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Post-Processing Techniques and Wavelet Applications for Hammerstein Integral Equations

Khomsan Neamprem
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POST-PROCESSING TECHNIQUES AND WAVELET
APPLICATIONS FOR HAMMERSTEIN INTEGRAL
EQUATIONS

by

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ABSTRACT

POST-PROCESSING TECHNIQUES AND WAVELET APPLICATIONS FOR HAMMERSTEIN INTEGRAL EQUATIONS

Khomsan Neamprem

Old Dominion University, 2010

Director: Dr. Hideaki Kaneko

This dissertation is focused on the varieties of numerical solutions of nonlinear Hammerstein integral equations. In the first part of this dissertation, several acceleration techniques for post-processed solutions of the Hammerstein equation are discussed. The post-processing techniques are implemented based on interpolation and extrapolation. In this connection, we generalize the results in [29] and [28] to nonlinear integral equations of the Hammerstein type. Post-processed collocation solutions are shown to exhibit better accuracy. Moreover, an extrapolation technique for the Galerkin solution of Hammerstein equation is also obtained. This result appears new even in the setting of the linear Fredholm equation.

In the second half of this dissertation, the wavelet-collocation technique of solving nonlinear Hammerstein integral equation is discussed. The main objective is to establish a fast wavelet-collocation method for Hammerstein equation by using a ‘linearization’ technique. The sparsity in the Jacobian matrix takes place in the fast wavelet-collocation method for Hammerstein equation with smooth as well as weakly singular kernels. A fast algorithm is based upon the block truncation strategy which was recently proposed in [10]. A multilevel augmentation method for the linearized Hammerstein equation is subsequently proposed which further accelerates the solution process while maintaining the order of convergence. Numerical examples are given throughout this dissertation.

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CHAPTER I

INTRODUCTION

Many physical problems in our daily life can be formulated in terms of mathematical models via differential or integral equations. In most cases, a solution of a differential equation or an integral equation is found by using a numerical method. This dissertation is focused on solving a class of nonlinear integral equations called Hammerstein equations. An integral equation occurs as a reformulation of a differential equation. In particular, when a two-point boundary value problem contains a nonlinear forcing term, its integral reformulation is the Hammerstein equation. Also, the Green's formula is one of the well-known reformulation techniques to convert a differential equation to an integral equation. Integral equations can be classified into two types, linear and nonlinear. As a basic background and for the purpose of introduction and motivation, the linear Fredholm integral equation of the second kind will be presented first. This will be followed by the nonlinear integral Hammerstein equation. New solution techniques presented in this dissertation are applied to this equation.

This dissertation presents two approaches, broadly classified, to solve nonlinear integral equations numerically. First, we present several methods to improve the accuracy of numerical solution by post-processing techniques. Each one of the post-processing techniques results in numerical solutions with higher order accuracies and they can be obtained with a relatively inexpensive numerical cost. Second in this dissertation, we focus our attention on the issue pertaining to the reduction of the computational cost both in terms of the computing time and the computer memory. Here, a class of wavelet basis will be applied. A multiresolution structure embedded in the wavelet basis allows us to employ a truncation strategy which eliminates a large number of elements in the corresponding Jacobian matrix. We will also establish an augmentation method for the nonlinear Hammerstein equation. A multilevel augmentation method, whose idea was originally presented in the paper [7], uses the multiresolution structure of the wavelet basis to obtain a numerical solution at a finer resolution level by correcting a coarser level solution by adding correction term. This results in an overall reduction of computational cost to the solution process while maintaining the order of accuracy.

A general form of the *Fredholm* integral equation of the second kind is

$$u(t) - \int_D k(s, t)u(s)ds = f(t), \quad t \in D \quad (1)$$

where D is a closed bounded set in \mathbb{R}^m , $m \geq 1$. The kernel k and the forcing term f are known functions and u is the unknown function to be determined. The historical background of (1) can be found in Fredholm [13] and Bernkopf [5].

During the last four decades or so, various numerical methods for solving (1) were developed. Among them, the well-known and frequently used numerical approaches are the methods of degenerate kernel, collocation, Galerkin and Nyström. Their methodologies are compiled in a recent monograph by Atkinson [3]. Also, additional expositions in both theoretical and analytical details of (1) can be found in Kress [22], Atkinson and Han [4] and Vanikko [34]. In this dissertation, we will be concerned with the collocation and Galerkin methods.

In order to enhance the order of accuracy of numerical solutions of the linear Fredholm equation, a number of techniques was presented in [28] and [29]. These techniques deal directly with the numerical solution which is obtained by a traditional method and by post-processing it by way of interpolation and extrapolation. Post-processing by interpolation of the collocation solutions of the equation (1) was constructed and also global extrapolation of the collocation method was obtained in [29]. Additionally, the interpolation of the Galerkin solutions of the Volterra integral equations can be found in the same paper. Lin et al. [28] constructed the extrapolation of iterated collocation solutions of the Fredholm equation. Post-processed solutions show higher rates of convergence, a phenomenon commonly known as *superconvergence*. The results in [29] and [28] motivated the author to extend the same techniques to the nonlinear Hammerstein equation. This will be done in Chapter III. We note that an extrapolation technique to accelerate the accuracy of the iterated Galerkin solution for the nonlinear Hammerstein equation presented in Chapter III appears new even in the setting of the linear Fredholm equation.

In approximating the solution of a linear integral equation using a numerical method mentioned above, it is always the case that a large system of linear equations must be solved. A matrix involved in the system is generally dense and therefore it is expensive to solve, particularly in the case of a multivariate integral equation.

Moreover, if one requires a higher order accuracy in the numerical solution, a standard way to achieve this is to increase the order of the basis functions. This results in a much larger system to solve which leads to a higher computational cost. In a series of recent papers, ([1], [2], [30] and references cited therein), a class of wavelet functions was used and applied to solving the equation (1). One of the advantages of the use of wavelet functions in this environment is due to the fact that the application of the wavelets results in a coefficient matrix which is sparse. Further work in this area can be found in Micchelli and Xu [31] and in Chen et al. [6], which is followed by a paper [10] by Chen et al. in which a fast collocation algorithm for solving (1) is presented. The multiscale piecewise linear, quadratic and cubic polynomial functions and the corresponding multiscale collocation functionals were constructed in [10]. In this dissertation, the piecewise linear polynomial wavelet will be used exclusively to demonstrate the validity of new theorems.

The integral equation which this dissertation primarily addresses is the nonlinear *Hammerstein* integral equation

$$u(t) - \int_D k(s, t)\psi(s, u(s))ds = f(t), \quad t \in D \quad (2)$$

where the functions k , f and ψ are known and u is an unknown function to be determined. As stated earlier, this equation arises as a reformulation of two-point boundary value problems with a certain nonlinear boundary condition. In equation (2), note that solution u appears under the nonlinear term ψ . The degenerate kernel method [20], the collocation method [25], the Galerkin method [21] and the Nyström method [26] methods were successfully used to find numerical solutions of this nonlinear equation. When applying these methods to equation (2), a nonlinear system of algebraic equation will take place and implementation of solving this nonlinear system is much more difficult than solving the linear system for the Fredholm case. We will show in Chapter IV of this dissertation that sparsity can be preserved in the Jacobian matrices which are part of the Newton method.

As part of the post-processing techniques, Kaneko et al. established the iterative methods for the Galerkin method and the collocation method in [21] and [18], respectively. The iterative method, when applied to the collocation as well as to the Galerkin method, double the order of accuracy provided that the solution and the kernel of the integral equations are sufficiently smooth. Iterative solutions are

obtained by iterations of a numerical solution through an integral operator. This is an excellent method to improve the accuracy of numerical approximation, since, as stated earlier, doubling the order of basis functions to attain the same order of accuracy results in a much larger system of nonlinear equations. On the other hand, the iterative methods present accuracy enhancement techniques which avoid this larger system and thus are computationally more efficient. In this study, the post-processing techniques based upon the interpolation and extrapolation are developed and applied to nonlinear equation (2). These techniques result in numerical solutions with higher accuracy but also are obtained with less expensive computational cost than the iterative method. In addition, we also discuss two other techniques as parts of the post-processing methods. We establish asymptotic orders of the iterated solutions, both the Galerkin and collocation method, for the nonlinear Hammerstein equation (2). Such results on the asymptotic errors lend naturally to the Richardson extrapolation method of accelerating the order of accuracy. We will see that this additional step of extrapolation brings an enhancement of the order of accuracy by two for the Galerkin and the collocation iterated methods.

In a recent paper, Chen et al. [9] established a new approach to solving an operator equation which is called multilevel augmentation method. Using a class of wavelet functions as a basis of computation and taking advantage of its multiresolution structure, the new method calls for obtaining a numerical solution at a finer resolution level by adding a correction term to a solution at a coarser level. The approach produces a numerical solution of high accuracy without solving a large system of linear equations. Following the paper [9], Chen et al. introduced a similar multilevel augmentation method to approximate the nonlinear Hammerstein equation in [11]. They obtained an order of convergence which is optimal and the computing time of the proposed method is less than the computing time of the traditional projection methods. In the second half of this dissertation, we implement a fast wavelet collocation algorithm to solve the nonlinear Hammerstein equation and establish a multilevel augmentation method for the equation. What is done differently in this paper from [11] is that, due to a ‘linearization’ of equation (2), the sparsity which was obtained by the application of the wavelets to linear equations now manifest in the Jacobian matrices and the corresponding block truncation strategy proposed in [10] can be implemented to obtain a fast wavelet-collocation algorithm for the Hammerstein equation. This point will be demonstrated both on the Hammerstein

equations with smooth as well as weakly singular kernels.

In summary, the main goals of this dissertation are as follows. A number of post-processing techniques which can be used to enhance the accuracy of numerical solutions of nonlinear integral equations of the Hammerstein type will be established. One type of the methods relies on interpolation and another relies on extrapolation. A second phase of this dissertation is a wavelet collocation method to acquire a fast algorithm to solve the Hammerstein equation. The class of piecewise polynomial wavelets constructed by Chen et al. [10] will be used. Next, a multilevel augmentation scheme is created in a similar way to [11] but this method is derived from a different operator form and it is a fast algorithm due to the preserving of the sparsity structure within the framework of a nonlinear solver.

This dissertation is organized into five chapters. After the current introductory chapter, we introduce necessary theoretical background in Chapter II which helps the reader to follow the materials in the ensuing chapters. This includes a method of construction of piecewise linear wavelets and that of the corresponding collocation functionals. Moreover, the collocation and the Galerkin methods which are parts of the projection method are also discussed in Chapter II. In Chapter III, we present several post-processing techniques of the collocation solutions. Additionally, the interpolation and extrapolation techniques for the Galerkin solutions of the Hammerstein equation are also obtained. In Chapter IV, the wavelets defined in Chapter II are used to approximate the solution of the Hammerstein equation. A multilevel augmentation is also discussed. Numerical results are interspersed throughout the Chapters III and IV. Finally, some conclusions, discussions and future research work will be presented in Chapter V.

Define $k_t(s) \equiv k(s, t)$ for $t, s \in [0, 1]$ to be the t section of k . We assume throughout this paper, unless stated otherwise, the following conditions on k, f and ψ :

(A1) $\lim_{t \rightarrow \tau} \|k_t - k_\tau\|_\infty = 0, \tau \in [0, 1]$;

(A2) $M \equiv \sup_{t \in [0, 1]} \int_0^1 |k(s, t)| ds < \infty$;

(A3) $f \in C[0, 1]$;

(A4) $\psi(s, x)$ is continuous in $s \in [0, 1]$ and Lipschitz continuous in $x \in (-\infty, \infty)$, i.e., there exists a constant $C_1 > 0$ for which

$$|\psi(s, x_1) - \psi(s, x_2)| \leq C_1 |x_1 - x_2| \text{ for all } x_1, x_2 \in (-\infty, \infty);$$

(A5) the partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that

$$|\psi^{(0,1)}(t, x_1) - \psi^{(0,1)}(t, x_2)| \leq C_2 |x_1 - x_2| \text{ for all } x_1, x_2 \in (-\infty, \infty);$$

(A6) for $x \in C[0, 1]$ and $\psi(\cdot, x(\cdot)), \psi^{(0,1)}(\cdot, x(\cdot)) \in C[0, 1]$.

Additional assumptions will be given later as needed.

CHAPTER II

NUMERICAL BACKGROUNDS AND WAVELETS

This chapter is used to review two standard methods used to approximate the solution of linear and nonlinear integral equations as well as to review the recent results in the area of wavelets which is pertinent to this dissertation. The collocation and the Galerkin methods are two of the most commonly used methods of approximating the solution of integral equations. They are both classified as special cases of the projection method. In the area of wavelet analysis and its application to integral equations, the review is particularly focused on the recent discoveries in [6], [8] and [10]. In these papers, interpolation functionals and a class of wavelet functions are established. Subsequently, they are applied to obtain a fast wavelet-collocation algorithm for solving the Fredholm integral equation.

II.1 NUMERICAL METHODS FOR INTEGRAL EQUATIONS

The Fredholm integral equation of the second kind is written in the general form as

$$u(t) - \int_0^1 k(s, t)u(s)ds = f(t), \quad t \in [0, 1], \quad (3)$$

and the Hammerstein integral equation takes the following form

$$u(t) - \int_0^1 k(s, t)\psi(s, u(s))ds = f(t), \quad t \in [0, 1]. \quad (4)$$

Here the kernel $k(s, t)$, the forcing term f and the nonlinear term ψ are known and the function u is to be determined.

Denote

$$Ku(t) := \int_0^1 k(s, t)u(s)ds,$$

and

$$\Psi(u)(t) := \psi(t, u(t)).$$

Then equations (3) and (4) can be written in the operator form as

$$u - Ku = f, \quad (5)$$

and

$$u - K\Psi(u) = f. \quad (6)$$

With the conditions described at the end of Chapter I, $K, K\Psi$ map a Banach space X into X . Usually, we choose X to be $L^\infty[0, 1]$ or $C[a, b]$ for the collocation analysis and $L^2[a, b]$ to formulate the Galerkin method.

To describe the method of projection, we let $\{X_n\}$ be a sequence of subspaces in X with the closure of the union of all X_n equal to X , i.e.

$$\overline{\bigcup_{n \geq 1} X_n} = X.$$

and define $P_n : X \rightarrow X_n$ be a projection. Unless the solution of (3) or (4) belongs to X_n , there would be a residual associated with an approximation u_n . Hence, for each $u_n \in X_n$, residuals r_n defined below are not zero;

$$r_n := f - (I - K)u_n,$$

and

$$r_n := f - (I - K\Psi)u_n.$$

The projection method finds a solution by requiring the residual to disappear under the projection P_n , i.e.,

$$P_n r_n = 0. \tag{7}$$

If P_n is an interpolation projection, then it is the collocation method, whereas, if P_n is an orthogonal projection, then it defines the Galerkin method. Equation (7) reduces to

$$u_n - P_n K u_n = P_n f, \quad u_n \in X_n,$$

for (5) and

$$u_n - P_n K \Psi u_n = P_n f, \quad u_n \in X_n,$$

for (6).

Let $\{b_1, b_2, \dots, b_n\}$ be a basis for X_n and assume that an approximate solution $u_n \in X_n$ of both integral equations can be written in the form

$$u_n(t) = \sum_{i=1}^n c_i b_i(t), \quad t \in [0, 1], \tag{8}$$

where $\{c_i\}_{i=1}^n$ is a set of constants to be determined.

II.1.1 Collocation Method

For collocation, we set $X = C[0, 1]$ and $X_n = \text{span}\{b_1, b_2, \dots, b_n\}$ where b_i has an interpolation property. Namely, there exists a set of points $\{t_j\}_{j=1}^n$ for which

$$b_i(t_j) = \delta_{ij}, \quad i = 1, \dots, n, \quad j = 1, \dots, n.$$

This implies that for each $x \in X$,

$$P_n x(t) = \sum_{i=1}^n x(t_i) b_i(t).$$

Thus, equation (7) becomes

$$r_n(t_j) = 0, \quad j = 1, 2, \dots, n. \quad (9)$$

When we substitute (8) into the Fredholm equation and apply (9) we get

$$\sum_{i=1}^n c_i b_i(t_j) - \sum_{i=1}^n c_i \int_0^1 k(s, t_j) b_i(s) ds = f(t_j), \quad j = 1, 2, \dots, n$$

or

$$c_j - \sum_{i=1}^n c_i \int_0^1 k(s, t_j) b_i(s) ds = f(t_j), \quad j = 1, 2, \dots, n.$$

This is a system of n linear equations with n unknown coefficients $\{c_i\}_{i=1}^n$,

$$Ac = f,$$

where $a_{ij} = b_i(t_j) - \int_0^1 k(s, t_j) b_i(s) ds$. It is known that this linear system has a unique solution if and only if $\det[b_i(t_j)] \neq 0$.

When the interpolation projection is applied to the nonlinear Hammerstien equation we obtain

$$\sum_{i=1}^n c_i b_i(t_j) - \int_0^1 k(s, t_j) \psi \left(s, \sum_{i=1}^n c_i b_i(s) \right) ds = f(t_j), \quad j = 1, 2, \dots, n$$

or

$$c_j - \int_0^1 k(s, t_j) \psi \left(s, \sum_{i=1}^n c_i b_i(s) \right) ds = f(t_j), \quad j = 1, 2, \dots, n.$$

This is a system of n nonlinear equations with n unknown coefficients $\{c_i\}_{i=1}^n$.

A nonlinear solver is used to find the solution $\{c_i\}_{i=1}^n$.

II.1.2 Galerkin Method

To describe the Galerkin method, we let X be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. In this subsection, we assume $X = L^2[0, 1]$. Let $X_n \subset X$ be a finite dimensional subspace of X and $P_n : X \rightarrow X_n$ an orthogonal projection. Assume that X_n is spanned by $\{b_j, j = 1, 2, \dots, n\}$. The Galerkin method, being a projection method, seeks solution by requiring (7), where P_n is the orthogonal projection onto X_n . This is equivalent to having the residual in the orthogonal complement of X_n in X . That is

$$\langle r_n, b_j \rangle = 0, \quad j = 1, 2, \dots, n.$$

Equivalently, we have

$$\sum_{i=1}^n c_i \langle b_i, b_j \rangle - \sum_{i=1}^n c_i \langle K b_i, b_j \rangle = \langle f, b_j \rangle, \quad j = 1, 2, \dots, n$$

which is

$$\sum_{i=1}^n c_i \int_0^1 b_i(t) b_j(t) dt - \sum_{i=1}^n c_i \int_0^1 \int_0^1 k(s, t) b_i(s) b_j(t) ds dt = \int_0^1 f(t) b_j(t) dt, \\ 1 \leq i, j \leq n.$$

If $\{b_1, b_2, \dots, b_n\}$ is an orthonormal basis then

$$\langle b_i, b_j \rangle = \int_0^1 b_i(t) b_j(t) dt = \delta_{ij}$$

and therefore the system of linear equations becomes

$$c_j - \sum_{i=1}^n c_i \int_0^1 \int_0^1 k(s, t) b_i(s) b_j(t) ds dt = \int_0^1 f(t) b_j(t) dt, \quad 1 \leq i, j \leq n.$$

For the nonlinear Hammerstein equation, a discretized Galerkin equation becomes

$$\sum_{i=1}^n c_i \int_0^1 b_i(t) b_j(t) dt - \int_0^1 \int_0^1 k(s, t) \psi \left(s, \sum_{i=1}^n c_i b_i(s) \right) ds b_j(t) dt \\ = \int_0^1 f(t) b_j(t) dt, \quad 1 \leq i, j \leq n.$$

If $\{b_j, j = 1, \dots, n\}$ is an orthonormal basis then the nonlinear system can be given as

$$c_j - \int_0^1 \int_0^1 k(s, t) \psi \left(s, \sum_{i=1}^n c_i b_i(s) \right) ds b_j(t) dt = \int_0^1 f(t) b_j(t) dt, \quad 1 \leq i, j \leq n.$$

This nonlinear system is solved by a nonlinear system solver, such as the Newton method or the quasi-Newton method.

Remarks: Matrices associated with the collocation method and the Galerkin method, when spline bases are used, are in general dense and computationally expensive to establish and its corresponding systems of equations, linear and nonlinear, are expensive to solve. Matrices associated with the Galerkin method are more expensive to build than those of the collocation method, due primarily to the fact that each component of the matrix involves an calculation of a double integral. To attenuate the computational cost, we will see in Chapter IV that certain sparseness can be obtained if wavelets are used to generate these matrices.

II.2 WAVELETS

In this section, we review necessary preliminary information which will be used to establish the wavelet collocation method for the nonlinear Hammerstein equation in Chapter IV. The wavelets have been used to solve a number of problems in several areas of science and engineering such as signal processing, image processing, computer graphics and approximation theory. The original construction of a class of wavelets by using translation and contraction of a function $\psi(t)$ was introduced by Grossman and Morlet [14], i.e.

$$\psi_{a,b}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0.$$

Each wavelet construction is based on a multiresolution analysis.

In the recent paper of Chen et al. [6], the concept of a refinable set was introduced which led to a set-theoretic multiresolution analysis. Subsequently, a construction of multiscale piecewise polynomial functions and its corresponding multiscale collocation functionals are established. It is helpful here to describe the definition of a multiresolution analysis and that of a refinable set.

II.2.1 Multiresolution analysis and Refinable set

Definition II.2.1. [Multiresolution analysis]

The function ϕ is a multiresolution analysis of $L^2(\mathbb{R})$ if there exists a nested sequence of closed subspaces $\{X_n\}_{n \in \mathbb{Z}}$ such that

1. there exists $\phi(t) \in X_0$ such that $\{\phi(t-l) : l \in \mathbb{Z}\}$ is an orthonormal basis of X_0 ,
2. $X_n \subset X_{n+1}$ for all $n \in \mathbb{Z}$, i.e., $\dots \subset X_{-1} \subset X_0 \subset X_1 \subset \dots$,
3. $f(t) \in X_n \Leftrightarrow f(2t) \in X_{n+1}$ for all $n \in \mathbb{Z}$,
4. $\bigcap_{n=-\infty}^{\infty} X_n = \{0\}$,
5. $\bigcup_{n=-\infty}^{\infty} X_n$ is dense in $L^2(\mathbb{R})$, i.e., $\overline{\bigcup_{n=-\infty}^{\infty} X_n} = L^2(\mathbb{R})$.

The function ϕ is called a *scaling function*. We define W_n to be the orthogonal complement of X_n in X_{n+1} , i.e.,

$$X_{n+1} = X_n \oplus W_n, \quad n \in \mathbb{Z},$$

and then

$$W_n \perp W_{n'}, \quad \text{if } n \neq n'.$$

Since X_n is a nested subspace it follows that

$$X_N = X_n \oplus \bigoplus_{m=0}^{N-n-1} W_{n+m}, \quad \text{for } n < N,$$

and that

$$L^2(\mathbb{R}) = X_0 \oplus \bigoplus_{n=0}^{\infty} W_n.$$

From the multiresolution analysis, there exists another function ψ in W_0 such that $\{\psi(t-l) : l \in \mathbb{Z}\}$ is an orthonormal basis for W_0 . The function ψ is called a *wavelet generator*.

Following [6], the definition of a refinable set is given next. The notion of refinable sets enables us to construct a class of wavelets and its corresponding functionals, together give the collocation method which exhibit the multiresolution properties. Proofs of the following theorems and examples can be found in [6].

Definition II.2.2. [Refinable set]

A subset V of X is said to be *refinable* relative to the mappings Φ if $V \subseteq \Phi(V)$.

Examples of a refinable set are shown in examples below when we define the mapping Φ as follows: with $X = \mathbb{R}$ and an integer $\mu > 1$,

$$\phi_\epsilon(t) = \frac{t + \epsilon}{\mu}, t \in \mathbb{R}, \epsilon \in Z_\mu, \quad (10)$$

$$Z_\mu := \{0, 1, \dots, \mu - 1\},$$

with a positive integer μ and

$$Z_\mu^k := Z_\mu \times \dots \times Z_\mu, \quad k \text{ times.}$$

Define the family of contractive mappings

$$\Phi := \{\phi_\epsilon : \epsilon \in Z_\mu\}.$$

Example II.2.3. The set $V_0 := \{\frac{j}{k} : j \in Z_{k+1}\}$ is refinable relative to the mappings Φ .

Example II.2.4. The set $V_0 = \{\frac{j}{k+1} : j - 1 \in Z_k\}$ is refinable relative to the mappings Φ if and only if μ and $k + 1$ are relatively prime.

II.2.2 Basic Tools

In what follows in this section, we take directly from [8], [10] and assume that $E = [0, 1]$. To understand the wavelet collocation method, we first introduce a family of contractive mappings $\Phi_\mu := \{\phi_\epsilon : \epsilon \in Z_\mu\}$, $Z_\mu = \{0, 1, \dots, \mu - 1\}$, with a fixed positive integer μ on the interval $I := [0, 1]$ defined by

$$\phi_\epsilon(t) = \frac{t + \epsilon}{\mu}, \quad t \in I, \quad \epsilon \in Z_\mu,$$

where

$$I := \bigcup_{\epsilon \in Z_\mu} \phi_\epsilon(I), \text{ and } \text{meas}\{\phi_\epsilon(I) \cap \phi_{\epsilon'}(I)\} = 0, \quad \epsilon \neq \epsilon'.$$

Let X_0 be the space of polynomials of order $k > 0$ on I and $\mathcal{T}_\epsilon : L^\infty(I) \rightarrow L^\infty(I)$ linear operators which are defined as follows;

$$\mathcal{T}_\epsilon x := x \circ \phi_\epsilon^{-1} \chi_{\phi_\epsilon(I)}, \quad \text{for } x \in L^\infty(I), \quad (11)$$

where χ_A is the characteristic function of A for a subset $A \subset I$. A sequence of subspaces \mathbb{X}_n can be defined by using the operator in (11) as follows;

$$\mathbb{X}_{n+1} = \bigoplus_{e \in Z_\mu} \mathcal{T}_e \mathbb{X}_n, \quad n \in \mathbb{N}_0,$$

where $\mathbb{N}_0 := \{0, 1, \dots\}$ and $A \oplus B$ denotes a direct sum of spaces A and B . It is easy to see that the following statements are true:

- The sequence of the subspaces is nested, i.e., $\mathbb{X}_n \subseteq \mathbb{X}_{n+1}$, $n \in \mathbb{N}_0$.
- \mathbb{X}_n is the space of piecewise polynomials of order k on I with knots at j/μ^n , $j = 1, 2, \dots, \mu^n - 1$.
- $\dim(\mathbb{X}_n) = k\mu^n$, $n \in \mathbb{N}_0$.

Because the subspaces have the nested property, we can obtain

$$\mathbb{X}_n = \mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \dots \oplus \mathbb{W}_n,$$

with $\mathbb{W}_0 := \mathbb{X}_0$ and \mathbb{W}_i is the orthogonal complement of \mathbb{X}_{i-1} in \mathbb{X}_i , $i = 1, 2, \dots, n$, i.e.,

$$\mathbb{X}_i = \mathbb{X}_{i-1} \oplus \mathbb{W}_i.$$

It is known from [31] that

$$\mathbb{W}_{n+1} = \bigoplus_{e \in Z_\mu} \mathcal{T}_e \mathbb{W}_n, \quad n \in \mathbb{N},$$

when the initial space \mathbb{W}_1 is constructed. For $i \in \mathbb{N}_0$, with $\omega(i) := \dim \mathbb{W}_i$, we have

$$\omega(n) = k(\mu - 1)\mu^{n-1}, \quad n \in \mathbb{N}.$$

The spaces $\{\mathbb{W}_i\}_{i=0}^n$ serve as *multiscale* subspaces of $L^\infty(I)$. Next, wavelet bases and corresponding collocation functionals on $L^\infty(I)$ must be constructed. To define a base for \mathbb{W}_1 , we use the fact that \mathbb{W}_1 is subspace of \mathbb{X}_1 and every element $w \in \mathbb{W}_1$ must be *orthogonal* to $\mathbb{X}_0 = \mathbb{W}_0$, i.e.,

$$\langle w, w_{0j} \rangle = 0, \quad j \in Z_k, \quad (12)$$

where $\{w_{0j}, j \in Z_k\}$ is a basis for \mathbb{W}_0 . Note that the dimension of \mathbb{X}_1 is $k\mu$, and equation (12) gives $r := k(\mu - 1)$ linearly independent solutions w_{1j} , $j \in Z_r$ which

form a basis for \mathbb{W}_1 . To generate bases for the subsequent wavelet spaces $\mathbb{W}_i, i > 1$, we define a composition operator \mathcal{T}_e by

$$\mathcal{T}_e := \mathcal{T}_{e_0} \circ \cdots \circ \mathcal{T}_{e_{i-1}},$$

where $\mathbf{e} := (e_0, \dots, e_{i-1}) \in Z_\mu^i = \underbrace{Z_\mu \times \cdots \times Z_\mu}_{i \text{ times}}$, $e_j \in Z_\mu$ for $j = 0, \dots, i-1$. With

$$\mu(\mathbf{e}) := \mu^{(i-1)}e_0 + \cdots + \mu e_{i-2} + e_{i-1},$$

and for $i = 2, 3, \dots, n$, we let

$$w_{ij} := \mathcal{T}_e w_{1l}, \quad j = \mu(\mathbf{e})r + l, \quad \mathbf{e} \in Z_\mu^{i-1}, \quad l \in Z_r, \quad (13)$$

and then $\mathbb{W}_i = \text{span}\{w_{ij} : j \in Z_{\omega(i)}\}$.

In order to facilitate the establishment of a wavelet collocation method, Chen, Micchelli and Xu ([6], [8]) defined the notion of a *refinable* set and consequently the collocation functionals in \mathbb{V}^* with $\mathbb{V} := C(I)$. Here, \mathbb{V}^* denotes the dual space of \mathbb{V} . Now select k distinct points $V_0 := \{t_j : j \in Z_k\}$ in I so that V_0 is *refinable* relative to the mappings Φ_μ , i.e., $V_0 \subseteq V_1 := \Phi_\mu(V_0)$. For example, it is shown in [6] that, when μ and $k+1$ are relatively prime, the set

$$V_0 := \left\{ \frac{j+1}{k+1} : j \in Z_k \right\}$$

is refinable relative to Φ_μ . For every point $t_j \in V_0$, we define $\ell_{0j} := \delta_{t_j}, j \in Z_k$, where $\ll \delta_{t_j}, x \gg := x(t_j)$ with $x \in C(I)$. We order the points in V_1 so that the first k points coincide the points in the set V_0 . For each $j \in Z_{k(\mu-1)}$, we find the vector $[c_{js} : s \in Z_{k\mu}]$ such that

$$\ell_{1j} := \sum_{s \in Z_{k\mu}} c_{js} \delta_{t_s}, \quad j \in Z_{k(\mu-1)} \quad (14)$$

satisfies the equations

$$\ll \ell_{1j}, w_{0j'} \gg = 0, \quad j' \in Z_k, \quad (15)$$

and

$$\ll \ell_{1j}, w_{1j'} \gg = \delta_{jj'}, \quad j' \in Z_r. \quad (16)$$

Equations (15) and (16) give rise to a unique solution ℓ_{1j} , which defines the interpolatory linear functionals ℓ_{1j} relative to the basis functions in \mathbb{W}_1 . To define linear

functionals with the interpolation property relative to basis $\{w_{ij}\}$, $i > 1$, we let $\mathcal{L}_e : \mathbb{V}^* \rightarrow \mathbb{V}^*$ be defined by the equation

$$\ll \mathcal{L}_e \ell, x \gg = \ll \ell, x \cdot \phi_e \gg, \quad \text{for all } x \in \mathbb{V} \text{ and } \ell \in \mathbb{V}^*, \quad (17)$$

and for $\mathbf{e} := (e_0, \dots, e_{n-1}) \in Z_\mu^n$, define

$$\mathcal{L}_e := \mathcal{L}_{e_0} \circ \dots \circ \mathcal{L}_{e_{n-1}}.$$

Finally, for $i = 2, 3, \dots, n$, define

$$\ell_{ij} := \mathcal{L}_e \ell_{iU}, \quad j = \mu(\mathbf{e})r + l, \quad \mathbf{e} \in Z_\mu^{i-1}, \quad l \in Z_r. \quad (18)$$

The basis functions w_{ij} and the collocation functionals ℓ_{ij} which are generated by using (17) and (18) have the following important properties.

- They have *vanishing moments* of order k , that is, for all element p in the space of polynomials of degree less than k on I ,

$$\ll \ell_{ij}, p \gg = 0, \quad \langle w_{ij}, p \rangle = 0, \quad j \in Z_{k(\mu-1)\mu^{i-1}}, \quad j \in \mathbb{N}. \quad (19)$$

- They satisfy the *semi-biorthogonality*, that is, for any $i, i' \in \mathbb{N}_0$

$$\ll \ell_{i'j'}, w_{ij} \gg = \delta_{ii'} \delta_{jj'}, \quad (i, j), (i', j') \in U, \quad i \leq i', \quad (20)$$

where $U := \{(i, j) : i \in \mathbb{N}_0, j \in Z_{w(i)}\}$.

- There exists a positive constant γ for which

$$\sum_{j \in Z_{w(i)}} |\ll \ell_{i'j'}, w_{ij} \gg| \leq \gamma, \quad (i, j), (i', j') \in U, \quad i > i'. \quad (21)$$

Next, the bases for the space \mathbb{X}_0 and \mathbb{W}_1 are shown for multiscale piecewise linear, quadratic and cubic polynomials and their corresponding multiscale collocation functionals.

1. Piecewise linear polynomials.

In the linear case, we set $k = 2$ and $\mu = 2$. Denote \mathbb{X}_n , $n \in N_0$ to be the space of piecewise linear polynomials on I with knots at $j/2^n$, $j = 1, 2, \dots, 2^n - 1$. Obviously, μ and $k + 1$ are relatively prime and $\Phi_2 := \{\phi_0, \phi_1\}$ where

$$\phi_0(t) = \frac{t}{2} \quad \text{and} \quad \phi_1(t) = \frac{t-1}{2}, \quad t \in I. \quad (22)$$

A refinable set relative to Φ_μ is given by

$$V_0 := \left\{ t_j = \frac{j+1}{k+1} : j \in Z_2 \right\} = \left\{ \frac{1}{3}, \frac{2}{3} \right\},$$

and $V_1 = \Phi_2(V_0) = \{\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}\}$. A basis for the space \mathbb{X}_0 is chosen to have the Lagrange interpolation linear property, i.e.,

$$w_{00}(t) = \frac{t - 2/3}{1/3 - 2/3} = 2 - 3t, \quad t \in I,$$

and

$$w_{01}(t) = \frac{t - 1/3}{2/3 - 1/3} = -1 + 3t, \quad t \in I.$$

From equation (12), a basis for space \mathbb{W}_1 having an orthogonal property can be given

$$w_{10}(t) = \begin{cases} 1 - \frac{9}{2}t, & t \in [0, \frac{1}{2}], \\ -1 + \frac{3}{2}t, & t \in (\frac{1}{2}, 1], \end{cases} \quad w_{11}(t) = \begin{cases} \frac{1}{2} - \frac{3}{2}t, & t \in [0, \frac{1}{2}], \\ -\frac{7}{2} + \frac{9}{2}t, & t \in (\frac{1}{2}, 1]. \end{cases}$$

Associated with the basis functions, we have the collocation functionals given by

$$\ell_{0j} = \delta_{t_j}, \quad j \in Z_2,$$

then

$$\ell_{00} = \delta_{\frac{1}{3}} \quad \text{and} \quad \ell_{01} = \delta_{\frac{2}{3}}.$$

To define the collocation functionals in V_1 , we have to find the vector $[c_j s : s \in Z_4]$ such that $\ell_{1j} = \sum_{s \in Z_4} c_j s \delta_{t_s}$, $j \in Z_2$ under the conditions (15) and (16), then we can obtain the unique solutions

$$\ell_{10} = \delta_{\frac{1}{6}} - \frac{3}{2}\delta_{\frac{1}{3}} + \frac{1}{2}\delta_{\frac{2}{3}}, \quad \ell_{11} = \frac{1}{2}\delta_{\frac{1}{3}} - \frac{3}{2}\delta_{\frac{2}{3}} + \delta_{\frac{5}{6}}.$$

2. Piecewise quadratic polynomials.

For the quadratic case, we choose $k = 3$ and $\mu = 3$. Define \mathbb{X}_n , $n \in \mathbb{N}_0$ to be the space of piecewise quadratic functions on I with knots at $j/3^n$, $j = 1, 2, \dots, 3^n - 1$, and $\Phi_3 := \{\phi_0, \phi_1, \phi_2\}$ where

$$\phi_0(t) = \frac{t}{3}, \quad \phi_1(t) = \frac{t+1}{3}, \quad \text{and} \quad \phi_2(t) = \frac{t+2}{3}, \quad t \in I.$$

A refinable set relative to Φ_3 is given by

$$V_0 := \left\{ t_j = \frac{j+1}{k+1} : j \in Z_3 \right\} = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right\},$$

and $V_1 = \Phi_3(V_0) = \{\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}\}$. A basis for space \mathbb{X}_0 is chosen to have a Lagrange interpolatory quadratic property, i.e.,

$$w_{00}(t) = (2t-1)(4t-3), \quad w_{01}(t) = -(4t-1)(4t-3), \quad \text{and} \quad w_{02}(t) = (4t-1)(2t-1).$$

A basis for space \mathbb{W}_1 is given by

$$\begin{aligned} w_{10}(t) &= \begin{cases} \frac{14}{9} - \frac{224}{9}t + 68t^2, & t \in [0, \frac{1}{3}], \\ -\frac{13}{9} + \frac{46}{9}t - 4t^2, & t \in (\frac{1}{3}, 1], \end{cases} \\ w_{11}(t) &= \begin{cases} \frac{46}{27} - \frac{544}{27}t + \frac{116}{3}t^2, & t \in [0, \frac{1}{3}], \\ -\frac{89}{27} + \frac{320}{27}t - \frac{28}{3}t^2, & t \in (\frac{1}{3}, 1], \end{cases} \\ w_{12}(t) &= \begin{cases} -\frac{22}{9} + \frac{224}{9}t - \frac{124}{3}t^2, & t \in [0, \frac{1}{3}], \\ \frac{113}{9} - \frac{370}{9}t + \frac{92}{3}t^2, & t \in (\frac{1}{3}, 1], \end{cases} \\ w_{13}(t) &= \begin{cases} \frac{19}{9} - \frac{182}{9}t + \frac{92}{3}t^2, & t \in [0, \frac{2}{3}], \\ -\frac{170}{9} + \frac{520}{9}t - \frac{124}{3}t^2, & t \in (\frac{2}{3}, 1], \end{cases} \\ w_{14}(t) &= \begin{cases} -\frac{7}{9} + \frac{184}{27}t - \frac{28}{3}t^2, & t \in [0, \frac{2}{3}], \\ \frac{182}{9} - \frac{1544}{27}t + \frac{116}{3}t^2, & t \in (\frac{2}{3}, 1], \end{cases} \\ w_{15}(t) &= \begin{cases} -\frac{1}{3} + \frac{26}{9}t - 4t^2, & t \in [0, \frac{2}{3}], \\ \frac{134}{3} - \frac{1000}{9}t + 68t^2, & t \in (\frac{2}{3}, 1]. \end{cases} \end{aligned}$$

The collocation functionals associated with the space \mathbb{X}_0 is given by

$$\ell_{00} = \delta_{\frac{1}{4}}, \quad \ell_{01} = \delta_{\frac{1}{2}}, \quad \ell_{02} = \delta_{\frac{3}{4}},$$

and those associated with space \mathbb{W}_1 is given by

$$\begin{aligned}\ell_{10} &= \delta_{\frac{1}{12}} - \frac{8}{3}\delta_{\frac{1}{6}} + 2\delta_{\frac{1}{4}} - \frac{1}{3}\delta_{\frac{5}{12}}, & \ell_{11} &= \delta_{\frac{1}{6}} - 2\delta_{\frac{1}{4}} + 2\delta_{\frac{5}{12}} - \delta_{\frac{1}{2}}, \\ \ell_{12} &= -\frac{1}{3}\delta_{\frac{1}{4}} + 2\delta_{\frac{5}{12}} - \frac{8}{3}\delta_{\frac{1}{2}} + \delta_{\frac{7}{12}}, & \ell_{13} &= \delta_{\frac{5}{12}} - \frac{8}{3}\delta_{\frac{1}{2}} + 2\delta_{\frac{7}{12}} - \frac{1}{3}\delta_{\frac{3}{4}}, \\ \ell_{14} &= -\delta_{\frac{1}{2}} + 2\delta_{\frac{7}{12}} - 2\delta_{\frac{3}{4}} + \delta_{\frac{5}{6}}, & \ell_{15} &= -\frac{1}{3}\delta_{\frac{7}{12}} + 2\delta_{\frac{3}{4}} - \frac{8}{3}\delta_{\frac{5}{6}} + \delta_{\frac{11}{12}}.\end{aligned}$$

3. Piecewise cubic polynomials.

In this case we take $k = 4$ and $\mu = 2$ and the mappings ϕ_0, ϕ_1 as defined by (22). Let \mathbb{X}_n be the space of piecewise cubic polynomials on I with knots at $j/2^n, j = 1, 2, \dots, 2^n - 1$ and $V_0 := \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$. A basis for space \mathbb{X}_0 is chosen to have a Lagrange interpolatory cubic property, i.e.,

$$\begin{aligned}w_{00}(t) &= -\frac{1}{6}(5t-2)(5t-3)(5t-4), & w_{01}(t) &= \frac{1}{2}(5t-1)(5t-3)(5t-4), \\ w_{02}(t) &= -\frac{1}{2}(5t-1)(5t-2)(5t-4), & w_{03}(t) &= \frac{1}{6}(5t-1)(5t-2)(5t-3).\end{aligned}$$

As described earlier, we construct a basis for space \mathbb{W}_1 given by

$$\begin{aligned}w_{10}(t) &= \begin{cases} \frac{85}{32} - \frac{725}{12}t + \frac{575}{2}t^2 - \frac{1475}{4}t^3, & t \in [0, \frac{1}{2}], \\ -\frac{235}{32} + \frac{575}{12}t - \frac{175}{2}t^2 + \frac{575}{12}t^3, & t \in (\frac{1}{2}, 1], \end{cases} \\ w_{11}(t) &= \begin{cases} \frac{1145}{288} - \frac{1775}{24}t + \frac{1675}{6}t^2 - \frac{4975}{18}t^3, & t \in [0, \frac{1}{2}], \\ -\frac{7495}{288} + \frac{3625}{24}t - \frac{525}{2}t^2 + \frac{2525}{18}t^3, & t \in (\frac{1}{2}, 1], \end{cases} \\ w_{12}(t) &= \begin{cases} \frac{805}{288} - \frac{375}{8}t + \frac{475}{3}t^2 - \frac{2525}{18}t^3, & t \in [0, \frac{1}{2}], \\ -\frac{19355}{288} + \frac{8275}{24}t - 550t^2 + \frac{4975}{18}, & t \in (\frac{1}{2}, 1], \end{cases} \\ w_{13}(t) &= \begin{cases} \frac{95}{96} - \frac{50}{3}t + \frac{225}{4}t^2 - \frac{575}{12}t^3, & t \in [0, \frac{1}{2}], \\ -\frac{13345}{96} + \frac{1775}{3}t - \frac{3275}{4}t^2 + \frac{1475}{4}t^3, & t \in (\frac{1}{2}, 1]. \end{cases}\end{aligned}$$

The collocation functionals associated with the space \mathbb{X}_0 have the form

$$\ell_{00} = \delta_{\frac{1}{5}}, \quad \ell_{01} = \delta_{\frac{2}{5}}, \quad \ell_{02} = \delta_{\frac{3}{5}}, \quad \ell_{03} = \delta_{\frac{4}{5}},$$

and those associated with the space \mathbb{W}_1 is given by

$$\begin{aligned}\ell_{10} &= \frac{2}{5}\delta_{\frac{1}{10}} - \frac{3}{2}\delta_{\frac{2}{10}} + 2\delta_{\frac{3}{10}} - \delta_{\frac{4}{10}} + \frac{1}{10}\delta_{\frac{6}{10}}, & \ell_{11} &= \frac{3}{10}\delta_{\frac{2}{10}} - \delta_{\frac{3}{10}} + \delta_{\frac{4}{10}} - \frac{1}{2}\delta_{\frac{6}{10}} + \frac{1}{5}\delta_{\frac{7}{10}}, \\ \ell_{12} &= \frac{1}{5}\delta_{\frac{3}{10}} - \frac{1}{2}\delta_{\frac{4}{10}} + \delta_{\frac{6}{10}} - \delta_{\frac{7}{10}} + \frac{3}{10}\delta_{\frac{8}{10}}, & \ell_{13} &= \frac{1}{10}\delta_{\frac{4}{10}} - \delta_{\frac{6}{10}} + 2\delta_{\frac{7}{10}} - \frac{3}{2}\delta_{\frac{8}{10}} + \frac{2}{5}\delta_{\frac{9}{10}}.\end{aligned}$$

Next, we focus on the construction of linear wavelet bases on any resolution level $\mathbb{W}_i, i > 1$ and that of corresponding linear functionals ℓ_{ij} . Then we use them to implement the wavelets collocation of the nonlinear equation (4) in the Chapter IV. The formulation to find the wavelets bases at finer resolution levels is given as follows:

$$w_{ij} := \mathcal{T}_{\mathbf{e}} w_{1l}, \quad j = \mu(\mathbf{e})r + l, \quad \mathbf{e} \in Z_{\mu}^{i-1}, \quad l \in Z_r,$$

and also the corresponding collocation functionals can be defined by

$$\ell_{ij} := \mathcal{L}_{\mathbf{e}} \ell_{1l}, \quad j = \mu(\mathbf{e})r + l, \quad \mathbf{e} \in Z_{\mu}^{i-1}, \quad l \in Z_r.$$

For example, when $i = 2$,

$$r = 2(2 - 1) = 2, \mathbf{e} = \{0, 1\} \in Z_2^1, \mu(\mathbf{e}_0) = 0, \mu(\mathbf{e}_1) = 1 \text{ and}$$

$$j = 0 = 0 \cdot 2 + 0,$$

$$j = 1 = 0 \cdot 2 + 1,$$

$$j = 2 = 1 \cdot 2 + 0,$$

$$j = 3 = 1 \cdot 2 + 1.$$

Then a basis for space \mathbb{W}_2 can be obtained by

$$\begin{aligned} w_{20} &= \mathcal{T}_{\mathbf{e}_0} w_{10} = w_{10} \circ \phi_0^{-1} \chi_{\phi_0(I)} \\ &= \begin{cases} 1 - \frac{9}{2}(2t) = 1 - 9t, & t \in [0, \frac{1}{4}], \\ -1 + \frac{3}{2}(2t) = -1 + 3t, & t \in (\frac{1}{4}, \frac{1}{2}], \end{cases} \\ w_{21} &= \mathcal{T}_{\mathbf{e}_0} w_{11} = w_{11} \circ \phi_0^{-1} \chi_{\phi_0(I)} \\ &= \begin{cases} \frac{1}{2} - \frac{3}{2}(2t) = \frac{1}{2} - 3t, & t \in [0, \frac{1}{4}], \\ -\frac{7}{2} + \frac{9}{2}(2t) = -\frac{7}{2} + 9t, & t \in (\frac{1}{4}, \frac{1}{2}], \end{cases} \\ w_{22} &= \mathcal{T}_{\mathbf{e}_1} w_{10} = w_{10} \circ \phi_1^{-1} \chi_{\phi_1(I)} \\ &= \begin{cases} 1 - \frac{9}{2}(2t - 1) = \frac{11}{2} - 9t, & t \in [\frac{1}{2}, \frac{3}{4}], \\ -1 + \frac{3}{2}(2t - 1) = -\frac{5}{2} + 3t, & t \in (\frac{3}{4}, 1], \end{cases} \\ w_{23} &= \mathcal{T}_{\mathbf{e}_1} w_{11} = w_{11} \circ \phi_1^{-1} \chi_{\phi_1(I)} \\ &= \begin{cases} \frac{1}{2} - \frac{3}{2}(2t - 1) = 2 - 3t, & t \in [\frac{1}{2}, \frac{3}{4}], \\ -\frac{7}{2} + \frac{9}{2}(2t - 1) = -8 + 9t, & t \in (\frac{3}{4}, 1], \end{cases} \end{aligned}$$

and the collocation functionals associated with space \mathbb{W}_2 is given by

$$\begin{aligned}
\ell_{20} &= \mathcal{L}_0 \ell_{10} \text{ where } \langle \mathcal{L}_0 \ell_{10}, x \rangle = \langle \ell_{10}, x \circ \phi_0 \rangle \\
&= \delta_{\frac{1}{12}} - \frac{3}{2} \delta_{\frac{1}{6}} + \frac{1}{2} \delta_{\frac{1}{3}}, \\
\ell_{21} &= \mathcal{L}_0 \ell_{11} \text{ where } \langle \mathcal{L}_0 \ell_{11}, x \rangle = \langle \ell_{11}, x \circ \phi_0 \rangle \\
&= \frac{1}{2} \delta_{\frac{1}{6}} - \frac{3}{2} \delta_{\frac{1}{3}} + \delta_{\frac{5}{12}}, \\
\ell_{22} &= \mathcal{L}_1 \ell_{10} \text{ where } \langle \mathcal{L}_1 \ell_{10}, x \rangle = \langle \ell_{10}, x \circ \phi_1 \rangle \\
&= \delta_{\frac{7}{12}} - \frac{3}{2} \delta_{\frac{2}{3}} + \frac{1}{2} \delta_{\frac{5}{6}}, \\
\ell_{23} &= \mathcal{L}_1 \ell_{11} \text{ where } \langle \mathcal{L}_1 \ell_{11}, x \rangle = \langle \ell_{11}, x \circ \phi_1 \rangle \\
&= \frac{1}{2} \delta_{\frac{2}{3}} - \frac{3}{2} \delta_{\frac{5}{6}} + \delta_{\frac{11}{12}}.
\end{aligned}$$

Similarly, when $i = 3$, $\mathbf{e} = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \in Z_2^2$,

$$\begin{aligned}
\mu(\mathbf{e}_0) &= 0 \cdot 2 + 0 = 0, \mu(\mathbf{e}_1) = 0 \cdot 2 + 1 = 1, \\
\mu(\mathbf{e}_2) &= 1 \cdot 2 + 0 = 2, \mu(\mathbf{e}_3) = 1 \cdot 2 + 1 = 3,
\end{aligned}$$

and

$$\begin{aligned}
j = 0 &= 0 \cdot 2 + 0, & j = 1 &= 0 \cdot 2 + 1 \\
j = 2 &= 1 \cdot 2 + 0, & j = 3 &= 1 \cdot 2 + 1, \\
j = 4 &= 2 \cdot 2 + 0, & j = 5 &= 2 \cdot 2 + 1, \\
j = 6 &= 3 \cdot 2 + 0, & j = 7 &= 3 \cdot 2 + 1.
\end{aligned}$$

The composition operators and their inverses are

$$\begin{aligned}
\phi_{(0,0)}(t) &= \phi_0 \circ \phi_0(t) = \frac{t}{4} \Rightarrow \phi_{(0,0)}^{-1}(t) = 4t, \\
\phi_{(0,1)}(t) &= \phi_0 \circ \phi_1(t) = \frac{t+1}{4} \Rightarrow \phi_{(0,1)}^{-1}(t) = 4t - 1, \\
\phi_{(1,0)}(t) &= \phi_1 \circ \phi_0(t) = \frac{t+2}{4} \Rightarrow \phi_{(1,0)}^{-1}(t) = 4t - 2, \\
\phi_{(1,1)}(t) &= \phi_1 \circ \phi_1(t) = \frac{t+3}{4} \Rightarrow \phi_{(1,1)}^{-1}(t) = 4t - 3.
\end{aligned}$$

Then a basis for space \mathbb{W}_3 can be obtained by

$$\begin{aligned}
w_{30} &= w_{10} \circ \phi_{(0,0)}^{-1} \chi_{\phi_{(0,0)}(I)} \\
&= \begin{cases} 1 - \frac{9}{2}(4t) = 1 - 18t, & t \in [0, \frac{1}{8}], \\ -1 + \frac{3}{2}(4t) = -1 + 6t, & t \in (\frac{1}{8}, \frac{1}{4}], \end{cases} \\
w_{31} &= w_{11} \circ \phi_{(0,0)}^{-1} \chi_{\phi_{(0,0)}(I)} \\
&= \begin{cases} \frac{1}{2} - \frac{3}{2}(4t) = \frac{1}{2} - 6t, & t \in [0, \frac{1}{8}], \\ -\frac{7}{2} + \frac{9}{2}(4t) = -\frac{7}{2} + 18t, & t \in (\frac{1}{8}, \frac{1}{4}], \end{cases}
\end{aligned}$$

$$\begin{aligned}
w_{32} &= w_{10} \circ \phi_{(0,1)}^{-1} \chi_{\phi_{(0,1)}(I)} \\
&= \begin{cases} 1 - \frac{9}{2}(4t-1) = \frac{11}{2} - 18t, & t \in [\frac{1}{4}, \frac{3}{8}], \\ -1 + \frac{3}{2}(4t-1) = -\frac{5}{2} + 6t, & t \in (\frac{3}{8}, \frac{1}{2}], \end{cases} \\
w_{33} &= w_{11} \circ \phi_{(0,1)}^{-1} \chi_{\phi_{(0,1)}(I)} \\
&= \begin{cases} \frac{1}{2} - \frac{3}{2}(4t-1) = 2 - 6t, & t \in [\frac{1}{4}, \frac{3}{8}], \\ -\frac{7}{2} + \frac{9}{2}(4t-1) = -8 + 18t, & t \in (\frac{3}{8}, \frac{1}{2}], \end{cases} \\
w_{34} &= w_{10} \circ \phi_{(1,0)}^{-1} \chi_{\phi_{(1,0)}(I)} \\
&= \begin{cases} 1 - \frac{9}{2}(4t-2) = 10 - 18t, & t \in [\frac{1}{2}, \frac{5}{8}], \\ -1 + \frac{3}{2}(4t-2) = -4 + 6t, & t \in (\frac{5}{8}, \frac{3}{4}], \end{cases} \\
w_{35} &= w_{11} \circ \phi_{(1,0)}^{-1} \chi_{\phi_{(1,0)}(I)} \\
&= \begin{cases} \frac{1}{2} - \frac{3}{2}(4t-2) = \frac{7}{2} - 6t, & t \in [\frac{1}{2}, \frac{5}{8}], \\ -\frac{7}{2} + \frac{9}{2}(4t-2) = -\frac{25}{2} + 18t, & t \in (\frac{5}{8}, \frac{3}{4}], \end{cases} \\
w_{36} &= w_{10} \circ \phi_{(1,1)}^{-1} \chi_{\phi_{(1,1)}(I)} \\
&= \begin{cases} 1 - \frac{9}{2}(4t-3) = \frac{29}{2} - 18t, & t \in [\frac{3}{4}, \frac{7}{8}], \\ -1 + \frac{3}{2}(4t-3) = -\frac{11}{2} + 6t, & t \in (\frac{7}{8}, 1], \end{cases} \\
w_{37} &= w_{11} \circ \phi_{(1,1)}^{-1} \chi_{\phi_{(1,1)}(I)} \\
&= \begin{cases} \frac{1}{2} - \frac{3}{2}(4t-3) = 5 - 6t, & t \in [\frac{3}{4}, \frac{7}{8}], \\ -\frac{7}{2} + \frac{9}{2}(4t-3) = -17 + 18t, & t \in (\frac{7}{8}, 1], \end{cases}
\end{aligned}$$

and the collocation functionals associated with space \mathbb{W}_3 is given by

$$\begin{aligned}
\ell_{30} &= \mathcal{L}_{(0,0)} \ell_{10} \text{ where } \langle\langle \mathcal{L}_{(0,0)} \ell_{10}, \mathbf{x} \rangle\rangle = \langle\langle \ell_{10}, \mathbf{x} \circ \phi_0 \circ \phi_0 \rangle\rangle \\
&= \delta_{\frac{1}{24}} - \frac{3}{2} \delta_{\frac{1}{12}} + \frac{1}{2} \delta_{\frac{1}{6}}, \\
\ell_{31} &= \mathcal{L}_{(0,0)} \ell_{11} \text{ where } \langle\langle \mathcal{L}_{(0,0)} \ell_{11}, \mathbf{x} \rangle\rangle = \langle\langle \ell_{11}, \mathbf{x} \circ \phi_0 \circ \phi_0 \rangle\rangle \\
&= \frac{1}{2} \delta_{\frac{1}{12}} - \frac{3}{2} \delta_{\frac{1}{6}} + \delta_{\frac{5}{24}}, \\
\ell_{32} &= \mathcal{L}_{(0,1)} \ell_{10} \text{ where } \langle\langle \mathcal{L}_{(0,1)} \ell_{10}, \mathbf{x} \rangle\rangle = \langle\langle \ell_{10}, \mathbf{x} \circ \phi_0 \circ \phi_1 \rangle\rangle \\
&= \delta_{\frac{7}{24}} - \frac{3}{2} \delta_{\frac{1}{3}} + \frac{1}{2} \delta_{\frac{5}{12}}, \\
\ell_{33} &= \mathcal{L}_{(0,1)} \ell_{11} \text{ where } \langle\langle \mathcal{L}_{(0,1)} \ell_{11}, \mathbf{x} \rangle\rangle = \langle\langle \ell_{11}, \mathbf{x} \circ \phi_0 \circ \phi_1 \rangle\rangle \\
&= \frac{1}{2} \delta_{\frac{1}{3}} - \frac{3}{2} \delta_{\frac{5}{12}} + \delta_{\frac{11}{24}}.
\end{aligned}$$

$$\begin{aligned}
\ell_{34} &= \mathcal{L}_{(1,0)}\ell_{10} \text{ where } \langle\langle \mathcal{L}_{(1,0)}\ell_{10}, x \rangle\rangle = \langle\langle \ell_{10}, x \circ \phi_1 \circ \phi_0 \rangle\rangle \\
&= \delta_{\frac{13}{24}} - \frac{3}{2}\delta_{\frac{7}{12}} + \frac{1}{2}\delta_{\frac{2}{3}}, \\
\ell_{35} &= \mathcal{L}_{(1,0)}\ell_{11} \text{ where } \langle\langle \mathcal{L}_{(1,0)}\ell_{11}, x \rangle\rangle = \langle\langle \ell_{11}, x \circ \phi_1 \circ \phi_0 \rangle\rangle \\
&= \frac{1}{2}\delta_{\frac{7}{12}} - \frac{3}{2}\delta_{\frac{2}{3}} + \delta_{\frac{17}{24}}, \\
\ell_{36} &= \mathcal{L}_{(1,1)}\ell_{10} \text{ where } \langle\langle \mathcal{L}_{(1,1)}\ell_{10}, x \rangle\rangle = \langle\langle \ell_{10}, x \circ \phi_1 \circ \phi_1 \rangle\rangle \\
&= \delta_{\frac{19}{24}} - \frac{3}{2}\delta_{\frac{5}{6}} + \frac{1}{2}\delta_{\frac{11}{12}}, \\
\ell_{37} &= \mathcal{L}_{(1,1)}\ell_{11} \text{ where } \langle\langle \mathcal{L}_{(1,1)}\ell_{11}, x \rangle\rangle = \langle\langle \ell_{11}, x \circ \phi_1 \circ \phi_1 \rangle\rangle \\
&= \frac{1}{2}\delta_{\frac{5}{6}} - \frac{3}{2}\delta_{\frac{11}{12}} + \delta_{\frac{23}{24}}.
\end{aligned}$$

The figures of these multiscale piecewise linear functions and the corresponding multiscale collocation functionals are shown in FIG. 1 - FIG. 8.

Eventually, the formulation of multiscale piecewise linear wavelets and their corresponding collocation functionals can be summarized in the following forms.

For any $(i, j) \in U$,

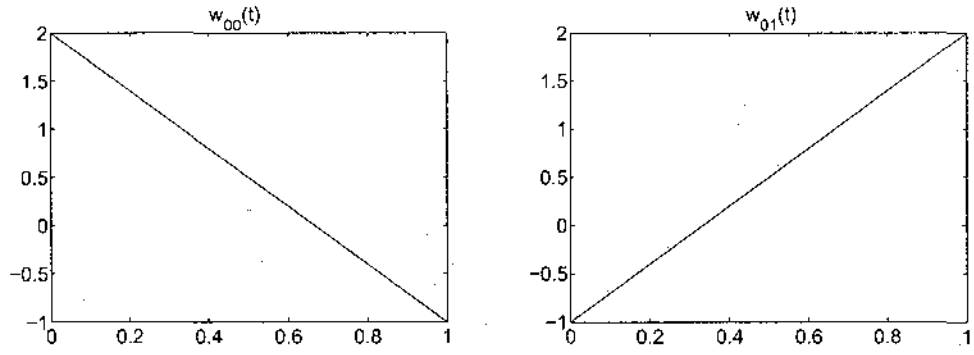
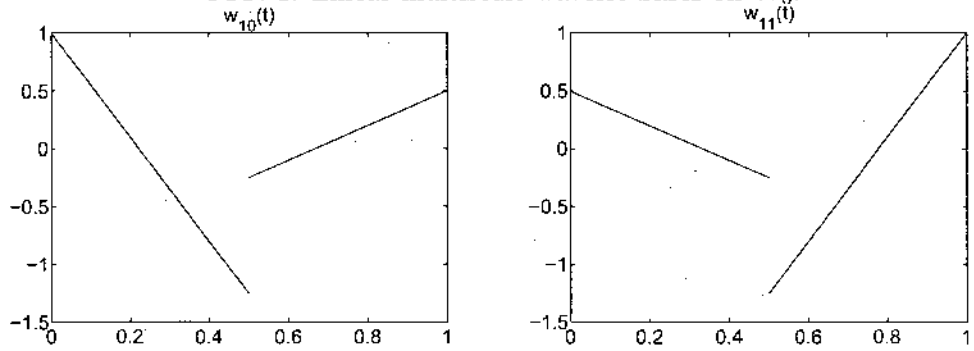
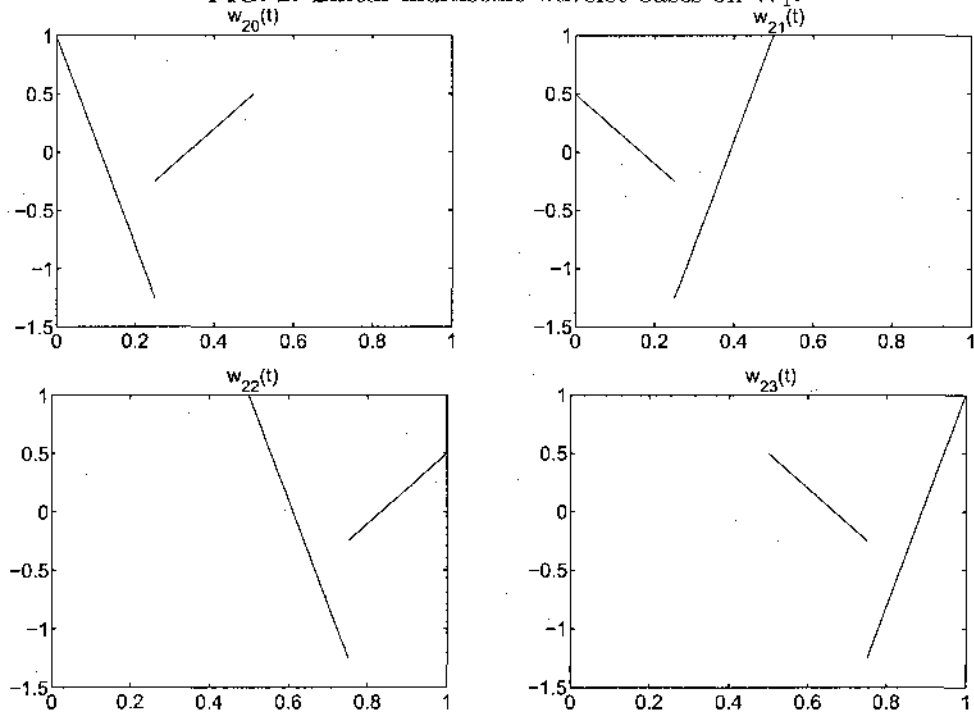
$$w_m(t) := w_{ij}(t) = \begin{cases} a + bt, & t \in [I_1, I_2], \\ c + dt, & t \in (I_2, I_3], \end{cases} \quad \forall i, j, \quad \text{if } i \geq 1,$$

and

$$\ell_m := \ell_{ij} = \begin{cases} \delta_{t_1}, & \text{if } i = 0, \\ \delta_{t_1} - \frac{3}{2}\delta_{t_2} + \frac{1}{2}\delta_{t_3}, & \text{if } i \neq 0 \text{ and } j \text{ is even,} \\ \frac{1}{2}\delta_{t_1} - \frac{3}{2}\delta_{t_2} + \delta_{t_3}, & \text{if } i \neq 0 \text{ and } j \text{ is odd.} \end{cases}$$

where the coefficients are collected in the TABLE 1 and TABLE 3.

The wavelet functions and the corresponding collocation functionals are the bases of $\mathbb{W}_0 \oplus \mathbb{W}_2 \oplus \dots \oplus \mathbb{W}_9$ that amount is 1024. These bases are used in Chapter IV to obtain the fast wavelet-collocation scheme for Fredholm and Hammerstein equations. For convenience of calculation, all of these coefficients are stored in database by Microsoft Excel and are recalled by the *xlsread* command of the Matlab program.

FIG. 1: Linear multiscale wavelet bases on W_0 .FIG. 2: Linear multiscale wavelet bases on W_1 .FIG. 3: Linear multiscale wavelet bases on W_2 .

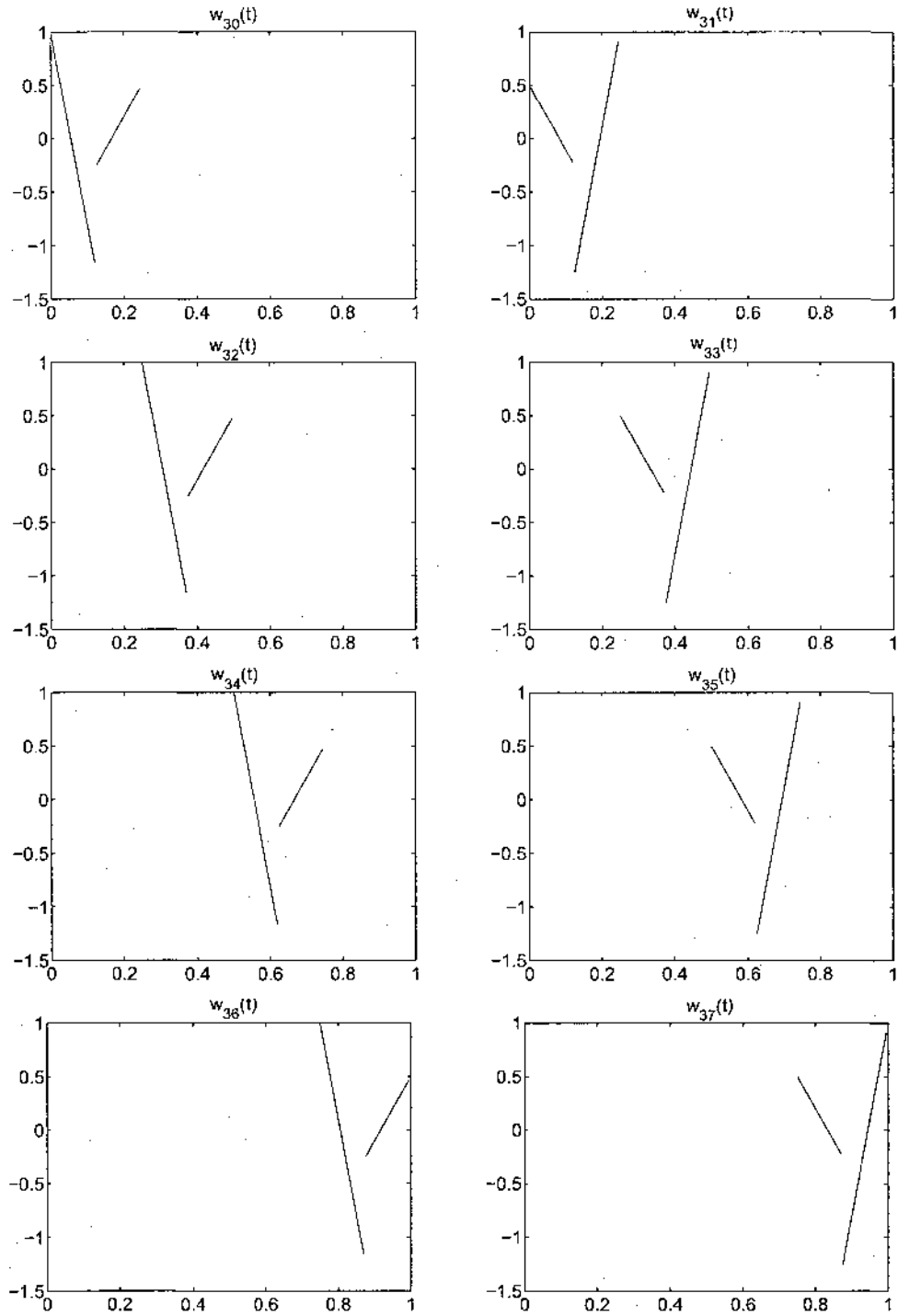
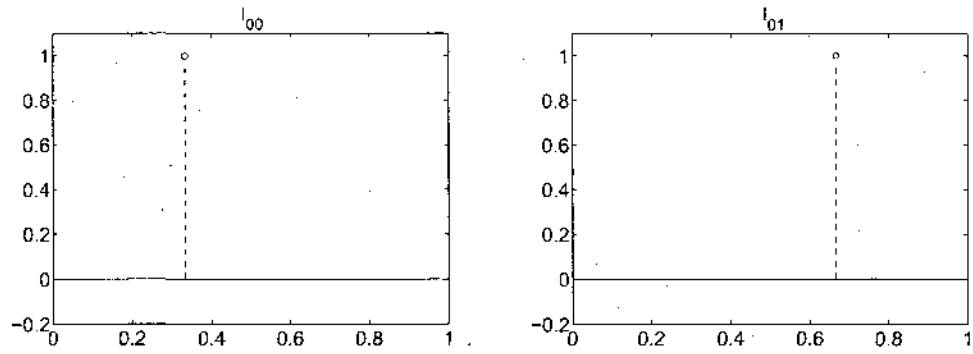
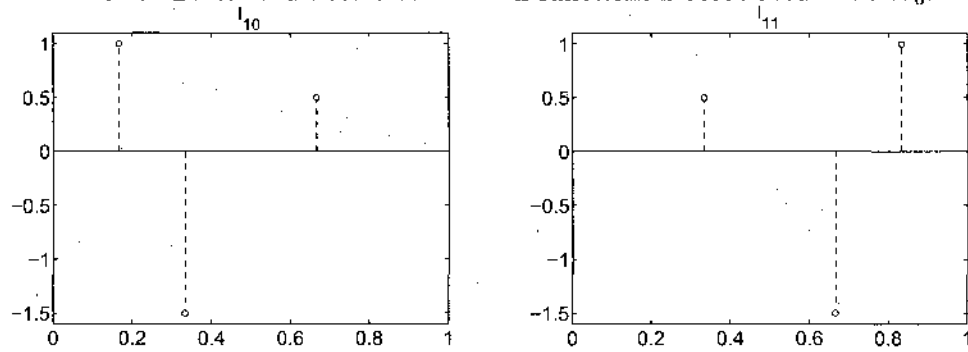
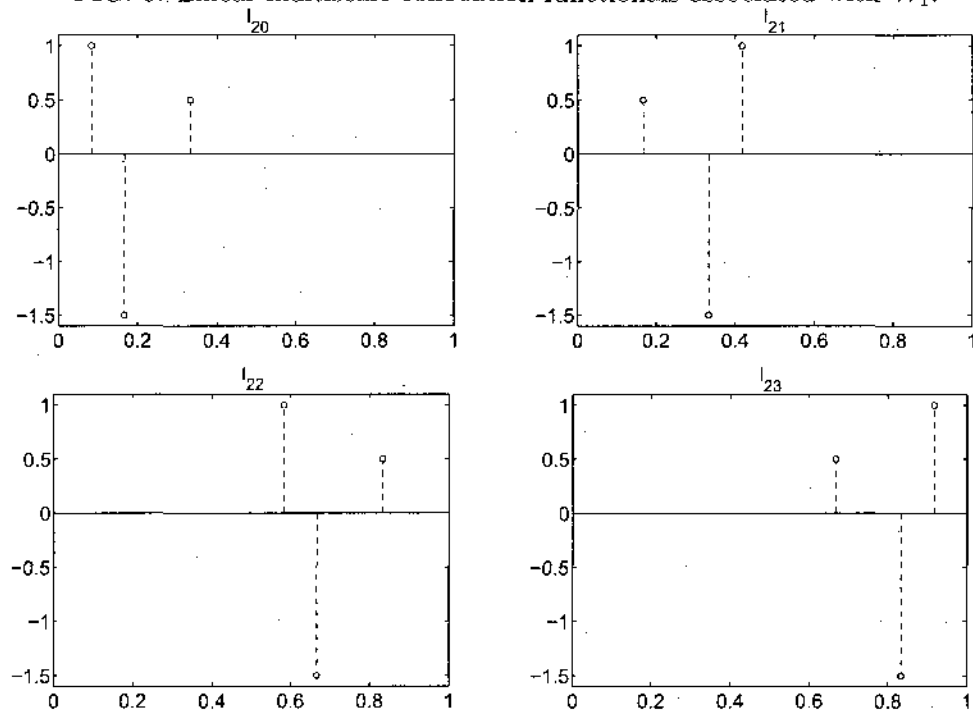


FIG. 4: Linear multiscale wavelet bases on W_3 .

FIG. 5: Linear multiscale collocation functionals associated with W_0 .FIG. 6: Linear multiscale collocation functionals associated with W_1 .FIG. 7: Linear multiscale collocation functionals associated with W_2 .

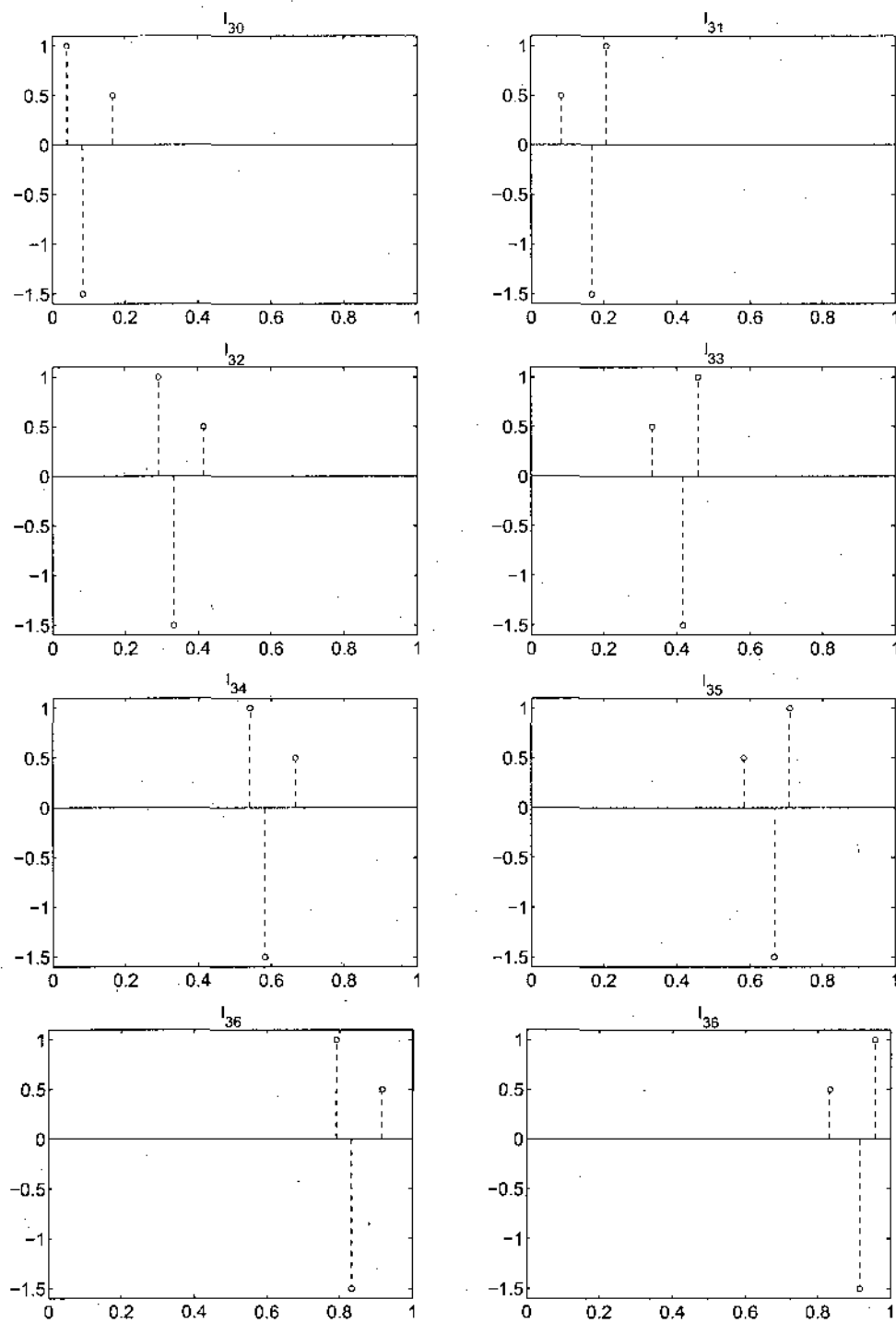


FIG. 8: Linear multiscale collocation functionals associated with W_3 .

TABLE 1: Coefficients of piecewise linear wavelet functions w_{ij} .

	(i, j)	m	Wavelet bases						
			a	b	c	d	I_1	I_2	I_3
W_0	(0,0)	1	2	-3	-	-	0	1	-
	(0,1)	2	-1	3	-	-	0	1	-
W_1	(1,0)	3	1	-9/2	-1	3/2	0	1/2	1
	(1,1)	4	1/2	-3/2	-7/2	9/2	0	1/2	1
W_2	(2,0)	5	1	-9	-1	3	0	1/4	1/2
	(2,1)	6	1/2	-3	-7/2	9	0	1/4	1/2
	(2,2)	7	11/2	-9	-5/2	3	1/2	3/4	1
	(2,3)	8	2	-3	-8	9	1/2	3/4	1
W_3	(3,0)	9	1	-18	-1	6	0	1/8	1/4
	(3,1)	10	1/2	-6	-7/2	18	0	1/8	1/4
	(3,2)	11	11/2	-18	-5/2	6	1/4	3/8	1/2
	(3,3)	12	2	-6	-8	18	1/4	3/8	1/2
	(3,4)	13	10	-18	-4	6	1/2	5/8	3/4
	(3,5)	14	7/2	-6	-25/2	18	1/2	5/8	3/4
	(3,6)	15	29/2	-18	-11/2	6	3/4	7/8	1
(3,7)	16	5	-6	-17	18	3/4	7/8	1	
W_4	(4,0)	17	1	-36	-1	12	0	1/16	1/8
	(4,1)	18	1/2	-12	-7/2	36	0	1/16	1/8
	(4,2)	19	11/2	-36	-5/2	12	1/8	3/16	1/4
	(4,3)	20	2	-12	-8	36	1/8	3/16	1/4
	(4,4)	21	10	-36	-4	12	1/4	5/16	3/8
	(4,5)	22	7/2	-12	-25/2	36	1/4	5/16	3/8
	(4,6)	23	29/2	-36	-11/2	12	3/8	7/16	1/2
	(4,7)	24	5	-12	-17	36	3/8	7/16	1/2
	(4,8)	25	19	-36	-7	12	1/2	9/16	5/8
	(4,9)	26	13/2	-12	-43/2	36	1/2	9/16	5/8
	(4,10)	27	47/2	-36	-17/2	12	5/8	11/16	3/4
	(4,11)	28	8	-12	-26	36	5/8	11/16	3/4
	(4,12)	29	28	-36	-10	12	3/4	13/16	7/8
	(4,13)	30	19/2	-12	-61/2	36	3/4	13/16	7/8
	(4,14)	31	65/2	-36	-23/2	12	7/8	15/16	1
(4,15)	32	11	-12	-35	36	7/8	15/16	1	

TABLE 2: Coefficients of piecewise linear wavelet function w_{ij} [Cont.].

	(i, j)	m	Wavelet bases						
			a	b	c	d	I_1	I_2	I_3
W_5	(5,0)	33	1	-72	-1	24	0	1/32	1/16
	(5,1)	34	1/2	-24	-7/2	72	0	1/32	1/16
	(5,2)	35	11/2	-72	-5/2	24	1/16	3/32	1/8
	(5,3)	36	2	-24	-8	72	1/16	3/32	1/8
	(5,4)	37	10	-72	-4	24	1/8	5/32	3/16
	(5,5)	38	7/2	-24	-25/2	72	1/8	5/32	3/16
	(5,6)	39	28/2	-72	-11/2	24	3/16	7/32	1/4
	(5,7)	40	5	-24	-17	72	3/16	7/32	1/4
	(5,8)	41	19	-72	-7	24	1/4	9/32	5/16
	(5,9)	42	13/2	-24	-43/2	72	1/4	9/32	5/16
	(5,10)	43	47/2	-72	-17/2	24	5/16	11/32	3/8
	(5,11)	44	8	-24	-26	72	5/16	11/32	3/8
	(5,12)	45	28	-72	-10	24	3/8	13/32	7/16
	(5,13)	46	19/2	-24	-61/2	72	3/8	13/32	7/16
	(5,14)	47	65/2	-72	-23/2	24	7/16	15/32	1/2
	(5,15)	48	11	-24	-35	72	7/16	15/32	1/2
	(5,16)	49	37	-72	-13	24	1/2	17/32	9/16
	(5,17)	50	25/2	-24	-79/2	72	1/2	17/32	9/16
	(5,18)	51	83/2	-72	-29/2	24	9/16	19/32	5/8
	(5,19)	52	14	-24	-44	72	9/16	19/32	5/8
	(5,20)	53	46	-72	-16	24	5/8	21/32	11/16
	(5,21)	54	31/2	-24	-97/2	72	5/8	21/32	11/16
	(5,22)	55	101/2	-72	-35/2	24	11/16	23/32	3/4
	(5,23)	56	17	-24	-53	72	11/16	23/32	3/4
	(5,24)	57	55	-72	-19	24	3/4	25/32	13/16
	(5,25)	58	37/2	-24	-115/2	72	3/4	25/32	13/16
	(5,26)	59	119/2	-72	-41/2	24	13/16	27/32	7/8
	(5,27)	60	20	-24	-62	72	13/16	27/32	7/8
	(5,28)	61	64	-72	-22	24	7/8	29/32	15/16
	(5,29)	62	43/2	-24	-133/2	72	7/8	29/32	15/16
	(5,30)	63	137/2	-72	-47/2	24	15/16	31/32	1
(5,31)	64	23	-24	-71	72	15/16	31/32	1	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
W_9	(9,512)	1024	383	-384	-1151	1152	255/256	511/512	1

TABLE 3: Coefficients of piecewise linear collocation functional ℓ_{ij} .

	(i, j)	m	Collocation functionals				(i, j)	m	Collocation functionals		
			t_1	t_2	t_3				t_1	t_2	t_3
W_0	(0,0)	1	1/3	-	-	W_5	(5,0)	33	1/96	1/48	1/24
	(0,1)	2	2/3	-	-		(5,1)	34	1/48	1/24	5/96
W_1	(1,0)	3	1/6	1/3	2/3	(5,2)	35	7/96	1/12	5/48	
	(1,1)	4	1/3	2/3	5/6	(5,3)	36	1/12	5/48	11/96	
W_2	(2,0)	5	1/12	1/6	1/3	(5,4)	37	13/96	7/48	1/6	
	(2,1)	6	1/6	1/3	5/12	(5,5)	38	7/48	1/6	17/96	
	(2,2)	7	7/12	2/3	5/6	(5,6)	39	19/96	5/24	11/48	
	(2,3)	8	2/3	5/6	11/12	(5,7)	40	5/24	11/48	23/96	
W_3	(3,0)	9	1/24	1/12	1/6	(5,8)	41	25/96	13/48	7/24	
	(3,1)	10	1/12	1/6	5/24	(5,9)	42	13/48	7/24	29/96	
	(3,2)	11	7/24	1/3	5/12	(5,10)	43	31/96	1/3	17/48	
	(3,3)	12	1/3	5/12	11/24	(5,11)	44	1/3	17/48	35/96	
	(3,4)	13	13/24	7/12	2/3	(5,12)	45	37/96	19/48	5/12	
	(3,5)	14	7/12	2/3	17/24	(5,13)	46	19/48	5/12	41/96	
	(3,6)	15	19/24	5/6	11/12	(5,14)	47	43/96	11/24	23/48	
	(3,7)	16	5/6	11/12	23/24	(5,15)	48	11/24	23/48	47/96	
W_4	(4,0)	17	1/48	1/24	1/12	(5,16)	49	49/96	25/48	13/24	
	(4,1)	18	1/24	1/12	5/48	(5,17)	50	25/48	13/24	53/96	
	(4,2)	19	7/48	1/6	5/24	(5,18)	51	55/96	7/12	29/48	
	(4,3)	20	1/6	5/24	11/48	(5,19)	52	7/12	29/48	59/96	
	(4,4)	21	13/48	7/24	1/3	(5,20)	53	61/96	31/48	2/3	
	(4,5)	22	7/24	1/3	17/48	(5,21)	54	31/48	2/3	65/96	
	(4,6)	23	19/48	5/12	11/24	(5,22)	55	67/96	17/24	35/48	
	(4,7)	24	5/12	11/24	23/48	(5,23)	56	17/24	35/48	71/96	
	(4,8)	25	25/48	13/24	23/48	(5,24)	57	73/96	37/48	19/24	
	(4,9)	26	13/24	7/12	29/48	(5,25)	58	37/48	19/24	77/96	
	(4,10)	27	31/48	2/3	17/24	(5,26)	59	79/96	5/6	41/48	
	(4,11)	28	2/3	17/24	35/48	(5,27)	60	5/6	41/48	83/96	
	(4,12)	29	37/48	19/24	5/6	(5,28)	61	85/96	43/48	11/12	
	(4,13)	30	19/24	5/6	41/48	(5,29)	62	43/48	11/12	89/96	
	(4,14)	31	43/48	11/12	23/24	(5,30)	63	91/96	23/24	47/48	
(4,15)	32	11/12	23/24	47/48	(5,31)	64	23/24	47/48	95/96		
	\vdots		\vdots	\vdots		\vdots	\vdots	\vdots	\vdots	\vdots	
W_9	(9,512)	1024	$\frac{383}{384}$	$\frac{767}{768}$	$\frac{1535}{1536}$						

CHAPTER III

POST-PROCESSED TECHNIQUES

In this chapter, we investigate a number of post-processing techniques which can be used to enhance the accuracy of numerical solutions of nonlinear integral equations of the Hammerstein type. Post-processing techniques discussed here can be classified into two groups, one based upon an interpolation and another based upon an extrapolation. Motivation for this research originates in the recent papers [29] and [28] in which similar results were obtained for linear integral equations of the Fredholm type. One of the goals of this study is to extend the results in [29] and [28] to a class of nonlinear equations. We represent that the post-processing techniques based upon an interpolation and an extrapolation results in even simpler processing of numerical solutions which results in less expensive computational cost in improving the accuracy of numerical solutions than the original projection methods.

III.1 PRELIMINARY

In this preliminary section, we gather together several results which already exist in the area of post-processing techniques for the linear as well as nonlinear integral equations. Included are the technique based upon an iteration, an interpolation and an extrapolation. The iterative methods were investigated by many authors, e.g., see Atkinson [3], Graham et al. [15], Sloan and Thomas [33], Kaneko and Xu [20], and Kaneko et al. [18]. Several post-processing techniques based upon an interpolation of the collocation solution and that of the Galerkin solutions were studied by Lin et al. [29]. We will see that the post-processing technique based upon the interpolation is more cost effective computationally than the iterative methods. Finally, an extrapolation of iterated collocation solutions and global extrapolation for the Fredholm equation discovered by Lin et al. [28] and [29] respectively are presented. We extend many existing theorems on the post-processing techniques to nonlinear Hammerstein equations. This will be done in Sections III.2 - III.6.

Recall the Fredholm equation of the second kind is written as

$$u(t) - \int_0^1 k(s,t)u(s)ds = f(t), \quad t \in I \equiv [0,1], \quad (23)$$

or in the operator form

$$u - Ku = f.$$

Denote S^h to be the space of piecewise polynomials of degree $\leq r$, i.e.,

$$S^h = \{v \in L^2(I) : v|_{e_i} \in P_r, 0 \leq i \leq N-1\},$$

where P_r denotes the space of all polynomials of degree $\leq r$.

III.1.1 Iterative method for Fredholm equation

Under the projection operator P_h , either interpolation projection or an orthogonal projection, the solution of equation (23) is approximated by solving the projection equation;

$$u^h - P_h K u^h = P_h f, \quad u_n \in X_n. \quad (24)$$

The iterated solution is then obtained by calculating u_{it}^h from

$$u_{it}^h = f + K u^h. \quad (25)$$

Applying P_h to both sides of (25), we obtain

$$P_h u_{it}^h = P_h f + P_h K u^h. \quad (26)$$

Using (24) and (26), we see that

$$P_h u_{it}^h = u^h.$$

Thus, the solution u_{it}^h satisfies

$$u_{it}^h = f + K P_h u_{it}^h.$$

It is proved in [3] that if $f \in C^{2r+2}(I)$ and $k \in C^{2r+2}(I \times I)$, then

$$\|u - u_{it}^h\| = O(h^{2r+2}),$$

whereas $\|u - u^h\| = O(h^{r+1})$.

Therefore, the iterative method, when applied to the collocation method as well as to the Galerkin method, doubles the order of accuracy of a numerical solution, provided that the solution and the kernel of the integral equation are sufficiently smooth. This is an excellent method to improve the accuracy of numerical solution.

We note that, in order to double the rate of convergence of numerical approximation by the collocation or the Galerkin method, it is necessary that the number of basis functions must be doubled. This results in the solution process which involves a much larger system of linear or nonlinear equations. The iterative methods, on the other hand, presents an accuracy enhancement technique which avoids a larger system and thus is computationally more efficient.

III.1.2 Interpolation method for Fredholm equation

A post-processing technique based upon interpolation appeared in a 1998 paper by Lin et al. [29]. The paper is concerned with the interpolation of the collocation solutions of (23).

To describe briefly this method, let T^h be a uniform partition on I with mesh size h and then define T^{2h} with mesh size $2h$ by subdividing each element of T^h into two elements. Moreover, define the collocation points to be the zeros of the Legendre polynomial of degree $(r + 1)$ mapped to each subinterval. Then using these collocation points, a higher interpolation operator I_{2h}^{2r+1} of degree $(2r + 1)$ over two consecutive subintervals is defined. The following superconvergence result is obtained,

$$\|u(t) - I_{2h}^{2r+1}(t)\| = O(h^{2r+2}).$$

We note that, in order to establish this result, one crucial property called *superclose* plays a critical role. It is shown that the interpolation method can be used to attain the same convergence rate as the iterated method with less computational cost.

A global superconvergence of the Galerkin solution of Volterra integral equation by interpolation is also reported in [29]. It is obtained that if $u \in C^{r+2}(I)$,

$$u(t) - I_{2h}^{r+1}u(t) = O(h^{r+2}),$$

where I_{2h}^{r+1} is an interpolation operator of degree of $(r + 1)$ associated with the mesh T^{2h} .

III.1.3 Extrapolation method for Fredholm equation

A post-processing technique based upon the extrapolation of the iterated collocation method for (23) reported by Lin et. al. [28] is now discussed. This extrapolation technique can be derived by using the following theorem in [28] which derives an asymptotic expansion of the error of the iterated collocation solution u_{it}^h ,

$$u(t) - u_{it}^h(t) = \sum_{k=1}^N h_k^{2r+2} \int_{e_k} b(s, t) ds + O(h^{2r+4}), \quad t \in I,$$

where $k \in C^{r+3}(I \times I)$ and $u \in C^{2r+4}(I)$.

Using this formula, the Richardson extrapolation gives

$$u(t) - \bar{u}_{it}^{h/2}(t) = O(h^{2r+4}),$$

where

$$\bar{u}_{it}^{h/2}(t) \equiv \frac{2^{2r+2} u_{it}^{h/2}(t) - u_{it}^h(t)}{2^{2r+2} - 1}.$$

We see that the rate of extrapolation method is higher than the rate of iterated method. This leads to another efficient scheme to obtain more accuracy of numerical approximation.

A global extrapolation approximation was also introduced in [29] by using a similar idea to the extrapolation scheme just described. Here, the Richardson extrapolation is performed on the interpolated collocation solution of the Fredholm equation. More specifically, we compute

$$\frac{2^{2r+2} I_{3h/2}^{2r+3} u^{h/2}(t) - I_{3h}^{2r+3} u^h(t)}{2^{2r+2} - 1} - u(t) = O(h^{2r+4}),$$

where I^{2r+3} is the interpolation operator of degree $(2r + 3)$ over three consecutive subintervals.

In the subsequent sections, we show how these post-processing techniques for the Fredholm equation can be extended to solve nonlinear Hammerstein integral equation. In Section III.2, we study a global superconvergence property of the interpolation post-processing technique for the collocation method for the Hammerstein equation. The case of the weakly singular Hammerstein equation is also included in this section. After this manuscript was completed, the paper by Huang and Zhang [16] was brought to the authors' attention which proved the same superconvergence

result for the post-processing collocation method. Despite this discovery, we decided to keep Section III.2 in this study since the proof provided here is different from the one given in [16] and it is more concise. Moreover, we present, in Section III.2, a numerical example exhibiting that the same superconvergence of the post-processed collocation method can be obtained for two-dimensional Hammerstein equations. Numerical experiments conducted on the two-dimensional Hammerstein reveals that there exist a number of interesting and important issues which must be addressed for a successful implementation of the post-processing technique via interpolation for multi-dimensional integral equations. They will be discussed in future research. In Section III.2, we also briefly mention a post-processing technique by interpolation to enhance the order of accuracy of a numerical solution of one-dimensional weakly singular Hammerstein equation. A numerical example using constant basis functions is reported in [16]. We include in Section III.2 two additional numerical examples for weakly singular Hammerstein equations using linear basis functions. We consider both types of weak singularities; logarithmic and algebraic.

An extrapolation technique for the iterated collocation solution of the Hammerstein equation is discussed in Section III.3. Results in Section III.3 play a critical role in establishing the global extrapolation method presented in Section III.4. The final two sections, Section III.5 and III.6, are concerned with superconvergence of the Galerkin method by the post-processing techniques. In Section III.5, we apply the interpolation technique explored in Section III.2 to the Galerkin method. We achieve a superconvergence result but the rate of acceleration is not as great as that of the collocation case of Section III.2. This result is consistent with a similar result for the Volterra equation reported in [29]. An extrapolation of the iterated Galerkin method is treated in the final section, Section III.6. Results in Section III.6 appear new even in the setting of the linear Fredholm equation.

III.2 GLOBAL SUPERCONVERGENCE FOR HAMMERSTEIN EQUATION BY COLLOCATION METHOD

In this section, we consider the following Hammerstein equation,

$$u(t) - \int_0^1 k(s, t)\psi(s, u(s))ds = f(t), \quad t \in I \equiv [0, 1], \quad (27)$$

where k , f and ψ are known functions and u is the function to be determined.

First, we introduce two kinds of norm

$$\|v\|_{m, \infty} \equiv \max_{0 \leq i \leq m} \{\|v^{(i)}\|_{\infty}\},$$

and

$$\|v\|_{m, 2} \equiv \left\{ \int_I \sum_{i=0}^m (v^{(i)}(x))^2 dx \right\}^{1/2}$$

where m is a nonnegative integer.

Let T^h be a partition of I :

$$0 = t_0 < t_1 < \cdots < t_N = 1,$$

and $e_i \equiv [t_i, t_{i+1})$, $i = 0, 1, \dots, N-2$ and $e_{N-1} = [t_{N-1}, t_N]$, $h_i = t_{i+1} - t_i$ and $h = \max_i h_i$. We denote by S^h the space of piecewise polynomials of degree $\leq r$, i.e.,

$$S^h = \{v \in L^2(I) : v|_{e_i} \in P_r, 0 \leq i \leq N-1\},$$

where P_r denotes the space of all polynomials of degree $\leq r$. Let $B = B_{r+1}$ consist of zeroes of $r+1$ degree Legendre polynomial located in $[-1, 1]$. Define $\Phi_i: [-1, 1] \rightarrow e_i$, $i = 0, \dots, N-1$, by

$$\Phi_i(t) = \frac{1+t}{2}t_{i+1} + \frac{1-t}{2}t_i, \quad t \in [-1, 1],$$

and

$$A = \bigcup_{i=0}^{N-1} \Phi_i(B),$$

so that A contains the collocation points. The collocation approximation $u^h \in S^h$ is obtained under the assumption that the residual

$$R^h(t) \equiv u^h(t) - \int_0^1 k(s, t)\psi(s, u^h(s))ds - f(t)$$

disappears under the interpolation projection i_h^r of $C(I)$ onto S^h . Here $i_h^r: C(I) \rightarrow S^h$ is defined by

$$i_h^r u|_{e_i} \in P_r, \quad i_h^r u(t) = u(t), \quad \text{for } t \in \Phi_i(B).$$

Equivalently,

$$u^h(t) - \int_0^1 k(s, t) \psi(s, u^h(s)) ds = f(t), \quad \text{for all } t \in A. \quad (28)$$

To describe equations (27) and (28) in operator form, we let

$$K\Psi(u)(t) \equiv \int_0^1 k(s, t) \psi(s, u(s)) ds, \quad t \in I,$$

and

$$\Psi(u)(s) = \psi(s, u(s)).$$

Then (27) and (28) can be written, respectively, as

$$u - K\Psi(u) = f, \quad (29)$$

and

$$u^h - i_h^r K\Psi(u^h) = i_h^r f. \quad (30)$$

We now establish a *superclose* estimate for $u^h - i_h^r u$ in relation to the Hammerstein equation. From (29),

$$i_h^r u - i_h^r K\Psi(u) = i_h^r f. \quad (31)$$

Let

$$g(t, s, i_h^r u(s), u^h(s), \theta) \equiv k(s, t) \psi^{(0,1)}(s, i_h^r u(s) + \theta(u^h(s) - i_h^r u(s))),$$

where $0 < \theta < 1$ and

$$G_h u(s) \equiv \int_0^1 g(t, s, i_h^r u(s), u^h(s), \theta) u(s) ds.$$

Here we assume that 1 is not an eigenvalue of the operator G_h , so that $I - G_h$ is invertible. Using (30) and (31), we obtain

$$\begin{aligned} u^h - i_h^r u &= i_h^r K\Psi(u^h) - i_h^r K\Psi(u) \\ &= i_h^r K[\Psi(u^h) - \Psi(i_h^r u) + \Psi(i_h^r u) - \Psi(u)] \end{aligned}$$

or

$$[u^h - i_h^r u] - i_h^r K[\Psi(u^h) - \Psi(i_h^r u)] = i_h^r K[\Psi(i_h^r u) - \Psi(u)],$$

and

$$u^h - i_h^r u = (I - i_h^r G_h)^{-1} i_h^r K[\Psi(i_h^r u) - \Psi(u)]. \quad (32)$$

An estimate on the right side of (32) was investigated in [18] and in the present setting, it is proved that

$$(I - i_h^r G_h)^{-1} i_h^r K[\Psi(i_h^r u) - \Psi(u)] = O(h^{2r+2}),$$

which gives the *superclose* identity

$$u^h - i_h^r u = O(h^{2r+2}). \quad (33)$$

The remaining analysis for obtaining global superconvergence by interpolation post-processing technique is the same as that for the linear case. First, we obtain a collocation solution u^h over the partition T^h where it is assumed that the total number N of intervals is even. u^h is then interpolated at the collocation points over two consecutive intervals $e_i \cup e_{i+1}$ by a polynomial of degree $2r+1$. In this connection, we define an interpolation operator I_{2h}^{2r+1} as follows:

$$I_{2h}^{2r+1} u|_{e_i \cup e_{i+1}} \in P_{2r+1}, \quad i = 0, 2, \dots, N-2,$$

and

$$I_{2h}^{2r+1} u(t) = u(t), \quad t \in \Psi_i(B) \cup \Psi_{i+1}(B).$$

If $u \in C^{2r+2}(I)$, then

$$\begin{aligned} \|I_{2h}^{2r+1} u^h - u\|_{0,\infty} &\leq \|I_{2h}^{2r+1}(u^h - i_h^r u)\|_{0,\infty} + \|I_{2h}^{2r+1} i_h^r u - u\|_{0,\infty} \\ &\leq C \|u^h - i_h^r u\|_{0,\infty} + \|I_{2h}^{2r+1} u - u\|_{0,\infty}, \end{aligned}$$

here we used the obvious fact that

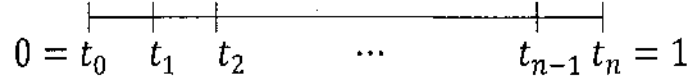
$$I_{2h}^{2r+1} i_h^r = I_{2h}^{2r+1}, \quad \|I_{2h}^{2r+1}\|_{0,\infty} < C, \quad \text{for some } C > 0.$$

This together with (33) gives

$$\|I_{2h}^{2r+1} u^h - u\|_{0,\infty} = O(h^{2r+2}). \quad (34)$$

We note that a similar estimate to (34) under L^2 norm is also valid.

In all numerical experiments of this Chapter, unless otherwise stated, we use the following general settings:

FIG. 9: Uniform meshes on I .

- The space of piecewise linear polynomials are considered, i.e.- $r = 1$.
- The interval $[0,1]$ is divided by uniform meshes, that is $h = \frac{1}{N}$ then $t_i = ih$, $i = 0, 1, \dots, N$ and $e_i = [t_i, t_{i+1}]$.
- The collocation points $\{t_{i,j}^*\}$ are chosen from the zeros of Legendre polynomial of degree two transformed by linear mapping to each subinterval e_i , that is, $B = \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$ then $t_{i,j}^* = \Phi_i(B)$, $i = 0, 1, \dots, N$ and $j = 1, 2$.

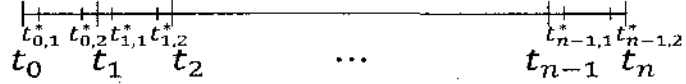
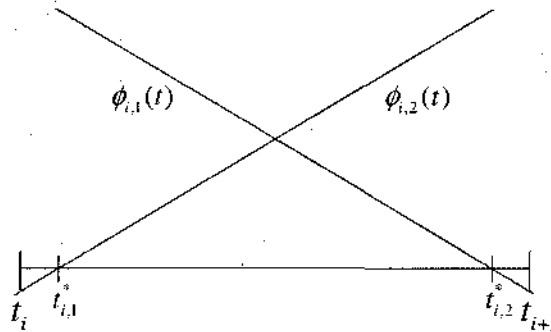


FIG. 10: Location of collocation points.

- The following linear spline bases are obtained as

$$\phi_{i,1}(t) = \begin{cases} \frac{t_{i,2}^* - t}{t_{i,2}^* - t_{i,1}^*}, & t \in [t_i, t_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

$$\phi_{i,2}(t) = \begin{cases} \frac{t - t_{i,1}^*}{t_{i,2}^* - t_{i,1}^*}, & t \in [t_i, t_{i+1}] \\ 0, & \text{otherwise.} \end{cases}$$

FIG. 11: Linear spline bases on each e_i .

Therefore, the projection solution u^h is defined by

$$u^h(t) := \sum_{i=1}^N \sum_{j=1}^2 c_{ij} \phi_{i,j}(t)$$

- The Lagrange interpolation is employed over two consecutive intervals to obtain the post-processed solution, see more details of interpolation in Appendix A.2. Note that for our specific problem, since the linear spline basis is used, the solution is interpolated by a cubic piecewise polynomial.

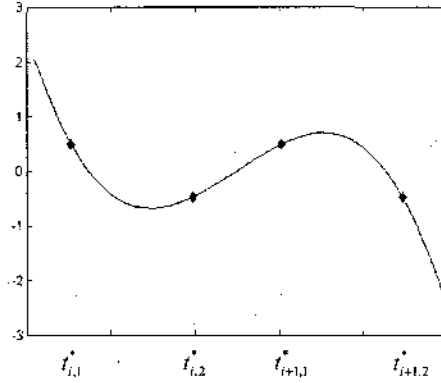


FIG. 12: Cubic interpolation on $e_i \times e_{i+1}$.

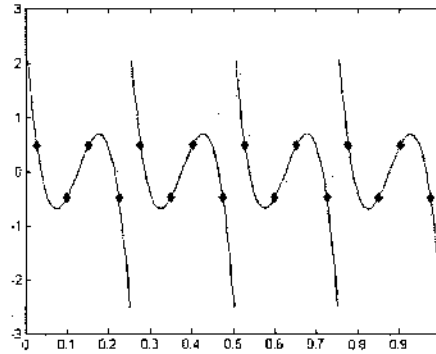


FIG. 13: Global cubic interpolation on I .

Moreover, the corresponding nonlinear systems are solved by two different iteration schemes. The original Newton-Raphson iteration method is used first with a sufficiently close initial element. Another approach is Quasi-Newton obtained by

$$c_{k+1} = c_k - J^{-1}(c_k) F(c_k)$$

where c_k is an unknown coefficient to be determined the solution is u , k is the number of iteration times and k' is a fixed number of iteration times. That is, the Jacobian is fixed throughout the iterations in the Quasi-Newton method. A stopping criteria is taken within fourteen digit accuracy, i.e., tolerance $\epsilon < 10^{-14}$.

Example III.2.1. Consider the equation

$$u(t) - \int_0^1 \sin(\pi(s+t))u^2(s)ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \sin(\pi t)$.

Notice that we define

$$\begin{aligned} e_h &= \|u - u^h\|_{0,\infty}, & R_h &= \log_2 \left(\frac{e_h}{e_{h/2}} \right), \\ \bar{e}_h &= \|u - I_{2h}^{2r+1} u^h\|_{0,\infty}, & \bar{R}_h &= \log_2 \left(\frac{\bar{e}_h}{\bar{e}_{h/2}} \right). \end{aligned}$$

We use NI for the total number of iterations, and CT is the CPU time for solving the collocation method of each scheme.

TABLE 4: Computational results of interpolation with smooth kernel.

N	Collocation		Newton		Quasi-Newton		Interpolation	
	e_h	R_h	NI	CT	NI	CT	\bar{e}_h	\bar{R}_h
2	7.5734e-2		5	0.12	7	0.09	9.0912e-2	
4	2.3643e-2	1.68	5	0.19	8	0.15	4.4949e-3	4.34
8	6.2950e-3	1.91	5	0.67	8	0.58	3.3793e-4	3.73
16	1.5982e-3	1.98	5	2.54	8	2.00	2.2050e-5	3.94
32	4.0108e-4	1.99	5	9.77	8	7.50	1.3928e-6	3.98
64	1.0037e-4	2.00	6	46.19	8	29.47	8.7283e-8	4.00
128	2.5098e-5	2.00	6	183.00	9	124.41	5.4588e-9	4.00

We can see the results confirm the estimate described in (34). See [16] for additional examples. Although the Quasi-Newton method requires more iterations than the Newton's method, the computing time of Quasi-Newton method is less. This difference is more pronounced in the Galerkin based computation which will be presented in Sections III.5 and III.6.

In the case of the Hammerstein equations with weakly singular kernels, the superconvergence result of (34) by the interpolation post-processing technique can also be obtained. Here, the kernel is assumed to be of the type

$$k(s, t) = k^* \left(\frac{t}{s} \right) \frac{1}{s},$$

where, with $\sigma = \frac{t}{s}$ and D^ℓ denoting the differential operator of order ℓ ,

$$\int_0^1 \sigma^\ell |D^\ell k^*(\sigma)| \frac{d\sigma}{\sigma} \leq \begin{cases} C_0 < 1, & \ell = 0, \\ C < \infty, & \ell \geq 1. \end{cases}$$

It is known that the optimal order of convergence of the collocation method for weakly singular Hammerstein equations can be obtained by use of a graded mesh, see, -e.g., [3], [19]. For example, if the spline of degree r is used in computation, one may select a partition T_h , $0 = t_0 < t_1 < \dots < t_N = 1$, with $t_i = (\frac{i}{N})^q$ and $q \geq r + 1$ to preserve the optimal order of convergence. To attain a similar super-convergence result for the numerical solution of weakly singular equations by the interpolation post-processing technique, we simply select the partition by defining $t_i = (\frac{i}{N})^q$, $q \geq 2r + 2$ and perform post-processing by the interpolation described above over the intervals beginning with t_1 . Note that a selection of $t_1 = N^{-q}$ guarantees the size of the first interval $[0, t_1]$ small enough so that the approximation error from this interval is consistent with the errors from the subsequent intervals despite the fact that the solution may not be differentiable over $[0, t_1]$. A numerical experiment is reported in [16] demonstrating the effectiveness of this approach using a constant basis whereas we present two additional examples using the linear basis.

Example III.2.2. Consider the equation

$$u(t) - \int_0^1 \log |s - t| u^2(s) ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^2$.

TABLE 5: Computational results of interpolation with logarithmic singular kernel.

N	Collocation		Interpolation	
	e_h	R_h	\bar{e}_h	\bar{R}_h
4	1.006500e-2		8.492456e-3	
8	2.606709e-3	1.9490	7.007576e-4	3.5992
16	6.518285e-4	1.9997	5.297109e-4	3.7256
32	1.628305e-4	2.0011	3.722810e-6	3.8307
64	4.069555e-5	2.0004	2.388927e-7	3.9620
128	1.017298e-5	2.0001	1.431229e-8	4.0610

Example III.2.3. Consider the equation

$$u(t) - \int_0^1 \frac{1}{\sqrt{|s-t|}} u^2(s) ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^2$.

TABLE 6: Computational results of interpolation with algebraic singular kernel.

N	Collocation		Interpolation	
	e_h	R_h	\bar{e}_h	\bar{R}_h
4	3.202584e-1		4.562162e-1	
8	8.081319e-3	5.3085	7.209956e-3	5.9836
16	1.035501e-3	2.9643	5.693164e-4	3.6627
32	1.897584e-4	2.4481	4.070014e-5	3.8061
64	4.280083e-5	2.1485	3.377003e-6	3.5912
128	1.038697e-5	2.0429	2.746804e-7	3.6199

Now we are ready to exhibit an example of multi-variable Hammerstein equation to demonstrate that the post-processing technique based upon the interpolation can be applied to multi-variable Hammerstein equation. For $\bar{s}, \bar{t} \in \mathbb{R}^2$, we consider

$$u(\bar{t}) - \int_0^1 \int_0^1 k(\bar{s}, \bar{t}) \psi(\bar{s}, u(\bar{s})) d\bar{s} = f(\bar{t}), \quad \bar{s}, \bar{t} \in I \times I. \quad (35)$$

We use the tensor product of S^h with itself, $S^h \otimes S^h$, as our approximating space. Note that $\cup_{h>0} (S^h \otimes S^h)$ is essentially dense in $C(I \times I)$, see [27]. With $\vec{i}_h^r \equiv \vec{i}_h^r \otimes \vec{i}_h^r$, $\vec{i}_h^r: C(I \times I) \rightarrow S^h \otimes S^h$ satisfies

$$\vec{i}_h^r u|_{e_i \times e_j} \in P_r \otimes P_r, \quad \vec{i}_h^r u(\bar{t}) = u(\bar{t}), \quad \text{for } \bar{t} \in \Phi_i(B) \times \Phi_j(B).$$

The collocation method is to solve

$$u^h(\bar{t}) - \int_0^1 \int_0^1 k(\bar{s}, \bar{t}) \psi(\bar{s}, u^h(\bar{s})) d\bar{s} = f(\bar{t}), \quad \bar{t} \in \cup_{0 \leq i, j \leq N-1} \Phi_i(B) \times \Phi_j(B). \quad (36)$$

Once $u^h(\bar{t})$ is obtained, one may interpolate its values at the collocation points over four squares $e_i \cup e_{i+1} \times e_j \cup e_{j+1}$ by the two-dimensional polynomial in the form

$$u^h(t, t') = a_1 + a_2 t' + a_3 t + a_4 t t' + a_5 t'^2 + a_6 t^2 + a_7 t t'^2 + a_8 t^2 t' + a_9 t'^3 + a_{10} t^3. \quad (37)$$

Here, recall that linear splines are used to discretise the solution in each direction. To double the order of accuracy, we require a polynomial of degree three in two variables

t, t' as indicated in (37). Our numerical experiment indicates that the location of these ten interpolation points influences much in achieving the superconvergence of a desired accuracy. More discussions on the post-processing technique for multi-variable Hammerstein equations will be made in future research.

Example III.2.4. Consider the following two-dimensional equation

$$u(t, t') - \int_0^1 \int_0^1 (s-t)(s'-t')(s+s'+u(s, s'))^2 ds ds' = f(t, t'), \quad (t, t') \in I \times I,$$

where f is chosen so that the exact solution is $u(t, t') = \exp(t + t')$. Ten points are selected from four contiguous squares $e_i \cup e_{i+1} \times e_j \cup e_{j+1}$, $i, j = 0, 2, \dots, 2N - 2$ and they are circled in FIG. 14, or $(t_{i,1}^*, t_{j,1}^*), (t_{i+1,1}^*, t_{j,1}^*), (t_{i+1,2}^*, t_{j,1}^*), (t_{i,2}^*, t_{j,2}^*), (t_{i+1,2}^*, t_{j,2}^*), (t_{i,1}^*, t_{j+1,1}^*), (t_{i,2}^*, t_{j+1,1}^*), (t_{i+1,1}^*, t_{j+1,1}^*), (t_{i,2}^*, t_{j+1,2}^*),$ and $(t_{i+1,2}^*, t_{j+1,2}^*)$.

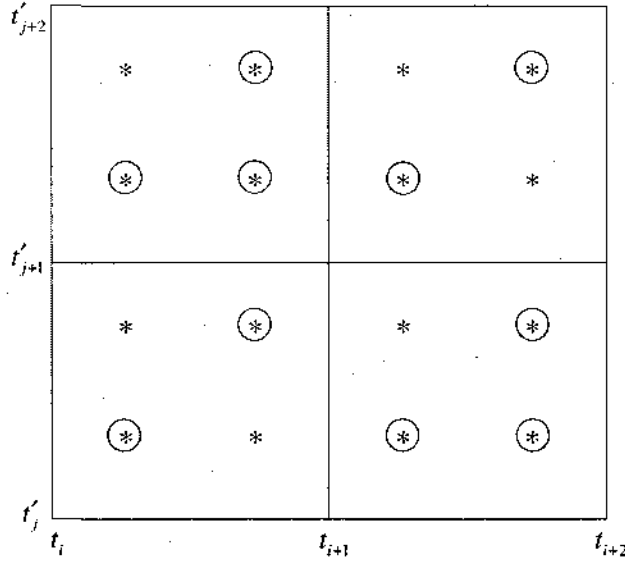


FIG. 14: Location of ten interpolating points.

In TABLE 7, NC is the total number of collocation points in domain $I \times I$. With the ten interpolation points described, our numerical experiment confirms the same superconvergence as the one-dimensional problem in Example III.2.1.

TABLE 7: Computational results of interpolation for two dimensional equation.

(N, N)	NC	Collocation		Interpolation	
		e_h	R_h	\bar{e}_h	\bar{R}_h
(2,2)	16	2.604955e-1		5.303465e-2	
(4,4)	64	7.078941e-2	1.8797	4.782085e-3	3.4712
(8,8)	256	1.845359e-2	1.9396	3.604376e-4	3.7298
(16,16)	1024	4.710457e-3	1.9700	2.412651e-5	3.9011
(32,32)	4096	1.189914e-3	1.9850	1.376186e-6	4.1319

III.3 EXTRAPOLATION OF ITERATED COLLOCATION SOLUTION FOR HAMMERSTEIN EQUATION

In this section, we generalize the result obtained in [28] concerning an extrapolation technique for the iterated collocation method. The iterated collocation solution u_{it}^h for the Hammerstein integral equation is defined as follows: for a collocation solution u^h of (30),

$$u_{it}^h = f(t) + \int_0^1 k(s, t)\psi(s, u^h(s))ds,$$

or, in the operator form,

$$u_{it}^h = f + K\Psi(u^h). \quad (38)$$

From equation (38),

$$i_h^r u_{it}^h = i_h^r f + i_h^r K\Psi(u^h), \quad (39)$$

and (30) and (39) yield

$$i_h^r u_{it}^h = u^h.$$

Thus, (38) becomes

$$u_{it}^h = f + K\Psi(i_h^r u_{it}^h). \quad (40)$$

It is proved in [18] that if $f \in C^{2r+2}(I)$ and $k \in C^{2r+2}(I \times I)$ then

$$\|u - u_{it}^h\|_{0,\infty} = O(h^{2r+2}).$$

The following theorem, which generalizes Theorem 1 of [28], establishes the basic fact which underlines the extrapolation technique for the numerical solution of Hammerstein equation.

Theorem III.3.1. Assume that $k \in C^{r+3}([0, 1] \times [0, 1])$ and the solution u of (27) satisfies $u \in C^{2r+4}[0, 1]$. Also assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(u)$. Then there exists $b \in C([0, 1] \times [0, 1])$, independent of the partition, such that

$$u(t) - u_{it}^h(t) = \sum_{k=1}^N h_k^{2r+2} \int_{e_k} b(s, t) ds + O(h^{2r+4}), \quad t \in [0, 1].$$

Proof. From (29) and (40),

$$\begin{aligned} u - u_{it}^h &= K\Psi(u) - K\Psi(i_h^r u_{it}^h) \\ &= K\Psi(u) - K\Psi(i_h^r u) + K\Psi(i_h^r u) - K\Psi(i_h^r u_{it}^h). \end{aligned} \quad (41)$$

Now, recall from the previous section,

$$G_h u(s) \equiv \int_0^1 g(t, s, i_h^r u(s), i_h^r u_{it}^h(s), \theta) u(s) ds,$$

where g is also defined in Section III.2. Then

$$K\Psi(i_h^r u) - K\Psi(i_h^r u_{it}^h) = G_h i_h^r (u - u_{it}^h).$$

Equation (41) becomes

$$u - u_{it}^h = K\Psi(u) - K\Psi(i_h^r u) + G_h i_h^r (u - u_{it}^h). \quad (42)$$

Arguing as in [18] and using assumptions (A2), (A5) and (A6), we can show that $\{G_h i_h^r\}$ is a family of collectively compact operators and $G_h i_h^r \rightarrow G \equiv (K\Psi)'(u)$ pointwise as $h \rightarrow 0$. Since G is compact and $(I - G)^{-1}$ exists by assumption, from a theory of compact operators (see, -e.g., [3]), $(I - G_h i_h^r)^{-1}$ exists and is uniformly bounded. This shows that

$$\|u - u_{it}^h\|_{0,\infty} \leq C \|K\Psi(u) - K\Psi(i_h^r u)\|_{0,\infty} = O(h^{2r+2}), \quad (43)$$

which establishes the superconvergence of the iterated collocation solution [18]. For present purposes, we require the following. Since

$$(I - G_h i_h^r)^{-1} = (I - G_h)^{-1} - (I - G_h i_h^r)^{-1} G_h (I - i_h^r) (I - G_h)^{-1}, \quad (44)$$

using (42) and (44), we obtain

$$\begin{aligned} u - u_{it}^h &= (I - G_h i_h^r)^{-1} \{K\Psi(u) - K\Psi(i_h^r u)\} \\ &= (I - G_h)^{-1} w^h - (I - G_h i_h^r)^{-1} G_h (I - i_h^r) (I - G_h)^{-1} w^h \\ &= v^h - (I - G_h i_h^r)^{-1} G_h (I - i_h^r) v^h, \end{aligned} \quad (45)$$

where

$$w^h \equiv K\Psi(u) - K\Psi(i_h^r u),$$

and

$$v^h \equiv (I - G_h)^{-1} w^h = (I - G_h)^{-1} K[\Psi(u) - \Psi(i_h^r u)].$$

Let $L \equiv (I - G_h)^{-1} K$ where L is an integral operator with a kernel $l^*(s, t)$ with the same smoothness properties as $k(s, t)$.

Then

$$v^h(t) = L[\Psi(u) - \Psi(i_h^r u)] = \int_0^1 l^*(s, t) [\psi(s, u(s)) - \psi(s, i_h^r u(s))] ds.$$

Using the mean value theorem, when $0 < \theta < 1$, we have

$$\begin{aligned} v^h(t) &= \int_0^1 l^*(s, t) \frac{\partial \psi}{\partial u}(s, (u + \theta(i_h^r u - u))(s)) (u - i_h^r u)(s) ds \\ &= \int_0^1 l(s, t) (u - i_h^r u)(s) ds \\ &= \sum_{k=1}^N \int_{E_k} l(s, t) (u - i_h^r u)(s) ds, \end{aligned}$$

where $l(s, t) \equiv l^*(s, t) \frac{\partial \psi}{\partial u}(s, (u + \theta(i_h^r u - u))(s))$.

By applying the results of Lemma 3 of [28] to each subinterval E_k (and noting that the change of scale introduces a factor $(h_k/2)^j$ for the j th derivative), we obtain

$$v^h(t) = \sum_{k=1}^N \left(\frac{h_k}{2}\right)^{2r+2} \sum_{\substack{i=r+1 \\ j+1=2r+2}}^{2r+2} c_{ji} \int_{E_k} D_s^j l(s, t) D^i u(s) ds + O(h^{2r+4}) \|u\|_{C^{2r+4}},$$

where D_s denotes the partial derivative with respect to s . The result may be rewritten as

$$v^h(t) = \sum_{k=1}^N h_k^{2r+2} \int_{E_k} b(s, t) ds + O(h^{2r+4}), \quad (46)$$

where

$$b(s, t) = 2^{-(2r+2)} \sum_{\substack{i=r+1 \\ j+1=2r+2}}^{2r+2} c_{ji} D_s^j l(s, t) D^i u(s).$$

Also,

$$\begin{aligned} \|(I - G_h i_h^r)^{-1} G_h (I - i_h^r) v^h\|_{0, \infty} &\leq C \|G_h (I - i_h^r) v^h\|_{0, \infty} \\ &\leq C \|(I - i_h^r) v^h\|_{0, \infty} \\ &\leq Ch^2 \|v^h\|_{0, \infty} = O(h^{2r+4}). \end{aligned} \quad (47)$$

Equations (45)-(47) give the desired result. \square

Theorem III.3.1 lends naturally to an extrapolation of the iterated collocation method for Hammerstein equation. Let $T^{h/2}$ be a partition of I :

$$0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \cdots < t_{N-1/2} < t_N = 1,$$

where

$$t_{k-1/2} = \frac{t_{k-1} + t_k}{2}, \quad k = 1, \dots, N.$$

Let $u^{h/2}$ and $u_{it}^{h/2}$ denote the collocation and iterated collocation approximation for Hammerstein equation with respect to this new partition. Theorem III.3.1 yields

$$u(t) - u_{it}^{h/2}(t) = 2^{-(2r+2)} \sum_{k=1}^N h_k^{2r+2} \int_{e_k} b(s, t) ds + O(h^{2r+4}).$$

Richardson extrapolation gives a new approximation

$$\bar{u}_{it}^{h/2}(t) \equiv \frac{2^{2r+2} u_{it}^{h/2} - u_{it}^h(t)}{2^{2r+2} - 1}.$$

It is straightforward that

$$u(t) - \bar{u}_{it}^{h/2}(t) = O(h^{2r+4}). \quad (48)$$

Example III.3.2. Consider the equation

$$u(t) - \int_0^1 e^{\pi(s-t)} u^2(s) ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \cos(t)$.

Notice that we define

$$\begin{aligned} \hat{e}_h &= \|u - u_{it}^h\|_{0,\infty}, & \hat{R}_h &= \log_2 \left(\frac{\hat{e}_h}{\hat{e}_{h/2}} \right), \\ \hat{\hat{e}}_h &= \|u - \bar{u}_{it}^{h/2}\|_{0,\infty}, & \hat{\hat{R}}_h &= \log_2 \left(\frac{\hat{\hat{e}}_h}{\hat{\hat{e}}_{h/2}} \right). \end{aligned}$$

TABLE 8: Computational results of extrapolation of iterated collocation solutions.

N	Collocation		Newton		Quasi-Newton		Post-Processing			
	e_h	R_h	NI	CT	NI	CT	\hat{e}_h	\hat{R}_h	\hat{e}_h	\hat{R}_h
2	1.1667e-2		5	0.11	10	0.08	1.2547e-3			
4	2.7077e-3	2.11	5	0.19	10	0.19	7.1920e-5	4.12	1.7389e-5	
8	6.5885e-4	2.04	5	0.67	10	0.64	4.4079e-6	4.03	2.3326e-7	6.22
16	1.6330e-4	2.01	5	2.48	10	2.29	2.7419e-7	4.01	3.4813e-9	6.07
32	4.7258e-5	1.79	5	9.64	10	8.58	1.7117e-8	4.00	5.3746e-11	6.02
64	1.0175e-5	2.21	5	38.17	10	33.26	1.0695e-9	4.00	8.4259e-13	5.99
128	2.5433e-6	2.00	5	151.04	10	131.60	6.6836e-11	4.00	1.7541e-14	5.59

III.4 GLOBAL EXTRAPOLATION FOR HAMMERSTEIN EQUATION

Theorem III.3.1 plays, once again, a critical role in establishing another method of improving the accuracy of numerical solution of the Hammerstein equation. Here, we examine a global extrapolation method for the Hammerstein equations. From (46),

$$\begin{aligned}
v^h(t) &= \sum_{k=1}^N h_k^{2r+2} \int_{E_k} b(s, t) ds + O(h^{2r+4}) \\
&= h^{2r+2} \sum_{k=1}^N \left(\frac{h_k}{h}\right)^{2r+2} \int_{E_k} b(s, t) ds + O(h^{2r+4}) \\
&= h^{2r+2} w(t) + O(h^{2r+4}),
\end{aligned}$$

where

$$w(t) \equiv \sum_{k=1}^N \left(\frac{h_k}{h}\right)^{2r+2} \int_{E_k} b(s, t) ds.$$

Equivalently,

$$(I - G_h)^{-1} G_h (u - i_h^r u)(t) = h^{2r+2} w(t) + O(h^{2r+4}). \quad (49)$$

Applying (32) and using the fact that

$$(I - i_h^r G_h)^{-1} = (I - G_h)^{-1} - (I - i_h^r G_h)^{-1} (I - i_h^r) G_h (I - G_h)^{-1},$$

we get

$$\begin{aligned}
u^h - i_h^r u &= (I - G_h)^{-1} i_h^r G_h (i_h^r u - u) \\
&\quad - (I - i_h^r G_h)^{-1} (I - i_h^r) G_h (I - G_h)^{-1} i_h^r G_h (i_h^r u - u) \\
&= (I - G_h)^{-1} G_h (i_h^r u - u) + (I - G_h)^{-1} (I - i_h^r) G_h (u - i_h^r u) \\
&\quad - (I - i_h^r G_h)^{-1} (I - i_h^r) G_h (I - G_h)^{-1} i_h^r G_h (i_h^r u - u). \quad (50)
\end{aligned}$$

By virtue of the fact that $u - i_h^r u = 0$ on the Gaussian points and arguing similarly to [29], it can be shown that

$$(I - G_h)^{-1}(I - i_h^r)G_h(u - i_h^r u) = O(h^{3r+3}), \quad (51)$$

and

$$(I - i_h^r G_h)^{-1}(I - i_h^r)G_h(I - G_h)^{-1}i_h^r G_h(i_h^r u - u) = O(h^{3r+3}). \quad (52)$$

Equations (49)-(52) yield the *superclose* identity with asymptotic error terms.

$$u^h(t) - i_h^r u(t) = \begin{cases} h^{2r+2}w(t) + O(h^{2r+4}), & r \geq 1, \\ h^2w(t) + O(h^3), & r = 0. \end{cases} \quad (53)$$

Equations in (53) lead naturally to the following extrapolation method. The theory follows exactly the same way as the one given in [29]. We include it for completeness. Let N be the number of elements of T^h and assume that it is a multiple of 3. Define an interpolation operator I_{3h}^{2r+3} mapping into a space of polynomials of degree $2r+3$, $r \geq 1$, as follows:

$$\begin{aligned} I_{3h}^{2r+3}u|_{e_{i-1} \cup e_i \cup e_{i+1}} &\in P_{2r+3}, \quad i = 3\ell + 1, \ell = 0, 1, \dots, \frac{N}{3} - 1, \\ I_{3h}^{2r+3}u(t) &= u(t), \quad t \in \Phi_{i-1}(B) \cup \Phi_{i+1}(B) \cup \{s_i^0, s_i^r\}, \end{aligned}$$

where $\Phi_i(B) = \{s_i^0, \dots, s_i^r\}$. Using (53) and

$$I_{3h}^{2r+3}i_h^r = I_{3h}^{2r+3},$$

and arguing as in [29], we obtain

$$I_{3h}^{2r+3}u^h - u = h^{2r+2}w + O(h^{2r+4}). \quad (54)$$

Equation (54) leads naturally to a global extrapolation method for the solution of the Hammerstein equation. In order to implement the global extrapolation, let $S^{h/2}$ be the space of piecewise polynomials of degree less than or equal to r with partition points

$$T^{h/2}: 0 = t_0 < t_{\frac{1}{2}} < t_1 < t_{\frac{3}{2}} < \dots < t_{N-\frac{1}{2}} < t_N = 1,$$

where

$$t_{i-\frac{1}{2}} = \frac{t_{i-1} + t_i}{2}, \quad i = 1, \dots, N.$$

Denote the collocation approximation and interpolation operator of degree $2r+3$ with respect to the partition $T^{h/2}$ by $u^{h/2}$ and $I_{3h/2}^{2r+3}$ so that

$$I_{3h/2}^{2r+3}u^{h/2} - u = \left(\frac{h}{2}\right)^{2r+2} w + O(h^{2r+4}).$$

The standard Richardson extrapolation gives an approximation with higher order of accuracy, namely

$$I_{3h/2}^{2r+3}\bar{u}^{h/2} - u = O(h^{2r+4}),$$

where

$$I_{3h/2}^{2r+3}\bar{u}^{h/2} \equiv \frac{2^{2r+2}I_{3h/2}^{2r+3}u^{h/2} - I_{3h}^{2r+3}u^h}{2^{2r+2} - 1}.$$

Example III.4.1. Consider the equation

$$u(t) - \int_0^1 e^{(s-t)}u^2(s)ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = e^t$.

Notice that we define

$$\begin{aligned} \bar{e}_h &= \|u - I_{3h}^{2r+3}u\|_{0,\infty}, & \bar{R}_h &= \log_2\left(\frac{\bar{e}_h}{\bar{e}_{h/2}}\right), \\ \hat{e}_h &= \|u - I_{3h}^{2r+3}\bar{u}^{h/2}\|_{0,\infty}, & \hat{R}_h &= \log_2\left(\frac{\hat{e}_h}{\hat{e}_{h/2}}\right). \end{aligned}$$

TABLE 9: Computational results of global extrapolation technique.

N	Collocation		Newton		Quasi-Newton		Post-Processing			
	e_h	R_h	NI	CT	NI	CT	\bar{e}_h	\bar{R}_h	\hat{e}_h	\hat{R}_h
3	2.2399e-2		6	0.20	13	0.14	4.2392e-4			
6	5.9444e-3	1.91	6	0.44	13	0.45	2.7526e-5	3.94	4.8714e-7	
12	1.5295e-3	1.96	6	1.70	14	1.65	1.7389e-6	3.98	7.7383e-9	5.98
24	3.8781e-4	1.98	6	6.38	14	6.08	1.0906e-7	3.99	1.2189e-10	5.99
48	9.7635e-5	1.99	6	24.82	14	23.23	6.8247e-9	4.00	1.9062e-12	6.00
96	2.4494e-5	1.99	6	98.93	14	91.31	4.2677e-10	4.00	3.2196e-14	5.89

III.5 GLOBAL SUPERCONVERGENCE FOR HAMMERSTEIN EQUATION BY GALERKIN METHOD:

In this section, we examine global superconvergence of the post-processed Galerkin method by interpolation: In other words, we apply the technique in Section III.2 to the Galerkin method. We denote by P_h the orthogonal projection of $L^2(I)$ onto S^h . More precisely,

$$\langle u - P_h u, v \rangle = 0, \quad \text{for all } v \in S^h. \quad (55)$$

Then the Galerkin method in solving (29) can be written as

$$u^h - P_h K \Psi(u^h) = P_h f, \quad u^h \in S^h. \quad (56)$$

The weak forms of (29) and (56) are

$$\langle u, v \rangle - \langle K\Psi(u), v \rangle = \langle f, v \rangle, \quad \text{for all } v \in L^2(I), \quad (57)$$

and

$$\langle u^h, v \rangle - \langle K\Psi(u^h), v \rangle = \langle f, v \rangle, \quad \text{for all } v \in S^h. \quad (58)$$

Using (57) and (58) along with (55), it is obtained that

$$\langle u^h - P_h u, v \rangle - \langle K\Psi(u^h) - K\Psi(P_h u), v \rangle = \langle K\Psi(P_h u) - K\Psi(u), v \rangle, \quad \text{for all } v \in S^h,$$

which can be further reduced to

$$\langle u^h - P_h u, v \rangle - \langle (K\Psi)'(\xi)(u^h - P_h u), v \rangle = \langle K\Psi(P_h u) - K\Psi(u), v \rangle, \quad \text{for all } v \in S^h, \quad (59)$$

where $\xi = \theta u^h + (1 - \theta)P_h u$ for some $\theta \in (0, 1)$. The standing conditions (A4)-(A6) described in Section III.2 guarantee that $(K\Psi)'(\xi)$ is a compact linear operator and $P_h \rightarrow I$ pointwise as $h \rightarrow 0$. A standard argument shows that $(I - P_h(K\Psi)'(\xi))^{-1}$ exists for sufficiently small h . Using the strong form of (59) and its rearrangements of terms, we see with $\xi_1 = \theta P_h u + (1 - \theta)u$ that

$$\begin{aligned} u^h - P_h u &= (I - P_h(K\Psi)'(\xi))^{-1}[P_h(K\Psi)'(\xi_1)(P_h u - u)] \\ &= (I - (K\Psi)'(\xi))^{-1}(K\Psi)'(\xi_1)(P_h u - u) + (I - (K\Psi)'(\xi))^{-1}(I - P_h)(K\Psi)'(\xi_1)(u - P_h u) \\ &\quad - (I - P_h(K\Psi)'(\xi))^{-1}(I - P_h)(K\Psi)'(\xi)(I - (K\Psi)'(\xi))^{-1}P_h(K\Psi)'(\xi_1)(P_h u - u). \end{aligned} \quad (60)$$

Conditions (A4)-(A6) once again guarantees that $(K\Psi)'(\xi_1)$ is a compact linear operator and we assume that it is in the form

$$(K\Psi)'(\xi_1)u(t) = \int_I k^*(s, t)u(s)ds,$$

with $k^* \in C^{r+2}(I \times I)$. Since for each $t \in I$,

$$\int_I P_h(s)k^*(s, t)(P_h u - u)(s)ds = 0,$$

we obtain

$$\begin{aligned} (K\Psi)'(\xi_1)(P_h u - u) &= \sum_{i=0}^{N-1} \int_{e_i} k^*(s, t)(P_h u - u)(s)ds \\ &= \sum_{i=0}^{N-1} \int_{e_i} (I - P_h(s))k^*(s, t)(P_h u - u)(s)ds \\ &= O(h^{r+2})\|u\|_{r+1, \infty}. \end{aligned}$$

Hence,

$$\|(I - (K\Psi)'(\xi))^{-1}(K\Psi)'(\xi_1)(P_h u - u)\|_{0,q} = O(h^{r+2})\|u\|_{r+1,q}, \quad (61)$$

where $q = 2, \infty$. For the second and third terms in (60), we proceed as follows;

$$\begin{aligned} \|(I - (K\Psi)'(\xi))^{-1}(I - P_h)(K\Psi)'(\xi_1)(u - P_h u)\|_{0,q} \\ \leq C\|(I - P_h)(K\Psi)'(\xi_1)(u - P_h u)\|_{0,q} \\ \leq Ch\|k^*(s, t)\|_{1,q}\|u - P_h u\|_{0,q} \leq Ch^{r+2}\|u\|_{r+1,q}, \end{aligned} \quad (62)$$

$$\begin{aligned} \|(I - P_h(K\Psi)'(\xi))^{-1}(I - P_h)(K\Psi)'(\xi)(I - (K\Psi)'(\xi))^{-1}P_h(K\Psi)'(\xi_1)(P_h u - u)\|_{0,q} \\ \leq Ch\|(I - (K\Psi)'(\xi))^{-1}P_h(K\Psi)'(\xi_1)(P_h u - u)\|_{0,q} \\ \leq Ch\|P_h u - u\|_{0,q} \\ \leq Ch^{r+2}\|u\|_{r+1,q}. \end{aligned} \quad (63)$$

When (61), (62) and (63) are combined with (60), we obtain

$$\|u^h - P_h u\|_{0,q} = O(h^{r+2})\|u\|_{r+1,\infty}. \quad (64)$$

In order to utilize (64) and obtain a global superconvergence of the Galerkin method by interpolation, it is necessary to define an interpolation operator I_{2h}^{r+1} as follows. In relation with the mesh T^{2h} ,

$$\begin{aligned} I_{2h}^{r+1}u|_{e_i \cup e_{i+1}} \in P_{r+1}, \quad i = 0, 2, \dots, N-2, \quad \text{such that} \\ \int_{e_i} I_{2h}^{r+1}uds = \int_{e_i} uds, \quad \int_{e_{i+1}} I_{2h}^{r+1}uds = \int_{e_{i+1}} uds \end{aligned}$$

and

$$\int_{e_i \cup e_{i+1}} v I_{2h}^{r+1}uds = \int_{e_i \cup e_{i+1}} vuds, \quad \text{for all } v \in P_r(e_i \cup e_{i+1}).$$

Using

$$\begin{aligned} I_{2h}^{r+1}P_h = I_{2h}^{r+1}, \\ \|I_{2h}^{r+1}v\|_{0,q} \leq C\|v\|_{0,q}, \quad \text{for all } v \in S^h, \\ \|I_{2h}^{r+1}v - v\|_{0,q} \leq Ch^{r+2}\|v\|_{r+2,q}, \end{aligned}$$

with $q = 2, \infty$, the global superconvergence of the Galerkin method by interpolation for the Hammerstein equation is now attained from

$$\begin{aligned} \|I_{2h}^{r+1}u^h - u\|_{0,q} &\leq \|I_{2h}^{r+1}u - I_{2h}^{r+1}P_h u\|_{0,q} + \|I_{2h}^{r+1}P_h u - u\|_{0,q} \\ &\leq C\|u^h - P_h u\|_{0,q} + \|I_{2h}^{r+1}u - u\|_{0,q} \\ &= O(h^{r+2})(\|u\|_{r+1,\infty} + \|u\|_{r+2,q}). \end{aligned}$$

Example III.5.1. Consider the equation

$$u(t) - \int_0^1 stu^2(s)ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \exp(t)$.

Notice that we define

$$\begin{aligned} e'_h &= \|u - u^h\|_{0,2}, & R'_h &= \log_2 \left(\frac{e'_h}{e'_{h/2}} \right), \\ \bar{e}'_h &= \|u - I_{2h}^{r+1} u^h\|_{0,2} \text{ and} & \bar{R}'_h &= \log_2 \left(\frac{\bar{e}'_h}{\bar{e}'_{h/2}} \right). \end{aligned}$$

TABLE 10: Computational results of interpolation Galerkin technique.

N	Galerkin		Newton		Quasi-Newton		Interpolation	
	e'_h	R'_h	NI	CT	NI	CT	\bar{e}'_h	\bar{R}'_h
2	1.6482e-2		5	0.11	7	0.06	2.1565e-3	
4	4.1484e-3	1.99	5	0.26	7	0.17	1.8942e-4	3.51
8	1.0398e-3	2.00	5	1.03	7	0.69	2.0170e-5	3.23
16	2.6013e-4	2.00	5	3.98	7	2.62	2.3970e-6	3.07
32	6.5044e-5	2.00	5	15.57	7	10.30	2.9560e-7	3.02
64	1.6262e-5	2.00	5	61.70	7	40.95	3.6823e-8	3.00
128	4.0655e-6	2.00	8	396.36	7	164.33	4.5988e-9	3.00

We point out a much shorter computational time with Quasi-Newton algorithm in this example.

III.6 EXTRAPOLATION OF ITERATED GALERKIN SOLUTION FOR HAMMERSTEIN EQUATION

In this final section, we explore the extrapolation technique developed in Section III.3 for the iterated collocation method for Hammerstein equations and extend it to accelerate further the rate of convergence of the iterated Galerkin method. The results reported in this section appear new even for the linear Fredholm equations. The iterated Galerkin solution, u_{it}^h , is obtained by

$$u_{it}^h = f + K\Psi(u^h), \quad (65)$$

where w^h is the solution of the Galerkin method (see (56)),

$$u^h - P_h K\Psi(u^h) = P_h f, \quad u^h \in S^h,$$

and P_h is the orthogonal projection of $L_2(I)$ onto S^h . From (65),

$$P_h u_{tt}^h = P_h f + P_h K \Psi(u^h). \quad (66)$$

From (66) and (56), we see that

$$P_h u_{tt}^h = u^h. \quad (67)$$

It is shown in [21] that if $f \in C^{2r+2}(I)$ and $k \in C^{2r+2}(I)$, then

$$\|u - u_{tt}^h\|_{0,2} = O(h^{2r+2}).$$

In order to successfully complete the current extrapolation method, it is necessary to establish an asymptotic error expansion for the iterated Galerkin solution which is analogous to Theorem III.3.1. A proof can be made similar to the proof of Theorem III.3.1 but the interpolation projection i_h^r must be replaced by the orthogonal projection P_h . Some differences which must be incorporated are highlighted in the proof of Lemma III.6.2 below.

Theorem III.6.1. *Assume that $k \in C^{r+3}([0, 1] \times [0, 1])$ and the solution u of (27) satisfies $u \in C^{2r+4}[0, 1]$. Also assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(u)$. Then there exists $b \in C([0, 1] \times [0, 1])$, independent of the partition, such that*

$$u(t) - u_{tt}^h(t) = \sum_{k=1}^N h_k^{2r+2} \int_{e_k} b(s, t) ds + O(h^{2r+4}), \quad t \in [0, 1].$$

Theorem III.6.1 is based upon the following lemma.

Lemma III.6.2. *Assume that $\chi, z \in C^{r+4}[-1, 1]$ and let P_h is the orthogonal projection of $L^2[-1, 1]$ onto S_h . Then there exists a constant $c_{r+1, r+1}$, independent of h and z , such that*

$$\int_{-1}^1 \chi(z - P_h z) ds = c_{r+1, r+1} \int_{-1}^1 D^{r+1} \chi D^{r+1} z ds + O(1) \sum_{i+j \geq 2r+4} \|D^i \chi\| \|D^j z\|.$$

Proof. Expand χ and z in Maclaurin series to get

$$\chi(s) = \sum_{j=0}^{r+3} \frac{1}{j!} D^j \chi(0) \phi_j + O(D^{r+4} \chi).$$

$$z(s) = \sum_{i=0}^{r+3} \frac{1}{i!} D^i z(0) \phi_i + O(D^{r+4} z),$$

where $\phi_j(s) = s^j$. Since $P_h: L^2[-1, 1] \rightarrow S^h$, $P_h \phi_j = \phi_j$ for $0 \leq j \leq r$ and thus

$$z - P_h z = \sum_{i=r+1}^{r+3} \frac{1}{i!} D^i z(0) (\phi_i - P_h \phi_i) + O(D^{r+4} z).$$

Also noting that

$$\int_{-1}^1 \varphi(s) (z - P_h z)(s) ds = 0, \quad \text{for all } \varphi \in S^h,$$

and thus

$$\begin{aligned} \int_{-1}^1 \chi(s) (z - P_h z)(s) ds &= \sum_{j=r+1}^{r+3} \sum_{i=r+1}^{r+3} \frac{1}{i! j!} D^j \chi(0) D^i z(0) \int_{-1}^1 \phi_j(s) (\phi_i - P_h \phi_i)(s) ds \\ &\quad + O(1) (\|D^{r+4} z\| \sum_{j=r+1}^{r+3} \|D^j \chi\| + \|D^{r+4} \chi\| \sum_{j=r+1}^{r+3} \|D^j z\|) \\ &= \sum_{j=r+1}^{r+3} \sum_{i=r+1}^{r+3} c_{ij} D^j \chi(0) D^i z(0) + O(1) (\|D^{r+4} z\| \sum_{j=r+1}^{r+3} \|D^j \chi\| \\ &\quad + \|D^{r+4} \chi\| \sum_{j=r+1}^{r+3} \|D^j z\|), \end{aligned} \quad (68)$$

where $c_{ij} = \frac{1}{i! j!} \int_{-1}^1 \phi_j(s) (\phi_i - P_h \phi_i)(s) ds$. Note that, in the first term of the last expression, $c_{r+1, r+2} = c_{r+2, r+1} = 0$. To see this, note that, if $r+1$ is odd, then $\phi_{r+1} - P_h \phi_{r+1}$ is also odd, since, with $(u, v) \equiv \int_{-1}^1 uv ds$,

$$P_h \phi_{r+1} = \sum_{i=0}^r b_i \phi_i, \quad \text{where } b_i = \frac{(\phi_{r+1}, \phi_i)}{(\phi_i, \phi_i)},$$

and thus $b_i = 0$ whenever i is even for in this case $\phi_{r+1} \phi_i$ becomes an odd function. Hence, $P_h \phi_{r+1}$ is odd and thus $\phi_{r+1} - P_h \phi_{r+1}$ is also odd. Under the assumption that $r+1$ is odd, ϕ_{r+2} an even function which in turn makes $\phi_{r+2}(\phi_{r+1} - P_h \phi_{r+1})$ odd, providing the result that $c_{r+1, r+2} = 0$. $c_{r+2, r+1} = 0$ is similar. Returning to (68),

$$\begin{aligned} \int_{-1}^1 \chi(s) (z - P_h z)(s) ds &= 2c_{r+1, r+1} D^{r+1} \chi(0) D^{r+1} z(0) \\ &\quad + c_{r+1, r+3} D^{r+3} \chi(0) D^{r+1} z(0) + c_{r+3, r+1} D^{r+1} \chi(0) D^{r+3} z(0) \\ &\quad + O(1) (\|D^{r+4} z\| \sum_{j=r+1}^{r+3} \|D^j \chi\| + \|D^{r+4} \chi\| \sum_{j=r+1}^{r+3} \|D^j z\|) \\ &= c_{r+1, r+1} \int_{-1}^1 D^{r+1} \chi D^{r+1} z ds + O(1) \sum_{i+j \geq 2r+4} \|D^i \chi\| \|D^j z\|, \end{aligned}$$

where the second order Maclaurin expansion was used in the last step. \square

Proof. (Theorem III.6.1) Arguing exactly the same way as between (41) and (44) with the interpolation projection i_h^r replaced by the orthogonal projection P_h , we obtain

$$u - u_{it}^h = v^h - (I - G_h P_h)^{-1} G_h (I - P_h) v^h, \quad (69)$$

where

$$G_h u(s) \equiv \int_0^1 g(t, s, P_h u(s), P_h u_{it}^h(s), \theta) u(s) ds,$$

where g is defined in Section III.2,

$$w^h \equiv K\Psi(u) - K\Psi(P_h u),$$

and

$$v^h \equiv (I - G_h)^{-1} w^h = (I - G_h)^{-1} K[\Psi(u) - \Psi(P_h u)].$$

Let $L \equiv (I - G_h)^{-1} K$ so that L is an integral operator with a kernel $l^*(s, t)$ with the same smoothness properties as $k(s, t)$.

Then

$$v^h(t) = L[\Psi(u) - \Psi(P_h u)] = \int_0^1 l^*(s, t) [\psi(s, u(s)) - \psi(s, P_h u(s))] ds.$$

Using the mean value theorem as was done before in the proof of Theorem III.3.1, we obtain

$$v^h(t) = \sum_{k=1}^N \int_{E_k} l(s, t) (u - P_h u)(s) ds,$$

where $l(s, t) \equiv l^*(s, t) \frac{\partial \psi}{\partial u}(s, (u + \theta P_h u)(s))$.

By applying Lemma III.6.2 to each subinterval E_k and noting that the change of scale introduce a factor $(h_k/2)^j$ for the j th derivative, we obtain

$$v^h(t) = \sum_{k=1}^N \left(\frac{h_k}{2}\right)^{2r+2} c_{r+1} \int_{E_k} D_s^{r+1} l(s, t) D^{r+1} u(s) ds + O(h^{2r+4}) \|u\|_{c^{2r+4}}, \quad (70)$$

where D_s denotes the partial derivative with respect to s . The result may be rewritten as

$$v^h(t) = \sum_{k=1}^N h_k^{2r+2} \int_{E_k} b(s, t) ds + O(h^{2r+4}), \quad (71)$$

where

$$b(s, t) = 2^{-(2r+2)} c_{r+1} D_s^{r+1} l(s, t) D^{r+1} u(s).$$

We already know that $\|v^h\| = O(h^{2r+2})$. Also,

$$\begin{aligned} \|(I - G_h P_h)^{-1} G_h (I - P_h) v^h\| &\leq C \|G_h (I - P_h) v^h\| \\ &\leq C \|(I - P_h) v^h\| \\ &\leq C h^2 \|v^h\| = O(h^{2r+4}). \end{aligned} \quad (72)$$

Equations (69), (71) and (72) give the desired result. \square

Theorem III.6.1 engenders the extrapolation of the iterated Galerkin method for the Hammerstein equation. The process is the same with the extrapolation of the iterated collocation method, namely, we use the classical Richardson extrapolation technique. Let $T^{h/2}$ be a partition of I :

$$0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \dots < t_{N-1/2} < t_N = 1,$$

where

$$t_{k-1/2} = \frac{t_{k-1} + t_k}{2}, \quad k = 1, \dots, N.$$

Let $u^{h/2}$ and $u_{it}^{h/2}$ denote the Galerkin and iterated Galerkin approximation for Hammerstein equation with respect to this new partition. Theorem III.6.1 yields

$$u(t) - u_{it}^{h/2}(t) = 2^{-(2r+2)} \sum_{k=1}^N h_k^{2r+2} \int_{e_k} b(s, t) ds + O(h^{2r+4}).$$

An extrapolation gives a new approximation

$$\bar{u}_{it}^{h/2}(t) \equiv \frac{2^{2r+2} u_{it}^{h/2} - u_{it}^h(t)}{2^{2r+2} - 1}.$$

It is straightforward that

$$u(t) - \bar{u}_{it}^{h/2}(t) = O(h^{2r+4}).$$

Example III.6.3. Consider the equation

$$u(t) - \int_0^1 stu^2(s)ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = e^t$.

Notice that we define

$$\begin{aligned} \hat{e}'_h &= \|u - u_{ii}^h\|_{0,2}, & \hat{R}'_h &= \log_2 \left(\frac{\hat{e}'_h}{\hat{e}'_{h/2}} \right), \\ \tilde{e}'_h &= \|u - \tilde{u}_{ii}^{h/2}\|_{0,2} \text{ and} & \tilde{R}'_h &= \log_2 \left(\frac{\tilde{e}'_h}{\tilde{e}'_{h/2}} \right). \end{aligned}$$

TABLE 11: Computational results of extrapolation of iterated Galerkin solutions.

N	Galerkin		Newton		Quasi-Newton		Post-Processing			
	e'_h	R'_h	NI	CT	NI	CT	\hat{e}'_h	\hat{R}'_h	\tilde{e}'_h	\tilde{R}'_h
2	1.6482e-2		5	0.11	7	0.06	1.6450e-3			
4	4.1484e-3	1.99	5	0.26	7	0.17	1.0573e-4	3.96	5.3988e-6	
8	1.0398e-3	2.00	5	1.03	7	0.69	6.6550e-6	3.99	8.6269e-8	5.96
16	2.6013e-4	2.00	5	3.98	7	2.62	4.1667e-7	4.00	1.3552e-9	5.99
32	6.5044e-5	2.00	5	15.57	7	10.30	2.6053e-8	4.00	2.1203e-11	6.00
64	1.6262e-5	2.00	5	61.70	7	40.95	1.6285e-9	4.00	3.3129e-13	6.00
128	4.0655e-6	2.00	8	396.36	7	164.33	1.0179e-10	4.00	7.9936e-15	5.37

As with the example III.5.1, we note that the Quasi-Newton method results in less than half of the computing time than that of the Newton's method.

CHAPTER IV

WAVELETS-COLLOCATION METHOD

In this chapter, we construct a wavelet-collocation method for Hammerstein integral equations. A wavelet-collocation method for Fredholm equation is recently established in [10]. Our first objective is to establish a fast wavelet-collocation method for Hammerstein equation by using the idea of ‘linearization’ technique described in [25] and [17]. We use the multiscale wavelet bases constructed by [10], which was introduced in Chapter II, to solve the nonlinear integral equation. We show that the sparsity of coefficient in the Jacobian matrices occurs for equations with either smooth or weakly singularity kernel. We use the block truncation strategy of [10] to attain a fast algorithm.

The second objective is to create a multilevel augmentation method for the Hammerstein equation. It is based upon the transformed linearized nonlinear equation and it is different from, even though greatly inspired by, the multilevel augmentation method recently established in the paper [11]. The main goal of a multilevel augmentation method is to avoid solving a large nonlinear system. In accomplishing this goal, two steps have to be implemented. The first step is to solve the nonlinear equation at a lower resolution level and secondly the error is compensated by adding a correction term from higher resolution level. The method leads us to a faster numerical technique while still preserving the order of convergence of approximation.

IV.1 PRELIMINARY

There have been many papers written in the recent years which establish numerical techniques for finding an approximation of a solution of the Hammerstein equation (see, -e.g., [11], [18], [19], [21], [20], [23], [24], [25]). In each of these methods, numerical solution is found by solving a system of nonlinear equations using a nonlinear solver such as the Newton method, the secant method or the quasi-Newton method. In executing its iterative process, the Newton’s method as well as the secant method require updating of the Jacobian matrix which is normally dense. The quasi-Newton’s method eliminates the need for the update by fixing a Jacobian matrix in computation. This reduces the overall cost of computation despite the fact that more iterations may be needed, since a large portion of the computing time is used for the assembly of

the Jacobian matrices. We point out that the Jacobian matrices used in the Newton method, the secant method and the quasi-Newton method are dense when spline bases are used. In general, to advance one iteration in the Newton's method or in the quasi-Newton's method requires $O(N^2)$ operation counts in an $N \times N$ system, and thus if one desires a higher accuracy in approximation, computational complexity increases with N . In [11], a class of wavelets was used in approximating a solution of Hammerstein equation which uses a fast multilevel augmentation method. The method is a generalization of the multilevel method for linear operator equations which was first established in [7] and when it was applied to Hammerstein equation, the complexity of computation decreases to $O(N \log N)$.

It was established in [1] that a class of wavelet basis can be applied to approximating a solution of the Fredholm integral equation of the second kind which produces a linear system with sparse structure. This has had a significant implication in the reduction of an overall computational expense in approximating a solution of the Fredholm equation, since, as stated earlier, a standard spline basis results in a linear system which is dense. This discovery was greatly expanded and generalized in the recent years in a series of papers, [32], [35], [8], [10], where truncation strategies were established to produce sparse systems, leading to fast algorithms. The latter three papers deals with the wavelet collocation methods for the Fredholm integral equations of the second kind.

IV.1.1 Wavelet collocation method for the second kind Fredholm equation

Using the functions w_{ij} and the functionals $\ell_{i'j'}$ thus constructed, fast wavelet collocation methods were developed in [8], [10]. The Fredholm integral equations of the second kind is written as

$$u(t) - \int_0^1 k(s, t)u(s)ds = f(t), \quad t \in I, \quad (73)$$

where k and $f \in C(I)$ are known functions and u is the function to be determined. In the operator form, (73) is

$$u - Ku = f. \quad (74)$$

In the wavelet collocation method, an approximate solution u_n for u in (74) is found in the form

$$u_n := \sum_{(i,j) \in U_n} u_{ij} w_{ij}$$

where, with

$$\mathbf{u}_n := [u_{ij} : (i, j) \in U_n]^T$$

and

$$U_n = \{(i, j) : j \in Z_{\omega(i)}, i \in Z_{n+1}\},$$

u_{ij} are found by solving

$$\ll \ell_{i'j'}, u_n(t) \gg = \ll \ell_{i'j'}, (f + Ku_n)(t) \gg, \quad (i', j') \in U_n. \quad (75)$$

Moreover, using the lexicographic ordering on $Z_{n+1} \times Z_{n+1}$, equation (75) yields

$$(\mathbf{E}_n - \mathbf{K}_n)\mathbf{u}_n = \mathbf{f}_n, \quad (76)$$

where

$$\begin{aligned} \mathbf{E}_n &:= [\ll \ell_{i'j'}, w_{ij} \gg : (i', j'), (i, j) \in U_n], \\ \mathbf{K}_n &:= [\ll \ell_{i'j'}, Kw_{ij} \gg : (i', j'), (i, j) \in U_n], \end{aligned}$$

and $\mathbf{f}_n := [\ll \ell_{i'j'}, f \gg : (i', j') \in U_n]^T$.

Recently, Chen et. al. ([8], [10]) estimated the size of the components in the wavelet collocation matrix for the Fredholm integral equation of the second kind. More specifically, with $K_{i'j',ij} := \ll \ell_{i'j'}, Kw_{ij} \gg$ and under the assumption that

the kernel $k(s, t)$ is weakly singular, i.e., $k(s, t)$ has continuous partial derivatives $\frac{\partial^\alpha}{\partial s^\alpha} \frac{\partial^\beta}{\partial t^\beta} k(s, t)$ for $\alpha \leq k, \beta \leq k$, when $s, t \in I$ with $s \neq t$ and there exists $0 < \sigma < 1$ and θ such that

$$\left| \frac{\partial^\alpha}{\partial s^\alpha} \frac{\partial^\beta}{\partial t^\beta} k(s, t) \right| \leq \frac{\theta}{|s - t|^{\sigma+2k}}. \quad (77)$$

Denote by S_{ij} the support of w_{ij} and

$$d_i := \max\{\text{diam}(S_{ij}) : j \in Z_{\omega(i)}\}, \quad i = 0, 1, \dots$$

then the conditions on the functionals $\ell_{i'j'}$ and the wavelet functions w_{ij} guarantee the following lemma (Lemma 3.1 [8]) which serves as a foundation of the truncation strategy.

Lemma IV.1.1. [8] *If there is a constant $r > 1$ such that*

$$\text{dist}(S_{ij}, S_{i'j'}) \geq r(d_i + d_{i'}),$$

then there exists a positive constant c such that

$$|K_{i'j',ij}| \leq c(d_i d_{i'})^k \sum_{s \in \tilde{S}_{i'j'}} \int_{S_{ij}} \frac{1}{|s - t|^{2k+\sigma}} dt.$$

To use Lemma IV.1.1, partition $\mathbf{K}_n := [K_{i'j',ij}]_{(i',j'),(i,j) \in U_{n+1}}$ into

$$\mathbf{K}_n = [\mathbf{K}_{i'i}]_{i', i \in Z_{n+1}},$$

where

$$\mathbf{K}_{i'i} = [K_{i'j',ij}]_{j' \in Z_{\omega(i')}, j \in Z_{\omega(i)}}.$$

A truncation parameter $\epsilon_{i'i} = \epsilon(i', i, n)$ is chosen to form a matrix

$$\mathbf{K}(\epsilon)_{i'i} = [K(\epsilon)_{i'j',ij}]_{j' \in Z_{\omega(i')}, j \in Z_{\omega(i)}},$$

where

$$K(\epsilon)_{i'j',ij} = \begin{cases} K_{i'j',ij}, & \text{dist}(S_{i'j'}, S_{ij}) \leq \epsilon_{i'i}, \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem was shown in [8] (Theorem 4.6) which serves as a basis for a fast wavelet collocation algorithm. Here $N(A)$ denotes the number of nonzero elements in A .

Theorem IV.1.2. [8] Let b and b' be real numbers not larger than one and the truncation parameter $\epsilon_{i'i} = c(i', i, n)$, $i', i \in Z_{n+1}$, be chosen such that

$$\epsilon_{i'i} = \max\{a\mu^{[-n+b(n-i)+b'(n-i')]/d}, r(d_i + d_{i'})\}, \quad i', i \in Z_{n+1},$$

for some constants $a > 0$ and $r > 1$. Then

$$N(\mathbf{E}_n - \mathbf{K}(\epsilon)) = O(f(n) \log^\tau f(n)),$$

where $f(n) = \dim(\mathbf{X}_n)$, $\tau = 1$ except for $b = b' = 1$, in which case $\tau = 2$.

In the above theorem, we take $d = 1$ in this study. It was also shown in [8] that the choice of parameter ϵ not only generate a sparse matrix $\mathbf{K}(\epsilon)$ but also preserve the optimal order of convergence of numerical solution.

Recently, Theorem 3.3 was improved by Chen, Wu and Xu in [10] to a more practical form in which its implementation is much easier. In the practical block truncation strategy, each rectangular block $\mathbf{K}_{i'i}$ in \mathbf{K}_n is further partitioned into a block having the same number of row sub-blocks and column sub-blocks, i.e.,

$$\mathbf{K}_{i'i} = [\mathbf{K}_{q'q}^{i'i} : q', q \in Z_{\mu^{i_0-1}}],$$

where $i_0 = \min\{i', i\}$ and the following lemma help to classify the entries of each sub-block.

Lemma IV.1.3. [10] For $i', i \in Z_{n+1}$, $j' = \mu(e')r + l'$, $j = \mu(e)r + l$, $e' \in Z_\mu^{i'-1}$, $e \in Z_\mu^{i-1}$, $l', l \in Z_r$, $K_{i'j', i, j}$ is an entry of $\mathbf{K}_{q'q}^{i'i}$ if and only if

$$q' = \mu(e'), \quad \text{and} \quad q = \left\lfloor \frac{\mu(e)}{\mu^{i'-i'}} \right\rfloor \quad \text{when } i \geq i',$$

and

$$q' = \left\lfloor \frac{\mu(e')}{\mu^{i'-i}} \right\rfloor, \quad \text{and} \quad q = \mu(e), \quad \text{when } i < i'.$$

The block truncation strategy is then to select a family of parameters according to Theorem IV.1.4 below so that the order of convergence and computational complexity are preserved. Specifically, the block truncation strategy is to define

$$\tilde{\mathbf{K}}_{i'i} = [\mathbf{K}(p)_{q'q}^{i'i} : q', q \in Z_{\mu^{i_0-1}}],$$

and

$$\mathbf{K}(p)_{q'q}^{i'i} := \begin{cases} \mathbf{K}_{q'q}^{i'i}, & |q - q'| \leq p_{i'i}, \\ 0 & \text{otherwise,} \end{cases} \quad (78)$$

where parameters $p_{i'i}$ are specified in the following theorem.

Theorem IV.1.4. [10] Suppose that $u \in W^{k,\infty}(I)$. For any constant $a > 0$ and $\nu > 1$, for $i' < i$, let

$$p_{i'} := \{1 + \max\{a\mu^{b'(n-i')-i+i'+1}, \nu(\mu^{-i+i'} + 1)\}\},$$

and for $i' \geq i$ let

$$p_{i'} := \{1 + \max\{a\mu^{b'(n-i')-1}, \nu(\mu^{-i'+i} + 1)\}\},$$

where $\frac{k-\sigma'}{2k-\sigma'} < b' \leq 1$, with $0 < \sigma' < 1 - \sigma$. Then, there exists a positive constant c such that

$$\|u - \tilde{u}\|_{0,\infty} \leq cf(n)^{-k} \log^\tau f(n) \|u\|_{k,\infty},$$

where \tilde{u} denotes the solution to the Fredholm integral equation of the second kind under the truncation strategy and τ is either 1 or 2 and

$$N(\mathbf{E}_n - \tilde{\mathbf{K}}_n) = O(f(n) \log f(n)).$$

We note that the choice for $p_{i'}$ defined in Theorem IV.1.4 for the block truncation strategy ensures that

$$\text{dist}(S_{i'j'}, S_{ij}) \leq \epsilon_{i'j},$$

where $\epsilon_{i'j}$ is defined in Theorem IV.1.2, and hence

$$\tilde{K}_{i'j',ij} = K_{i'j',ij}$$

is guaranteed.

Remark: We note that the constants $a > 0$ and $\nu > 1$ can be any number for the wavelet collocation method to converge at the optimal rate, but our numerical experiments show that CPU times vary significantly with different values of $a > 0$ and $\nu > 1$.

Throughout all numerical experiments reported below, we choose the linear multiscale wavelets to be the basis functions.

Example IV.1.5. Consider the equation

$$u(t) - \int_0^1 \frac{u(s)}{\sqrt{|s-t|}} ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^2$.

TABLE 12: CPU time CT and number of nonzeros N for solving the wavelet collocation method.

$f(n)$	CT	\widetilde{CT}_1	\widetilde{CT}_2	\widetilde{CT}_3	\widetilde{CT}_4	\widetilde{CT}_5
32	0.208	0.190	0.203	0.208	0.212	0.216
64	0.706	0.549	0.569	0.608	0.705	0.707
128	2.677	1.613	1.642	1.787	2.717	2.714
256	11.903	5.006	5.019	5.424	11.306	11.621
512	93.018	17.090	17.148	18.180	44.662	63.459
1024	1486.343	64.806	64.976	68.580	212.690	530.878

$f(n)$	$N(A)$	$N(\widetilde{A}_1)$	$N(\widetilde{A}_2)$	$N(\widetilde{A}_3)$	$N(\widetilde{A}_4)$	$N(\widetilde{A}_5)$
32	1024	912	928	976	1024	1024
64	4096	2816	2832	3136	4096	4096
128	16384	7856	7872	8816	16384	16384
256	65536	20448	20464	22816	61152	63512
512	262131	50704	50720	56016	167000	212336
1024	1045230	121408	121424	132736	360088	616136

where CT and $N(A)$ is the results without the block truncation strategy, \widetilde{CT}_1 and $N(\widetilde{A}_1)$ is the results with $a = 0.01, \nu = 1.01$ and $b' = 0.8$, \widetilde{CT}_2 and $N(\widetilde{A}_2)$ is the results with $a = 0.25, \nu = 1.01$ and $b' = 0.8$, \widetilde{CT}_3 and $N(\widetilde{A}_3)$ is the results with $a = 0.5, \nu = 1.5$ and $b' = 0.5$, \widetilde{CT}_4 and $N(\widetilde{A}_4)$ is the results with $a = 20, \nu = 1.01$ and $b' = 0.8$, \widetilde{CT}_5 and $N(\widetilde{A}_5)$ is the results with $a = 0.25, \nu = 20$ and $b' = 0.8$.

Next, some numerical experiments of wavelet collocation method of the Fredholm equation are shown. Denote

$$e_h = \|u - u_n^h\|_{0,\infty}, \quad R_h = \log_2 \left(\frac{e_h}{e_{h/2}} \right).$$

Example IV.1.6. Consider the equation

$$u(t) - \int_0^1 \sin(s+t)u(s)ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = e^t$.

TABLE 13: Computational results of Fredholm equation with smooth kernel.

$f(n)$	e_h	R_h	CT
2	4.816643e-1		0.009
4	1.239859e-1	1.9578	0.018
8	3.198254e-2	1.9548	0.042
16	8.154885e-3	1.9716	0.137
32	2.060935e-3	1.9844	0.473
64	5.181485e-4	1.9919	1.754
128	1.299022e-4	1.9959	6.820
256	3.251324e-5	2.0007	26.929
512	8.124553e-6	1.9983	108.058
1024	2.022185e-6	2.0064	438.780

Note that we use the numerical quadrature scheme in Appendix A to calculate integral term of smooth kernel case. Recall that the numerical error of collocation method is $O(h^{r+1})$. Therefore, in this case, we see that the expected convergence rate is 2.

Example IV.1.7. Consider the equation

$$u(t) - \int_0^1 \frac{u(s)}{\sqrt{|s-t|}} ds = f(t), \quad t \in [0, 1]$$

and $f(t)$ is chosen so that the exact solution is $u(t) = t^2$.

Note that we use analytic integration to calculate the integral term in the weakly singular kernel case. From FIG. 15, we see that the sparsity of corresponding matrix for solving linear system occurs when using wavelet bases for the weakly singular kernel case.

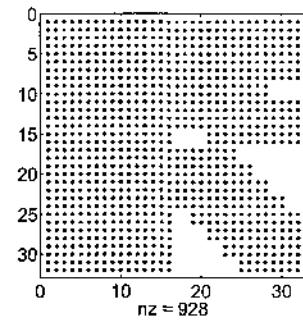
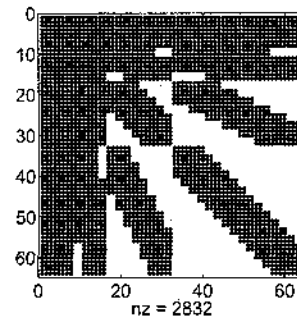
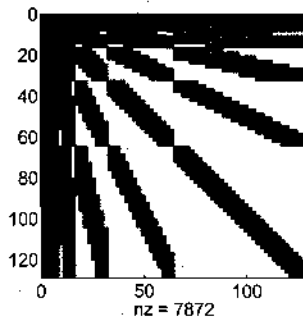
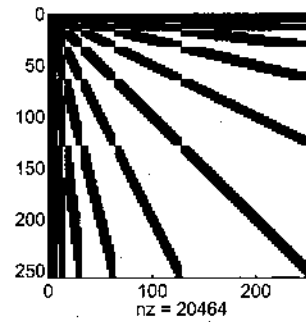
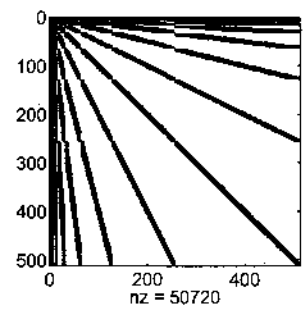
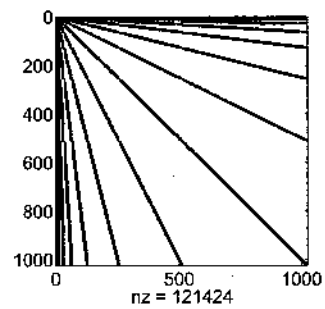
(a) $f(n) = 32$ (b) $f(n) = 64$ (c) $f(n) = 128$ (d) $f(n) = 256$ (e) $f(n) = 512$ (f) $f(n) = 1024$

FIG. 15: Sparsity of corresponding matrix of Fredholm equation.

TABLE 14: Computational results of Fredholm equation with weakly singular kernel.

$f(n)$	e_h	R_h	CT	\bar{e}_h	\bar{R}_h	\bar{CT}
2	1.704129e-1		0.007	1.704129e-1		0.007
4	5.012502e-2	1.7654	0.011	5.012502e-2	1.7654	0.012
8	1.276033e-2	1.9739	0.040	1.276033e-2	1.9739	0.027
16	3.099368e-3	2.0416	0.079	3.099368e-3	2.0416	0.069
32	7.733665e-4	2.0027	0.208	7.734493e-4	2.0026	0.203
64	1.943866e-4	1.9922	0.706	1.945008e-4	1.9915	0.569
128	4.886239e-5	1.9921	2.677	4.897603e-5	1.9896	1.642
256	1.243263e-5	1.9746	11.903	1.246557e-5	1.9741	5.019
512	3.151628e-6	1.9800	93.018	3.182451e-6	1.9697	17.148
1024	7.704299e-7	2.0324	1486.343	8.773358e-7	1.8589	64.976

One of the goals of this Chapter is to extend the previous implementation to the wavelet collocation method for the Hammerstein equation. We show, in Section IV.2, that by ‘linearlizing’ the Hammerstein equation, our wavelet collocation method produces Jacobian matrices which are sparse. This generalizes a similar result to the Petrov-Galerkin method for the Hammerstein equation [17]. In Section IV.3, a multilevel augmentation method applied directly to the linearized Hammerstein equation is discussed. The new fast multilevel augmentation method is similar to the one established in [11], but also takes an advantage of the sparse structure of the Jacobian matrix which leads to an added reduction in the overall computational cost.

IV.2 WAVELET COLLOCATION METHOD FOR HAMMERSTEIN EQUATION

The purpose of this section is to establish a wavelet collocation method for the Hammerstein equation written as

$$u(t) - \int_0^1 k(s, t)\psi(s, u(s))ds = f(t), \quad t \in I. \quad (79)$$

In the case of nonlinear Hammerstein equation, a similar fast algorithm based upon the same truncation strategy described in section IV.1.1 must be modified, since the basis functions w_{ij} appear under the nonlinear term ψ . To circumvent this difficulty, we first transform (79) as follows: Define

$$z(t) = \psi(t, u(t)), \quad (80)$$

and substituting it into (79), we get

$$u(t) = f(t) + \int_0^1 k(t, s)z(s)ds. \quad (81)$$

Equations (80) and (81) give

$$z(t) = \psi(t, f(t) + \int_0^1 k(t, s)z(s)ds). \quad (82)$$

As in the Fredholm case, we seek a solution z_n of (82) in the form

$$z_n := \sum_{(i,j) \in U_n} z_{ij} w_{ij},$$

and z_{ij} are found by requiring that

$$\ll \ell_{i'j'}, z_n(t) \gg = \ll \ell_{i'j'}, \psi(t, f(t) + Kz_n(t)) \gg, \quad (i', j') \in U_n. \quad (83)$$

Once z_n is found, the approximate solution u_n of (79) can be found from (81), namely

$$u_n(t) = f(t) + \int_0^1 k(t, s)z_n(s)ds.$$

It should be pointed out that the substitution techniques described between (80) and (82) were first introduced in [25], [23] and [24], and also used in conjunction with some other methods in [17] and [21]. Let P_n be an interpolation projection mapping $C(I)$ onto X_n which is defined by

$$P_n(x)(t) := \sum_{(i,j) \in U_n} \ll \ell_{ij}, x \gg w_{ij}(t). \quad (84)$$

Then,

$$P_n x \rightarrow x, \quad \text{as } n \rightarrow \infty \text{ and for } x \in C(I)$$

from which we obtain by the uniform bounded principle,

$$\sup_n \|P_n\|_{0,\infty} < M, \quad \text{where } M \text{ is independent of } n.$$

With

$$T(x)(t) := f(t) + Kx(t), \quad x \in L^\infty(I), t \in I,$$

equation (82) can be written as

$$z = \Psi T(z). \quad (85)$$

whereas equation (83) can be written in operator form as

$$z_n = P_n \Psi T(z_n), \quad z_n \in \mathbb{X}_n. \quad (86)$$

Under the assumptions (A1)-(A5) along with the condition that 1 is not an eigenvalue of the linear operator $(\Psi T)'(z)$, the Fréchet derivative of ΨT at z , it was shown [25] that equation (85) has an isolated solution. Similarly, one may invoke theorem 1 of [25] to show the existence of a solution z_n of (86). Alternatively, one may utilize theorem 2 of Vainikko [34], as was done in [21], to obtain the following.

Theorem IV.2.1. *Let z be an isolated solution of (85). Assume that 1 is not an eigenvalue of the linear operator $(\Psi T)'(z)$, where $(\Psi T)'(z)$ denotes the Fréchet derivative of ΨT at z . Then the wavelet collocation approximation equation (86) has a unique solution z_n in a ball $B(z, \delta) := \{c \in C(I) : \|x - z\|_{0,\infty} \leq \delta\}$ for some $\delta > 0$ and for sufficiently large n . Moreover, there exists a constant $0 < q < 1$, independent of n , such that*

$$\frac{\alpha_n}{1+q} \leq \|z_n - z\|_{0,\infty} \leq \frac{\alpha_n}{1-q}, \quad (87)$$

where $\alpha_n := \|(I - (P_n \Psi T)'(z))^{-1}((P_n \Psi T)(z) - \Psi T(z))\|_{0,\infty}$. Finally

$$E_n(z) \leq \|z_n - z\|_{0,\infty} \leq C E_n(z), \quad (88)$$

where C is a constant independent of n and $E_n(z) = \inf_{u \in \mathbb{X}_n} \|z - u\|_{0,\infty}$.

It is noted that under the current polynomial wavelets,

$$E_n(z) = O\left(\frac{1}{\mu^{nk}}\right). \quad (89)$$

Now, in order to find the solution z_n of (86), one must more likely use an iterative method. Here we consider the Newton's method for an illustration. First, for each $(i', j') \in U_n$ and $\mathbf{z}_n := [z_{ij}]_{(i,j) \in U_n}$ with $z_n = \sum_{(i,j) \in U_n} z_{ij} w_{ij}$, we let

$$\begin{aligned} F_{i'j'}(\mathbf{z}_n) &:= \ll \ell_{i'j'}, z_n(t) \gg - \ll \ell_{i'j'}, \psi(t, f + K z_n(t)) \gg \\ &= \ll \ell_{i'j'}, \sum_{(i,j) \in U_n} z_{ij} w_{ij}(t) \gg - \ll \ell_{i'j'}, \psi(t, f + K \sum_{(i,j) \in U_n} z_{ij} w_{ij}(t)) \gg. \end{aligned}$$

Entries of the Jacobian matrix are computed from

$$\begin{aligned} J_{i'j', \alpha\beta}(\mathbf{z}_n) &:= \frac{\partial F_{i'j'}(\mathbf{z}_n)}{\partial z_{\alpha\beta}} \\ &= \ll \ell_{i'j'}, w_{\alpha\beta}(t) \gg - \ll \ell_{i'j'}, \psi^{(0,1)}(t, f + \sum_{(i,j) \in U_n} z_{ij} K w_{ij}(t)) \cdot K w_{\alpha\beta}(t) \gg, \end{aligned} \quad (90)$$

with $j' \in \omega(i')$, $\beta \in \omega(\alpha)$; $i', \alpha \in Z_{n+1}$. The first term in the last equation is simplified as $\ll \ell_{i'j'}, w_{\alpha\beta}(t) \gg = \delta_{i'\alpha} \delta_{j'\beta}$, $\alpha \leq i'$. The second term points to the significance of the 'linearized' equation (82) or equivalently (85). Notice that wavelets w_{ij} and $w_{\alpha\beta}$ appear directly under the linear operator K . This enables the present method to use the truncation strategies of Chen et. al. described in the last section. More specifically, recalling (14), the second term of (90) can be written as

$$\begin{aligned} & \ll \ell_{i'j'}, \psi^{(0,1)}(t, f + \sum_{(i,j) \in U_n} z_{ij} K w_{ij}(t)) \cdot K w_{\alpha\beta}(t) \gg \\ &= \sum_{s \in Z_{k\mu}} c_{js} \psi^{(0,1)}(t_s, f(t_s) + \sum_{(i,j) \in U_n} z_{ij} K w_{ij}(t_s)) \cdot K w_{\alpha\beta}(t_s) \\ &= \sum_{s \in Z_{k\mu}} c_{js} \psi^{(0,1)}(t_s, f(t_s)) + \sum_{(i,j) \in U_n} z_{ij} \int_0^1 k(t_s, s) w_{ij}(s) ds \cdot \int_0^1 k(t_s, s) w_{\alpha\beta}(s) ds. \end{aligned} \quad (91)$$

By assumption (A5), we may assume, for $M > 0$,

$$|\psi^{(0,1)}(t, y)| \leq M, \quad t, y \in (-\infty, \infty). \quad (92)$$

Lemma IV.2.2. *If there is a constant $r > 1$ such that*

$$\text{dist}(S_{ij}, S_{i'j'}) \geq r(d_i + d_{i'}),$$

then there exists a positive constant c such that

$$|J_{i'j', \alpha\beta}(\mathbf{z}_n)| \leq c(d_i d_{i'})^k \sum_{s \in \tilde{S}_{i'j'}} \int_{S_{ij}} \frac{1}{|s-t|^{2k+\sigma}} dt, \quad (93)$$

where $c = \frac{M\theta\theta_1^2 e^{\frac{2d}{1-r}}}{k^2(1-r^{-1})^\sigma}$ in which θ and σ are defined in (77) and $\|\ell_{ij}\| + \|w_{ij}\|_{0,\infty} \leq \theta_1$.

Proof. This follows immediately from Lemma IV.1.1 (lemma 3.1, [8]) and (92). \square

Once the estimate (93) for the entries $J_{i'j', \alpha\beta}(\mathbf{z}_n)$ of the Jacobian $\mathbf{J}(\mathbf{z}_n)$ of $\mathbf{F}(\mathbf{z}_n^{(k)}) := [F_{ij}(\mathbf{z}_n^{(k)})]_{j \in \omega(i), i \in Z_{n+1}}$ are given, the block truncation strategy of Chen et. al. [10], also described in section IV.1.1, can be applied to this Jacobian. Namely, let $p = \{p_{i'i} : i', i \in Z_{n+1}\}$ be the sequence defined in Theorem IV.1.4 and let $\mathbf{J}(\mathbf{z}_n) = [\mathbf{J}_{i'\alpha}(\mathbf{z}_n) : i', \alpha \in Z_{n+1}]$ where $\mathbf{J}_{i'\alpha}(\mathbf{z}_n) := [J_{i'j', \alpha\beta}(\mathbf{z}_n) : j' \in \omega(i'), \beta \in \omega(\alpha)]$. Define $\tilde{\mathbf{J}}(\mathbf{z}_n) := [\tilde{\mathbf{J}}_{i', \alpha}(\mathbf{z}_n) : i', \alpha \in Z_{n+1}]$ where

$$\tilde{\mathbf{J}}_{i', \alpha}(\mathbf{z}_n) := [\mathbf{J}(\mathbf{z}_n)(p)_{q'q}^{i'\alpha} : q', q \in Z_{\mu^{\min\{i', \alpha\}-1}}],$$

and

$$\mathbf{J}(\mathbf{z}_n)(p)_{q'q}^{i'\alpha} = \begin{cases} \mathbf{J}(\mathbf{z}_n)_{q'q}^{i'\alpha} & |q' - q| \leq p^{i'\alpha}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{J}(\mathbf{z}_n)(p)_{q'q}^{i'\alpha}$ is defined similarly to the method in [10] which was stated in section IV.1.1.

Lemma IV.2.3. *Give $\epsilon > 0$. Then parameters p as defined in Theorem IV.1.4 can be chosen so that for $\mathbf{z} \in \mathbb{R}^{f(n)}$,*

$$\|\mathbf{J}(\mathbf{z}) - \tilde{\mathbf{J}}(\mathbf{z})\|_{0,\infty} \leq \epsilon.$$

Proof. Note that lemma 3.2 [8] ensures that there exists a positive constant c such that for all $i', \alpha \in Z_{n+1}$ and all $\mathbf{z} \in \mathbb{R}^{f(n)}$,

$$\|\mathbf{J}_{i'\alpha}(\mathbf{z}) - \tilde{\mathbf{J}}_{i'\alpha}(\mathbf{z})\|_{0,\infty} \leq c\epsilon_{i'\alpha}^{-(2k-\sigma)} \mu^{-k(i'+\alpha)}. \quad (94)$$

Once estimate (94) is obtained, then arguing in the same way as in the proof of lemma 4.2 (equation (4.12) [8]),

$$\|(\mathbf{J}(\mathbf{z}) - \tilde{\mathbf{J}}(\mathbf{z}))\mathbf{v}\|_{0,\infty} \leq c\mu^{-\sigma'n/d} \|\mathbf{v}\|_{0,\infty},$$

where $\mathbf{v} \in \mathbb{R}^{f(n)}$, c is a constant independent of n and $0 < \sigma' < \min\{2k, d - \sigma\}$. Here recall that σ is the parameter of weak singularity. Hence, this lemma is proved by selecting n so that $|c\mu^{-\sigma'n/d}| < \epsilon$. \square

In Lemma IV.2.3, for the current discussion, $d = 1$ so that $\sigma' = 2k$. The Newton's method finds the solution z_n of (86) as follows: Starting with an initial vector $\mathbf{z}_n^{(0)} = [z_{ij}^0]$, which we assume to be sufficiently close to the solution $\mathbf{z}_n = [z_{ij}]$ of (86) or equivalently $\mathbf{z}_n = \mathbf{F}(\mathbf{z}_n)$ so that the Newton's method converges, we compute

$$\mathbf{z}_n^{(k+1)} = \mathbf{z}_n^{(k)} - \mathbf{J}^{-1}(\mathbf{z}_n^{(k)})\mathbf{F}(\mathbf{z}_n^{(k)}). \quad (95)$$

Of course, one does not invert the Jacobian $\mathbf{J}(\mathbf{z}_n^{(k)})$ directly for each n , but rather solve for $\mathbf{y}_n^{(k)}$ in

$$\mathbf{J}(\mathbf{z}_n^{(k)})\mathbf{y}_n^{(k)} = \mathbf{F}(\mathbf{z}_n^{(k)}), \quad (96)$$

and (95) is implemented as

$$\mathbf{z}_n^{(k+1)} = \mathbf{z}_n^{(k)} - \mathbf{y}_n^{(k)}. \quad (97)$$

Now, we are ready to propose a *fast wavelet-collocation method* for Hammerstein equation based upon (86). Let equations (96) and (97) are replaced by

$$\tilde{\mathbf{J}}(\tilde{\mathbf{z}}_n^{(k)})\tilde{\mathbf{y}}_n^{(k)} = \mathbf{F}(\tilde{\mathbf{z}}_n^{(k)}), \quad (98)$$

and

$$\tilde{\mathbf{z}}_n^{(k+1)} = \tilde{\mathbf{z}}_n^{(k)} - \tilde{\mathbf{y}}_n^{(k)}, \quad (99)$$

respectively, where $\tilde{\mathbf{z}}_n^{(0)} = \mathbf{z}_n^{(0)}$. First, we note the following lemma.

Lemma IV.2.4. *For each $n \in \mathbb{N}$, $\mathbf{z}_n^{(0)} \in \mathbb{R}^{J(n)}$ and $\epsilon > 0$, then parameters p defined in Theorem IV.1.4 can be chosen so that, for all $k \in \mathbb{N}$, we have*

$$\|\tilde{\mathbf{z}}_n^{(k)} - \mathbf{z}_n^{(k)}\|_{0,\infty} < \epsilon. \quad (100)$$

Proof. We prove this by induction. With $k = 1$, since $\tilde{\mathbf{z}}_n^{(0)} = \mathbf{z}_n^{(0)}$,

$$\begin{aligned} \tilde{\mathbf{z}}_n^{(1)} - \mathbf{z}_n^{(1)} &= \mathbf{y}_n^{(0)} - \tilde{\mathbf{y}}_n^{(0)} \\ &= \left[\mathbf{J}^{-1}(\mathbf{z}_n^{(0)}) - \tilde{\mathbf{J}}^{-1}(\tilde{\mathbf{z}}_n^{(0)}) \right] \mathbf{F}(\mathbf{z}_n^{(0)}) \\ &= \mathbf{J}^{-1}(\mathbf{z}_n^{(0)})[\tilde{\mathbf{J}}(\tilde{\mathbf{z}}_n^{(0)}) - \mathbf{J}(\mathbf{z}_n^{(0)})]\tilde{\mathbf{J}}^{-1}(\tilde{\mathbf{z}}_n^{(0)})\mathbf{F}(\mathbf{z}_n^{(0)}). \end{aligned}$$

Taking the norm on both sides and noting that $\|\tilde{\mathbf{J}}(\tilde{\mathbf{z}}_n^{(0)}) - \mathbf{J}(\mathbf{z}_n^{(0)})\|_{0,\infty}$ can be made arbitrarily small by Lemma IV.2.3, we prove the case when $k = 1$. Now assume that (100) is true for $k - 1$. Then

$$\begin{aligned} \tilde{\mathbf{z}}_n^{(k)} - \mathbf{z}_n^{(k)} &= \tilde{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n^{(k-1)} - (\tilde{\mathbf{y}}_n^{(k-1)} - \mathbf{y}_n^{(k-1)}) \\ &= \tilde{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n^{(k-1)} + [\mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)}) - \tilde{\mathbf{J}}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)})]\mathbf{F}(\tilde{\mathbf{z}}_n^{(k-1)}) \\ &\quad + \mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)})[\mathbf{F}(\tilde{\mathbf{z}}_n^{(k-1)}) - \mathbf{F}(\mathbf{z}_n^{(k-1)})] + [\mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)}) - \mathbf{J}^{-1}(\mathbf{z}_n^{(k-1)})]\mathbf{F}(\mathbf{z}_n^{(k-1)}) \\ &:= I + II + III + IV. \end{aligned}$$

For *II*,

$$\|[\mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)}) - \tilde{\mathbf{J}}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)})]\mathbf{F}(\tilde{\mathbf{z}}_n^{(k-1)})\|_{0,\infty} \leq c_1 \|\tilde{\mathbf{J}}(\tilde{\mathbf{z}}_n^{(k-1)}) - \mathbf{J}(\tilde{\mathbf{z}}_n^{(k-1)})\|_{0,\infty}, \quad (101)$$

where $c_1 = \|\mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)})\|_{0,\infty} \|\tilde{\mathbf{J}}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)})\|_{0,\infty} \|\mathbf{F}(\tilde{\mathbf{z}}_n^{(k-1)})\|_{0,\infty}$. The right side of (101) can be made $< \frac{\epsilon}{4}$ by Lemma IV.2.3. For *III*,

$$\|\mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)})[\mathbf{F}(\tilde{\mathbf{z}}_n^{(k-1)}) - \mathbf{F}(\mathbf{z}_n^{(k-1)})]\|_{0,\infty} \leq c_2 \|\tilde{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n^{(k-1)}\|_{0,\infty}, \quad (102)$$

where $c_2 = \|\mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)})\|_{0,\infty} \|\mathbf{J}(\tilde{\mathbf{z}}_n^{(k-1)}) + \theta(\tilde{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n^{(k-1)})\|_{0,\infty}$ with $0 < \theta < 1$. Applying induction, the right side of (102) can be made $< \frac{\epsilon}{4}$. Finally for *IV*,

$$\|[\mathbf{J}^{-1}(\tilde{\mathbf{z}}_n^{(k-1)}) - \mathbf{J}^{-1}(\mathbf{z}_n^{(k-1)})]\mathbf{F}(\mathbf{z}_n^{(k-1)})\|_{0,\infty} \leq c_3 \|\tilde{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n^{(k-1)}\|_{0,\infty}, \quad (103)$$

where $c_3 = \|\mathbf{F}(\mathbf{z}_n^{(k-1)})\|_{0,\infty} \|\mathbf{H}(\bar{\mathbf{z}}_n^{(k-1)} + \beta(\bar{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n^{(k-1)}))\|_{0,\infty}$ with $0 < \beta < 1$ and $\mathbf{H}(\mathbf{a})$ denotes the Hessian of \mathbf{F} at $\mathbf{a} \in \mathbb{R}^{f(n)}$. Another application of the induction hypothesis and considering (101)-(103), we complete the proof of this lemma. \square

The following theorem establishes the convergence of the fast wavelet-collocation method for Hammerstein equation described in (98) and (99).

Theorem IV.2.5. *Assume that $\mathbf{z}_n = [z_{ij}]$ is a solution of (86), i.e., $\mathbf{z}_n = \mathbf{F}(\mathbf{z}_n)$. Also let $\mathbf{z}_n^{(0)} \in \mathbb{R}^{f(n)}$ be such that the Newton iteration given in (95)-(97) generate a sequence $\mathbf{z}_n^{(k)}$ that converges to \mathbf{z}_n as $k \rightarrow \infty$. Then the parameter p in the block truncation strategy can be chosen so that the fast wavelet-collocation method described in (98)-(99) generates a sequence $\tilde{\mathbf{z}}_n^{(k)}$ which converges to \mathbf{z}_n as $k \rightarrow \infty$.*

Proof. This follows by noting that

$$\tilde{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n = \tilde{\mathbf{z}}_n^{(k-1)} - \mathbf{z}_n^{(k)} + \mathbf{z}_n^{(k)} - \mathbf{z}_n,$$

and apply Lemma IV.2.4 to first two terms and the convergence assumption to the remaining two terms. \square

Next, the numerical results of wavelet collocation scheme of solving the linearized Hammerstein equation are shown. The comparison between the full wavelet collocation solutions and the compressed wavelet collocation solutions is presented. Here, the word ‘full’ means solving the corresponding system without the truncation strategy and ‘compressed’ means solving the corresponding system with the block truncation strategy, setting $a = 0.25, \nu = 1.01$ and $b' = 0.8$.

Example IV.2.6. Consider the equation

$$u(t) - \int_0^1 e^{s+t} u^2(s) ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \cos(t)$.

TABLE 15: Computational results of Hammerstein equation for Example IV.2.6.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\tilde{e}_h	\tilde{R}_h	\tilde{CT}
2	2.779527e-2		0.005	2.779527e-2		0.004
4	5.779759e-3	2.27	0.014	5.779759e-3	2.27	0.015
8	1.386280e-3	2.06	0.070	1.386280e-3	2.06	0.064
16	3.431079e-4	2.01	0.281	3.431079e-4	2.01	0.271
32	8.556368e-5	2.00	1.127	8.556261e-5	2.00	1.027
64	2.137762e-5	2.00	4.526	2.137565e-5	2.00	3.515
128	5.343552e-6	2.00	18.205	5.341373e-6	2.00	11.709
256	1.335814e-6	2.00	73.153	1.333582e-6	2.00	38.913
512	3.339281e-7	2.00	293.516	3.316833e-7	2.01	133.359
1024	8.345797e-8	2.00	1177.954	8.980856e-8	1.88	474.664

Example IV.2.7. Consider the equation

$$u(t) - \int_0^1 \cos(s+t) e^{s+u(s)} ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = 1$.

TABLE 16: Computational results of Hammerstein equation for Example IV.2.7.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\tilde{e}_h	\tilde{R}_h	\tilde{CT}
2	5.877689e-2		0.004	5.877689e-2		0.003
4	1.662155e-3	1.82	0.021	1.662155e-3	1.82	0.014
8	4.284816e-3	1.96	0.066	4.284816e-3	1.96	0.070
16	1.079426e-3	1.99	0.278	1.079426e-3	1.99	0.278
32	2.703726e-4	2.00	1.104	2.703743e-4	2.00	1.051
64	6.762578e-5	2.00	4.475	6.762899e-5	2.00	3.485
128	1.690883e-5	2.00	17.936	1.691238e-5	2.00	11.608
256	4.227701e-6	2.00	74.831	4.231332e-6	2.00	38.693
512	1.057301e-6	2.00	297.746	1.060952e-6	2.00	132.352
1024	2.646933e-7	2.00	1172.390	2.693742e-7	1.98	472.071

Example IV.2.8. Consider the equation

$$u(t) - \int_0^1 \frac{u^2(s)}{\sqrt{|s-t|}} ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^2$.

TABLE 17: Computational results of Hammerstein equation for Example IV.2.8.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\tilde{e}_h	\tilde{R}_h	\tilde{CT}
2	2.917026e-1		0.004	2.917026e-1		0.005
4	1.606610e-1	0.86	0.024	1.606610e-1	0.86	0.020
8	4.716359e-2	1.77	0.050	4.716359e-2	1.77	0.051
16	1.597451e-2	1.56	0.215	1.597451e-2	1.56	0.201
32	3.360508e-3	2.25	0.730	3.360610e-3	2.25	0.720
64	8.634525e-4	1.96	2.937	8.638484e-4	1.96	2.696
128	2.198411e-4	1.97	11.439	2.203948e-4	1.97	10.055
256	5.411619e-5	2.02	46.018	5.471957e-5	2.01	39.429
512	1.326851e-5	2.03	184.283	1.388284e-5	1.98	154.570
1024	3.300703e-6	2.01	740.624	3.358074e-6	2.05	600.178

Example IV.2.9. Consider the equation

$$u(t) - \int_0^1 \log|s-t| u^2(s) ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^2$.

TABLE 18: Computational results of Hammerstein equation for Example IV.2.9.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\tilde{e}_h	\tilde{R}_h	\tilde{CT}
2	1.848228e-1		0.004	1.848228e-1		0.003
4	4.479195e-2	2.04	0.012	4.479195e-2	2.04	0.016
8	1.048831e-2	2.09	0.038	1.048831e-2	2.09	0.041
16	2.277151e-3	2.20	0.147	2.277151e-3	2.20	0.143
32	4.831762e-4	2.24	0.598	4.831753e-4	2.24	0.562
64	1.090730e-4	2.15	2.255	1.090665e-4	2.15	2.060
128	2.676196e-5	2.03	9.044	2.675423e-5	2.03	7.827
256	6.516920e-6	2.04	36.575	6.508463e-6	2.04	30.108
512	1.613282e-6	2.01	167.634	1.604425e-6	2.02	116.387
1024	4.997735e-7	1.69	672.429	5.217983e-7	1.62	539.528

TABLE 19: Number of zeros of compressed Jacobian matrix.

$f(n)$	32	64	128	256	512	1024
number of zeros	96	1264	8512	45072	211424	927152
ratio of zeros (in %)	9.375	30.8594	51.9531	68.7744	80.6519	88.4201

In summary, the computing times of compressed solutions are less than the computing time of full collocation solutions, especially in the smooth kernel case. From TABLE 19 and FIG. 16 is shown the sparsity of the Jacobian matrices in the fast wavelet-collocation method. These show that the number of non-zeros components of the Jacobian matrix is $O(f(n) \log f(n))$ which is consistent to Theorem IV.1.4. In the next section, we present yet another numerical technique, called the multilevel augmentation method, which further improve the solution process of the wavelet-collocation method for Hammerstein equation in terms of CPU time and computer memory.

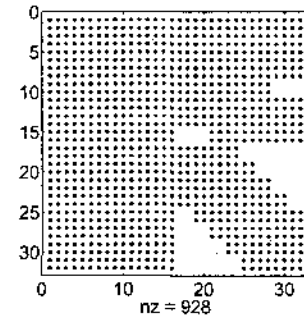
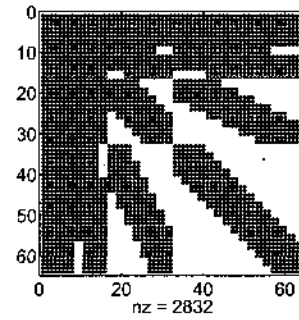
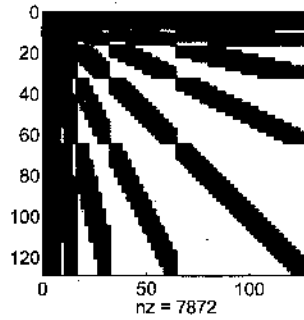
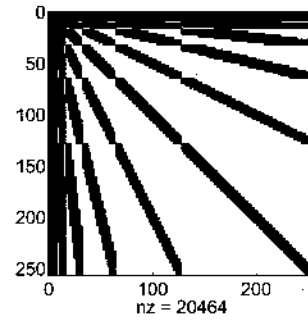
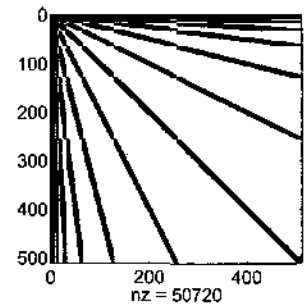
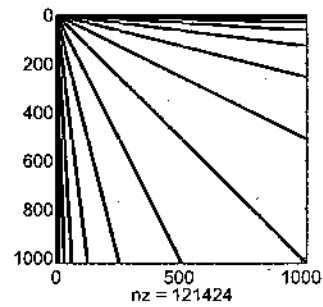
(a) $f(n) = 32$ (b) $f(n) = 64$ (c) $f(n) = 128$ (d) $f(n) = 256$ (e) $f(n) = 512$ (f) $f(n) = 1024$

FIG. 16: Sparsity of compressed Jacobian matrix of linearized Hammerstein equation.

IV.3 MULTILEVEL AUGMENTATION METHOD FOR HAMMERSTEIN EQUATION

A first form of a multilevel augmentation method appeared in [7] in connection with the Fredholm integral equation of the second kind. Multilevel augmentation methods are used with a basis having a multiresolution structure such as wavelet functions. Each multilevel augmentation method solves an underlying equation at a low dimensional subspace and enhances the accuracy of its approximation by adding successively to the solution corrected terms which can be obtained by solving systems of high dimension. The method was recently extended to obtain a fast algorithm for a class of nonlinear Hammerstein equations [9]. In this section, we present another multilevel augmentation method for the Hammerstein which is based upon the 'linearization' technique which was explored in Section IV.2. The block truncation strategy is also used in constructing the Jacobian matrices. Numerical examples are provided to demonstrate the convergence of the new multilevel augmentation method and the effectiveness of the truncation strategy.

Recall the decomposition of the subspace \mathbb{X}_n for $L^\infty(E)$, i.e., with $n = k + m$,

$$\mathbb{X}_{k+m} = \mathbb{X}_k \oplus \mathbb{W}_{k+1} \oplus \dots \oplus \mathbb{W}_{k+m}.$$

Our method goes as follows: first, we obtain an approximation of the solution of the Hammerstein equation in the space \mathbb{X}_k by solving (86) exactly to obtain the solution z_k . The next step is to obtain an approximation of the solution z_{k+1} of equation (86) with $n = k + 1$. For this purpose, we decompose

$$z_{k+1} = z_{k+1}^L + z_{k+1}^H, \quad \text{with } z_{k+1}^L \in \mathbb{X}_k \text{ and } z_{k+1}^H \in \mathbb{W}_{k+1}$$

so that (86) becomes

$$z_{k+1}^L + z_{k+1}^H = P_{k+1}\Psi(f + Kz_{k+1}).$$

This equation can be transformed into

$$z_{k+1}^L = P_k\Psi(f + Kz_{k+1}) + (P_{k+1} - P_k)\Psi(f + Kz_{k+1}) - z_{k+1}^H, \quad (104)$$

or equivalently,

$$P_k(z_{k+1}^L + z_{k+1}^H) = P_k\Psi(f + Kz_{k+1}) + (P_{k+1} - P_k)\Psi(f + Kz_{k+1}) - z_{k+1}^H. \quad (105)$$

Here we used the fact that $P_k z_{k+1}^H = 0$ as $z_{k+1}^H \in \mathbb{W}_{k+1}$ and P_k is a projection onto \mathbb{X}_k . Next, we compute

$$z_{k,1}^H := (P_{k+1} - P_k)\Psi(f + Kz_k), \quad (106)$$

and note that $z_{k,1}^H \in \mathbb{W}_{k+1}$. In (104), we replace z_{k+1}^H and the second term in the right hand side by $z_{k,1}^H$ to obtain an equation for $z_{k,1}^L \in \mathbb{X}_k$:

$$z_{k,1}^L = P_k \Psi(f + K(z_{k,1}^L + z_{k,1}^H)). \quad (107)$$

The element $z_{k,1}^L$ turns out to be a good approximation to z_{k+1}^L . We then obtain an approximation to the solution z_{k+1} of equation (86) by setting

$$z_{k,1} := z_{k,1}^L + z_{k,1}^H. \quad (108)$$

Note that $z_{k,1}^L$ and $z_{k,1}^H$, respectively, represent the lower and higher frequency components of $z_{k,1}$.

We continue this process to find an approximation of the solution of equation (86) with $n = k + 2$. Specifically, we compute

$$z_{k,2}^H := (P_{k+2} - P_k)\Psi(f + Kz_{k,1})$$

using the approximate solution $z_{k,1}$ obtained in the previous step, and solve for $z_{k,2}^L \in \mathbb{X}_k$ from the equation

$$z_{k,2}^L = P_k \Psi(f + K(z_{k,2}^L + z_{k,2}^H)).$$

An approximation to the solution z_{k+2} of equation (86) with $n = k + 2$ is hence obtained by setting

$$z_{k,2} := z_{k,2}^L + z_{k,2}^H.$$

This procedure is repeated to obtain an approximation $z_{k,m}$ of the solution z_{k+m} of equation (86) with $n = k + m$. Once $z_{k,m}$ is obtained, then let

$$u_{k,m} = f + Kz_{k,m} \quad (109)$$

which approximates the solution u_n of (81) with $n = k + m$. The preceding steps can be summarized in the following algorithm.

Algorithm1: The Multilevel Augmentation Method: An Operator Form

Let k be a fixed positive integer.

Step1: Find the solution $z_k \in \mathbb{X}_k$ of the equation (86) with $n := k$. Set $z_{k,0} := z_k$ and $\ell := 1$.

Step2: Compute

$$z_{k,\ell}^H := (P_{k+\ell} - P_k)\Psi(f + Kz_{k,\ell-1}). \quad (110)$$

Step3: Solve for $z_{k,\ell}^L \in \mathbb{X}_k$ from the equation

$$z_{k,\ell}^L = P_k\Psi(f + K(z_{k,\ell}^L + z_{k,\ell}^H)). \quad (111)$$

Step4: Let

$$z_{k,\ell} := z_{k,\ell}^L + z_{k,\ell}^H. \quad (112)$$

Set $\ell \leftarrow \ell + 1$ and go back to Step 2 until $\ell = m$.

Step5: Obtain the approximate solution of u_n in (81) by $u_{k,m} := f + Kz_{k,m}$.

The existence of the solution $z_{k,m}^L$ of (111) can be guaranteed similarly as in Theorem IV.2.1. Let

$$T(a)(u)(t) := f(t) + K(u + a)(t), \quad u, a \in L^\infty(I),$$

so that (111) with $\ell = m$ can be written as

$$z_{k,m}^L = P_k\Psi T(z_{k,\ell}^H)(z_{k,m}^L). \quad (113)$$

Lemma IV.3.1. *Let z be an isolated solution of (85). Assume that 1 is not an eigenvalue of the linear operator $(\Psi T(a))'(z)$, where $(\Psi T(a))'(z)$ denotes the Fréchet derivative of $\Psi T(a)$ at z with $a \in L^\infty(I)$. Then the equation (113) has a unique solution $z_{k,m}^L$ in a ball $B(z, \delta) := \{c \in C(I) : \|c - z\|_{0,\infty} \leq \delta\}$ for some $\delta > 0$ and for sufficiently large k . Moreover, there exists a constant $0 < q < 1$, independent of k , such that*

$$\frac{\alpha_k}{1+q} \leq \|z_{k,m}^L - z\|_{0,\infty} \leq \frac{\alpha_k}{1-q},$$

where $\alpha_n := \|(J - (P_n\Psi T(a))'(z))^{-1}((P_k\Psi T(a))(z) - \Psi T(a)(z))\|_{0,\infty}$. Finally

$$E_k(z) \leq \|z_{k,m}^L - z\|_{0,\infty} \leq CE_k(z),$$

where C is a constant independent of k and $E_k(z) = \inf_{u \in \mathbb{X}_k} \|z - u\|_{0,\infty}$.

Lemma IV.3.2. *Let z be an isolated solution of (85). Assume that 1 is not an eigenvalue of $(\Psi T)'(z)$. Then there exists a sequence of positive numbers $\alpha_{k,m}$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ with $\lim_{k \rightarrow \infty} \alpha_{k,m} = 0$ uniformly for $m \in \mathbb{N}_0$ and a positive integer N such that for all $k \geq N$ and $m \in \mathbb{N}_0$,*

$$\|z_{k,m} - z_{k+m}\| \leq \alpha_{k,m} \|z_{k,m-1} - z_{k+m}\|.$$

Proof. From (110), (111) and (112),

$$\begin{aligned} z_{k,m} &= z_{k,m}^H + z_{k,m}^L \\ &= (P_{k+m} - P_k)\Psi(f + Kz_{k,m-1}) + P_k\Psi(f + Kz_{k,m}). \end{aligned} \quad (114)$$

Using (114) and (86) with $n = k + m$,

$$\begin{aligned} z_{k,m} - z_{k+m} &= (P_{k+m} - P_k)(\Psi T(z_{k,m-1}) - \Psi T(z_{k+m})) \\ &= (P_{k+m} - P_k)(\Psi T)'(z_{k+m} + \theta(z_{k+m} - z_{k,m-1}))(z_{k+m} - z_{k,m-1}), \end{aligned}$$

where $0 < \theta < 1$. Let $\alpha_{k,m} := \|((P_{k+m} - P_k)(\Psi T)'(z_{k+m} + \theta(z_{k+m} - z_{k,m-1})))\|$. Assumptions (A1), (A2) and (A5) guarantee that $\alpha_{k,m} \rightarrow 0$ uniformly in $m \in \mathbb{N}_0$ as $k \rightarrow \infty$ and finally

$$\|z_{k,m} - z_{k+m}\| \leq \alpha_{k,m} \|z_{k,m-1} - z_{k+m}\|.$$

□

The rate of convergence of the multilevel augmentation method is now discussed. We utilize the idea of a *majorization* sequence introduced in [11]. A sequence of nonnegative numbers γ_n , $n \in \mathbb{N}_0$, is called a majorization sequence of E_n , $n \in \mathbb{N}_0$, if $\gamma_n \geq E_n$ for all $n \in \mathbb{N}_0$ and there exists a positive integer N_0 and a positive constant σ such that for $n \geq N_0$,

$$\frac{\gamma_{n+1}}{\gamma_n} \geq \sigma.$$

For the wavelet collocation method described in Section IV.1.1, it is known (theorem 5.2 [11]) that the majorization constant can be selected as

$$\gamma_n := c \frac{1}{2^{rn}} \|z\|_{r,2}, \quad (115)$$

where $z \in W_{r,2}(E)$, r is the order of the wavelet and c is independent of n such that $E_n \leq \gamma_n$. Proof of the following theorem is included for completeness (see Theorem 3.3 [11]).

Theorem IV.3.3. *Let z be an isolated solution of (85) and let γ_n , $n \in \mathbb{N}_0$ be a majorization sequence of E_n , $n \in \mathbb{N}_0$. Assume that 1 is not an eigenvalue of $(\Psi T)'(z)$. Then there exists a positive constant ρ and a positive constant N such that for all $k \geq N$ and $m \in \mathbb{N}_0$,*

$$\|z - z_{k,m}\| \leq (\rho + 1)\gamma_{k+m}. \quad (116)$$

Proof. We apply induction on m . The estimate clearly holds with $m = 0$. Suppose that (116) holds for $m-1$. Then, using the inductive hypothesis and the majorization property,

$$\begin{aligned} \|z_{k,m-1} - z_{k+m}\| &\leq \|z_{k,m-1} - z\| + \|z_{k+m} - z\| \\ &\leq \left(\rho + \frac{\rho+1}{\sigma}\right) \gamma_{k+m}. \end{aligned}$$

Choosing N large enough that for $k > N$, Lemma IV.2.2 holds and that $\alpha_{k,m}(\rho + \frac{\rho+1}{\sigma}) < 1$, we get

$$\|z_{k,m} - z_{k+m}\| \leq \gamma_{k+m}.$$

Finally,

$$\|z_{k,m} - z\| \leq \|z_{k,m} - z_{k+m}\| + \|z_{k+m} - z\| \leq (\rho + 1)\gamma_{k+m}.$$

□

IV.3.1 Discrete Multilevel Augmentation method for Hammerstein equation:

In this subsection, a discrete version of the multilevel augmentation method is described. Recall the functionals $\ell_{ij} \in X^*$, the dual space of $X = L^\infty(I)$, and the corresponding interpolation functions $w_{ij} \in X$. Here, using the Hahn-Banach theorem, we may assume X^* to be the dual of $C(I)$, see [8]. We let

$$\mathbb{X}_0 = \text{span}\{w_{0,j} : j \in \mathbb{Z}_{w(0)}\}, \quad \mathbb{L}_0 = \text{span}\{\ell_{0,j} : j \in \mathbb{Z}_{w(0)}\},$$

$$\mathbb{W}_i = \text{span}\{w_{i,j} : j \in \mathbb{Z}_{w(i)}\}, \quad \mathbb{V}_i = \text{span}\{\ell_{i,j} : j \in \mathbb{Z}_{w(i)}\}, \quad i > 0,$$

and

$$\mathbb{X}_n = \text{span}\{w_{i,j} : (i,j) \in U_n\}, \quad \mathbb{L}_n = \text{span}\{\ell_{i,j} : (i,j) \in U_n\}, \quad n > \mathbb{N}_0.$$

With the projection P_n defined in (84), for all $x \in \mathbb{X}$ and for all $\ell \in \mathbb{L}_n$, we have $\langle\langle \ell, x - P_n x \rangle\rangle = 0$. This implies the decomposition

$$\mathbb{L}_{n+1} = \mathbb{L}_n \oplus \mathbb{V}_{n+1},$$

and

$$\mathbb{L}_{k+m} = \mathbb{L}_k \oplus \mathbb{V}_{k,m},$$

where $\mathbb{V}_{k,m} = \mathbb{V}_{k+1} \oplus \cdots \oplus \mathbb{V}_{k+m}$.

For any $v \in \mathbb{X}_{k+m}$, we have a unique expansion

$$v = \sum_{(i,j) \in U_{k+m}} v_{i,j} w_{i,j}.$$

The vector $\mathbf{v} := [v_{i,j} : (i,j) \in U_{k+m}]^T$ represents v . Thus, for solution $z_{k,m}$ of (113), its representation vector is given by $\mathbf{z}_{k,m} := [(z_{k,m})_{i,j} : (i,j) \in U_{k+m}]^T$. Setting $U_{k,m} := U_{k+m} \setminus U_k$, we obtain that

$$U_{k,m} = \{(i,j) : j \in \mathbb{Z}_{w(i)}, i \in \mathbb{Z}_{k+m+1} \setminus \mathbb{Z}_{k+1}\}.$$

Consequently, we have the representations

$$z_{k,m}^L := \sum_{(i,j) \in U_k} (z_{k,m})_{i,j} w_{i,j} \quad \text{and} \quad z_{k,m}^H := \sum_{(i,j) \in U_{k,m}} (z_{k,m})_{i,j} w_{i,j}.$$

Algorithm2: The Multilevel Augmentation Method: A Discrete Form

Let k be a fixed positive integer.

Step1: Solve the nonlinear system

$$\ll \ell_{i',j'}, \sum_{(i,j) \in U_k} (z_k)_{i,j} w_{i,j} \gg = \ll \ell_{i',j'}, \Psi \left(f + K \left(\sum_{(i,j) \in U_k} (z_k)_{i,j} w_{i,j} \right) \right) \gg,$$

for $(i',j') \in U_k$, and for $z_k := [(z_k)_{i,j} : (i,j) \in U_k]^T$.

Let $z_{k,0} := z_k$ and $l := 1$.

Step2: Solve the linear system

$$\ll \ell_{i',j'}, \sum_{(i,j) \in U_{k,l}} (z_{k,l}^h)_{i,j} w_{i,j} \gg = \ll \ell_{i',j'}, \Psi(f + K z_{k,l-1}) \gg, \quad (i',j') \in U_{k,l}$$

to obtain $z_{k,l}^h$ and define $z_{k,l}^H := \sum_{(i,j) \in U_{k,l}} (z_{k,l}^h)_{i,j} w_{i,j}$.

Step3: Solve the nonlinear system

$$\ll \ell_{i',j'}, \sum_{(i,j) \in U_k} (z_{k,l}^L)_{i,j} w_{i,j} + z_{k,l}^H \gg = \ll \ell_{i',j'}, \Psi \left(f + K \left(\sum_{(i,j) \in U_k} (z_{k,l}^L)_{i,j} w_{i,j} + z_{k,l}^H \right) \right) \gg,$$

$(i', j') \in U_k$ to obtain $z_{k,l}^L := [(z_{k,l}^L)_{i,j} : (i, j) \in U_k]^T$.

Define

$$z_{k,l}^L := \sum_{(i,j) \in U_k} (z_{k,l}^L)_{i,j} w_{i,j}$$

and

$$z_{k,l} = z_{k,l}^L + z_{k,l}^H.$$

Step4: Set $l \leftarrow l + 1$ and go back to Step 2 until $l = m$.

Step5: Obtain the approximation solution of u_n from

$$u_n := f + K z_{k,m}.$$

In order to solve the nonlinear equation in Step 3, we use the Newton method and apply the block truncation strategy described in section IV.2. This establishes a fast multilevel augmentation method for Hammerstein equations.

The numerical experiments of the multilevel augmentation method to solve the Hammerstein equation are presented below.

Denote

$$e'_h = \|u - u_{k,m}\|_{0,\infty}, \quad R'_h = \log_2 \left(\frac{e'_h}{e'_{h/2}} \right),$$

$$\tilde{e}'_h = \|u - \tilde{u}'_{k,m}\|_{0,\infty}, \quad \tilde{R}'_h = \log_2 \left(\frac{\tilde{e}'_h}{\tilde{e}'_{h/2}} \right),$$

where $u_{k,m}$ is the multilevel augmentation solution with fixed k and $\tilde{u}'_{k,m}$ is the compressed multilevel augmentation solution.

Example IV.3.4. Consider the equation

$$u(t) - \int_0^1 \sin(s+t) \cos(\log(u(s))) ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \exp(t)$.

TABLE 20: Computational results of Hammerstein equation for Example IV.3.4.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\tilde{e}_h	\tilde{R}_h	\widetilde{CT}
16	2.907214e-4		0.267	2.907214e-4		0.290
32	7.271394e-5	2.00	1.031	7.271390e-5	2.00	0.949
64	1.818057e-5	2.00	4.219	1.818049e-5	2.00	3.212
128	4.545257e-6	2.00	16.766	4.545178e-6	2.00	10.453
256	1.136307e-6	2.00	67.557	1.136227e-6	2.00	34.085
512	2.840621e-7	2.00	271.659	2.839810e-7	2.00	113.927
1024	7.100113e-8	2.00	1088.084	7.394344e-8	1.94	398.177

Multilevel Augmentation schemes when $k = 3$							
m	$f(n)$	full multilevel solution			compressed multilevel solution		
		e'_h	R'_h	CT	\tilde{e}'_h	\tilde{R}'_h	\widetilde{CT}
0	16	2.907214e-4		0.294	2.907214e-4		0.301
1	32	7.274118e-5	2.00	0.934	7.274116e-5	2.00	0.864
2	64	1.818908e-5	2.00	3.334	1.818901e-5	2.00	2.342
3	128	4.547493e-6	2.00	12.735	4.547414e-6	2.00	6.240
4	256	1.136873e-6	2.00	50.140	1.136792e-6	2.00	16.194
5	512	2.842040e-7	2.00	200.264	2.841229e-7	2.00	41.490
6	1024	7.404369e-8	1.94	800.396	7.397767e-8	1.94	105.701

Example IV.3.5. Consider the equation

$$u(t) - \int_0^1 \sin(\pi(s+t))u^2(s)ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \sin(\pi t)$.

TABLE 21: Computational results of Hammerstein equation for Example IV.3.5.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\tilde{e}_h	\tilde{R}_h	\widetilde{CT}
16	3.119330e-3		0.293	3.119330e-3		0.286
32	7.737703e-4	2.01	1.123	7.733855e-4	2.01	1.025
64	1.930683e-4	2.00	4.492	1.922919e-4	2.01	3.536
128	4.824368e-5	2.00	17.964	4.737521e-5	2.02	11.879
256	1.205938e-5	2.00	72.516	1.117841e-5	2.08	38.117
512	3.014675e-6	2.00	288.766	2.167331e-6	2.37	129.749
1024	7.536243e-7	2.00	1161.637	4.791776e-7	2.18	457.543

Multilevel Augmentation schemes when $k = 3$							
m	$f(n)$	full multilevel solution			compressed multilevel solution		
		e'_h	R'_h	CT	\tilde{e}'_h	\tilde{R}'_h	\widetilde{CT}
0	16	3.119330e-3		0.310	3.119330e-3		0.307
1	32	7.474600e-4	2.06	1.004	7.471047e-4	2.06	0.894
2	64	1.853357e-4	2.01	3.460	1.845625e-4	2.02	2.433
3	128	4.623852e-5	2.00	13.016	4.538137e-5	2.02	6.419
4	256	1.155368e-5	2.00	51.414	1.068286e-5	2.09	16.655
5	512	2.887972e-6	2.00	205.567	2.055478e-6	2.38	42.378
6	1024	7.130420e-7	2.02	822.314	5.026841e-7	2.03	107.972

For Example IV.3.4 and IV.3.5, we see that the computing time of the compressed multilevel augmentation solutions is less than one-tenth of the computing time of the full collocation solutions when solving the nonlinear equation with smooth kernel. It is the most efficient numerical method among the ones compared and it also preserves the order of accuracy.

Example IV.3.6. Consider the equation

$$u(t) - \int_0^1 \log|s-t|u^4(s)ds = f(t), \quad t \in [0,1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \sqrt{t}$.

TABLE 22: Computational results of Hammerstein equation for Example IV.3.6.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\bar{e}_h	\tilde{R}_h	\overline{CT}
16	8.489815e-4		0.156	8.489815e-4		0.158
32	1.949729e-4	2.12	0.619	1.949640e-4	2.12	0.588
64	4.655164e-5	2.07	2.439	4.653352e-5	2.07	2.228
128	1.157152e-5	2.01	9.697	1.154975e-5	2.01	8.242
256	2.857055e-6	2.02	38.861	2.834520e-6	2.03	31.851
512	7.097962e-7	2.01	155.001	6.876834e-7	2.04	123.318
1024	2.220791e-7	1.68	620.824	2.265562e-7	1.60	482.955

Multilevel Augmentation schemes when $k = 3$							
m	$f(n)$	full multilevel solution			compressed multilevel solution		
		e'_h	R'_h	CT	\bar{e}'_h	\tilde{R}'_h	\overline{CT}
0	16	8.489815e-4		0.157	8.489815e-4		0.152
1	32	1.763337e-4	2.27	0.395	1.763280e-4	2.27	0.375
2	64	4.569981e-5	1.95	0.976	4.568140e-5	1.95	0.789
3	128	1.118496e-5	2.03	2.948	1.116352e-5	2.03	1.765
4	256	2.832751e-6	1.98	10.798	2.810179e-6	1.99	4.588
5	512	6.997331e-7	2.02	46.039	6.776315e-7	2.05	17.527
6	1024	2.475609e-7	1.50	215.279	2.415896e-7	1.49	90.497

Example IV.3.7. Consider the equation

$$u(t) - \int_0^1 \log |s - t| u^2(s) ds = f(t), \quad t \in [0, 1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^{3/2}$.

TABLE 23: Computational results of Hammerstein equation for Example IV.3.7.

$f(n)$	full collocation solution			compressed collocation solution		
	e_h	R_h	CT	\tilde{e}_h	\tilde{R}_h	\widetilde{CT}
16	1.339395e-3		0.152	1.339395e-3		0.153
32	2.926967e-4	2.19	0.600	2.926941e-4	2.19	0.580
64	6.783471e-5	2.11	2.445	6.782715e-5	2.11	2.253
128	1.632947e-5	2.05	9.526	1.631710e-5	2.05	8.397
256	4.018497e-6	2.02	38.435	4.007189e-6	2.02	32.336
512	9.960398e-7	2.01	154.430	9.841026e-7	2.02	124.945
1024	3.567405e-7	1.48	615.768	3.723903e-7	1.40	491.206

Multilevel Augmentation schemes when $k = 3$							
m	$f(n)$	full multilevel solution			compressed multilevel solution		
		e'_h	R'_h	CT	\tilde{e}'_h	\tilde{R}'_h	\widetilde{CT}
0	16	1.339395e-3		0.157	1.339395e-3		0.156
1	32	2.411157e-4	2.47	0.398	2.411149e-4	2.47	0.382
2	64	6.724324e-5	1.84	1.016	6.723707e-5	1.84	0.789
3	128	1.531932e-5	2.13	2.992	1.530542e-5	2.13	1.773
4	256	3.953759e-6	1.95	10.836	3.942501e-6	1.96	4.580
5	512	9.808793e-7	2.01	46.077	9.689500e-7	2.02	17.473
6	1024	3.797881e-7	1.37	215.179	3.738184e-7	1.37	90.798

For Example IV.3.6 and IV.3.7, we see that the computing time of the compressed multilevel augmentation solutions is less than one-fifth of the computing time of the full collocation solutions when solving the nonlinear equation with weakly singular kernel. Therefore, the multilevel augmentation method is the fastest numerical algorithm.

CHAPTER V

CONCLUSIONS AND FUTURE WORK

In this dissertation, two separately major topics concerning the nonlinear Hammerstein equation have been investigated. First, we discussed several acceleration techniques based upon the interpolation and extrapolation of the numerical solution of the Hammerstein equation arising out of the projection methods; collocation and Galerkin. Numerical examples confirm the validity of the acceleration techniques. Most of the acceleration techniques reported in this dissertation provide a means to obtain more accurate approximation to the solution of a nonlinear equation without increasing the size of nonlinear system. A second topic which we discussed in this dissertation is the solution process for the nonlinear Hammerstein equation based on the linearization technique along with a class of multiscale wavelets bases. This led us to fast wavelet-collocation method of Chapter IV. The fast wavelet-collocation method is based upon the block truncation strategy and it was explored in concert with the multilevel augmentation method.

V.1 CONCLUSIONS

TABLE 24 and TABLE 25 recapitulate the superconvergence results in the post-acceleration techniques described in Chapter III when the technique is applied to a collocation solution and a Galerkin solution, respectively, when the kernel is smooth. We recall that the last item *Extrapolation Scheme* corresponds to the global extrapolation or the extrapolation of iterated projection solutions.

TABLE 24: Summary of the collocation techniques.

Numerical Implementation	Numerical Approximation Error
Collocation Scheme	$\ u^h - \bar{u}\ _{0,\infty} = O(h^{r+1})$
Iterative Scheme	$\ u_{it}^h - \bar{u}\ _{0,\infty} = O(h^{2r+2})$
Interpolation Scheme	$\ I_{2h}^{2r+1}u^h - \bar{u}\ _{0,\infty} = O(h^{2r+2})$
Extrapolation Scheme	$\ I_{2h}^{2r+3}u^h - \bar{u}\ _{0,\infty} = O(h^{2r+4})$ or $\ \bar{u}_{it}^{h/2} - \bar{u}\ _{0,\infty}$

Note that \bar{u} is an isolated solution of the Hammerstein equation.

TABLE 25: Summary of the Galerkin techniques.

Numerical Implementation	Numerical Approximation Error
Galerkin Scheme	$\ u^h - \bar{u}\ _{0,2} = O(h^{r+1})$
Iterative Scheme	$\ u_{it}^h - \bar{u}\ _{0,2} = O(h^{2r+2})$
Interpolation Scheme	$\ I_{2h}^{r+1}u^h - \bar{u}\ _{0,2} = O(h^{r+2})$
Extrapolation Scheme	$\ \bar{u}_{it}^{h/2} - \bar{u}\ _{0,2} = O(h^{2r+4})$

In addition, an acceleration technique based upon the interpolation of weakly singular Hammerstein equations has also been presented, see numerical results in Example III.2.2 and Example III.2.3. Furthermore, the post-processed interpolation techniques was extended to two-dimensional nonlinear integral equations. Similar numerical improvement are also found in this case. For more details, see Example III.2.4. This brings a number of interesting and important issues of post-processing technique via interpolation for the multi-dimensional integral equation. These issues will be addressed in the future. Brief comparisons of the post-processing techniques discussed in Chapter III are given below.

First, we found that the iterative method has a higher computational complexity than the interpolation method due primarily to the fact that the iterative process requires the calculation of the solution under a nonlinear integral term. This must be done numerically utilizing efficient quadratures. For an equation with weakly singular kernel, a class of graded meshes generated by the singularity of the kernel has to be used to approximate the nonlinear integral and this increases greatly the complexity of the calculation involved in the iterative methods.

For methods based upon interpolation, post-processing techniques are designed to obtain an approximation by interpolating the existing numerical solution over two successive intervals. This method therefore needs only the addition and multiplication of simple closed forms and thus there is no need for numerical quadratures. Therefore, even though the interpolation and iterated post-processing methods have the same convergence rate, the interpolation technique gives simpler calculation than the iterated technique to attain the same accuracy in approximation. Numerical results for the collocation and Galerkin solutions based upon the interpolation are shown in Example III.2.1 and Example III.5.1, respectively.

Finally, we studied the post-processing technique based upon the extrapolation of the projection solution. This approach requires an additional calculation performed

on the iterative or on the interpolation solutions. The cost of the additional calculation is minimal. The global extrapolation requires first an interpolation of the projection solution over three consecutive intervals. Then a post-processed solution is obtained by using the Richardson extrapolation scheme. This extrapolation scheme has been derived from Theorem III.3.1. An extrapolation of the iterated solution also requires Richardson extrapolation as established from Theorem III.6.1. Computational complexities of completing various extrapolation schemes were found to be dependent on the difficulty of either interpolation or iteration technique associated with the extrapolation. In summary, Table 24, Table 25 and the numerical results in Chapter III (see Example III.3.2, III.4.1 and III.6.3) reveal that the extrapolation of the projection solutions gives the highest order of accuracy.

In the second half of this dissertation, we addressed the issue pertaining to an efficient solution process of solving numerically for a solution of nonlinear Hammerstein equation. When the numerical accuracy must be enhanced, one is required to use approximate subspaces of higher dimensions. Thus, it demands a significantly large amount of computational effort. Therefore, in order to reduce the computational cost of both computing time and computer memory, we established fast algorithms using wavelet bases. The multiscale piecewise polynomial wavelets and the corresponding collocation functionals were reviewed in Chapter II. The fast wavelet-collocation method was implemented to solve linear as well as nonlinear integral equations. As for the linear equation with weakly singular kernel, the sparsity of the corresponding matrix occurs when using the block truncation strategy with truncation parameters defined by Theorem IV.1.4. The numerical results show that the collocation method and the fast wavelet-collocation method have the same optimal rate of convergence. Here we also discovered that computing time varies according to the size of parameters a and ν in Theorem IV.1.4 (see Example IV.1.5). Example IV.1.7 shows that the fast wavelet-collocation method is much quicker to execute than the traditional collocation method.

In Chapter IV, the wavelet-collocation method for the Hammerstein equation was established which uses a linearization technique. The Newton method is used to solve the system of nonlinear equations. A block truncation strategy was used in building the Jacobian matrix at each iteration of the Newton method. This gives each Jacobian matrix a sparse structure which results in a fast wavelet-collocation algorithm. Examples IV.2.3 - IV.2.6 exhibit the effectiveness of the algorithm and

FIG. 16 shows the structure of this sparsity. When the corresponding nonlinear system is much larger, it was shown that the multilevel augmentation scheme reduces the problem to a sequence of calculations with smaller scales. In other words, the multilevel augmentation technique is to solve the nonlinear system at a lower dimension with fixed Jacobian matrix and correct the solution by adding a corrected term from a higher resolution level. This method leads us to a faster numerical technique while still preserving the order of convergence of the approximation. Uniqueness and existence of the solution of the wavelet-collocation method and the multilevel augmentation method are stated in Theorem IV.2.1 and Lemma IV.3.1, respectively. Also, the convergence of the the wavelet-collocation method and the multilevel augmentation method are proved in Theorem IV.2.5 and Theorem IV.3.3, respectively. In numerical experiments reported in Examples IV.3.4 - IV.3.7, we see that the proposed fast multilevel augmentation method gives the fastest computing time when compared with the traditional collocation method and the multilevel augmentation method without the compression strategy.

All numerical experiments in this dissertation were done with the Matlab program and they were run on a personal computer with 2.0GHz CPU and 4 GB memory.

V.2 FUTURE WORK

We list below several interesting research topics which arose while the author was engaged in this dissertation work. These topics will be pursued in the future.

1. We intend to obtain global extrapolation results for nonlinear Hammerstien integral equation with weakly singular kernel.
2. We intend to implement the post-processing techniques to solve the multi-dimensional Hammerstein equation.
3. We intend to extend the wavelet-collocation method, based on the linearization technique, to multi-dimensional Hammerstein equations.
4. We intend to study how to implement the post-processing techniques on wavelet collocation solutions.
5. We intend to apply multiscale wavelets to solving nonlinear integral equations by using other numerical schemes such as the degenerate kernel method and the Nyström method.

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APPENDIX A

NUMERICAL SCHEMES

A.1 GAUSSIAN QUADRATURE

A Gaussian quadrature approximates the definite integral of a function.

In particular, the integral of the form $\int_{-1}^1 f(x)dx$, can be approximated efficiently with the weight function $W(x) = 1$ and using the Legendre Gaussian quadrature.

An n -point Legendre Gaussian quadrature formula is obtained as

$$\int_{-1}^1 f(x)dx = \sum_{i=1}^n w_i f(x_i) + \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi), \quad \xi \in (a, b),$$

when the abscissas or zeros of Legendre polynomials x_i and weight factors w_i are listed below.

TABLE 26: Lagrange Quadrature coefficients for $n = 4$ to 8.

n	$\pm x_i$	w_i
4	0.339981043584856	0.652145154862546
	0.861136311594053	0.347854845137454
5	0.000000000000000	0.568888888888889
	0.538469310105683	0.478628670499366
	0.906179845938664	0.236926885056189
6	0.238619186083197	0.467913934572691
	0.661209386466265	0.360761573048139
	0.932469514203152	0.171324492379170
7	0.000000000000000	0.417959183673469
	0.405845151377397	0.381830050505119
	0.741531185599394	0.279705391489277
	0.949107912342759	0.129484966168870
8	0.183434642495650	0.362683783378362
	0.525532409916329	0.313706645877887
	0.796666477413627	0.222381034453374
	0.960289856497536	0.101228536290376

An integral over an arbitrary interval $[a, b]$ can be computed by Gaussian quadrature by simply transforming the interval $[a, b]$ into $[-1, 1]$. More specifically,

$$\int_a^b f(x)dx = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) dx,$$

and thus

$$\int_a^b f(x)dx = \frac{b-a}{2} \sum_{i=1}^n w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right) + E_n,$$

with the error term

$$E_n = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi), \quad \xi \in (a, b).$$

Next, to apply Gaussian quadrature to double integral written in the form

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx$$

first requires translating, for each x in $[a, b]$, the interval $[c(x), d(x)]$ to $[-1, 1]$ and then applying Gaussian quadrature. This results in the formula

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx \approx \int_a^b \frac{d(x) - c(x)}{2} \sum_{j=1}^n w_j f\left(x, \frac{(d(x) - c(x))x_j + d(x) + c(x)}{2}\right) dx, \quad (117)$$

where, as before, the abscissas x_j and coefficients w_j come from TABLE 26. Now the interval $[a, b]$ is translated to $[-1, 1]$ and Gaussian quadrature is applied to approximate the integral on the right side of equation (117).

A.2 LAGRANGE INTERPOLATION

The Lagrange interpolation polynomial $P_{n-1}(x)$ is the polynomial of degree $n - 1$ that passes through n prescribed points. Assuming that $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ are given distinct points, there exists the unique polynomial P_{n-1} which satisfies

$$P_{n-1}(x_j) = f(x_j), \quad \forall j = 1, 2, \dots, n.$$

The formula is given by

$$P_{n-1}(x) = \sum_{j=1}^n y_j L_j(x)$$

where

$$L_j(x) = \prod_{\substack{k=1 \\ k \neq j}}^n \frac{x - x_k}{x_j - x_k}$$

Note that $L_j(x)$ has the property that

$$L_j(x_k) = \delta_{jk} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}, \quad \forall k = 1, \dots, n.$$

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