{1}{2}\left[2\beta + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{4\beta - d_0^2 g_0^2 + 4\beta d_0\sqrt{\nu^2 - g_0^2} + d_0^2(\nu^2 - g_0^2) - 4c\beta + 4N}\right]$$
$$= \frac{1}{2}\left[2\beta + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{4\beta(1 + d_0\sqrt{\nu^2 - g_0^2}) - d_0^2 g_0^2 + d_0^2(\nu^2 - g_0^2) - 4c\beta + 4N}\right]$$
$$= \frac{1}{2}\left[2\beta + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{4c\beta - d_0^2 g_0^2 + d_0^2(\nu^2 - g_0^2) - 4c\beta + 4N}\right]$$
$$= \frac{1}{2}\left[2\beta + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{d_0^2(\nu^2 - g_0^2) - d_0^2 g_0^2 + 8g_0^2\alpha^{-2}}\right]$$
$$= \frac{1}{2}\left[2\beta + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{d_0^2(\nu^2 - g_0^2) - d_0^2 g_0^2 + 2d_0^2 g_0^2}\right]$$
$$= \frac{1}{2}\left[2\beta + d_0\sqrt{\nu^2 - g_0^2} + d_0\nu\right]$$
$$= \beta + \frac{d_0}{2}\left[\nu + \sqrt{\nu^2 - g_0^2}\right].$$

As a result, it is found that

$$b = 2\beta - \beta - \frac{d_0}{2}\left[\nu + \sqrt{\nu^2 - g_0^2}\right] + d_0\sqrt{\nu^2 - g_0^2}$$

$$= \beta - \frac{d_0}{2}\left[\nu - \sqrt{\nu^2 - g_0^2}\right].$$

We note that

$$\frac{(c-1)\alpha + 1}{2} = \frac{2\sqrt{\nu^2 - g_0^2} + 1}{2} = \frac{1}{2} + \sqrt{\nu^2 - g_0^2}$$

and

$$\frac{A}{2} = \beta.$$

Combining all of the above, we obtain that the radial eigenfunctions for fields of magnetic-type for Profile 2 are given by

$$\begin{aligned} S_n^{(1)}(x) = x^{\sqrt{\nu^2 - g_0^2} - \frac{1}{2}}(1 + x^\alpha)^\beta \\ \times {}_2F_1\Bigg(\beta + \frac{d_0}{2}\left[\nu + \sqrt{\nu^2 - g_0^2}\right], \beta - \frac{d_0}{2}\left[\nu - \sqrt{\nu^2 - g_0^2}\right]; \\ 1 + d_0\sqrt{\nu^2 - g_0^2}; -x^\alpha\Bigg). \end{aligned} \tag{243}$$

IV.3.2 Radial Eigenfunctions for Fields of Electric-Type

We now determine the radial eigenfunctions for fields of electric-type for Profile 2. From [21], the radial eigenfunctions for fields of electric-type, in terms of r, are given by

$$u(r) = \left[b_0^{\frac{1}{\alpha}} r\right]^{\frac{1+(c-1)\alpha}{2}} (1 + b_0 r^\alpha)^{\frac{a-b+1}{2}} {}_2F_1(a, c-b; c; -b_0 r^\alpha).$$

Writing this in terms of x, and renaming the resulting function $v(x)$, we have that

$$\begin{aligned} v(x) &= x^{\frac{1+(c-1)\alpha}{2}}(1 + x^\alpha)^{\frac{a-b+1}{2}} {}_2F_1(a, c-b; c; -x^\alpha) \\ &= x^{\frac{1}{2}[(c-1)\alpha - 1]} x^1 (1 + x^\alpha)^1 (1 + x^\alpha)^{\frac{a-b-1}{2}} {}_2F_1(a, c-b; c; -x^\alpha) \\ &= c_0[R_2(x)]^{-1} T_n^{(1)}(x), \end{aligned}$$

where

$$T_n^{(1)}(x) = x^{\frac{1}{2}[(c-1)\alpha - 1]}(1 + x^\alpha)^{\frac{a-b-1}{2}} {}_2F_1(a, c-b; c; -x^\alpha)$$

and [21]

$$\begin{aligned} L &= \frac{1}{4} + \left\{a_0^2 - \frac{1}{4}(2n+1)^2\right\}\alpha^{-2} \\ &= \frac{1}{4} + \left\{(c_0 k\hat{a})^2 - \nu^2\right\}\alpha^{-2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4} + \left\{g_0^2 - \nu^2\right\} \alpha^{-2}, \\
M &= -\frac{3}{4} - \alpha^{-1} - \frac{1}{4}(2n+1)^2\alpha^{-2} \\
&= -\frac{3}{4} - \alpha^{-1} - \nu^2\alpha^{-2}, \\
N &= -\frac{1}{2}(2n+1)^2\alpha^{-2} - \frac{1}{2} - \alpha^{-1} \\
&= -2\nu^2\alpha^{-2} - \frac{1}{2} - \alpha^{-1}.
\end{aligned}
$$

We find that

$$
\begin{aligned}
c &= 1 \pm \sqrt{1-4L} = 1 + \sqrt{4\alpha^{-2}(\nu^2 - g_0^2)} = 1 + \frac{2}{\alpha}\sqrt{\nu^2 - g_0^2} \\
&= 1 + d_0\sqrt{\nu^2 - g_0^2}, \\
A &= 1 \pm \sqrt{1-4M} = 1 + \sqrt{1 + 3 + 4\alpha^{-1} + 4\alpha^{-2}\nu^2} = 1 + 2\sqrt{1 + \alpha^{-1} + \alpha^{-2}\nu^2} \\
&= 1 + 2\sqrt{1 + \frac{d_0}{2} + \frac{d_0^2}{4}\nu^2} \equiv 1 + 2\mu_1, \\
b &= A - a + c - 1 = A - a + d_0\sqrt{\nu^2 - g_0^2}, \\
2N &= cA - 2ab = cA - 2a\left[A - a + d_0\sqrt{\nu^2 - g_0^2}\right].
\end{aligned}
$$

Thus it is seen that

$$
2a^2 - 2a\left[A + d_0\sqrt{\nu^2 - g_0^2}\right] + cA - 2N = 0,
$$

yielding that

$$
\begin{aligned}
a &= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{\left(A + d_0\sqrt{\nu^2 - g_0^2}\right)^2 - 2(cA - 2N)}\right\} \\
&= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{A^2 + 2Ad_0\sqrt{\nu^2 - g_0^2} + d_0^2(\nu^2 - g_0^2) - 2(cA - 2N)}\right\} \\
&= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{A^2 + 2A(c-1) + d_0^2(\nu^2 - g_0^2) - 2cA + 4N}\right\} \\
&= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{A(A-2) + 4N + d_0^2(\nu^2 - g_0^2)}\right\} \\
&= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{(1 + 2\mu_1)(-1 + 2\mu_1) + 4N + d_0^2(\nu^2 - g_0^2)}\right\} \\
&= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{4\mu_1^2 + 4N - 1 + d_0^2(\nu^2 - g_0^2)}\right\}.
\end{aligned}
$$

We note that

$$4\mu_1^2 = 4 + 2d_0 + d_0^2\nu^2$$

and

$$4N = -2 - 2d_0 - 2d_0^2\nu^2.$$

Then, it is determined that

$$4\mu_1^2 + 4N - 1 + d_0^2(\nu^2 - g_0^2) = 1 - d_0^2 g_0^2.$$

Hence, it follows that

$$a = \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{1 - d_0^2 g_0^2}\right\} = \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} - \sqrt{1 - d_0^2 g_0^2}\right\}$$
$$= \beta + \mu_1 + \frac{d_0}{2}\sqrt{\nu^2 - g_0^2}$$

and

$$b = A - a + c - 1 = 1 + 2\mu_1 - \beta - \mu_1 - \frac{d_0}{2}\sqrt{\nu^2 - g_0^2} + d_0\sqrt{\nu^2 - g_0^2}$$
$$= 1 - \beta + \mu_1 + \frac{d_0}{2}\sqrt{\nu^2 - g_0^2}.$$

We have that

$$\frac{(c-1)\alpha - 1}{2} = \frac{2\sqrt{\nu^2 - g_0^2} - 1}{2},$$
$$\frac{a - b - 1}{2} = \frac{\beta + \mu_1 + \frac{d_0}{2}\sqrt{\nu^2 - g_0^2} - 1 + \beta - \mu_1 - \frac{d_0}{2}\sqrt{\nu^2 - g_0^2} - 1}{2}$$
$$= \beta - 1,$$
$$c - b = 1 + d_0\sqrt{\nu^2 - g_0^2} - 1 + \beta - \mu_1 - \frac{d_0}{2}\sqrt{\nu^2 - g_0^2}$$
$$= \beta - \mu_1 + \frac{d_0}{2}\sqrt{\nu^2 - g_0^2}.$$

Therefore it is obtained that the radial eigenfunctions for fields of electric-type for Profile 2 are given by

$$T_n^{(1)}(x) = x^{\frac{2\sqrt{\nu^2 - g_0^2} - 1}{2}}(1 + x^\alpha)^{\beta - 1}$$
$$\times {}_2F_1\bigg(\beta + \mu_1 + \frac{d_0}{2}\sqrt{\nu^2 - g_0^2}, \beta - \mu_1 + \frac{d_0}{2}\sqrt{\nu^2 - g_0^2};$$
$$1 + d_0\sqrt{\nu^2 - g_0^2}; -x^\alpha\bigg). \tag{244}$$

IV.4 PROFILE 3

We will now consider one final refractive index profile that is based on the hypergeometric equation and derive the radial eigenfunctions for fields of magnetic- and electric-type. Like the previous refractive index profile, it is hoped that the analysis that was carried out for the refractive index profile $R(x)$ given by equation (142) may be extended to this refractive index profile in future work. First, consider the wavenumber

$$m(r) = \frac{a_0}{r\sqrt{1 + b_0 r^\alpha}}. \tag{245}$$

Then, writing (245) in terms of x, it is seen that

$$m(x) = \frac{a_0}{x\hat{a}\sqrt{1 + x^\alpha}},$$

where we have taken $b_0 = \hat{a}^{-\alpha}$. Taking the refractive index profile to be, which we call Profile 3,

$$R_3(x) = \frac{c_0}{x\sqrt{1 + x^\alpha}},$$

we must have that $a_0 = c_0 k\hat{a}$. As a consequence, we determine that

$$m(x) = \frac{c_0 k}{x\sqrt{1 + x^\alpha}}.$$

IV.4.1 Radial Eigenfunctions for Fields of Magnetic-Type

From [21], the radial eigenfunctions for fields of magnetic-type for Profile 3, in terms of r, are given by

$$u(r) = \left[b_0^{\frac{1}{\alpha}} r\right]^{\frac{1+(c-1)\alpha}{2}} (1 + b_0 r^\alpha)^{\frac{A}{2}} \,{}_2F_1(a, b; c; -b_0 r^\alpha).$$

Writing this in terms of x, we obtain that

$$S_n^{(1)}(x) = x^{\frac{1+(c-1)\alpha}{2}} (1 + x^\alpha)^{\frac{A}{2}} \,{}_2F_1(a, b; c; -x^\alpha), \tag{246}$$

where [21]

$$\begin{aligned} L &= \frac{1}{4} + \left\{a_0^2 - \frac{1}{4}(2n+1)^2\right\}\alpha^{-2} \\ &= \frac{1}{4} + \left\{(c_0 k\hat{a})^2 - \nu^2\right\}\alpha^{-2} \\ &= \frac{1}{4} + \{g_0^2 - \nu^2\}\alpha^{-2}, \end{aligned}$$

$$M = 0.$$
$$N = a_0^2\alpha^{-2} = g_0^2\alpha^{-2}.$$

From the equation $c^2 - 2c + 4L = 0$, we have that

$$c = 1 \pm \sqrt{1-4L} = 1 + d_0\sqrt{\nu^2 - g_0^2}.$$

Using that $M = 0$, solving the equation $A^2 - 2A + 4M = 0$ for A yields that either $A = 0$ or $A = 2$. As a result of our problem, we must have that $A = 2$. We also have that $b = A - a + c - 1 = 2 - a + c - 1 = 1 - a + c$. Then it is determined that

$$2N = cA - 2ab = 2c - 2a(1-a+c) = 2a^2 - 2a(c+1) + 2c.$$

Hence $2a^2 - 2a(c+1) + 2c - 2N = 0$ and as a result $a^2 - a(c+1) + c - N = 0$. Because of this, we see that

$$\begin{aligned}
a &= \frac{1}{2}\left\{c + 1 \pm \sqrt{(c+1)^2 - 4(c-N)}\right\} \\
&= \frac{1}{2}\left\{c + 1 \pm \sqrt{c^2 + 2c + 1 - 4c + 4N}\right\} \\
&= \frac{1}{2}\left\{c + 1 \pm \sqrt{c^2 - 2c + 1 + d_0^2 g_0^2}\right\} \\
&= \frac{1}{2}\left\{c + 1 \pm \sqrt{(c-1)^2 + d_0^2 g_0^2}\right\} \\
&= \frac{1}{2}\left\{c + 1 \pm \sqrt{d_0^2(\nu^2 - g_0^2) + d_0^2 g_0^2}\right\} \\
&= \frac{1}{2}\left\{c + 1 + d_0\nu\right\} \\
&= 1 + \frac{d_0}{2}\left(\nu + \sqrt{\nu^2 - g_0^2}\right).
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
b &= A - a + c - 1 = 1 - a + c = c - \frac{d_0}{2}\left(\nu + \sqrt{\nu^2 - g_0^2}\right) \\
&= 1 - \frac{d_0}{2}\left(\nu - \sqrt{\nu^2 - g_0^2}\right).
\end{aligned}$$

We note that

$$\frac{1 + (c-1)\alpha}{2} = \frac{2\sqrt{\nu^2 - g_0^2} + 1}{2} = \sqrt{\nu^2 - g_0^2} + \frac{1}{2}$$

and

$$\frac{A}{2} = 1$$

Therefore, we obtain that the radial eigenfunctions for fields of magnetic-type for Profile 3 are given by

$$\begin{aligned} S_n^{(1)}(x) &= x^{\sqrt{\nu^2 - g_0^2} + \frac{1}{2}} (1 + x^\alpha) \\ &\quad \times {}_2F_1\left(1 + \frac{d_0}{2}\left(\nu + \sqrt{\nu^2 - g_0^2}\right), 1 - \frac{d_0}{2}\left(\nu - \sqrt{\nu^2 - g_0^2}\right);\right. \\ &\qquad \left. 1 + d_0\sqrt{\nu^2 - g_0^2}; -x^\alpha\right). \end{aligned} \tag{247}$$

IV.4.2 Radial Eigenfunctions for Fields of Electric-Type

From [21], the radial eigenfunctions for fields of electric-type for Profile 3, in terms of r, are given by

$$u(r) = \left[b_0^{\frac{1}{\alpha}} r\right]^{\frac{1+(c-1)\alpha}{2}} (1 + b_0 r^\alpha)^{\frac{a-b+1}{2}} {}_2F_1(a, c-b; c; -b_0 r^\alpha).$$

Writing this in terms of x, and renaming the resulting function $w(x)$, yields that

$$\begin{aligned} w(x) &= x^{\frac{1+(c-1)\alpha}{2}} (1 + x^\alpha)^{\frac{a-b+1}{2}} {}_2F_1(a, c-b; c; -x^\alpha) \\ &= x^{\frac{(c-1)\alpha-1}{2}} x^1 (1 + x^\alpha)^{\frac{1}{2}} (1 + x^\alpha)^{\frac{a-b}{2}} {}_2F_1(a, c-b; c; -x^\alpha) \\ &= c_0[R_3(x)]^{-1} T_n^{(1)}(x), \end{aligned}$$

where

$$T_n^{(1)}(x) = x^{\frac{(c-1)\alpha-1}{2}} (1 + x^\alpha)^{\frac{a-b}{2}} {}_2F_1(a, c-b; c; -x^\alpha)$$

and [21]

$$\begin{aligned} L &= \frac{1}{4} + \left\{a_0^2 - \frac{1}{4}(2n+1)^2\right\} \alpha^{-2} \\ &= \frac{1}{4} + \left\{g_0^2 - \nu^2\right\} \alpha^{-2} \\ &= \frac{1}{4} + \left\{g_0^2 - \nu^2\right\} \frac{d_0^2}{4}, \\ M &= -\frac{1}{4}(2n+1)^2 \alpha^{-2} - \frac{1}{2}\alpha^{-1} \\ &= -\nu^2 \alpha^{-2} - \frac{1}{2}\alpha^{-1} \end{aligned}$$

$$= -\frac{d_0^2\nu^2 + d_0}{4},$$

$$N = \left\{a_0^2 - \frac{1}{2}(2n+1)^2\right\}\alpha^{-2} - \frac{1}{2}\alpha^{-1}$$
$$= \frac{d_0^2(g_0^2 - 2\nu^2) - d_0}{4}.$$

Since our L for electric-type fields is the same as that for magnetic-type fields for Profile 3, we have that

$$c = 1 + d_0\sqrt{\nu^2 - g_0^2}.$$

It is found that

$$A = 1 \pm \sqrt{1 - 4M} = 1 \pm \sqrt{1 + d_0 + d_0^2\nu^2}$$
$$\equiv 1 + \mu_2.$$

We have that

$$b = A - a + c - 1 = A - a + d_0\sqrt{\nu^2 - g_0^2}$$

and

$$2N = cA - 2ab = cA - 2a\left(A - a + d_0\sqrt{\nu^2 - g_0^2}\right)$$
$$= 2a^2 - 2a\left(A + d_0\sqrt{\nu^2 - g_0^2}\right) + cA.$$

From the second equation, we see that

$$2a^2 - 2a\left(A + d_0\sqrt{\nu^2 - g_0^2}\right) + cA - 2N = 0.$$

Solving for a, we obtain that

$$a = \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{A^2 + 2Ad_0\sqrt{\nu^2 - g_0^2} + d_0^2(\nu^2 - g_0^2) - 2(cA - 2N)}\right\}$$
$$= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{A^2 + 2A(c-1) + d_0^2(\nu^2 - g_0^2) - 2cA + 4N}\right\}$$
$$= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{A(A-2) + 4N + d_0^2(\nu^2 - g_0^2)}\right\}$$
$$= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{(1+\mu_2)(-1+\mu_2) + 4N + d_0^2(\nu^2 - g_0^2)}\right\}$$
$$= \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2} \pm \sqrt{\mu_2^2 + 4N - 1 + d_0^2(\nu^2 - g_0^2)}\right\}.$$

We note that

$$\mu_2^2 + 4N - 1 + d_0^2(\nu^2 - g_0^2)$$
$$= 1 + d_0 + d_0^2\nu^2 + (g_0^2 - 2\nu^2)d_0^2 - d_0 - 1 + d_0^2\nu^2 - d_0^2 g_0^2$$
$$= 0.$$

Consequently, it is determined that

$$a = \frac{1}{2}\left\{A + d_0\sqrt{\nu^2 - g_0^2}\right\} = \frac{1}{2}\left\{1 + \mu_2 + d_0\sqrt{\nu^2 - g_0^2}\right\} = \frac{1}{2}\left[1 + d_0\sqrt{\nu^2 - g_0^2}\right] + \frac{\mu_2}{2}.$$

Hence it follows that

$$b = A - a + d_0\sqrt{\nu^2 - g_0^2} = 2a - a = a.$$

We note that

$$\frac{(c-1)\alpha - 1}{2} = \sqrt{\nu^2 - g_0^2} - \frac{1}{2},$$
$$a - b = 0,$$
$$c - b = 1 + d_0\sqrt{\nu^2 - g_0^2} - \frac{1}{2}\left[1 + d_0\sqrt{\nu^2 - g_0^2}\right] - \frac{\mu_2}{2}.$$
$$= \frac{1}{2}\left[1 + d_0\sqrt{\nu^2 - g_0^2}\right] - \frac{\mu_2}{2}.$$

Therefore we obtain that the radial eigenfunctions for fields of electric-type for Profile 3 are given by

$$T_n^{(1)}(x) = x^{\sqrt{\nu^2 - g_0^2} - \frac{1}{2}} \times {}_2F_1\left(\frac{1}{2}\left[1 + d_0\sqrt{\nu^2 - g_0^2}\right] + \frac{\mu_2}{2}, \frac{1}{2}\left[1 + d_0\sqrt{\nu^2 - g_0^2}\right] - \frac{\mu_2}{2}; 1 + d_0\sqrt{\nu^2 - g_0^2}; -x^\alpha\right). \qquad (248)$$

CHAPTER V

CONCLUSIONS

In this dissertation, we have analyzed aspects of both ray and wave formulations of electromagnetic scattering from radially inhomogeneous spheres and cylinders. The deviation of an incident ray from its original direction upon the sphere at angle of incidence i is related to a ray path integral where the integrand is singular at the lower limit. We have evaluated this integral for ten refractive index profiles and noted that evaluation of the integral could be extremely complicated for even simple refractive index profiles. For example, even a linear profile leads to a ray path integral evaluated in terms of incomplete elliptic integrals of the first and third kinds. Once this integral was calculated, we could calculate the deviation undergone by a ray that is directly transmitted through the sphere (in other words, the internal reflection is ignored). In some cases, we were able to determine restrictions on the refractive index profile parameters that allow transmission to occur.

With respect to the wave-theoretic formulation, the leading term of the backscattered field of a plane electromagnetic wave is obtained by using a modified Watson transformation of the exact solution for a specific refractive index profile. This analysis lead to the construction of the exact electromagnetic solutions for the asymptotic backscattered field produced by an incident plane wave. The radial eigenfunctions were evaluated exactly and asymptotically for the TE and TM modes. Consequently, the high-frequency backscattered field could be determined. In the appendices, we considered a variety of topics such as evaluating two integrals that were necessary for the calculation of the ray path integral for several profiles, verification of solutions that were utilized in the wave-theoretic analysis, and coupling two differential equations.

There is much potential for future work associated with the topics addressed here. In order to determine necessary and sufficient conditions on a refractive index profile that enable a zero-order bow (or higher-order bows) to exist, it appears that a much more general result for the derivative of the ray path integral is required. Obtaining such a result is a challenge due to the nature of the improper integral and its complicated dependence on the angle of incidence of the incoming ray. It is believed that deeper results from the theory of integral equations are required for such a result. It is hoped that the analysis presented here will be valuable in determining the properties of the refractive index profile that allow for the existence of a direction transmission (and higher-order) bows in radially inhomogeneous media. It is important to note that the wave analysis in Chapter 4 can readily be extended (in principle at least) to the radial eigenfunctions of the two other profiles that are based on the hypergeometric equation that we considered in Chapter 4. Furthermore, this analysis may also be applied to refractive index profiles that are based on other well-known equations such as Bessel's equation and Whittaker's equation.

REFERENCES

[1] J. A. Adam, 'Rainbows' in homogeneous and radially inhomogeneous spheres: connections with ray, wave, and potential scattering theory. *Advances in interdisciplinary mathematical research: applications to engineering, physical and life sciences*, Springer Proceedings in Mathematics & Statistics, vol. 37, Ed. Bourama Toni. New York, NY, USA: Springer, 2013.

[2] J. A. Adam, "Zero-order bows in radially inhomogeneous spheres: direct and inverse problems," *Applied Optics*, vol. 50, no. 28, pp. F50-F59, 2011.

[3] U. Leonhardt and T. Philbin, *Geometry and Light: The Science of Invisibility*. Mineola, NY, USA: Dover Publications, Inc., 2010.

[4] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*. Cambridge, United Kingdom: Cambridge University Press, 1999.

[5] M. Born and E. Wolf, *Principles of Optics*, 6th ed. Cambridge, United Kingdom: Cambridge University Press, p. 147, 1980.

[6] M. Born and E. Wolf, *Principles of Optics*, 7th ed. Cambridge, United Kingdom: Cambridge University Press, pp. 759-789, 1999.

[7] G. Mie, *Annalen der Physik*, vol. 330, no. 3, pp. 377-445, 1908.

[8] M. R. Vetrano, J. P. A. J. van Beeck, and M. L. Riethmuller, "Generalization of the rainbow Airy theory to nonuniform spheres," *Optics Letters*, vol. 30, no. 6, pp. 658-660, 2005.

[9] S. Kedenburg, M. Vieweg, T. Gissibl, and H. Giessen, "Linear refractive index and absorption measurements of nonlinear optical liquids in the visible and near-infrared spectral region," *Journal of the Optical Society of America*, vol. 2, no. 11, pp. 1588-1611, 2012.

[10] C. L. Brockman, Jr., "High Frequency Electromagnetic Wave Backscattering from Radially Inhomogeneous Dielectric Spheres," University of California, Los Angeles, PhD, Electrical Engineering, 1974.

[11] C. L. Brockman, Jr. and N. G. Alexopoulos, "Geometric optics of inhomogeneous particles: glory ray and the rainbow revisted," *Applied Optics*, vol. 16, no. 1, pp. 166-174, 1977.

[12] N. G. Alexopoulos, "Scattering from inhomogeneous cylindrically symmetric lenses with a line infinity in the index of refraction." *Journal of the Optical Society of America*, vol. 62, no. 9, pp. 1088-1094, 1972.

[13] N. G. Alexopoulos, "On the refractive properties of media with poles or zeros in the index of refraction," *IEEE Transactions on Antennas and Propagation*, vol. 22, no. 2, pp. 242-251, 1974.

[14] N. G. Alexopoulos, "Scattering from inhomogeneous lenses with singular points," *Journal of the Optical Society of America*, vol. 61, no. 7, pp. 876-878, 1971.

[15] G. Bekefi, *Radiation Processes in Plasmas*. New York, NY, USA: Wiley, p. 214, 1966.

[16] R. N. Gould and R. Burman, "Some electromagnetic wave functions for propagation in stratified media," *Journal of Atmospheric and Terrestrial Physics*, vol. 26, pp. 335-340, 1964.

[17] J. A. Adam and P. Laven, "Rainbows from inhomogeneous transparent spheres: a ray-theoretic approach," *Applied Optics*, vol. 46, no. 6, pp. 922-929, 2007.

[18] J. A. Lock, "Scattering of an electromagnetic plane wave by a Luneburg lens. I. Ray theory," *Journal of the Optical Society of America A*, vol. 25, no. 12, pp. 2971-2979, 2008.

[19] M. J. Adams, *An Introduction to Optical Waveguides*. New York, NY, USA: Wiley, 1981.

[20] W. Magnus, F. Oberhettinger, and R. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*. New York, NY, USA: Springer, 1966.

[21] B. S. Westcott, "Electromagnetic wave propagation in spherically stratified isotropic media," *Electronic Letters*, vol. 4, no. 25, pp. 572-573, 1968.

[22] P. L. E. Uslenghi and V. H. Weston, "High-frequency scattering from a metal-like dielectric lens," *Applied Sciences Research*, vol. 23, pp. 147-163, 1970.

[23] P. L. E. Uslenghi, "Electromagnetic and optical behavior of two classes of dielectric lenses," *IEEE Transactions on Antennas and Propagation*, vol. 17, no. 2, pp. 235-236, 1969.

[24] J. M. C. Scott, "On asymptotic series for the radar scattering cross section of a spherical target," Report T/M 30, Atomic Energy Research Establishment, Harwell, Berks., England, 1949.

[25] S. N. Samaddar, "Scattering of plane electromagnetic waves by radially inhomogeneous infinite cylinders," *Il Nuovo Cimento B*, vol. 66, no. 1, pp. 33-50, 1970.

APPENDIX A

DERIVATION OF TWO INTEGRALS

We derive formulae for the evaluation of two integrals. First, consider

$$f(r) = \frac{1}{r\sqrt{Ar^2 + Br + C}}.$$

We have that

$$\begin{aligned}
f(r) &= \frac{2\sqrt{-C}}{r}\frac{1}{\sqrt{-4C(Ar^2 + Br + C)}} \\
&= \frac{2\sqrt{-C}}{r}\frac{1}{\sqrt{-4ACr^2 - 4BCr - 4C^2}} \\
&= \frac{2\sqrt{-C}}{r}\frac{1}{\sqrt{r^2(B^2 - 4AC) - 4C^2 - 4BCr - B^2r^2}} \\
&= \frac{2\sqrt{-C}}{r}\frac{1}{\sqrt{r^2(B^2 - 4AC) - (2C + Br)^2}} \\
&= \frac{1}{\sqrt{-C}}\frac{1}{\sqrt{1 - \frac{(2C+Br)^2}{r^2(B^2-4AC)}}}\frac{1}{\sqrt{B^2 - 4AC}}\left[-\frac{2C}{r^2}\right] \\
&= \frac{1}{\sqrt{-C}}\frac{1}{\sqrt{1 - \left[\frac{2C+Br}{r\sqrt{B^2-4AC}}\right]^2}}\frac{1}{\sqrt{B^2 - 4AC}}\left[-\frac{2C}{r^2}\right]. \qquad \text{(A.1)}
\end{aligned}$$

Integration of equation (A.1) with respect to r gives us that

$$\begin{aligned}
I(r) &= \int f(r)dr \\
&= \frac{1}{\sqrt{-C}}\int \left\{ \frac{1}{\sqrt{1 - \left[\frac{2C+Br}{r\sqrt{B^2-4AC}}\right]^2}}\frac{1}{\sqrt{B^2 - 4AC}}\left[-\frac{2C}{r^2}\right] \right\} dr. \qquad \text{(A.2)}
\end{aligned}$$

Let

$$v = \frac{2C + Br}{r\sqrt{B^2 - 4AC}}. \qquad \text{(A.3)}$$

Then it follows that

$$dv = \frac{1}{\sqrt{B^2 - 4AC}}\left[\frac{r(B) - (2C + Br)}{r^2}\right]dr = \frac{-2C}{r^2\sqrt{B^2 - 4AC}}dr. \tag{A.4}$$

Utilizing equations (A.3) and (A.4), equation (A.2) becomes

$$\begin{aligned} I(r) &= \frac{1}{\sqrt{-C}}\int \frac{dv}{\sqrt{1 - v^2}} = \frac{1}{\sqrt{-C}} \arcsin v + C_1 \\ &= \frac{1}{\sqrt{-C}} \arcsin\left[\frac{2C + Br}{r\sqrt{B^2 - 4AC}}\right] + C_1, \end{aligned} \tag{A.5}$$

where C_1 is a constant.

Next, consider

$$g(r) = \frac{1}{\sqrt{Dr^2 + Er + F}}.$$

We have that

$$\begin{aligned} g(r) &= \sqrt{D}\frac{1}{2\sqrt{D(Dr^2 + Er + F)} + 2Dr + E}\left[\frac{2Dr + E + 2\sqrt{D(Dr^2 + Er + F)}}{\sqrt{D(Dr^2 + Er + F)}}\right] \\ &= \frac{1}{\sqrt{D}}\frac{1}{2\sqrt{D(Dr^2 + Er + F)} + 2Dr + E}\left[\frac{D(2Dr + E)}{\sqrt{D(Dr^2 + Er + F)}} + 2D\right]. \end{aligned} \tag{A.6}$$

Let

$$w = 2\sqrt{D(Dr^2 + Er + F)} + 2Dr + E. \tag{A.7}$$

Then we see that

$$dw = \left[\frac{D(2Dr + E)}{\sqrt{D(Dr^2 + Er + F)}} + 2D\right]dr. \tag{A.8}$$

Utilizing equations (A.7) and (A.8), we find that

$$\begin{aligned} J(r) &= \int g(r)dr = \frac{1}{\sqrt{D}}\int \frac{dw}{w} = \frac{1}{\sqrt{D}} \log w + C_2 \\ &= \frac{1}{\sqrt{D}} \log\left[2\sqrt{D(Dr^2 + Er + F)} + 2Dr + E\right] + C_2, \end{aligned} \tag{A.9}$$

where C_2 is a constant.

APPENDIX B

SOLUTION OF A SYSTEM OF NON-LINEAR EQUATIONS

Consider the following nonlinear system:

$$ax^2 + bxy + c = 0, \tag{B.1a}$$

$$ay^2 + bxy + d = 0. \tag{B.1b}$$

Applying the quadratic formula to equation (B.1b) to solve for y, we determine that

$$y = \frac{-bx \pm \sqrt{b^2x^2 - 4ad}}{2a}. \tag{B.2}$$

Utilizing equation (B.2) in equation (B.1a) yields that

$$ax^2 + bx\left[\frac{-bx \pm \sqrt{b^2x^2 - 4ad}}{2a}\right] + c = 0.$$

Then it follows that

$$\begin{aligned}
2a^2x^2 - b^2x^2 \pm bx\sqrt{b^2x^2 - 4ad} + 2ac &= 0 \\
\pm bx\sqrt{b^2x^2 - 4ad} &= b^2x^2 - 2a^2x^2 - 2ac \\
b^2x^2(b^2x^2 - 4ad) &= (b^2x^2 - 2a^2x^2 - 2ac)^2 \\
b^4x^4 - 4ab^2dx^2 &= b^4x^4 - 2a^2b^2x^4 - 2ab^2cx^2 \\
&\quad - 2a^2b^2x^4 + 4a^4x^4 + 4a^3cx^2 \\
&\quad - 2ab^2cx^2 + 4a^3cx^2 + 4a^2c^2
\end{aligned}$$

$$a(b^2 - a^2)x^4 + (b^2c - b^2d - 2a^2c)x^2 - ac^2 = 0. \tag{B.3}$$

Applying the quadratic formula to equation (B.3), it is found that

$$x^2 = \frac{-(b^2c - b^2d - 2a^2c) \pm \sqrt{(b^2c - b^2d - 2a^2c)(b^2c - b^2d - 2a^2c) + 4a^2c^2(b^2 - a^2)}}{2a(b^2 - a^2)}. \tag{B.4}$$

We will simplify the expression under the radical in equation (B.4):

$$(b^2c - b^2d - 2a^2c)(b^2c - b^2d - 2a^2c) + 4a^2c^2(b^2 - a^2)$$

$$\begin{aligned}
&= b^4c^2 - b^4cd - 2a^2b^2c^2 - b^4cd + b^4d^2 + 2a^2b^2cd - 2a^2b^2c^2 \\
&\qquad + 2a^2b^2cd + 4a^4c^2 + 4a^2c^2(b^2 - a^2) \\
&= b^4c^2 - 2b^4cd + b^4d^2 + 4a^2b^2cd \\
&= b^2(b^2c^2 - 2b^2cd + b^2d^2 + 4a^2cd).
\end{aligned}$$

Using this result in equation (B.4) gives us that

$$x^2 = \frac{-(b^2c - b^2d - 2a^2c) + b\sqrt{4a^2cd + b^2c^2 - 2b^2cd + b^2d^2}}{2a(b^2 - a^2)}$$

$$x^2 = \frac{b^2c - b^2d - 2a^2c - b\sqrt{4a^2cd + b^2c^2 - 2b^2cd + b^2d^2}}{2a(a^2 - b^2)}. \tag{B.5}$$

Therefore, we discover that

$$x = \sqrt{\frac{b^2c - b^2d - 2a^2c - b\sqrt{4a^2cd + b^2c^2 - 2b^2cd + b^2d^2}}{2a(a^2 - b^2)}}, \tag{B.6}$$

where we have taken the positive root since x is assumed to be positive in our problem.

Now we find y. Multiplying equation (B.1b) by -1 and adding equation (B.1a) to the resulting equation yields that

$$a(x^2 - y^2) + c - d = 0.$$

Then it is seen that

$$x^2 - y^2 = \frac{d - c}{a}.$$

Thus, we obtain that

$$y^2 = x^2 - \frac{d - c}{a} = x^2 + \frac{c - d}{a}. \tag{B.7}$$

We will use equation (B.5) in equation (B.7). To that end, we have that

$$y^2 = \frac{b^2c - b^2d - 2a^2c + 2(a^2 - b^2)(c - d) - b\sqrt{4a^2cd + b^2c^2 - 2b^2cd + b^2d^2}}{2a(a^2 - b^2)}$$

$$y^2 = \frac{b^2c - b^2d - 2a^2c + 2a^2c - 2b^2c - 2a^2d + 2b^2d - b\sqrt{4a^2cd + b^2c^2 - 2b^2cd + b^2d^2}}{2a(a^2 - b^2)}$$

$$y^2 = \frac{b^2d - b^2c - 2a^2d - b\sqrt{4a^2cd + b^2c^2 - 2b^2cd + b^2d^2}}{2a(a^2 - b^2)}. \tag{B.8}$$

As a result, it is determined that

$$y = \sqrt{\frac{b^2d - b^2c - 2a^2d - b\sqrt{4a^2cd + b^2c^2 - 2b^2cd + b^2d^2}}{2a(a^2 - b^2)}}. \tag{B.9}$$

where we have taken the positive root since y is assumed to be positive in our problem.

We note that adding equations (B.1a) and (B.1b) gives us that

$$a(x^2 + y^2) + 2bxy + c + d = 0.$$

Consequently, it is found that

$$a(x^2 + y^2) + 2bxy = -(c + d). \tag{B.10}$$

APPENDIX C

VERIFICATION OF SOLUTIONS FROM WESTCOTT (1968)

In a paper by Westcott [21], he provides solutions for several wavenumbers $m(r)$ in the medium to the differential equation

$$\frac{d^2u}{dr^2} + \left\{ m_{eff}^2(r) - \frac{n(n+1)}{r^2} \right\} u = 0, \tag{C.1}$$

where

$$m_{eff}^2(r) = m^2(r) \qquad \text{for fields of magnetic-type, and} \tag{C.2a}$$

$$m_{eff}^2(r) = m^2(r) - m(r)\frac{d^2}{dr^2}\left\{\frac{1}{m(r)}\right\} \qquad \text{for fields of electric-type.} \tag{C.2b}$$

We note that $m(r) = k\tilde{n}(r)$, where k is the free space wavenumber and $\tilde{n}(r)$ is the refractive index profile. In this appendix, we will verify that the solutions given by Westcott satisfy equation (C.1). Since many of the second derivative calculations of the provided solutions are rather long, we will only summarize the results for the second derivative that we derived. We will denote solutions of equation (C.1) for fields of electric-type by $u_E(r)$ and for fields of magnetic-type by $u_M(r)$.

C.1 PROFILE BASED ON BESSEL'S EQUATION

Consider the wavenumber [21]

$$m(r) = ar^b, \tag{C.3}$$

where a and b are constants. Equation (C.1) with equation (C.3) has solution [21]

$$u(r) \propto r^{\frac{1}{2}} Z_\nu(z), \tag{C.4}$$

where Z_ν denotes any solution of Bessel's equation of order ν and $z = \frac{a}{1+b} r^{b+1}$. Differentiating equation (C.4) twice, we obtain that

$$u''(r) \propto r^{-2}\left[(b+1)^2\nu^2 - (b+1)^2 z^2 - \frac{1}{4}\right] u(r). \tag{C.5}$$

The order ν is different for fields of electric- and magnetic-type.

C.1.1 Fields of Electric-Type

For this case, we have that [21]

$$\nu^2 = \frac{b}{1+b} + \left\{\frac{2n+1}{2(1+b)}\right\}^2. \tag{C.6}$$

First, it is noted that (using equation (C.2b))

$$m_{eff}^2(r) = a^2 r^{2b} - \frac{b(b+1)}{r^2}. \tag{C.7}$$

Using equations (C.5) and (C.6), it is found that

$$\begin{aligned} u_E''(r) &\propto r^{-2}\left[b(b+1) + \frac{(2n+1)^2}{4} - a^2 r^{2(b+1)} - \frac{1}{4}\right] u_E(r) \\ &= r^{-2}[b(b+1) + n(n+1) - a^2 r^{2(b+1)}] u_E(r). \end{aligned} \tag{C.8}$$

Using equations (C.7) and (C.8) in equation (C.1), we discover that

$$u_E''(r) + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\} u_E(r) \propto 0,$$

which verifies that the solution of equation (C.1) for the wavenumber in equation (C.3) is given by equation (C.4) where ν is defined by equation (C.6) for fields of electric-type.

C.1.2 Fields of Magnetic-Type

For this case, it is given that [21]

$$\nu = \frac{n+\frac{1}{2}}{b+1}. \tag{C.9}$$

Using equations (C.2a) and (C.3), we find that

$$m_{eff}^2(r) = m^2(r) = a^2 r^{2b}. \tag{C.10}$$

Upon using equations (C.5) and (C.9), it is determined that

$$u_M''(r) \propto r^{-2}\left[\left(n+\frac{1}{2}\right)^2 - a^2 r^{2(b+1)} - \frac{1}{4}\right] u_M(r) = r^{-2}[n(n+1) - a^2 r^{2(b+1)}] u_M(r). \tag{C.11}$$

Utilizing equations (C.10) and (C.11) in equation (C.1), we obtain that

$$u''_M(r) + \left\{ m^2_{eff}(r) - \frac{n(n+1)}{r^2} \right\} u_M(r) \propto 0,$$

which verifies that the solution of equation (C.1) for the wavenumber in equation (C.3) is given by equation (C.4) where ν is defined by equation (C.9) for fields of magnetic-type.

C.2 PROFILES BASED ON WHITTAKER'S EQUATION

C.2.1 Profile 1

First, consider the wavenumber [21]

$$m(r) = \frac{a}{r \ln br}, \tag{C.12}$$

where a and b are constants.

For fields of electric-type, independent solutions are given by [21]

$$u_E(r) \propto r^{\frac{1}{2}} W_{\pm c,d}\{\pm(2n+1)\ln br\}, \tag{C.13}$$

where $c = -(2n+1)^{-1}, d = \sqrt{\frac{1}{4} - a^2}$, and $W_{\pm c,d}(z)$ is Whittaker's function. Differentiating equation (C.13) twice with respect to r, we find that

$$\begin{aligned}
u''_E(r) &\propto r^{-2}\{\pm(2n+1)\ln br\}^{-1}\left\{(2n+1)^2\left[\mp c + \left(d^2 - \frac{1}{4}\right)\{\pm(2n+1)\ln br\}^{-1}\right]\right. \\
&\qquad \left. \pm\, n(n+1)(2n+1)\ln br\right\} u_E(r) \\
&= r^{-2}\{(2n+1)\ln br\}^{-1}\left\{(2n+1)^2\left[-c + \left(d^2 - \frac{1}{4}\right)\{(2n+1)\ln br\}^{-1}\right]\right. \\
&\qquad \left. +\, n(n+1)(2n+1)\ln br\right\} u_E(r) \\
&= r^{-2}\left\{(2n+1)\{\ln br\}^{-1}\left[-c + \left(d^2 - \frac{1}{4}\right)\{(2n+1)\ln br\}^{-1}\right]\right. \\
&\qquad \left. +\, n(n+1)\right\} u_E(r) \\
&= r^{-2}\left\{-\frac{(2n+1)c}{\ln br} + \frac{(d^2 - \frac{1}{4})}{(\ln br)^2} + n(n+1)\right\} u_E(r)
\end{aligned} \tag{C.14}$$

$$= r^{-2}\left\{\frac{1}{\ln br} - \frac{a^2}{(\ln br)^2} + n(n+1)\right\}u_E(r),$$

where the definitions for c and d from the previous page have been used. We also find that

$$m_{eff}^2(r) = \frac{a^2}{r^2(\ln br)^2} - \frac{1}{r^2 \ln br}.$$

Hence, it follows that

$$u_E''(r) + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\}u_E(r) \propto 0.$$

For fields of magnetic-type, independent solutions are given by [21]

$$u_M(r) \propto (r \ln br)^{\frac{1}{2}} Z_\nu\left\{\pm i\left(n + \frac{1}{2}\right)\ln br\right\}, \tag{C.15}$$

with $\nu = \sqrt{\frac{1}{4} - a^2}$ and $i = \sqrt{-1}$. Differentiating equation (C.15) twice with respect to r yields that

$$\begin{aligned} u_M''(r) &\propto (r \ln br)^{-2}\left[\left(\nu^2 - \frac{1}{4}\right) + n(n+1)(\ln br)^2\right]u_M(r) \\ &= \left[(r \ln br)^{-2}\left(\nu^2 - \frac{1}{4}\right) + n(n+1)r^{-2}\right]u_M(r) \\ &= [-a^2(r \ln br)^{-2} + n(n+1)r^{-2}]u_M(r). \end{aligned}$$

It is seen that

$$m_{eff}^2(r) = a^2(r \ln br)^{-2}.$$

Accordingly, it is found that

$$u_M''(r) + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\}u_M(r) \propto 0.$$

C.2.2 Profile 2

Now, consider the wavenumber [21]

$$m(r) = \frac{a}{r\sqrt{\ln br}}, \tag{C.16}$$

where a and b are constants.

For fields of electric-type, the independent solutions of equation (C.1) for the wavenumber in equation (C.16) are given by [21]

$$u_E(r) \propto r^{\frac{1}{2}} W_{\pm c,0}\{\pm(2n+1)\ln br\}, \tag{C.17}$$

with

$$c = \frac{a^2 - \frac{1}{2}}{2n+1}. \tag{C.18}$$

Using equation (C.14) with c given by equation (C.18) and $d = 0$, we obtain that the second derivative of equation (C.17) is

$$u_E''(r) \propto r^{-2}\left\{\frac{(\frac{1}{2} - a^2)}{\ln br} - \frac{1}{4(\ln br)^2} + n(n+1)\right\}u_E(r).$$

It is noted that (from equation (C.2b))

$$m_{eff}^2(r) = \frac{1}{r^2}\left\{\frac{a^2}{\ln br} - \frac{1}{2\ln br} + \frac{1}{4(\ln br)^2}\right\}.$$

As a result, we discover that

$$u_E''(r) + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\}u_E(r) \propto 0.$$

For fields of magnetic-type, the independent solutions of equation (C.1) for the wavenumber in equation (C.16) are [21]

$$u_M(r) \propto r^{\frac{1}{2}} W_{\pm c, \frac{1}{2}}\{\pm(2n+1)\ln br\}, \tag{C.19}$$

with

$$c = \frac{a^2}{2n+1}. \tag{C.20}$$

Upon utilizing equation (C.14), with c given by equation (C.20) and $d = \frac{1}{2}$, we find that the second derivative of equation (C.19) is

$$u_M''(r) \propto r^{-2}\left\{-\frac{a^2}{\ln br} + n(n+1)\right\}u_M(r).$$

We have that

$$m_{eff}^2(r) = m^2(r) = \frac{a^2}{r^2 \ln br}.$$

Accordingly, it is obtained that

$$u_M''(r) + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\}u_M(r) \propto 0.$$

C.3 PROFILES BASED ON THE HYPERGEOMETRIC EQUATION

For each of the three profiles that we will consider in this section, independent solutions for equation (C.1) for fields of electric- and magnetic-type, respectively, may be written as (where $z = -\beta r^\alpha$ with the constants α and β) [21]

$$
\begin{aligned}
u_E(r) \propto\ & a_1 r^{\frac{1+(c-1)\alpha}{2}} (1-z)^{\frac{a-b+1}{2}} {}_2F_1(a, c-b; c; z) \\
& + a_2 r^{\frac{1-(c-1)\alpha}{2}} (1-z)^{\frac{a-b+1}{2}} {}_2F_1(1+a-c, 1-b; 2-c; z) \qquad \text{(C.21a)} \\
u_M(r) \propto\ & a_1 r^{\frac{1+(c-1)\alpha}{2}} (1-z)^{\frac{A}{2}} {}_2F_1(a, b; c; z) \\
& + a_2 r^{\frac{1-(c-1)\alpha}{2}} (1-z)^{\frac{A}{2}} {}_2F_1(1+a-c, 1+b-c; 2-c; z), \qquad \text{(C.21b)}
\end{aligned}
$$

where a_1 and a_2 are constants and ${}_2F_1(a, b; c; z)$ is Gauss's hypergeometric function. We note that the constants a, b, and c will be different for each profile. Let

$$
\begin{aligned}
L &= \frac{1}{2}c - \frac{1}{4}c^2, \\
M &= \frac{1}{2}A - \frac{1}{4}A^2, \\
N &= \frac{1}{2}cA - ab, \\
A &= a + b - c + 1.
\end{aligned} \qquad \text{(C.22)}
$$

The constants a, b, and c in equations (C.21a) and (C.21b) may be determined by equation (C.22) and another set of equations for L, M, and N that will be different for each profile. Differentiating equations (C.21a) and (C.21b) with respect to r, after several steps, it is obtained that

$$
\begin{aligned}
u_E''(r) \propto r^{-2} u_E(r) \Bigg\{ & \frac{\left[\left(N - \frac{1}{2}\right)\alpha^2 + \frac{1}{2}\right] z + \left[\left(\frac{1}{4} - M\right)\alpha^2 - \frac{1}{4}\right] z^2}{(1-z)^2} \\
& - \frac{\left[\left(L - \frac{1}{4}\right)\alpha^2 + \frac{1}{4}\right]}{(1-z)^2} \Bigg\} \qquad \text{(C.23a)}
\end{aligned}
$$

and

$$
u_M''(r) \propto r^{-2} u_M(r) \left\{ \frac{[N - M]\alpha^2 z^2 - N\alpha^2 z}{(1-z)^2} + \left(\frac{1}{4} - L\right)\alpha^2 - \frac{1}{4} \right\}. \qquad \text{(C.23b)}
$$

We will now consider three profiles and show that the solutions of equation (C.1) are given by equations (C.21a) and (C.21b) for fields of electric- and magnetic-type, respectively.

C.3.1 Profile 1

First, consider the wavenumber [21]

$$m(r) = \frac{a_0}{r(1+\beta r^{\alpha})}, \tag{C.24}$$

where a_0 is a constant.

For fields of electric-type for the wavenumber in equation (C.24), we are given that [21]

$$L = \left\{a_0^2 - \frac{1}{4}\left(2n+1\right)^2\right\}\alpha^{-2} + \frac{1}{4},$$
$$M = -\frac{3}{4} - \alpha^{-1} - \frac{1}{4}\left(2n+1\right)^2\alpha^{-2},$$

and

$$N = -\frac{1}{2}\left(2n+1\right)^2\alpha^{-2} - \frac{1}{2} - \alpha^{-1}.$$

Using these definitions, it is found that (using equation (C.23a))

$$\begin{aligned} u_E''(r) &\propto \frac{r^{-2}u_E(r)}{(1-z)^2}\Big\{[-2n(n+1) - \alpha(\alpha+1)]z + [\alpha(\alpha+1) + n(n+1)]z^2 \\ &\qquad - a_0^2 + n(n+1)\Big\} \\ &= \frac{r^{-2}u_E(r)}{(1-z)^2}\Big\{-a_0^2 + n(n+1)[1-2z+z^2] + \alpha(\alpha+1)z(z-1)\Big\} \\ &= r^{-2}u_E(r)\left\{-\frac{a_0^2}{(1-z)^2} + n(n+1) - \frac{\alpha(\alpha+1)z}{1-z}\right\}. \end{aligned}$$

Using equation (C.2b), it is also found that

$$m_{eff}^2(r) = \frac{a_0^2}{r^2(1-z)^2} + \frac{\alpha(\alpha+1)z}{r^2(1-z)}.$$

Thus, it follows that

$$u_E''(r) + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\}u_E(r) \propto 0.$$

For fields of magnetic-type for the wavenumber in equation (C.24), we are given that [21]

$$L = \frac{1}{4} + \left\{a_0^2 - \frac{1}{4}(2n+1)^2\right\}\alpha^{-2},$$

$$M = a_0^2 \alpha^{-2},$$

and

$$N = 2a_0^2 \alpha^{-2}.$$

Using these definitions for L, M, N, and equation (C.23b), it is discovered that

$$u_M''(r) \propto r^{-2} u_M(r) \left\{ \frac{a_0^2 z^2 - 2a_0^2 z}{(1-z)^2} - a_0^2 + n(n+1) \right\}.$$

From equation (C.2a), it is also discovered that

$$m_{eff}^2(r) = \frac{a_0^2}{r^2(1-z)^2}.$$

Therefore we obtain that

$$u_M''(r) + \left\{ m_{eff}^2(r) - \frac{n(n+1)}{r^2} \right\} u_M(r) \propto 0.$$

C.3.2 Profile 2

Next, consider the wavenumber [21]

$$m(r) = \frac{a_0 r^{\frac{\alpha}{2}-1}}{1+\beta r^{\alpha}}, \tag{C.25}$$

where a_0 is a constant.

For fields of electric-type for the wavenumber in equation (C.25), we are given that [21]

$$L = \frac{1}{2}\alpha^{-1} - \frac{1}{4}(2n+1)^2 \alpha^{-2},$$
$$M = -\frac{1}{2}\alpha^{-1} - \frac{1}{4}(2n+1)^2 \alpha^{-2},$$

and

$$N = \left\{ a_0^2 \beta^{-1} - \frac{1}{2}(2n+1)^2 \right\} \alpha^{-2}.$$

Utilizing these definitions (and equation (C.23a)), it is obtained that

$$u_E''(r) \propto \frac{r^{-2}u_E(r)}{(1-z)^2}\Bigg\{\left[a_0^2\beta^{-1} - 2n(n+1) - \frac{1}{2}\alpha^2\right]z + \left[\frac{1}{4}\alpha^2 + \frac{1}{2}\alpha + n(n+1)\right]z^2 - \frac{1}{2}\alpha + n(n+1) + \frac{1}{4}\alpha^2\Bigg\}$$

$$= \frac{r^{-2}u_E(r)}{(1-z)^2}\Bigg\{\frac{1}{4}\alpha^2\left[1 - 2z + z^2\right] + n(n+1)[1-2z+z^2] + a_0^2\beta^{-1}z + \frac{1}{2}\alpha\left[z^2 - 1\right]\Bigg\}$$

$$= r^{-2}u_E(r)\left\{\frac{1}{4}\alpha^2 + n(n+1) + \frac{a_0^2\beta^{-1}z}{(1-z)^2} + \frac{\frac{1}{2}\alpha(z^2-1)}{(1-z)^2}\right\}$$

$$= r^{-2}u_E(r)\left\{\frac{1}{4}\alpha^2 + n(n+1) + \frac{a_0^2\beta^{-1}z}{(1-z)^2} - \frac{\frac{1}{2}\alpha(1+z)}{1-z}\right\}.$$

Using equation (C.2b), it is also obtained that

$$m_{eff}^2(r) = \frac{a_0^2 r^{\alpha} r^{-2}}{(1-z)^2} - \frac{r^{-2}}{1-z}\left\{-\left(\frac{\alpha^2}{4} + \frac{\alpha}{2}\right)z - \frac{\alpha}{2} + \frac{\alpha^2}{4}\right\}$$

$$= r^{-2}\left\{-\frac{a_0^2\beta^{-1}z}{(1-z)^2} - \frac{1}{1-z}\left[\frac{\alpha^2}{4}\left(1-z\right) - \frac{\alpha}{2}\left(z+1\right)\right]\right\}$$

$$= r^{-2}\left\{-\frac{a_0^2\beta^{-1}z}{(1-z)^2} - \frac{1}{4}\alpha^2 + \frac{\alpha}{2}\frac{(1+z)}{1-z}\right\}.$$

As a result, we see that

$$u_E''(r) + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\}u_E(r) \propto 0.$$

For fields of magnetic-type for the wavenumber in equation (C.25), we are given that [21]

$$L = \frac{1}{4} - \frac{1}{4}(2n+1)^2\alpha^{-2},$$
$$M = -a_0^2\beta^{-1}\alpha^{-2},$$

and

$$N = -a_0^2\beta^{-1}\alpha^{-2}.$$

Upon using these definitions for L, M, N, and equation (C.23b), it is found that

$$u''_M(r) \propto r^{-2} u_M(r) \left\{ \frac{a_0^2 \beta^{-1} z}{(1-z)^2} + n(n+1) \right\}.$$

We also find that

$$m_{eff}^2(r) = \frac{a_0^2 r^{\alpha} r^{-2}}{(1-z)^2} = -\frac{a_0^2 \beta^{-1} z}{(1-z)^2} r^{-2}.$$

Hence, it is determined that

$$u''_M(r) + \left\{ m_{eff}^2(r) - \frac{n(n+1)}{r^2} \right\} u_M(r) \propto 0.$$

C.3.3 Profile 3

Finally, consider the wavenumber [21]

$$m(r) = \frac{a_0}{r\sqrt{1+\beta r^{\alpha}}}, \tag{C.26}$$

where a_0, α, and β are constants.

For fields of electric-type for the wavenumber in equation (C.26), it is given that [21]

$$L = \left\{ a_0^2 - \frac{1}{4}(2n+1)^2 \right\} \alpha^{-2} + \frac{1}{4},$$
$$M = -\frac{1}{4}(2n+1)^2 \alpha^{-2} - \frac{1}{2}\alpha^{-1},$$

and

$$N = \left\{ a_0^2 - \frac{1}{2}(2n+1)^2 \right\} \alpha^{-2} - \frac{1}{2}\alpha^{-1}.$$

Using the above definitions (and equation (C.23a)), we obtain that

$$u''_E(r) \propto r^{-2} u_E(r) \left\{ \frac{\left[a_0^2 - 2n(n+1) - \frac{1}{2}\alpha(\alpha+1) \right] z + \left[\frac{1}{4}\alpha^2 + n(n+1) + \frac{1}{2}\alpha \right] z^2}{(1-z)^2} + \frac{-a_0^2 + n(n+1)}{(1-z)^2} \right\}$$

$$= r^{-2} u_E(r) \left\{ -\frac{a_0^2}{1-z} + n(n+1) + \frac{\left[\frac{1}{4}\alpha^2 + \frac{1}{2}\alpha \right] z^2 - \frac{1}{2}\alpha(\alpha+1) z}{(1-z)^2} \right\}$$

$$= r^{-2}u_E(r)\left\{-\frac{a_0^2}{1-z}+n(n+1)-\frac{\frac{1}{2}\alpha z}{1-z}+\frac{\frac{1}{4}\alpha^2 z^2-\frac{1}{2}\alpha^2 z}{(1-z)^2}\right\}.$$

Upon using equation (C.2b), it is also obtained that

$$\begin{aligned} m_{eff}^2(r) &= r^{-2}\left\{\frac{a_0^2}{1-z}+\frac{\frac{1}{4}\alpha^2 z^2}{(1-z)^2}+\frac{\frac{1}{2}\alpha z(\alpha+1)}{1-z}\right\} \\ &= r^{-2}\left\{\frac{a_0^2}{1-z}+\frac{\frac{1}{2}\alpha z}{1-z}+\frac{\frac{1}{4}\alpha^2 z^2+\frac{1}{2}\alpha^2 z(1-z)}{(1-z)^2}\right\} \\ &= r^{-2}\left\{\frac{a_0^2}{1-z}+\frac{\frac{1}{2}\alpha z}{1-z}+\frac{\frac{1}{2}\alpha^2 z-\frac{1}{4}\alpha^2 z^2}{(1-z)^2}\right\}. \end{aligned}$$

Thus, it follows that

$$u_E''(r)+\left\{m_{eff}^2(r)-\frac{n(n+1)}{r^2}\right\}u_E(r)\propto 0.$$

For fields of magnetic-type for the wavenumber in equation (C.26), we are given that [21]

$$L=\frac{1}{4}+\left\{a_0^2-\frac{1}{4}(2n+1)^2\right\}\alpha^{-2},$$
$$M=0,$$

and

$$N=a_0^2\alpha^{-2}.$$

Utilizing these definitions for L, M, N, and equation (C.23b), it is found that

$$\begin{aligned} u_M''(r) &\propto r^{-2}u_M(r)\left\{\frac{a_0^2(z^2-z)}{(1-z)^2}-a_0^2+n(n+1)\right\} \\ &= r^{-2}u_M(r)\left\{-\frac{a_0^2 z}{1-z}-a_0^2+n(n+1)\right\} \\ &= r^{-2}u_M(r)\left\{-\frac{a_0^2}{1-z}+n(n+1)\right\}. \end{aligned}$$

Using equation (C.2a), we also find that

$$m_{eff}^2(r)=m^2(r)=\frac{a_0^2 r^{-2}}{1-z}.$$

Consequently, we discover that

$$u_M''(r)+\left\{m_{eff}^2(r)-\frac{n(n+1)}{r^2}\right\}u_M(r)\propto 0.$$

APPENDIX D

RELATING WESTCOTT (1968) TO USLENGHI AND WESTON (1970)

In this appendix, we show how the set of differential equations found in [21] are related to equations (135) and (136) (which are the ones found in [22]). From [21], we have the following differential equation

$$\frac{d^2u}{dr^2} + \left\{ m_{eff}^2(r) - \frac{n(n+1)}{r^2} \right\} u = 0, \tag{D.1}$$

where

$$m_{eff}^2 = m^2(r) \qquad \text{for fields of magnetic-type and} \tag{D.2}$$

$$m_{eff}^2 = m^2(r) - m(r)\frac{d^2}{dr^2}\left\{ \frac{1}{m(r)} \right\} \qquad \text{for fields of electric-type.} \tag{D.3}$$

The above is in terms of r. We want to show that equation (D.1) with equations (D.2) and (D.3) is equivalent to equations (135) and (136), respectively. It is noted that equations (135) and (136) are in terms of x. Recall that

$$x = \frac{r}{\hat{a}}.$$

Then it follows that

$$\frac{d}{dr} = \frac{d}{dx}\frac{dx}{dr} = \frac{1}{\hat{a}}\frac{d}{dx}$$

and

$$\frac{d^2}{dr^2} = \frac{1}{\hat{a}^2}\frac{d^2}{dx^2}.$$

First, consider fields of magnetic-type and denote the solution of equation (D.1) as $u_M(r)$. Using equation (D.2), we have the differential equation

$$\frac{d^2u_M}{dr^2} + \left\{ m^2(r) - \frac{n(n+1)}{r^2} \right\} u_M(r) = 0. \tag{D.4}$$

Let

$$u_M(r) \equiv S_n(x) \tag{D.5}$$

and

$$m(r) \equiv kR(x). \tag{D.6}$$

Using equations (D.5) and (D.6), writing equation (D.4) in terms of x yields that

$$\frac{1}{\hat{a}^2}\frac{d^2 S_n}{dx^2} + \left\{[kR(x)]^2 - \frac{n(n+1)}{(x\hat{a})^2}\right\}S_n(x) = 0.$$

Hence, we have that

$$S_n''(x) + \left\{[k\hat{a}R(x)]^2 - \frac{n(n+1)}{x^2}\right\}S_n(x) = 0,$$

which is equation (135).

Next consider fields of electric-type and denote the solution of equation (D.1) by $u_E(r)$. We have the following differential equation

$$\frac{d^2 u_E}{dr^2} + \left\{m_{eff}^2(r) - \frac{n(n+1)}{r^2}\right\}u_E(r) = 0, \tag{D.7}$$

where $m_{eff}^2(r)$ is now given by equation (D.3). Let

$$u_E(r) \equiv c_0[R(x)]^{-1}T_n(x). \tag{D.8}$$

Using equations (D.3) and (D.6), writing $m_{eff}^2(r)$ in terms of x, it is seen that

$$\begin{aligned}
m_{eff}^2(r) &= m^2(r) - m(r)\frac{d^2}{dr^2}\left\{\frac{1}{m(r)}\right\} \\
&= [kR(x)]^2 - \frac{kR(x)}{\hat{a}^2}\frac{d^2}{dx^2}\left[\frac{1}{kR(x)}\right] \\
&= [kR(x)]^2 - \frac{R(x)}{\hat{a}^2}\frac{d^2}{dx^2}\left\{[R(x)]^{-1}\right\} \\
&= \frac{1}{\hat{a}^2}\left\{[k\hat{a}R(x)]^2 - R(x)\frac{d^2}{dx^2}\left\{[R(x)]^{-1}\right\}\right\}.
\end{aligned}$$

We note that

$$\frac{d^2}{dx^2}\left\{[R(x)]^{-1}\right\} = \frac{d}{dx}\{-R'R^{-2}\} = -[-2(R')^2R^{-3} + R''R^{-2}]$$

$$= R^{-1}\left[2(R'R^{-1})^2 - R''R^{-1}\right].$$

Hence, it is found that

$$m_{eff}^2(r) = \frac{1}{\hat{a}^2}\left\{[k\hat{a}R(x)]^2 + R''(x)[R(x)]^{-1} - 2(R'(x)[R(x)]^{-1})^2\right\}. \tag{D.9}$$

We have that

$$\begin{aligned}
\frac{d^2u_E}{dr^2} &= \frac{1}{\hat{a}^2}\frac{d^2}{dx^2}\left\{c_0[R(x)]^{-1}T_n(x)\right\} = \frac{c_0}{\hat{a}^2}\frac{d}{dx}\left[\frac{d}{dx}\{R^{-1}T_n\}\right] \\
&= \frac{c_0}{\hat{a}^2}\frac{d}{dx}\left[R^{-1}T_n' - R'R^{-2}T_n\right] = \frac{c_0}{\hat{a}^2}\frac{d}{dx}\left[R^{-1}\{T_n' - R'R^{-1}T_n\}\right] \\
&= \frac{c_0}{\hat{a}^2}\left[R^{-1}\left\{T_n'' - \left[R'R^{-1}T_n' + (-(R')^2R^{-2} + R''R^{-1})T_n\right]\right\}\right. \\
&\qquad \left. - R'R^{-2}\{T_n' - R'R^{-1}T_n\}\right] \\
&= \frac{c_0R^{-1}}{\hat{a}^2}\left[T_n'' - 2R'R^{-1}T_n' + \{2(R'R^{-1})^2 - R''R^{-1}\}T_n\right].
\end{aligned} \tag{D.10}$$

Substituting equations (D.8)-(D.10) into equation (D.7) gives us that

$$\begin{aligned}
c_0[R(x)]^{-1}&\left[T_n''(x) - 2R'(x)[R(x)]^{-1}T_n'(x)\right. \\
&\qquad \left. + \left\{2(R'(x)[R(x)]^{-1})^2 - R''(x)[R(x)]^{-1}\right\}T_n(x)\right] \\
&\qquad + \left\{[k\hat{a}R(x)]^2 + R''(x)[R(x)]^{-1}\right. \\
&\qquad\qquad \left. - 2(R'(x)[R(x)]^{-1})^2 - \frac{n(n+1)}{x^2}\right\}c_0[R(x)]^{-1}T_n(x) = 0.
\end{aligned}$$

Therefore, it is obtained that

$$T_n''(x) - 2\frac{R'(x)}{R(x)}T_n'(x) + \left\{[k\hat{a}R(x)]^2 - \frac{n(n+1)}{x^2}\right\}T_n(x) = 0.$$

which is equation (136). We have shown how the r-dependent differential equations of [21] relate to the x-dependent differential equations of [22].

APPENDIX E

COUPLING TWO EQUATIONS FROM SAMADDAR (1970)

In a paper by Samaddar [25], he discusses the oblique incidence scattering from a radially inhomogeneous cylindrical structure with cylindrical coordinates (ρ, ϕ, z). He begins with Maxwell's equations for the electric field **E** and the magnetic field **H**. This leads to the inhomogeneous wave equations for **E** and **H**. Then he studies the z-component for these wave equations, denoted by E_z and H_z. For an oblique incident plane electromagnetic wave, the fields E_z and H_z may be put in the form

$$E_z = e^{ihz}\hat{E}_z(\rho, \phi),$$
$$H_z = e^{ihz}\hat{H}_z(\rho, \phi),$$

where $h = K_0 \sin\theta$ with $K_0^2 = \omega^2\mu_0\epsilon_0$. θ being the angle which the incident wave vector makes with the normal to the radially inhomogeneous cylinder, ω being the angular frequency, ϵ_0 being the permittivity of free space, μ_0 being the permeability of free space, and $\epsilon(\rho)$ being the inhomogeneous dielectric constant. Using the above representations for E_z and H_z, the coupled equations for $\hat{E}_z$ and $\hat{H}_z$ inside the source-free inhomogeneous cylinder take the following form:

$$\left[\nabla_t^2 + K_0^2\epsilon(\rho) - h^2 - \frac{h^2\epsilon'(\rho)}{K_0^2\epsilon(\rho) - h^2}\frac{\partial}{\partial\rho}\right]\hat{E}_z = -\frac{h\omega\mu_0\epsilon'(\rho)}{(K_0^2\epsilon(\rho) - h^2)\rho}\frac{\partial\hat{H}_z}{\partial\phi}, \tag{E.1}$$

$$\left[\nabla_t^2 + K_0^2\epsilon(\rho) - h^2 - \frac{K_0^2\epsilon'(\rho)}{K_0^2\epsilon(\rho) - h^2}\frac{\partial}{\partial\rho}\right]\hat{H}_z = \frac{h\omega\epsilon_0\epsilon'(\rho)}{(K_0^2\epsilon(\rho) - h^2)\rho}\frac{\partial\hat{E}_z}{\partial\phi}, \tag{E.2}$$

where

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}.$$

If we assume that

$$\hat{E}_z = \sum_{n=-\infty}^{\infty} e^{in\phi}R_n(\rho)$$

and

$$\hat{H}_z = \sum_{n=-\infty}^{\infty} e^{in\phi} S_n(\rho),$$

equations (E.1) and (E.2) may be written as

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \eta^2(\rho) - \frac{n^2}{\rho^2} - \frac{h^2\epsilon'(\rho)}{\eta^2(\rho)}\frac{d}{d\rho}\right] R_n(\rho)$$
$$= -\frac{inh\omega\mu_0\epsilon'(\rho)}{\rho\eta^2(\rho)} S_n(\rho). \tag{E.3}$$

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \eta^2(\rho) - \frac{n^2}{\rho^2} - \frac{K_0^2\epsilon'(\rho)}{\eta^2(\rho)}\frac{d}{d\rho}\right] S_n(\rho)$$
$$= \frac{inh\omega\epsilon_0\epsilon'(\rho)}{\rho\eta^2(\rho)} R_n(\rho), \tag{E.4}$$

where

$$\eta^2(\rho) = K_0^2\epsilon(\rho) - h^2. \tag{E.5}$$

Introducing the functions $U_n(\rho) = \rho R_n(\rho)$ and $V_n(\rho) = \rho S_n(\rho)$, equations (E.3) and (E.4) can be expressed as

$$\frac{d^2}{d\rho^2}U_n(\rho) - \left\{\frac{1}{\rho} + \frac{h^2\epsilon'(\rho)}{\eta^2(\rho)}\right\}\frac{d}{d\rho}U_n(\rho) + \left\{\eta^2(\rho) + \frac{1-n^2}{\rho^2} + \frac{h^2\epsilon'(\rho)}{\rho\eta^2(\rho)}\right\}U_n(\rho)$$
$$= -\frac{inh\omega\mu_0\epsilon'(\rho)}{\rho\eta^2(\rho)} V_n(\rho), \tag{E.6}$$

$$\frac{d^2}{d\rho^2}V_n(\rho) - \left\{\frac{1}{\rho} + \frac{K_0^2\epsilon'(\rho)}{\eta^2(\rho)}\right\}\frac{d}{d\rho}V_n(\rho) + \left\{\eta^2(\rho) + \frac{1-n^2}{\rho^2} + \frac{K_0^2\epsilon'(\rho)}{\rho\eta^2(\rho)}\right\}V_n(\rho)$$
$$= \frac{inh\omega\epsilon_0\epsilon'(\rho)}{\rho\eta^2(\rho)} U_n(\rho). \tag{E.7}$$

Our goal in this appendix is to combine equations (E.6) and (E.7) into one differential equation for either $U_n(\rho)$ or $V_n(\rho)$.

From (E.6), we have that

$$V_n = \frac{i}{nh\omega\mu_0}\frac{\rho\eta^2(\rho)}{\epsilon'(\rho)}\left[U_n'' - \left\{\frac{1}{\rho} + \frac{h^2\epsilon'(\rho)}{\eta^2(\rho)}\right\}U_n' + \left\{\eta^2(\rho) + \frac{1-n^2}{\rho^2} + \frac{h^2\epsilon'(\rho)}{\rho\eta^2(\rho)}\right\}U_n\right], \tag{E.8}$$

where we note that the dependence of U_n and V_n on ρ has been assumed and the prime denotes differentiation with respect to ρ. Differentiating equation (E.5) with respect to ρ yields that

$$2\eta\eta' = K_0^2\epsilon'. \tag{E.9}$$

Differentiating equation (E.8) with respect to ρ gives us that

$$V_n' = \frac{i}{nh\omega\mu_o}\Bigg[\frac{\rho\eta^2}{\epsilon'}\Bigg[U_n''' - \left\{\frac{1}{\rho} + \frac{h^2\epsilon'}{\eta^2}\right\}U_n'' - \left\{-\frac{1}{\rho^2} + \frac{\eta^2h^2\epsilon'' - h^2\epsilon'(2\eta\eta')}{\eta^4}\right\}U_n'$$
$$+ \left\{\eta^2 + \frac{1-n^2}{\rho^2} + \frac{h^2\epsilon'}{\rho\eta^2}\right\}U_n'$$
$$+ \left\{2\eta\eta' - \frac{2(1-n^2)}{\rho^3} + \frac{\rho\eta^2h^2\epsilon'' - h^2\epsilon'(2\rho\eta\eta' + \eta^2)}{\rho^2\eta^4}\right\}U_n\Bigg]$$
$$+ \left\{\frac{\epsilon'(2\rho\eta\eta' + \eta^2) - \rho\eta^2\epsilon''}{(\epsilon')^2}\right\}$$
$$\times\left[U_n'' - \left\{\frac{1}{\rho} + \frac{h^2\epsilon'}{\eta^2}\right\}U_n' + \left\{\eta^2 + \frac{1-n^2}{\rho^2} + \frac{h^2\epsilon'}{\rho\eta^2}\right\}U_n\right]\Bigg].$$

Then we see that

$$V_n' = \frac{i}{nh\omega\mu_0}\Bigg[\frac{\rho\eta^2}{\epsilon'}U_n''' + \left\{-\frac{\eta^2}{\epsilon'} - h^2\rho + K_0^2\rho + \frac{\eta^2}{\epsilon'} - \frac{\rho\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n''$$
$$+ \Bigg\{\frac{\eta^2}{\rho\epsilon'} - \frac{h^2\rho\epsilon''}{\epsilon'} + \frac{h^2K_0^2\rho\epsilon'}{\eta^2} + \frac{\rho\eta^4}{\epsilon'} + \frac{\eta^2(1-n^2)}{\rho\epsilon'} + h^2$$
$$- K_0^2 - \frac{\eta^2}{\rho\epsilon'} + \frac{\eta^2\epsilon''}{(\epsilon')^2} - \frac{K_0^2\rho h^2\epsilon'}{\eta^2} - h^2 + \frac{h^2\rho\epsilon''}{\epsilon'}\Bigg\}U_n'$$
$$+ \Bigg\{K_0^2\rho\eta^2 - \frac{2\eta^2(1-n^2)}{\rho^2\epsilon'} + \frac{h^2\epsilon''}{\epsilon'} - \frac{h^2K_0^2\epsilon'}{\eta^2} - \frac{h^2}{\rho} + K_0^2\rho\eta^2 + \frac{\eta^4}{\epsilon'} - \frac{\rho\eta^4\epsilon''}{(\epsilon')^2}$$
$$+ \frac{K_0^2(1-n^2)}{\rho} + \frac{\eta^2(1-n^2)}{\rho^2\epsilon'} - \frac{\eta^2\epsilon''(1-n^2)}{\rho(\epsilon')^2}$$
$$+ \frac{K_0^2h^2\epsilon'}{\eta^2} + \frac{h^2}{\rho} - \frac{h^2\epsilon''}{\epsilon'}\Bigg\}U_n\Bigg].$$

Thus, it follows that

$$V_n' = \frac{i}{nh\omega\mu_0}\Bigg[\frac{\rho\eta^2}{\epsilon'}U_n''' + \left\{(K_0^2 - h^2)\rho - \frac{\rho\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n''$$
$$+ \left\{\frac{\rho\eta^4}{\epsilon'} + \frac{\eta^2(1-n^2)}{\rho\epsilon'} - K_0^2 + \frac{\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n'$$
$$\Bigg\{2K_0^2\rho\eta^2 - \frac{\eta^2(1-n^2)}{\rho^2\epsilon'} + \frac{\eta^4}{\epsilon'} - \frac{\rho\eta^4\epsilon''}{(\epsilon')^2}$$
$$+ \frac{K_0^2(1-n^2)}{\rho} - \frac{\eta^2\epsilon''(1-n^2)}{\rho(\epsilon')^2}\Bigg\}U_n\Bigg]. \tag{E.10}$$

Upon differentiating equation (E.10) with respect to ρ, it is found that

$$V_n'' = \frac{i}{nh\omega\mu_0}\Bigg[\frac{\rho\eta^2}{\epsilon'}U_n^{(iv)} + \left\{\frac{\epsilon'(\rho(2\eta\eta') + \eta^2) - \rho\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n'''$$

$$+ \left\{(K_0^2 - h^2)\rho - \frac{\rho\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n'''$$

$$+ \left\{K_0^2 - h^2 - \left(\frac{(\epsilon')^2(\rho(\eta^2\epsilon''' + 2\eta\eta'\epsilon'') + \eta^2\epsilon'') - 2\epsilon'(\epsilon'')^2\rho\eta^2}{(\epsilon')^4}\right)\right\}U_n''$$

$$+ \left\{\frac{\rho\eta^4}{\epsilon'} + \frac{\eta^2(1-n^2)}{\rho\epsilon'} - K_0^2 + \frac{\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n''$$

$$+ \left\{\frac{\epsilon'(4\eta^3\eta'\rho + \eta^4) - \rho\eta^4\epsilon''}{(\epsilon')^2} + (1-n^2)\left(\frac{2\eta\eta'\rho\epsilon' - \eta^2(\rho\epsilon'' + \epsilon')}{\rho^2(\epsilon')^2}\right)\right.$$

$$\left. + \frac{(\epsilon')^2(\eta^2\epsilon''' + 2\eta\eta'\epsilon'') - 2\epsilon'(\epsilon'')^2\eta^2}{(\epsilon')^4}\right\}U_n'$$

$$+ \left\{2K_0^2\rho\eta^2 - \frac{\eta^2(1-n^2)}{\rho^2\epsilon'} + \frac{\eta^4}{\epsilon'} - \frac{\rho\eta^4\epsilon''}{(\epsilon')^2}\right.$$

$$\left. + \frac{K_0^2(1-n^2)}{\rho} - \frac{\eta^2\epsilon''(1-n^2)}{\rho(\epsilon')^2}\right\}U_n'$$

$$+ \left\{2K_0^2(2\eta\eta'\rho + \eta^2) - (1-n^2)\left(\frac{2\eta\eta'\rho^2\epsilon' - \eta^2(\rho^2\epsilon'' + 2\rho\epsilon')}{\rho^4(\epsilon')^2}\right)\right.$$

$$+ \frac{4\eta^3\eta'\epsilon' - \eta^4\epsilon''}{(\epsilon')^2} - \frac{K_0^2(1-n^2)}{\rho^2}$$

$$- \left(\frac{(\epsilon')^2(\rho(\eta^4\epsilon''' + 4\eta^3\eta'\epsilon'') + \eta^4\epsilon'') - 2\epsilon'(\epsilon'')^2\rho\eta^4}{(\epsilon')^4}\right)$$

$$\left. - (1-n^2)\left(\frac{\rho(\epsilon')^2(\eta^2\epsilon''' + 2\eta\eta'\epsilon'') - \eta^2\epsilon''(2\rho\epsilon'\epsilon'' + (\epsilon')^2)}{\rho^2(\epsilon')^4}\right)\right\}U_n\Bigg].$$

Consequently, it is determined that

$$V_n'' = \frac{i}{nh\omega\mu_0}\Bigg[\frac{\rho\eta^2}{\epsilon'}U_n^{(iv)} + \left\{K_0^2\rho + \frac{\eta^2}{\epsilon'} - \frac{\rho\eta^2\epsilon''}{(\epsilon')^2} + K_0^2\rho - h^2\rho - \frac{\rho\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n'''$$

$$+ \left\{K_0^2 - h^2 - \frac{\eta^2\rho\epsilon'''}{(\epsilon')^2} - \frac{K_0^2\rho\epsilon''}{\epsilon'} - \frac{\eta^2\epsilon''}{(\epsilon')^2} + \frac{2\rho\eta^2(\epsilon'')^2}{(\epsilon')^3}\right.$$

$$\left. + \frac{\rho\eta^4}{\epsilon'} + \frac{\eta^2(1-n^2)}{\rho\epsilon'} - K_0^2 + \frac{\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n''$$

$$+ \left\{2K_0^2\rho\eta^2 + \frac{\eta^4}{\epsilon'} - \frac{\rho\eta^4\epsilon''}{(\epsilon')^2} + (1-n^2)\left(\frac{K_0^2}{\rho} - \frac{\eta^2\epsilon''}{\rho(\epsilon')^2} - \frac{\eta^2}{\rho^2\epsilon'}\right)\right.$$

$$+ \frac{\eta^2\epsilon'''}{(\epsilon')^2} + \frac{K_0^2\epsilon''}{\epsilon'} - \frac{2(\epsilon'')^2\eta^2}{(\epsilon')^3}$$

$$+ 2K_0^2\rho\eta^2 - \frac{\eta^2(1-n^2)}{\rho^2\epsilon'} + \frac{\eta^4}{\epsilon'}$$

$$
\begin{aligned}
&- \frac{\rho\eta^4\epsilon''}{(\epsilon')^2} + (1-n^2)\left(\frac{K_0^2}{\rho} - \frac{\eta^2\epsilon''}{\rho(\epsilon')^2}\right)\Bigg\}U_n' \\
&+ \Bigg\{2K_0^4\epsilon'\rho + 2K_0^2\eta^2 + (1-n^2)\left(\frac{\eta^2\epsilon''}{\rho^2(\epsilon')^2} + \frac{2\eta^2}{\rho^3\epsilon'} - \frac{K_0^2}{\rho^2}\right) \\
&\quad + 2K_0^2\eta^2 - \frac{\eta^4\epsilon''}{(\epsilon')^2} + \frac{2\rho\eta^4(\epsilon'')^2}{(\epsilon')^3} - \frac{\rho\eta^4\epsilon'''}{(\epsilon')^2} \\
&\quad - \frac{2K_0^2\rho\eta^2\epsilon''}{\epsilon'} - \frac{\eta^4\epsilon''}{(\epsilon')^2} - \frac{K_0^2(1-n^2)}{\rho^2} \\
&\quad + (1-n^2)\left(\frac{2\eta^2(\epsilon'')^2}{\rho(\epsilon')^3} + \frac{\eta^2\epsilon''}{\rho^2(\epsilon')^2} - \frac{\eta^2\epsilon'''}{\rho(\epsilon')^2} - \frac{K_0^2\epsilon''}{\rho\epsilon'}\right)\Bigg\}U_n\Bigg].
\end{aligned}
$$

Therefore, we obtain that

$$
\begin{aligned}
V_n'' = \frac{i}{nh\omega\mu_0}\Bigg[&\frac{\rho\eta^2}{\epsilon'}U_n^{(iv)} + \left\{(2K_0^2 - h^2)\rho + \frac{\eta^2}{\epsilon'} - \frac{2\rho\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n''' \\
&+ \left\{\frac{2\rho\eta^2(\epsilon'')^2}{(\epsilon')^3} - \frac{\eta^2\rho\epsilon'''}{(\epsilon')^2} - \frac{K_0^2\rho\epsilon''}{\epsilon'} + \frac{\rho\eta^4}{\epsilon'} + \frac{\eta^2(1-n^2)}{\rho\epsilon'} - h^2\right\}U_n'' \\
&+ \Bigg\{4K_0^2\rho\eta^2 + \frac{2\eta^4}{\epsilon'} - \frac{2\rho\eta^4\epsilon''}{(\epsilon')^2} - \frac{2(\epsilon'')^2\eta^2}{(\epsilon')^3} + \frac{\eta^2\epsilon'''}{(\epsilon')^2} + \frac{K_0^2\epsilon''}{\epsilon'} \\
&\quad + 2(1-n^2)\left[\frac{K_0^2}{\rho} - \frac{\eta^2\epsilon''}{\rho(\epsilon')^2} - \frac{\eta^2}{\rho^2\epsilon'}\right]\Bigg\}U_n' \\
&+ \Bigg\{2K_0^4\epsilon'\rho + 4K_0^2\eta^2 + \frac{2\rho\eta^4(\epsilon'')^2}{(\epsilon')^3} - \frac{\rho\eta^4\epsilon'''}{(\epsilon')^2} - \frac{2\eta^4\epsilon''}{(\epsilon')^2} - \frac{2K_0^2\rho\eta^2\epsilon''}{\epsilon'} \\
&\quad + (1-n^2)\Bigg[\frac{2\eta^2\epsilon''}{\rho^2(\epsilon')^2} + \frac{2\eta^2}{\rho^3\epsilon'} - \frac{2K_0^2}{\rho^2} \\
&\qquad + \frac{2\eta^2(\epsilon'')^2}{\rho(\epsilon')^3} - \frac{\eta^2\epsilon'''}{\rho(\epsilon')^2} - \frac{K_0^2\epsilon''}{\rho\epsilon'}\Bigg]\Bigg\}U_n\Bigg]. \qquad \text{(E.11)}
\end{aligned}
$$

We will now calculate the terms on the left-hand side of equation (E.7) using equations (E.10) and (E.11). We have that

$$
\begin{aligned}
-\frac{1}{\rho}V_n' = \frac{i}{nh\omega\mu_0}\Bigg[&-\frac{\eta^2}{\epsilon'}U_n''' + \left\{\frac{\eta^2\epsilon''}{(\epsilon')^2} - (K_0^2 - h^2)\right\}U_n'' \\
&+ \left\{\frac{K_0^2}{\rho} - \frac{\eta^4}{\epsilon'} - \frac{\eta^2(1-n^2)}{\rho^2\epsilon'} - \frac{\eta^2\epsilon''}{\rho(\epsilon')^2}\right\}U_n' \\
&+ \Bigg\{\frac{\eta^4\epsilon''}{(\epsilon')^2} - \frac{\eta^4}{\rho\epsilon'} - 2K_0^2\eta^2 \\
&\quad + (1-n^2)\left[\frac{\eta^2}{\rho^3\epsilon'} + \frac{\eta^2\epsilon''}{\rho^2(\epsilon')^2} - \frac{K_0^2}{\rho^2}\right]\Bigg\}U_n\Bigg] \qquad \text{(E.12)}
\end{aligned}
$$

and

$$-\frac{K_0^2\epsilon'}{\eta^2}V_n' = \frac{i}{nh\omega\mu_0}\Bigg[-K_0^2\rho U_n''' + \left\{\frac{K_0^2\rho\epsilon''}{\epsilon'} - \frac{K_0^2(K_0^2-h^2)\epsilon'\rho}{\eta^2}\right\}U_n'' + \left\{\frac{K_0^4\epsilon'}{\eta^2} - K_0^2\rho\eta^2 - \frac{K_0^2(1-n^2)}{\rho} - \frac{K_0^2\epsilon''}{\epsilon'}\right\}U_n' + \left\{\frac{K_0^2\rho\eta^2\epsilon''}{\epsilon'} - K_0^2\eta^2 - 2K_0^4\rho\epsilon' + (1-n^2)\left[\frac{K_0^2\epsilon''}{\rho\epsilon'} + \frac{K_0^2}{\rho^2} - \frac{K_0^4\epsilon'}{\rho\eta^2}\right]\right\}U_n\Bigg]. \tag{E.13}$$

Using equation (E.8), we also have the following relations:

$$\eta^2 V_n = \frac{i}{nh\omega\mu_0}\left[\frac{\rho\eta^4}{\epsilon'}U_n'' - \left\{\frac{\eta^4}{\epsilon'} + h^2\eta^2\rho\right\}U_n' + \left\{\frac{\rho\eta^6}{\epsilon'} + (1-n^2)\frac{\eta^4}{\rho\epsilon'} + h^2\eta^2\right\}U_n\right], \tag{E.14}$$

$$\frac{(1-n^2)}{\rho^2}V_n = \frac{i(1-n^2)}{nh\omega\mu_0}\left[\frac{\eta^2}{\rho\epsilon'}U_n'' - \left\{\frac{\eta^2}{\rho^2\epsilon'} + \frac{h^2}{\rho}\right\}U_n' + \left\{\frac{\eta^4}{\rho\epsilon'} + (1-n^2)\frac{\eta^2}{\rho^3\epsilon'} + \frac{h^2}{\rho^2}\right\}U_n\right], \tag{E.15}$$

and

$$\frac{K_0^2\epsilon'}{\rho\eta^2}V_n = \frac{i}{nh\omega\mu_0}\left[K_0^2U_n'' - \left\{\frac{K_0^2}{\rho} + \frac{K_0^2h^2\epsilon'}{\eta^2}\right\}U_n' + \left\{K_0^2\eta^2 + \frac{K_0^2(1-n^2)}{\rho^2} + \frac{K_0^2h^2\epsilon'}{\rho\eta^2}\right\}U_n\right]. \tag{E.16}$$

Utilizing equations (E.11) and (E.12)-(E.16) in the left-hand side of equation (E.7) yields that

$$V_n'' - \left\{\frac{1}{\rho} + \frac{K_0^2\epsilon'}{\eta^2}\right\}V_n' + \left\{\eta^2 + \frac{1-n^2}{\rho^2} + \frac{K_0^2\epsilon'}{\rho\eta^2}\right\}V_n$$
$$= \frac{i}{nh\omega\mu_0}\Bigg[\frac{\rho\eta^2}{\epsilon'}U_n^{(iv)} + \left\{(2K_0^2-h^2)\rho + \frac{\eta^2}{\epsilon'} - \frac{2\rho\eta^2\epsilon''}{(\epsilon')^2} - \frac{\eta^2}{\epsilon'} - K_0^2\rho\right\}U_n''' + \left\{\frac{2\rho\eta^2(\epsilon'')^2}{(\epsilon')^3} - \frac{\eta^2\rho\epsilon'''}{(\epsilon')^2} - \frac{K_0^2\rho\epsilon''}{\epsilon'} + \frac{\rho\eta^4}{\epsilon'} + \frac{\eta^2(1-n^2)}{\rho\epsilon'} - h^2 + \frac{\eta^2\epsilon''}{(\epsilon')^2} - K_0^2 + h^2 + \frac{K_0^2\rho\epsilon''}{\epsilon'} - \frac{K_0^4\epsilon'\rho}{\eta^2} + \frac{K_0^2h^2\epsilon'\rho}{\eta^2} + \frac{\rho\eta^4}{\epsilon'} + \frac{\eta^2(1-n^2)}{\rho\epsilon'} + K_0^2\right\}U_n'' + \Bigg\{4K_0^2\rho\eta^2 + \frac{2\eta^4}{\epsilon'} - \frac{2\rho\eta^4\epsilon''}{(\epsilon')^2} - \frac{2\eta^2(\epsilon'')^2}{(\epsilon')^3} + \frac{\eta^2\epsilon'''}{(\epsilon')^2} + \frac{K_0^2\epsilon''}{\epsilon'} + 2(1-n^2)\left[\frac{K_0^2}{\rho} - \frac{\eta^2\epsilon''}{\rho(\epsilon')^2} - \frac{\eta^2}{\rho^2\epsilon'}\right] + \frac{K_0^2}{\rho} - \frac{\eta^4}{\epsilon'} - \frac{\eta^2(1-n^2)}{\rho^2\epsilon'}$$

$$
\begin{aligned}
&-\frac{\eta^2\epsilon''}{\rho(\epsilon')^2}+\frac{K_0^4\epsilon'}{\eta^2}-K_0^2\rho\eta^2-\frac{K_0^2(1-n^2)}{\rho}-\frac{K_0^2\epsilon''}{\epsilon'}\\
&-\frac{\eta^4}{\epsilon'}-h^2\eta^2\rho-\frac{\eta^2(1-n^2)}{\rho^2\epsilon'}-\frac{h^2(1-n^2)}{\rho}-\frac{K_0^2}{\rho}-\frac{K_0^2h^2\epsilon'}{\eta^2}\bigg\}U_n'\\
+&\bigg\{2K_0^4\epsilon'\rho+4\eta^2K_0^2+\frac{2\rho\eta^4(\epsilon'')^2}{(\epsilon')^3}-\frac{\rho\eta^4\epsilon'''}{(\epsilon')^2}-\frac{2\eta^4\epsilon''}{(\epsilon')^2}-\frac{2K_0^2\rho\eta^2\epsilon''}{\epsilon'}\\
&+(1-n^2)\left[\frac{2\eta^2\epsilon''}{\rho^2(\epsilon')^2}+\frac{2\eta^2}{\rho^3\epsilon'}-\frac{2K_0^2}{\rho^2}+\frac{2\eta^2(\epsilon'')^2}{\rho(\epsilon')^3}-\frac{\eta^2\epsilon'''}{\rho(\epsilon')^2}-\frac{K_0^2\epsilon''}{\rho\epsilon'}\right]\\
&+\frac{\eta^4\epsilon''}{(\epsilon')^2}-\frac{\eta^4}{\rho\epsilon'}-2K_0^2\eta^2+(1-n^2)\left[\frac{\eta^2}{\rho^3\epsilon'}+\frac{\eta^2\epsilon''}{\rho^2(\epsilon')^2}-\frac{K_0^2}{\rho^2}\right]\\
&+\frac{K_0^2\rho\eta^2\epsilon''}{\epsilon'}-K_0^2\eta^2-2K_0^4\rho\epsilon'+(1-n^2)\left[\frac{K_0^2\epsilon''}{\rho\epsilon'}+\frac{K_0^2}{\rho^2}-\frac{K_0^4\epsilon'}{\rho\eta^2}\right]\\
&+\frac{\rho\eta^6}{\epsilon'}+(1-n^2)\frac{\eta^4}{\rho\epsilon'}+h^2\eta^2+(1-n^2)\frac{\eta^4}{\rho\epsilon'}+(1-n^2)^2\frac{\eta^2}{\rho^3\epsilon'}\\
&+(1-n^2)\frac{h^2}{\rho^2}+K_0^2\eta^2+(1-n^2)\frac{K_0^2}{\rho^2}+\frac{K_0^2h^2\epsilon'}{\rho\eta^2}\bigg\}U_n\Bigg].
\end{aligned}
$$

Simplifying the above, it is obtained that

$$
\begin{aligned}
&V_n''-\left\{\frac{1}{\rho}+\frac{K_0^2\epsilon'}{\eta^2}\right\}V_n'+\left\{\eta^2+\frac{1-n^2}{\rho^2}+\frac{K_0^2\epsilon'}{\rho\eta^2}\right\}V_n\\
&=\frac{i}{nh\omega\mu_0}\Bigg[\frac{\rho\eta^2}{\epsilon'}U_n^{(iv)}+\left\{(K_0^2-h^2)\rho-\frac{2\rho\eta^2\epsilon''}{(\epsilon')^2}\right\}U_n'''\\
&\quad+\bigg\{\frac{2\rho\eta^2(\epsilon'')^2}{(\epsilon')^3}-\frac{\eta^2\rho\epsilon'''}{(\epsilon')^2}+\frac{2\rho\eta^4}{\epsilon'}+(1-n^2)\frac{2\eta^2}{\rho\epsilon'}\\
&\qquad+\frac{\eta^2\epsilon''}{(\epsilon')^2}+\frac{K_0^2h^2\epsilon'\rho}{\eta^2}-\frac{K_0^4\epsilon'\rho}{\eta^2}\bigg\}U_n''\\
&\quad+\bigg\{3K_0^2\rho\eta^2-h^2\rho\eta^2-\frac{2\rho\eta^4\epsilon''}{(\epsilon')^2}-\frac{2\eta^2(\epsilon'')^2}{(\epsilon')^3}+\frac{\eta^2\epsilon'''}{(\epsilon')^2}-\frac{\eta^2\epsilon''}{\rho(\epsilon')^2}\\
&\qquad+\frac{K_0^4\epsilon'}{\eta^2}-\frac{K_0^2h^2\epsilon'}{\eta^2}+(1-n^2)\left[\frac{K_0^2}{\rho}-\frac{2\eta^2\epsilon''}{\rho(\epsilon')^2}-\frac{4\eta^2}{\rho^2\epsilon'}-\frac{h^2}{\rho}\right]\bigg\}U_n'\\
&\quad+\bigg\{2K_0^2\eta^2+\frac{2\rho\eta^4(\epsilon'')^2}{(\epsilon')^3}-\frac{\rho\eta^4\epsilon'''}{(\epsilon')^2}-\frac{\eta^4\epsilon''}{(\epsilon')^2}-\frac{K_0^2\rho\eta^2\epsilon''}{\epsilon'}+\frac{\rho\eta^6}{\epsilon'}\\
&\qquad+h^2\eta^2-\frac{\eta^4}{\rho\epsilon'}+\frac{K_0^2h^2\epsilon'}{\rho\eta^2}\\
&\qquad+(1-n^2)\bigg[\frac{3\eta^2\epsilon''}{\rho^2(\epsilon')^2}+\frac{3\eta^2}{\rho^3\epsilon'}-\frac{K_0^2}{\rho^2}+\frac{2\eta^2(\epsilon'')^2}{\rho(\epsilon')^3}-\frac{\eta^2\epsilon'''}{\rho(\epsilon')^2}-\frac{K_0^4\epsilon'}{\rho\eta^2}\\
&\qquad\qquad+\frac{2\eta^4}{\rho\epsilon'}+(1-n^2)\frac{\eta^2}{\rho^3\epsilon'}+\frac{h^2}{\rho^2}\bigg]\bigg\}U_n\Bigg].
\end{aligned}
\tag{E.17}
$$

Let

$$f(\rho) = \frac{\rho\eta^2}{\epsilon'}.$$

Then equation (E.17) becomes

$$\begin{aligned}
&V_n'' - \left\{\frac{1}{\rho} + \frac{K_0^2\epsilon'}{\eta^2}\right\}V_n' + \left\{\eta^2 + \frac{1-n^2}{\rho^2} + \frac{K_0^2\epsilon'}{\rho\eta^2}\right\}V_n \\
&\quad = \frac{i}{nh\omega\mu_0}\Bigg[fU_n^{(iv)} + \left\{(K_0^2-h^2)\rho - \frac{2f\epsilon''}{\epsilon'}\right\}U_n''' \\
&\qquad + \Bigg\{2f\left(\frac{\epsilon''}{\epsilon'}\right)^2 - f\frac{\epsilon'''}{\epsilon'} + 2f\eta^2 + 2(1-n^2)\frac{f}{\rho^2} \\
&\qquad\quad + \frac{f\epsilon''}{\rho\epsilon'} - \frac{K_0^2(\epsilon')^2 f}{\eta^4}(K_0^2-h^2)\Bigg\}U_n'' \\
&\qquad + \Bigg\{f\epsilon'(3K_0^2-h^2) - \frac{2f\eta^2\epsilon''}{\epsilon'} - \frac{2f}{\rho}\left(\frac{\epsilon''}{\epsilon'}\right)^2 \\
&\qquad\quad + \frac{f\epsilon'''}{\rho\epsilon'} - \frac{f\epsilon''}{\rho^2\epsilon'} + \frac{K_0^2(\epsilon')^2 f}{\rho\eta^4}(K_0^2-h^2) \\
&\qquad\quad + (1-n^2)\left[\frac{(K_0^2-h^2)}{\rho} - \frac{2f\epsilon''}{\rho^2\epsilon'} - \frac{4f}{\rho^3}\right]\Bigg\}U_n' \\
&\quad + \Bigg\{(2K_0^2+h^2)\eta^2 + 2f\eta^2\left(\frac{\epsilon''}{\epsilon'}\right)^2 - \frac{f\eta^2\epsilon'''}{\epsilon'} - \frac{f\eta^2}{\rho}\frac{\epsilon''}{\epsilon'} - K_0^2 f\epsilon'' \\
&\qquad + f\eta^4 - \frac{f\eta^2}{\rho^2} + \frac{K_0^2h^2(\epsilon')^2 f}{\rho^2\eta^4} \\
&\qquad + (1-n^2)\left[\frac{3f\epsilon''}{\rho^3\epsilon'} + \frac{3f}{\rho^4} - \frac{(K_0^2-h^2)}{\rho^2} + \frac{2f}{\rho^2}\left(\frac{\epsilon''}{\epsilon'}\right)^2 - \frac{f\epsilon'''}{\rho^2\epsilon'}\right. \\
&\qquad\quad \left. - \frac{K_0^4 f(\epsilon')^2}{\rho^2\eta^4} + \frac{2f\eta^2}{\rho^2} + (1-n^2)\frac{f}{\rho^4}\right]\Bigg\}U_n\Bigg]. \qquad \text{(E.18)}
\end{aligned}$$

Upon using equation (E.18) in equation (E.7) gives us that

$$\begin{aligned}
&U_n^{(iv)} + \left\{\frac{(K_0^2-h^2)\rho}{f} - \frac{2\epsilon''}{\epsilon'}\right\}U_n''' \\
&\quad + \left\{2\left(\frac{\epsilon''}{\epsilon'}\right)^2 - \frac{(\rho\epsilon'''-\epsilon'')}{\rho\epsilon'} + 2\left(\eta^2 + \frac{1-n^2}{\rho^2}\right) - \frac{K_0^2(\epsilon')^2}{\eta^4}(K_0^2-h^2)\right\}U_n'' \\
&\quad + \Bigg\{\epsilon'(3K_0^2-h^2) - 2\left(\eta^2\frac{\epsilon''}{\epsilon'} + \frac{1}{\rho}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right) + \frac{(\rho\epsilon'''-\epsilon'')}{\rho^2\epsilon'} \\
&\qquad + \frac{K_0^2(\epsilon')^2}{\rho\eta^4}(K_0^2-h^2) + \frac{1-n^2}{\rho^2}\left[(K_0^2-h^2)\frac{\rho}{f} - \frac{2\epsilon''}{\epsilon'} - \frac{4}{\rho}\right]\Bigg\}U_n' \\
&\quad + \Bigg\{(2K_0^2+h^2)\frac{\eta^2}{f} - K_0^2\epsilon'' + 2\eta^2\left(\frac{\epsilon''}{\epsilon'}\right)^2 - \frac{\eta^2(\rho\epsilon'''+\epsilon'')}{\rho\epsilon'}
\end{aligned}$$

$$
\begin{aligned}
&+ \eta^2\left(\eta^2 - \frac{1}{\rho^2}\right) + \frac{K_0^2 h^2 (\epsilon')^2}{\rho^2 \eta^4} \\
&+ \frac{1 - n^2}{\rho^2}\left[\frac{1}{\rho}\left(3\frac{\epsilon''}{\epsilon'} + \frac{4 - n^2}{\rho}\right) + 2\left(\left(\frac{\epsilon''}{\epsilon'}\right)^2 + \eta^2\right)\right. \\
&\qquad\qquad \left. - \frac{(K_0^2 - h^2)}{f} - \frac{\epsilon'''}{\epsilon'} - \frac{K_0^4 (\epsilon')^2}{\eta^4}\right] \\
&- \frac{(nh\omega)^2 \epsilon_0 \mu_0}{f^2}\Bigg\} U_n = 0. \qquad \text{(E.19)}
\end{aligned}
$$

We see that equation (E.19) is a fourth-order differential equation for the function $U_n(\rho)$ which is what we wanted to derive in this appendix.

VITA

Michael A. Pohrivchak
Department of Computational and Applied Mathematics
Old Dominion University
Norfolk, VA 23529

Education

- Ph.D. in Computational and Applied Mathematics, Old Dominion University, Norfolk, VA (May 2014).
- M.S. in Computational and Applied Mathematics, Old Dominion University, Norfolk, VA (May 2011).
- B.S. in Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA (May 2007).

Experience and Awards

- Adjunct Instructional Faculty, Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA (Fall 2013 - Spring 2014).
- Research Intern, Chemistry Division, Naval Research Lab, Washington, DC (May 2013 - July 2013).
- President of the Math/Stat Club, Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA (2010 - 2011).
- Treasurer of the Society for Industrial and Applied Mathematics, Old Dominion University, Norfolk, VA (2009 - 2010).

Typeset using LaTeX.